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Riemann Spaces of Class Two and their Algebraic Characterization.

Part I.

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We shall investigate in this paper a necessary and sufficient condition that an n-dimensional Riemann space $R_n(n\geqq 6)$ be of class two. Let the line element of R_n be a positive definite quadratic form

$$ds^2=g_{ij}dx^i dx^j\,;\quad (i,j,\ldots=1,2,\ldots,n);$$

where $g's$ are analytic functions of $x^1,\ldots,x^n$.

Consider, in an $(n+2)$-dimensional euclidean space E_{n+2}, an n-dimensional variety S_n defined by

$$y^a=\varphi^a(x^1,\ldots x^n)\qquad (a=1,\ldots,\ n+2)\ ;$$

where $y's$ are current coordinates of the point of S_n refered to a rectangular cartesian coordinate system in E_{n+2} and $\varphi's$ are analytic functions of $x^1,\ldots,x^n$. The line element along a curve on S_n is given by

$$ds^2=\sum_a (dy^a)^2=\sum_a B_i^{\cdot a}\ B_j^{\cdot a}\ dx^i dx^j=g_{ij}\ dx^i dx^j\,;$$

where

$$B_i^{\cdot a}=\frac{\partial y^a}{\partial x^i}.$$

Let $B_P^{\cdot a}(P=I,II)$ be the components of two mutually orthogonal unit vectors normal to S_n. The variation of $B_\lambda^{\cdot a}(a=1,\ldots,n+2\,;\ \lambda=1,\ldots n,\ I,\ II)$ along the curve can be written as

$$dB_\lambda^{\cdot a}=H_{\lambda i}^{\sigma}\ B_\sigma^{\cdot a}\ dx^i\quad (i=1,\ldots,\ n\,;\ \sigma,\lambda=1,\ldots,\ n,\ I,II\,;\ a=1,\ldots,n+2).$$

As a condition of integrability of these equations we get immediately that H_{jk}^i $(i,j,k=1,\ldots,n)$ are Christoffel's symbols and H_{ij}^P $(P=I,II\,;\ i,j=1,\ldots,n)$ are symmetric in i and j; and $H_{Qi}^P(P,Q=I,II\,;\ i=1,\ldots,n)$ are skew-symmetric in P and Q; those $H_{\lambda i}^{\sigma}$ satisfy the *Gauss equation*

$$(1)\qquad R_{ijkl}=H_{ik}^{P}\ H_{jl}^{P}-H_{il}^{P}\ H_{jk}^{P},$$

the *Codazzi equation*

(2) $$H^{P}_{ai,j}-H^{P}_{aj,i}=H^{Q}_{ai}\ H^{Q}_{Pj}-H^{Q}_{aj}\ H^{Q}_{Pj},$$

the *Ricci equation*

(3) $$H^{P}_{Qi,j}-H^{P}_{Qj,i}=g^{ab}(H^{Q}_{ai}H^{P}_{bj}-H^{Q}_{aj}H^{P}_{bi}),$$

and finally the equation

$$H_{Pj}^{\ \ i}=-g^{ai}H^{P}_{aj}.$$

In this paper we discuss the type number of a Riemann space $R_n(n\geqq 4)$ of class two. Making use of it, we give, in the forthcoming paper,[1] a necessary and sufficient condition that R_n $(n\geqq 6)$ be of class two.

We restrict oureselves the discussions in a domain of R_n, where g_{ij} are analytic.

§ I. Type number

We put

(1.1) $$L_{ijkl}=H^{I}_{ij}\ H^{II}_{kl}-H^{I}_{il}\ H^{II}_{jk}-H^{II}_{ij}\ H^{I}_{kl}+H^{II}_{il}\ H^{I}_{jk}.$$

If we define K_{ij} as

(1.2) $$K_{ij}=\frac{1}{2}\ g^{ab}\ L_{ajbi},$$

we have from (1.1)

(1.3) $$K_{ij}=g^{ab}(H^{II}_{ai}\ H^{I}_{bj}-H^{II}_{aj}\ H^{I}_{bi})\ ;$$

where K_{ij} is a skew-symmetric tensor. If we put

$$H^{I}_{IIi}=-H^{II}_{Ii}=H_i,$$

the Ricci equation (3) becomes

$$H_{i,j}-H_{j,i}=g^{ab}\ (H^{II}_{ai}\ H^{I}_{bj}-H^{II}_{aj}\ H^{I}_{bi}),$$

accordingly we have from (1.3)

