Journal of the Mathematical Society of Japan Vol. 2 Nos. 1-2, Sept., 1950.

On Betti Numbers of Riemannian Spaces.

Yasuro TOMONAGA.

(Received June. 15, 1949)

1. Recently H. Iwamoto [1] has proved the following

Theorem. *Let B_P be the p-th Betti number of an orientable compact positive definite Riemannian space and let B'_P be the maximum number of linearly independent skew-symmetric tensors of the degree p whose covariant derivatives vanish. Then we have a relation*

$$B_P \geqq B'_P.$$

We shall remark first that the proof of this theorem by H. Iwamoto, depending on the theorem of de Rham [2] may be simplified if we use the following theorem of Hodge [3].

The p-th Betti number of an orientable compact positive definite Riemannian space is equal to the maximum number of linearly independent harmonic tensors of the degree p. A tensor $\xi_{a_1 \cdots a_p}$ is said to be harmonic if (1) it is skew-symmetric, and (2) it satisfies the conditions

$$\text{(A)} \qquad \xi_{a_1 \cdots a_p ; r} = \sum_{q=1}^{p} \xi_{a_1 \cdots a_{q-1} r a_{q+1} \cdots a_p ; a_q}$$

and

$$\text{(B)} \qquad \xi_{a_1 \cdots a_p ; r} g^{a_p r} = 0,$$

where the semi-colon denotes the covariant derivative with regard to the Christoffel symbols.

By this Hodge's therem, we can prove the above theorem as follows. Let $\xi_{a_1 \cdots a_p}$ be a skew-symmetric tensor whose covariant derivative vanishes. Then $\xi_{a_1 \ldots a_p}$ satisfies evidently the conditions (A) and (B).

Hence it becomes a harmonic tensor. If there exist two linearly independent skew-symmetric tensors $\xi_{a_1 \ldots a_p}$ and $\eta_{a_1 \cdots a_p}$ whose covariant derivatives vanish, then their linear combinations are skew-symmetric and their covariant derivatives vanish also. Hence they are also harmonic. Thus we have

$$B_d \geqq B'_p. \qquad Q. E. D.$$

Let us now examine the case in which the equality $B_p = B'_p$ occurs. We shall prove that the equallity holds under some restriction in the case of the symmetric space of E. Cartan.

Theorem 1. *Let R_n be an orientable compact positive definite Riemannian space. If R_n has the properties:*

(1) $R_{ijkl;h}=0$; *(spmmetric space!)*

(2) *the quadratic form*

$$\{P(P-1)R_{ijkl}g_{st}+g_{kl}(2pR_{ijts}-pR_{ij}g_{st}-R_{st}g_{ij})\}\xi^{lis}\xi^{kjt}$$

with respect to ξ is negative definite, where we assume

$$\xi^{ijk}=-\xi^{jik}$$

then the covariant derivative of any harmonic tensor of the degree p vanishes, that is to say

$$B_p=B'_p.$$

Especially in the case $p=1$, the condition (2) becomes

(2′) *the quadratic form*

$$(2R_{ijts}-R_{ij}g_{st}-R_{st}g_{ij})\xi^{is}\xi^{jt}$$

is negative definite, where we assume

$$\xi^{ij}=\xi^{ji} \text{ and } \xi^{ij}\neq\rho g^{ij}.$$

Proof. Let $\xi_{a_1\ldots a_p}$ be any harmonic tensor, put

$$\varphi=\xi_{a_1\ldots a_p;r}\xi^{a_1\ldots a_p;r} \tag{1·1}$$

where

$$\xi^{a_1\ldots a_p;r}=g^{a_1b_1}\ldots g^{a_pb_p}g^{rs}\xi_{b_1\ldots b_p;s} \tag{1·2}$$

and consider a scalar defined by

$$\Delta=g^{ab}\varphi_{,a;b}. \tag{1·3}$$

By Green's theorem [4], we have

$$\int_{Rn} \Delta \mathrm{dv}=0, \tag{1.4}$$

where $\int dv$ denotes the volume integral over the whole space. On the other hand, we have

$$\Delta=2g^{bc}\xi_{a_1\ldots a_p;r;b;c}\,\xi^{a_1\ldots a_p;r}+2\xi_{a_1\ldots a_p;r;b}\,\xi^{a_1\ldots a_p;r;b} \tag{1.5}$$