(1.4) $$K_{ij}=H_{i,j}-H_{j,i}.$$

If we differentiate this equation covariantly with respect to x^k and sum three equations obtained by cyclic permutation of i, j and k, we have

(1.5) $$K_{ij,k}+K_{jk,i}+K_{ki,j}=0.$$

We write instead of (1.3)

(1.3′) $$K^{P}_{Q\cdot ij}=g^{cd}(H^{Q}_{ci}\ H^{P}_{dj}-H^{Q}_{cj}\ H_{di}),$$

and then we have immediately

$$K^{P}_{Q\cdot ij}=-K^{Q}_{P ij}=-K^{P}_{Q ji}.$$

When we multiply (1.3′) by H^{Q}_{ak}, sum for Q, and sum up those three equations obtained by cyclic permutaion of i, j and k, we have in consequence of (I)

$$H^{Q}_{a(i}K^{P}_{\ \ |Q|jk)}=H^{P}_{c(i}R_{|a|\cdot jk)}^{\ \ c}. \tag{1.6}$$

If multiplying (1.6) by H^{P}_{bl} and summing for P, we subtract three equations obtained by interchanging l with i, j, and k, we have in consequence of (I) and (1.1)

$$N_{abijkl}=L_{ai|b|(j}K_{kl)}+K_{i(j}L_{|a|k|b|l)}\ ; \tag{1.7}$$

where

$$-N_{abijkl}=R_{cbi(j}R_{|a|\cdot kl)}^{\ \ c}+R_{a\cdot i(j}^{\ \ c}R_{|cb|kl)}. \tag{1.8}$$

Contracting (1.7) by g^{ab} we have in consequence of (1.2)

$$M_{ijkl}=K_{ij}\ K_{kl}+K_{ik}\ K_{lj}+K_{il}\ K_{jk}\ ; \tag{1.9}$$

where

$$M_{ijkl}=-\frac{1}{2}\ g^{ab}\ N_{abijkl}=\frac{1}{2}\ R_{b\cdot i(j}^{\ \ a}\ R_{|a|\cdot kl)}^{\ \ b}. \tag{1.10}$$

The intrinsic tensor M_{ijpl} is skew-symmetric in its every two indices. We have from (1.9) the

Theorem I.I... *A necessary condition that a Riemann space* $R_n(n\geqq 4)$ *be of class two is that there is a skew-symmetric tensor* K_{ij} *which satisfies the algebraic equations* (1.9), *where* M*'s are defined by* (1. 10).

As K_{ij} is skew-symmetric, the rank of matrix $||K_{ij}||$, whose elements are K_{ij}, is even and we shall therefore define the type number of a Riemann space R_n of class two as follows :

Definition :...*A variety* S_n $(n\geqq 4)$ *in a euclidean space* E_{n+2} *will be said to be of type one if the rank of matrix* $||K||$ *is zero or two. It will be said to be of type* τ *if the rank of the above matrix is* 2τ.

We shall now prove that type number of S_n is determined by its intrinsic properties. According to the theory of the skew-symmetric determinant[2] we have

$$(\text{I. II}) \qquad (K_{i(j}K_{kl)})^2 = \begin{vmatrix} 0 & K_{ij} & K_{ik} & K_{il} \\ -K_{ij} & 0 & K_{jk} & K_{jl} \\ -K_{ik} & -K_{jk} & 0 & K_{kl} \\ -K_{il} & -K_{jl} & -K_{kl} & 0 \end{vmatrix} .$$

and if the rank is equal to 2τ, there is necessarily one 2τ-rowed principal minor which is not zero.

(A) Suppose that rank of $\|K\|$ is zero or two. The determinant of the right-hand member of (I. II) must be zero. Hence, it follows from (1.9) that all of M's are zero. Conversely, if all of M's are zero, we have

$$(1.\ 12) \qquad K_{i(j}K_{kl)} = 0 \qquad (i,\ j,\ k,\ l=1,\dots,n)$$

Suppose that the rank of $\|K\|$ is n (even), then, contracting (1. 12) by K^{kl} which is skew-symmetric in k and l[(3)], we have $(n-2)K_{ij}=0$. Accordingly all of K_{ij} are zero for $n \geqq 4$ in contradiction to the hypothesis on the rank of $\|K\|$. Next suppose that the rank of $\|K\|$ is 2τ $(n > 2\tau \geqq 4)$. Now transform the coordinate system in such a way that $\|K\|$ has the form

$$(1.\ 13) \qquad \|K\| = \left\| \begin{array}{cccc|c} 0 & K_{12} & \cdots & K_{1(2\tau)} & \\ -K_{12} & 0 & & \vdots & \\ \vdots & & \ddots & \vdots & 0 \\ -K_{1(2\tau)} & & & 0 & \\ \hline & & 0 & & 0 \end{array} \right\| .$$