To calculate the first term in the second member, putting

$$\Phi=g^{bc}\xi_{a_2\ldots a_p;r;b;c}\,\xi^{a_1\ldots a_p;r}$$

we have

$$\Phi=(\xi_{a_1\ldots a_p;b;r}-\sum_{S=1}^{p}R^{m}{}_{\cdot a_s r}\,\xi_{a_1\ldots \underset{(s)}{m}\ldots a_p})_{;c}\,g^{bc}\,\xi^{a_1\ldots a_p;r} \tag{1.6}$$

$$=(\xi_{a_1\ldots a_p;b;r;c}-\sum_{s=1}^{p}R^{m}{}_{\cdot a_s rb;c}\,\xi_{a_1\ldots \underset{(s)}{m}\ldots a_p}-\sum_{s=1}^{p}R^{m}{}_{\cdot a_s rb}\,\xi_{a_1\ldots \underset{(s)}{m}\ldots a_p;c})g^{bc}\,\xi^{a_1\ldots a_p;r}$$

from which, by virtue of the relation

$$R_{ijkl;h}=0,$$

we have

$$\Phi=(\xi_{a_1\ldots a_p;b;r;c}-\sum_{s=1}^{p}R^{m}{}_{\cdot a_s rb}\,\xi_{a_1\ldots \underset{(s)}{m}\ldots a_p;c})g^{bc}\,\xi^{a_1\ldots a_p;r}$$

$$=(\xi_{a_1\ldots a_p;b;c;r}-\sum_{s=1}^{p}R^{m}{}_{\cdot a_s rc}\,\xi_{a_1\ldots \underset{(s)}{m}\ldots a_p;b}-R^{m}{}_{\cdot brc}\,\xi_{a_1\ldots a_p;m}$$

$$-\sum_{s=1}^{p}R^{m}{}_{\cdot a_s rb}\,\xi_{a_1\ldots \underset{(s)}{m}\ldots a_p;c})g^{bc}\,\xi^{a_1\ldots a_p;r}. \tag{1.7}$$

From the conditions for the harmonic tensor, we obtain

$$\Phi=(\sum_{s=1}^{p}\xi_{a_1\ldots \underset{(s)}{b}\ldots a_p;a_s;c;r}-\sum_{s=1}^{p}R^{m}{}_{\cdot a_s rc}\,\xi_{a_1\ldots \underset{(s)}{m}\ldots a_p;b} \tag{1.8}$$

$$-R^{m}{}_{\cdot brc}\,\xi_{a_1\ldots a_p;m}-\sum_{s=1}^{p}R^{m}{}_{\cdot a_s rb}\,\xi_{a_1\ldots \underset{(s)}{m}\ldots a_p;c})g^{bc}\,\xi^{a_1\ldots a_p;r}$$

$$=\Big\{\sum_{s=1}^{p}\xi_{a_1\ldots \underset{(s)}{b} \ldots a_p;c;a_s;r}-\Big(\sum_{s=1}^{p}R^{m}{}_{\bullet a_t a_s c}\,\xi_{a_1\ldots \underset{(t)}{m}\ldots \underset{(s)}{b}\ldots a_p}+R^{m}{}_{\bullet b a_s c}\,\xi_{a_1\ldots m\ldots a_p}\Big)_{;r}$$

$$-\sum_{s=1}^{p}R^{m}{}_{\bullet a_s r c}\,\xi_{a_1\ldots \underset{(s)}{m}\ldots a_p;b}-R^{m}{}_{\bullet brc}\,\xi_{a_1\ldots a_p;m}-\sum_{s=1}^{p}R^{m}{}_{\bullet a_s r b}\,\xi_{a_1\ldots \underset{(s)}{m}\ldots a_p;c}\Big\}$$

$$\times g^{bc}\xi^{a_1\ldots a_p;r}$$

$$\Phi=-\Big\{-p(p-1)\,R^{m}{}_{\bullet a_p a_{p-1}c}\,\xi_{a_1\ldots a_{p-2}bm;r}+pR^{m}{}_{\bullet b a_p c}\,\xi_{a_1\ldots a_{p-1}m;r}$$

$$+2pR^{m}{}_{\bullet a_p rb}\,\xi_{a_1\ldots a_{p-1}m;c}+R^{m}{}_{\bullet brc}\,\xi_{a_1\ldots a_p;m}\Big\}g^{bc}\xi^{a_1\ldots a_p;r}$$

$$=-\Big\{p(p-1)\,R_{mjib}g_{kl}-pR_{mj}g_{bi}\,g_{kl}+2pR_{mjlk}g_{bi}-R_{kl}\,g_{bi}\,g_{mj}\Big\}$$

$$\times\xi_{a_1\ldots a_{p-2}}{}^{bm;k}\,\xi^{a_1\ldots a_{p-2}ij;l} \tag{1·9}$$