We consider the values of indices $i,\ j,\ k,\ l=1,\dots,\ 2\tau$ in (1. 12) and have similarly $K_{ij}=0$ $(i,j=1,\dots 2\tau)$ for $2\tau \geqq 4$. Accordingly the rank of $\|K\|$ is zero or two.

(B) Consider the following two systems of equations

$$(1.\ 14) \qquad K_{ij}\, v^i = 0,$$

$$(1.\ 15) \qquad M_{ijkl}\, v^i = 0 \qquad (i,\ j,\ k,\ l=1,\dots,n).$$

Suppose that the rank of $\|K\|$ is n (even) and the rank of $\|M\|$, i.e.

$$\left\| \begin{array}{c} M_{1abc} \cdots\cdots\cdots M_{nabc} \\ M_{1ijk} \cdots\cdots\cdots M_{nijk} \\ \cdots\ \cdots\cdots\cdots\ \cdots \\ M_{1pqr} \cdots\cdots\cdots M_{npqr} \end{array} \right\|$$

of coefficients of the system (1. 15) is $<n$. Then the system (1. 15) has a non-trivial solution v^i and it results from (1.9) that

$$\text{(I. 16)} \qquad K_{i(j}\ K_{kl)}\, v^i=0.$$

But since the determinant $|K|\fallingdotseq 0$ by hypothesis, it follows from (I. 16) by contracting with K^{kl} that all of v^i are zero; hence the rank of $\|M\|$ is also n. Conversely if the rank of $\|M\|$ is n and that of $\|K\|<n$, (I. 14) would have a non-trivial solution v^i satisfying (I. 15) according to (I. 16). This contradicts to the hypothesis on the rank of $\|M\|$. Hence the rank of $\|K\|$ is n if, and only if, the matrix $\|M\|$ has rank n.

(C) Consider finally the case in which the rank of $\|K\|$ is 2τ ($n>2\tau \geqq 4$). Now transform the coordinate system in such a way that $\|K\|$ has the form (I. 13). All of solutions of (I. 14) satisfy (I. 15) by means of (I. 16). Conversely, let any non-trivial solution of (I. 15) be v^i and putting indices i, j, k and l to be $1,\ldots,2\tau$ in (I. 16) and contracting by K^{kl} we have $v^1=\ldots=v^{2\tau}=0$; also we know that one of the quantities $v^{2\tau+1},\ldots$ $\ldots$, v^n is not zero. Since these v's satisfy the system (I. 14), and solution of (I. 15) is therefore a solution of (I. 14). Accordingly the rank of $\|K\|$ is equal to that of $\|M\|$. Hence we have the

Theorem 1.2 :...*The type number of a variety $S_n(n\geqq 4)$ of a euclidean space E_{n+2} is determined by its intrinsic properties*;

I) *the type number is equal to one if, and only if, the tensor M_{ijkl} is the zero tensor.*

II) *The type number is equal to τ if, and only if, the rank of the matrix $\|M\|$ is 2τ $(n>2\tau\geqq 4)$.*

For Riemann spaces of dimension less than four, tensor M_{ijkl} is constantly zero as is seen from (I. 10).

If S_n is immersible in an $(n+1)$-dimensional euclidean space E_{n+1}, the Gauss equation is

$$R_{ijkl}=H_{ik}\ H_{jl}-H_{il}\ H_{jk},$$

and then we can see immediately that the tensor M_{ijkl} is zero. Therefore S_n being of type $\geqq 2$ is not immersible in E_{n+1}, i.e. not of class one or zero.

C. B. Allendoerfer discussed Riemann spaces of class $p(\geqq 2)$[(4)]. He put

$$C_{ab\,|\,ij}=\begin{vmatrix} H^{\mathrm{I}}_{ai} & H^{\mathrm{II}}_{ai} \\ H^{\mathrm{I}}_{bj} & H^{\mathrm{II}}_{bj} \end{vmatrix}$$

According to (I. 3) we have

(I. 17) $$g^{ab}C_{ab\,|\,ij}=-K_{ij}.$$

Therefore contracting C_1 in his paper, i.e. $C_1=C_{ab\,|\,ij}\delta^{ij}_{rs}$, by g^{ab} we have from (I. 17)

(I. 18) $$-2K_{rs}=g^{ab}C_1.$$

Moreover contracting C_2, i.e. $C_2=C_{ab\,|\,ij}C_{cd\,|\,kl}\delta^{ijkl}_{rstu}$, by $g^{ab}g^{cd}$ we have from (I. 17)

(I. 19) $$(-1)^2\cdot 2^2\cdot 2!\cdot\sqrt{|K_2|}=g^{ab}g^{cd}C_2,$$

and so on; where $|K_2|$ is symbolically a 4-rowed principal minor of $\|K\|$ i.e.

$$\sqrt{|K_2|}=K_{r(s}\,K_{tu)}\,.$$

Thus we have in general

(I. 20) $$(-1)^{\tau}\cdot 2^{\tau}\cdot\tau!\cdot\sqrt{|K_{\tau}|}=g^{a_1b_1}\ldots\ldots g^{a_{\tau}b_{\tau}}C_{\tau}\,;$$

where $|K_{\tau}|$ is symbolically a 2τ-rowed principal minor of $\|K\|$.

He defined such a type number that a Riemann space R_n of class two is of type τ if there is one C_{τ} not zero and all of $C_{\tau+1}$ are zero.

If a R_n of class two is of type τ in the sense of this paper, we must have that $|K_{\tau}|$ is not zero. Hence, all of C_{τ} are not zero from (I. 20). As the result, *R_n of class two and of type τ in the sense of this paper is of type $\geqq\tau$ in the sense of the Allendoerfer's paper.*

Hence, if we interchange the Allendoerfer's definition of type number with that in this paper, the theorem I and II, and Lemma V of his paper become the following three theorems.

Theorem I. 3 :...*If a variety S_n $(n\geqq 6)$ in a euclidean space E_{n+2} is of type $\geqq 3$, S_n is intrinsically rigid.*

Theorem I. 4 :...*If in a Riemann space $R_n(n\geqq 6)$ of type $\geqq 3$ there are two* sets *of functions H^P_{ij} and H^P_{Qi} $(P, Q=I, II;\ i,j=1,\ldots, n)$ satisfying the Gauss and Codazzi equations, the Ricci equation is automatically satisfied.*

Theorem I. 5 :...*If in a Riemann space R_n $(n\geqq 8)$ of type $\geqq 4$ there is a set of functions $H^P_{ij}(p=I, II;\ i,j=1,\ldots,n)$ satisfying the Gauss equation, there is a set of functions H^P_{Qi} $(P,Q=I, II;\ i=1,\ldots,n)$ satisfying the Codazzi and Ricci equations.*

§2. Characters of a solution of the equations (I. 9)

Let the algebraic equations (I. 9) have a solution K's. We shall discuss the characters of the solution.

From the theorem I. 2, *if there are two systems of solutions K's and $\bar{K}$'s we have that the rank of $\|K\|$ is equal to that of $\|\bar{K}\|$.*

Now we shall prove the following theorem in relation to intrinsic rigidity :

Theorem 2. I:... *If a Riemann space $R_n (n \geqq 6)$ of class two is of type $\geqq 3$, a solution K's of (I. 9) is uniquely determined to within algebraic sign.*

The algebraic sign of K's can not be determined by intrinsic properties, because it changes by interchanging indices I and II of the normals as is seen from (I. 3).

Let K's and $\bar{K}$'s be two systems of solution and we put

$$\bar{K}_{ij} = K_{ij} + A_{ij} \qquad (i.\ j=1,\dots,n). \tag{2.1}$$

We have from (I. 9)

$$\bar{K}_{i(j} \bar{K}_{kl)} = K_{i(j} K_{kl)} . \tag{2.2}$$

Substituting (2. I) in (2.2) we have

$$K_{i(j} A_{kl)} + A_{i(j} K_{kl)} + A_{i(j} A_{kl)} = 0. \tag{2.3}$$

(A) Suppose det. $|K| \neq 0$ and $|A| \neq 0$. Contracting (2.3) by A^{kl} we have

$$(n-4) K_{ij} + (n-2+A^{ab} K_{ab}) A_{ij} = 0. \tag{2.4}$$

Moreover contracting (2.4) by A^{ij} we have $A^{ab} K_{ab} = -n/2$, and substituting this expression in (2.4), we have $A_{ij} = -2K_{ij}$ for $n \geqq 6$. Hence from (2. I) $\bar{K}_{ij} = -K_{ij}$ for $i, j=1,\dots,n$.