There exists always a coordinate system in which the metric tensor δ_{ij} takes the values δ_{ij} at a specified point. Then the quantity

$$\xi_{a_1\ldots a_{p-2}}{}^{bm;k}\,\xi^{a_1\ldots a_{p-2}ij;l}$$

takes the form

$$\sum_{a_1'\ldots a'_{p-2}}\xi_{a_1'\ldots a'_{p-2}\bullet\bullet}{}^{l'm';k'}\,\xi_{a_1'\ldots a'_{p-2}\bullet\bullet}{}^{i'j';l'}$$

Hence, if the quadratic form of ξ

$$\{p(p-1)R_{mjib}\,g_{kl}+g_{bi}(2pR_{mjlk}-pR_{mj}g_{kl}-R_{kl}g_{mj})\}\xi^{bmk}\xi^{ijl} \tag{1·10}$$

with the relation

$$\xi^{ijl}=-\xi^{jil}$$

is negative definite, then Φ becomes positive unless

$$\xi_{a_1\ldots a_p;r}=0.$$

On the other hand, from (1·5) we have

$$\Delta=2\Phi+2\xi_{a_1\ldots a_p;r;b}\,\xi^{a_1\ldots a_p;r;b} \tag{11·1}$$

Therefore, if $\xi_{a_1 \ldots a_p ; r}$ is not identically zero, then the scalar Δ is not negative and is positive at some point of the space. Hence we must have

$$\int_{Rn} \Delta \, dv > 0$$

But this inequality contradicts to (1·4), Hence it follows that

$$\xi_{a_1 \ldots a_p . r} = 0. \qquad \textit{Q. E. D.}$$

2. From Hodge's theorem, we get easily the following

Theorem 2. *Let R_n be an orientable compact positive definite Riemannian space. If R_n admits m linearly independent parallel vectors, then it follows that*

$$B^1 \geqq m,$$
$$B_{m'} \geqq 1, \quad (m' \leqq m)$$
$$B_{2k} \geqq 1, \quad \left(k=1, 2, \ldots, \frac{n}{2}-1\right), \quad (n \text{ even}).$$

Proof. By the assumption, there exist m linearly independent parallel vectors

$$\underset{(1)}{v^i}, \underset{(2)}{v^i}, \ldots\ldots, \underset{(m)}{v^i}$$

Putting

$$\underset{(m')}{v_i} = g_{ij} \underset{(m')}{v^j} \quad (m' = 1, 2 \ldots\ldots, m)$$

we have

$$\underset{(m')}{v_{i;j}} = 0$$

Therefore, these covariant vectors are harmonic. Hence, it follows from Hodge's theorem that

$$B_1 \geqq m.$$

Next, we consider an m'-vector such as

$$|\underset{(1)}{v_i}\ \underset{(2)}{v_i}\ \cdots\ \underset{(m')}{v_i}| \qquad (2\leqq m'\leqq m).$$

It is not identically zero and its covariant derivative vanishes. Therefore they are harmonic tensors of the degree m'. Hence it follows that

$$B_{m'}\geqq 1. \qquad (m'\leqq m)$$

Next, we construct a skew-symmetric tensor such that

$$[H_{i_1 i_2}\ H_{i_3 i_4}\cdots\cdots H_{i_{2k-1}i_{2k}}] \qquad (1\leqq k\leqq \frac{n}{2}-1),$$

where

$$H_{ij}=|\underset{1}{v_i}\,\underset{2}{v_j}-\underset{1}{v_j}\,\underset{2}{v_i}|.$$

Evidently its covariant derivative vanishes.