Next suppose det. $|A|=0$. Let v's be a non-trivial solution of the the system of equations $A_{ij} v^i = 0$ $(i, j=1,\dots,n)$. Contracting (2.3) by K^{ij} we have

$$(n-4+K^{ab} A_{ab}) A_{kl} + K^{ab} A_{ab} K_{kl} - K^{ij} (A_{ki} A_{lj} + A_{il} A_{kj}) = 0. \tag{2.5}$$

Since contracting (2.5) by v^k we have $(K^{ab} A_{ab}) K_{kl} v^k = 0$, we have $K^{ab} A_{ab} = 0$, because $|K|$ is not zero. Hence we have from (2.5)

$$(n-4) A_{kl} - K^{ij} (A_{ki} A_{lj} + A_{il} A_{kj}) = 0.$$

Substituting (2.1) in this equation we have

$$n \bar{K}_{kl} = (n-2) K_{kl} - K^{ij} (\bar{K}_{ik} \bar{K}_{lj} + \bar{K}_{il} \bar{K}_{jk}). \tag{2.6}$$

From $|\bar{K}| \neq 0$, we have similarly

$$(2.7) \qquad nK_{kl}=(n-2)\bar{K}_{kl}-\bar{K}^{ij}\ (K_{ik}K_{lj}+K_{il}K_{jk}).$$

Now from (2.6) and (2.7) we have

$$n\bar{K}_{kl}=(n-2)K_{kl}-K^{ij}\bar{K}_{ik}\left\{\frac{n}{n-2}K_{lj}+\frac{1}{n-2}\ \bar{K}^{ab}(K_{al}\ K_{jb}\right.$$
$$\left.+K_{aj}K_{bl})\right\}-K^{ij}\bar{K}_{il}\left\{\frac{n}{n-2}K_{jk}+\frac{1}{n-2}\bar{K}^{ab}(K_{aj}K_{kb}+K_{ak}K_{bj})\right\},$$

and we deduce $\bar{K}_{kl}=K_{kl}$ $(k,\ l=1,\dots,n)$ for $n\geqq 6$.

(B) Suppose that rank of $\|K\|=2\tau$ $(n>2\tau\geqq 6)$. Transform $\|K\|$ into the form (I. 13). Then $\|\bar{K}\|$ has also the similar form at the same time. In fact, putting $i,\ j,\ k,\ l=1,\dots,2\tau$ in (2.2) and contracting by K^{ij} we have

$$(2\tau-2)K_{kl}=C^h_k\ \bar{K}_{hl} \quad (h,\ k,\ l=1,\dots,2\tau);$$

where

$$C^h_k=(K^{ab}\bar{K}_{ab})\ \delta^h_k-2K^{ah}\bar{K}_{ak}.$$

Hence we have

$$(2\tau-2)\,|\,K_\tau\,|=|\,C\,|.\quad|\,\bar{K}_\tau|.$$

Accordingly in $\|\bar{K}\|$ we have

$$\begin{vmatrix} 0 & \bar{K}_{12}\cdots\cdots & \bar{K}_{1(2\tau)} \\ -\bar{K}_{12} & 0 & \vdots \\ \vdots & \ddots & \vdots \\ -\bar{K}_{1(2\tau)} & & \ddots 0 \end{vmatrix}\neq 0.$$

Next putting $i>2\tau$ and $j,\ k,\ l=1,\dots,2\tau$ in (2.2) we have

$$\bar{K}_{i(j}\ \bar{K}_{kl)}=0$$

and contracting this equation by $\bar{K}^{kl}$ we have $\bar{K}_{ij}=0$ for $i>2\tau$ and $j=1,\dots,2\tau$.

Finally putting $i,\ j>2\tau$ and $k,\ l=1,\dots,2\tau$ in (2.2) we have $\bar{K}_{ij}\ \bar{K}_{kl}=0$. We have therefore $\bar{K}_{ij}=0$ for $i,j>2\tau$. Accordingly, by the similar way as for (A), we have the theorem 2.I.

Now in relation to the equation (I.5) we shall prove the

Theorem 2.3 :... *When a Riemann space $R_n(n\geqq 8)$ of class two and of type $\geqq 4$, a solution K's of* (I. 9) *satisfies the equations* (I. 5).

We differentiate covariantly (I. 9) with respect to x^h and subtract four equations obtained by interchanging h with i, j, k and l. Making use of (I. 10) and the Bianchi's identity we have

$$\begin{aligned}(2.8)\qquad & K_{ij}\, K_{klh} + K_{ik}\, K_{jhl} + K_{il}\, K_{jkh} + K_{ih}\, K_{jlk} + K_{jk} K_{ilh} \\ & + K_{jl} K_{ihk} + K_{jh} K_{ikl} + K_{kl} K_{ijh} + K_{hk} K_{ilj} + K_{lh} K_{ijk} = 0 ;\end{aligned}$$

where K_{ijk} are left-hand member of (I. 5).

(A) Suppose det. $|K| \neq 0$. Contracting (2.8) by K^{lh} we have

$$(2.9)\qquad (n-6)K_{ijk} + K^{lh}(K_{ij}\, K_{lhk} + K_{ki}\, K_{lhj} + K_{jk}\, K_{lhi}) = 0,$$

and contracting (2.9) by K^{ij} we have $K^{ab}\, K_{abi} = 0$ for $n \geqq 6$. We can therefore deduce from (2.9) that all of K_{ijk} are zero for $n \geqq 8$.

(B) Suppose that the rank of $\|K\| = 2\tau$ $(n > 2\tau \geqq 8)$. Transform $\|K\|$ into (I. 13) and take $i, j, k, l, h = 1,\ldots,2\tau$ in (2.8). By the similar way as for (A) we have $K_{ijk} = 0$ for $i, j, k = 1,\ldots,2\tau$.

Next putting $k > 2\tau$ and $i, j, l, h = 1,\ldots, 2\tau$ in (2.8) we have

$$\begin{aligned}(2.\ 10)\qquad & K_{ij}\, K_{klh} + K_{il}\, K_{jkh} + K_{ih}\, K_{kjl} + K_{jl}\, K_{ihk} + K_{jh}\, K_{ikl} \\ & + K_{lh}\, K_{kij} = 0,\end{aligned}$$

and contracting (2. 10) by K^{lh} we have

$$(2.\ 11)\qquad (2\tau - 4)\ K_{ijk} + K_{ij}\ \ K^{lh}\, K_{lhk} = 0,$$

and contracting (2. 11) by K^{ij} we have $K^{lh}\, K_{lhk} = 0$ for $2\tau > 2$, and therefore from (2. 11) $K_{ijk} = 0$ for $i, j = 1,\ldots,2\tau$ and $k > 2\tau$ if $2\tau > 4$.

Next putting $j, k > 2\tau$ and $i, l, h = 1,\ldots,2\tau$ in (2.8) we have

$$(2.\ 12)\qquad K_{il}\ K_{jkh} + K_{ih}\ K_{jki} + K_{hi}\ K_{jkl} = 0,$$

and contracting (2. 12) by K^{lh} we have $K_{ijk} = 0$ for $i = 1,\ldots,2\tau$ and $j, k > 2\tau$.

Finally putting $i, j, k > 2\tau$ and $l, h = 1, \ldots,2\tau$ in (2.8) we have $K_{lh} K_{ijk} = 0$ and therefore $K_{ijk} = 0$ for $i, j, k > 2\tau$. Since K_{abc} is skew-symmentric in its every two indices, all of K_{abc} are zero and we have the theorem 2.2.

It is to be noted here that (1.5) is a necessary and sufficient condition of integrability of the differential equations (1.4) for the determination of the unknowns H_i (we omit here the proof).

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References

1) M. Matsumoto: Riemann spaces of class two and their algebraic charactezation (part II), J. Math. Soc. Japan.

2) G. Kowalewski: Einführung in die Determinantentheorie, PP. 140 and 145.

3) 1. c. P. 134.

4) Amer. J. Math., 61 (1939). This paper is called here simply the Allendoerfer's paper.

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