Hence it is a harmonic tensor of the degree $2k$, unless it is identically zero. Then, if the dimension of the space is even, it follows tat

$$B_{2k}\geqq 1\ \ (k=1,2,\ldots,\frac{n}{2}-1). \qquad \text{Q. E. D.}$$

3. Let R_n be an orientable compact positive definite Riemannian space. If our R_n admits a tensor $V^i_{\cdot jk}$ whose covariant derivative vanishes, i. e.

$$V^i_{\cdot jk;l}=0, \tag{3·1}$$

then we can construct the following skew-symmetric tensors

$$V^i_{\cdot ik},\ \delta\binom{k_1\,k_2\,k_3}{j_1\,j_2\,j_3}V^{i_1}_{\cdot i_2 k_1}V^{i_2}_{\cdot i_3 k_1}V^{i_3}_{\cdot i_1 k_3},\cdots\cdots,$$

$$\delta\binom{k_1\cdots k_m}{j_1\cdots j_m}V^{i_1}_{\cdot i_2 k_1}V^{i_2}_{\cdot i_3 k_2}\cdots\cdots V^{i_m}_{\cdot i_1 k_m} \quad (m<n),$$

where the symbol

$$\delta\binom{k_1\ \ldots\ k_m}{i_1\ \ldots\ i_m}$$

is equal to 1 or -1 according as $k_1\ldots k_m$ constitutes an even or odd permutation of $i_1\ldots i_m$ and is otherwise zero. Form (3·1) it follows that these tensors are harmonic. Hence, if the tensor of the degree p of (3·2) is not identically zero, then the p-th Betti number of R_n is not zero. Generally, if there exists a tensor $X^{i}_{\cdot jk_1\dots k_p}$ such that

$$\left.\begin{array}{ll}\text{(a)} & X^{i}_{\cdot jk_1\dots k_p} \text{ is not symmetric with respect to any} \\ & \text{pair of the indices } k_1\dots k_p, \\ \text{(b)} & X^{i}_{jk_1\dots k_p;\, r}=0,\end{array}\right\} \qquad (3\cdot3)$$

then we can construct the following tensors,

$$X^{i}_{\cdot ik_1\dots k_p}, \dots \delta\binom{k_1\dots k_{mp}}{j_1\dots j_{mp}} X^{i_1}_{\cdot i_2k_1\dots k_p} X^{i_2}_{\cdot\, i_3k_{p+1}\dots k_{2p}} \cdots$$
$$\times X^{i_m}_{\cdot i_1k_{mp-p+1}\dots k_{mp}}. \quad (mp<n) \qquad (3\cdot4)$$

These tensors are skew-symmetric and their covariant derivatives vanish. Therefore they are harmonic.

Hence we see that* if the tensor of the degree mp of (3·4) is not identically zero, then the mp-th Betti number of R_n is not zero.

Example. If our R_n is symmetric, i. e.

$$R^{i}_{\cdot jkl;\,h}=0, \qquad (3\cdot5)$$

then we can construct the following harmonic tensors,

$$\delta\binom{k_1\dots k_4}{j_1\dots j_4} R^{i_1}_{\cdot i_2k_1k_2} R^{i_2}_{\;\; i_1k_3k_4}, \dots\dots,$$
$$\delta\binom{k_1\dots k_{2m}}{j_1\dots j_{2m}} R^{i_1}_{\cdot i_2k_1k_2} R^{i_2}_{\cdot i_2k_1k_2} \dots\dots R^{i_m}_{\cdot i_1k_{2m}k_{2m}} \quad (2m<n). \qquad (3\cdot6)$$

Hence we have the

Theorem 3. *The m-th Betti number* $\left(2\leqq m<\frac{n}{2}\right)$ *of an orientable compact positive definite symmetric Riemannian space is not zeso.*

4. Moreover, if there exists a tensor $W_{k_1\dots k_p}$ with the properties:

---

* When $m=2$ and p is odd, the tensor of (3. 4) becomes identically zero.

$$\left.\begin{array}{ll}\text{(a)} & W_{k_1\ldots k_p;r}=0\\ \text{(b)} & p \text{ is even,}\\ \text{(c)} & W_{k_1\ldots k_p} \text{ is not symmetric with respect to any pair}\\ & \text{of the indices,}\end{array}\right\} \tag{4·1}$$

then we can construct the following harmonic tensors

$$\delta\binom{k_1\ldots k_{mp}}{j_1\ldots j_{mp}} W_{k_1\ldots k_p} \ldots W_{k_{mp-p+1}\ldots k_{mp}} (mp<n). \tag{4·2}$$

Therefore, if the tensor of degree mp of (4·2) is not identically zero, then the mp-th Betti number is not zero. Further, if there exist many tensors satisfying (4·1) or (3·3) and being linearly independent to each other, then their products of the form

$$\delta\binom{k_1\ldots k_{p+q}}{j_1\ldots j_{p+q}} R^{\,i_1}_{\cdot i_2 k_1\ldots k_p} S^{\,i_2}_{\cdot i_1 k_{p+1}\ldots k_{p+q}} \tag{4·4}$$

or

$$\delta\binom{k_1\ldots k_{m+l}}{j_1\ldots j_{m+l}} P_{k_1\ldots k_l} Q_{k_{l+1}\ldots k_{m+l}}$$

become also harmonic.

5. Previously, T. Y. Thomas [4] treated a tensor equation of the form

$$T_{a_1\ldots a_p;r;s}\, g^{rs} = c\,T_{a_1\ldots a_p}, \tag{5·1}$$

which, for the sake of brevity, we write as

$$\Delta T = cT, \tag{5·2}$$

where c is a constant. We shall now generalize above equation as follows:

$$\left.\begin{array}{ll}\text{(a)} & T_{a_1\ldots a_p;r;s}\,g^{rs} = K^{\cdots\cdots b_1\ldots b_p}_{a_1\ldots a_p}\, T_{b_1\ldots b_p},\\ \text{(b)} & T_{a_1\ldots a_p;r;s}\,g^{rs} = K^{\cdots\cdots b_1\ldots b_p}_{a_1\ldots a_p}\, T_{b_1\ldots b_p} + L_{a_1\ldots a_p},\end{array}\right\} \tag{5·3}$$

where K and L are given tensors. For the sake of brevity, we write (6·3) as follows :

$$\left.\begin{aligned}&\text{(a)}\quad \varDelta T=K\cdot T,\\ &\text{(b)}\quad \varDelta T=K\cdot T+L,\end{aligned}\right\} \qquad (5\cdot 4)$$

If (5·4) (a) has a solution T, then we have

$$\varDelta(T\cdot T)=(T_{a_1\ldots a_p}T^{a_1\ldots a_p})_{;r;s}g^{rs}=2\,(T_{a_1\ldots a_p;r;s}\,T^{a_1\ldots a_p}g^{rs})$$

$$+2T_{a_1\ldots a_p;r}T^{a_1\ldots a_p;r}=2(\varDelta T\cdot T)+2(\delta T\cdot\delta T),$$

where we have put

$$T^{a_1\ldots a_p}=g^{a_1b_1}\ldots\ldots g^{a_pb_p}\,T_{b_1\ldots b_p}$$

$$T^{a_1\ldots a_p;r}=g^{rs}\,T^{a_1\ldots a_p}{}_{;s}$$

$$\delta T=T_{a_1\ldots a_p;r}$$

By Green's theorem [5], we have

$$0=\int_{R_n}\varDelta(T\cdot T)dv=2\int_{R_n}(\varDelta T\cdot T)dv+2\int_{R_n}(\delta T\cdot\delta T)\,dv, \qquad (5\cdot 6)$$

where dv denotes the volume element. From (5·4) (a) and (5·6), we have

$$0=2\int_{R_n}(K\cdot T\cdot T)\,dv+2\int_{R_n}(\delta T\cdot\delta T)\,dv. \qquad (5\cdot 7)$$

As R_n is positive definite, the second term in the second member of (5·7) is not negative. Therefore, if the quadratic form K·T·T is positive definite, then the first term in the second member of (5·7) becomes positive unless $T\equiv 0$. Hence, if K·T·T is positive definite, the solution T must vanish identically.

Next, if K·T·T is everywhere positive semi-definite, then the second member of (5·7) must vanish. In this case we have

$$\delta T=0, \qquad (5\cdot 8)$$

that is to say

$$T_{a_1\ldots a_p;r}=0,$$

Moreover, from (5·4) (a) and (5·8), we have

$$K\cdot T=0. \tag{5·9}$$

Next, if (5·4) (b) has two solutions T and U, then their difference T—U must satisfy (5·4) (a). Then, if (5·4) (a) has no solution other than zero, the equation (5·4) (b) has not two solutions. Hence we have the

Theorem 4. *If the quadratic form $K\cdot T\cdot T$ is positive definite at every point of R_n, then the equation*

(A) $$\Delta T=K\cdot T$$

has no solution other than zero. In this case the equation

(B) $$\Delta T=K\cdot T+L$$

cannot have two solutions. If $K\cdot T\cdot T$ is everywhere positive semi-definite, the solution of the equation (A) must satisfy the following relations

$$\delta T=0 \text{ and } K\cdot T=0.$$

6. Above theorem has many interesting applications.

One of them is the problem of the infinitesimal collineation. An infinitesimal point transformation which carries any geodesic into a geodesic is called infinitesimal affine collineation, provided that the change of the parameter is linear. Let

$$\bar{x}^i=x^i+\varepsilon\xi^i(x)$$

be an infinitesimal affine collineation. Then the covariant components of the vector ξ^i must satisfy

$$\xi_{i;j;k}+R_{ijka}\xi^a=0. \quad \text{(K. Yano and Y. Tomonaga [6])} \tag{6·1}$$

Let ξ_i be a solution of (6·1). Then we have

$$\Delta\xi=\xi_{i;j;k}g^{jk}=-R_{ijka}g^{jk}\xi^a=-R_{ij}\xi^j.$$

We see that if $R_{ij}\xi^i\xi^j$ is negative definite at every point of R_n, then the solution of (6·1) must vanish identically and if $R_{ij}\xi^i\xi^j$ is everywhere negative semi-definite, then it follows that

$$\xi_{i;j}=0,$$

that is to say ξ^i is absolutely parallel. Hence we have the

Theorem 5. *If the quadratic form $R_{ij}\xi^i\xi^j$ is everywhere negative definite, then R_n admits no infinitesimal affine collineation of the class C''. If $R_{ij}\xi^i\xi^j$ is everywhere negative semi-definite, then the vector of the transformation, if it exists, is absolutely parallel.* Especially in the case of the infinitesimal motion we have the

Theorem. (S. Bochner. [7])

If $R_{ij}\xi^i\xi^j$ is everywhere negative definite, then R_n admits no infinitesimal motion.

7. Harmonic tensors.

A skew-symmetric tensor $\xi_{a_1 \ldots a_p}$ is a hrmonic tensor, if it satisfies the conditions

$$\left.\begin{aligned} &\text{(a)}\quad \xi_{a_1\ldots a_p;r}=\sum_{m=1}^{p}\xi_{a\ldots \underset{(m)}{r}\ldots a_p;a_m},\\ &\text{(b)}\quad \xi_{a_1\ldots a_p;s}\,g^{a_p s}=0. \end{aligned}\right\} \qquad (7\cdot1)$$

Then $\xi_{a_1\ldots a_p}$ satisfies the equation of the form

$$\varDelta\xi=K\cdot\xi, \qquad (6\cdot2)$$

where K is a complicated tensor. In this case we have

$$K\cdot\xi\cdot\xi=(pg_{a_1b_2}R_{a_1b_1}-p(p-1)R_{a_1b_2b_1a_2})\xi^{a_1a_2a_2a_p}\,\xi^{b_1b_2}_{\ldots a_2\ldots a_p} \qquad (7\cdot3)$$

Therefore, *if $K\cdot\xi\cdot\xi$ is everywhere positive definite, then it follows from Hodge's theorem* [3] *that the p-th Betti number of R_n is zero.* This fact was discovered by S. Bochner [7]. Moreover, if $K\cdot\xi\cdot\xi$ is everywhere positive semi-definite, then there exist follcwing two cases:

1. The p-th Betti number B_p is zero.
2. $B_p\neq0$. In this case there exists at least one harmonic tensor, say ξ. Then we have

$$\delta\xi=0 \text{ and } K\cdot\xi=0.$$

Hence, there exists at least one skew-symmetric tensor whose covariant derivative vanishes.

Literatures.

[1]. H. Iwamoto. The topology of Riemann spaces and the exact differential forms which are invariant under the groups of holonomy. (In Japanese.) Holonomy Gun Kenkyû, No. 7, 1948.

[2] de Rham. Sur l'analysis situs des variétes á n dimensions J. Math. Pures. Appl., Vol. 10, 1930, pp. 115–200.

[3] W.V.D. Hodge. The theory and applications of harmonic integrals, Camb. Univ. Press, 1941.

[4] T. Y. Thomas. Some simple applications of Green's theorem for compact Riemann space, Tohoku Math. J., Vol. 46, 1940, pp. 261–266.

[5] S. Bochner, Remark on the theorem of Green, Duke Math. J., 1937, pp. 334–338.

[6] K. Yano, K. Takano and Y. Tomonaga. On the infinitesimsal deformations of curves in the space with linear connection. Jap. J. of Math., Vol. 19, 1948, pp. 433–477.

[7] S. Bochner. Curvature and Betti numbers, Ann. of Math., 1948, pp. 379–385.