Journal of the Mathematical Society of Japan Vol. 2, Nos. 1–2, Sept., 1950.

On topological completeness

Jun-iti NAGATA

(Received May 30, 1949)

E. Čech has proved the following theorem[1]:
A metrizable space R is topologically complete if and only if it is completely metrizable.

In this paper we shall show that by making use of the theorem of N. A. Shanin,[2] we can simplify the proof of Čech's theorem and generalize it slightly.

We mean in this paper by a filter a family of closed sets having the finite intersection property, and we say that a filter $\{F_\alpha \mid A\}$ is vanishing when $\Pi\ F_\alpha=\phi$ holds.

N. A. Shanin's theorem. *In order that a T_1-space R can be represented as an intersection of at most $\mathfrak{n}$ (a cardinal number) open sets in Wallman's bicompactification $W(R)$ of R, it is necessary and sufficient that there exists a collection $\{\mathfrak{F}_\gamma\}$ of at most $\mathfrak{n}$ vanishing filters $\mathfrak{F}_\gamma$ of R with the property: For an arbitrary maximum vanishing filter $\mathfrak{F}$ of R, there exists a filter $\mathfrak{F}_\gamma$ of $\{\mathfrak{F}_\gamma\}$ such that $\mathfrak{F}_\gamma\subset\mathfrak{F}$.*

When we note that there exists a one-to-one correspondence between an open set of $W(R)$ containing R and a vanishing filter of R as well as between a point of $W(R)-R$ and a maximum vanishing filter of R, this theorem is almost obvious.

Proof of Čech's theorem. We begin with the necessity of the condition. Let R be a topologically complete and metrizable space. Since R is topologically complete, R is, as is well known, a G_δ-set in Čech's bicompactification $\beta(R)$ of R, *i.e.* an intersection of at most countable open sets of $\beta(R)$. Since R is metrizable, and accordingly normal, $\beta(R)$ and $w(R)$ are, as is well-known, identical. Therefore, when we use Shanin's theorem in the case of $\mathfrak{n}=\mathfrak{a}$, we get the family $\{\mathfrak{F}_n\}$ of at most a countable number of vaning filters $\mathfrak{F}_n$ mentioned in the theorem.
Let $\mathfrak{F}_n=\{F_{n,\alpha}\mid \alpha\in A_n\}$; then $\{F^c_{n,\alpha}\mid \alpha\in A_n\}=\mathfrak{M}_n$[3] $(n=1,2,\dots)$ are open coverings of R.

On the other hand, since R is metrizable, R has a base $\{\mathfrak{N}_m\}$ of uniform-

ity of a countable number of open coverings $\mathfrak{N}_m$ agreeing with its topology.
Now we put $\mu=\{\mathfrak{M}_n, \mathfrak{N}_m\}$ $(n, m=1,2,\ldots)$,
and $\triangle\mu=\{\mathfrak{P}\wedge\mathfrak{P}'|\mathfrak{P},\mathfrak{P}'\in\mu\}$.
Since R is metrizable, and accordingly fully normal, for an arbitrary open covering $\mathfrak{P}$ of R, there exists an open $\triangle$-refinement $\mathfrak{Q}$ $(\mathfrak{P})$[4)]
Therefore we put $\mu\triangle=\{\mathfrak{Q}(\mathfrak{P}) \mid \mathfrak{P}\in\mu\}$,
and successively

$$\mu_1=\mu+\triangle\mu+\mu\triangle,$$
$$\mu_2=\mu_1+\triangle\mu_1+\mu_1\triangle,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

and finally $\nu=\mu_1+\mu_2+\ldots\ldots$;

then ν has the countable cardinal number.
Since ν contains $\{\mathfrak{N}_m\}$, it is obviously a base of a uniformity agreeing with the topology of R. By using ν, we introduce a metric agreeing with ν and accordingly with the topopology of R.

Now we can show that the uniformity ν is complete, *i. e.* this metric space R is complete.

For this purpose, we shall show that no Cauchy filter can be vanishing. Assume that the assertion is false, *i. e.* there exists a vanishing Cauchy filter ; then by constructing a maximum filter which contains this filter, we get a maximum vanishing Cauchy filter $\mathfrak{F}=\{F_\beta\}$. For this $\mathfrak{F}$, we can choose an element $\mathfrak{F}_n$ of $\{\mathfrak{F}_n\}$ such that

$$\mathfrak{F}\supset\mathfrak{F}_n=\{F_{n,\alpha}\}.$$

Let $\{F_{n,\alpha}^c\}=\mathfrak{M}_n$; then there exists an open covering $\mathfrak{P}$ such that

$$\mathfrak{P}\in\nu,\ \mathfrak{P}^\triangle<\mathfrak{M}_n.$$

On the other hand, since $\mathfrak{F}$ is a Cauchy filter, there exist F_β and a such that

$$F_\beta\in\mathfrak{F},\ a\in R;\quad F_\beta\subset S\ (a,\ \mathfrak{P}).$$[5)]

Since $\mathfrak{P}^\triangle<\mathfrak{M}_n$, there exists an element $F_{n,\alpha}^c$ of $\mathfrak{M}_n$ such that $S(a,\ \mathfrak{P})\subset F_{n,\alpha}^c$.
Therefore it must be $F_\beta\cap F_{n,\alpha}=\phi$.
Since $F_{n,\alpha}\in\mathfrak{F}$, $\mathfrak{F}$ would not be a filter, contradicting the assumption. Therefore the uniformity ν must be complete.

Next, we shall prove the sufficiency.

Let R be a complete metric space.
If R is totally bounded; then R is bicompact and the problem is trivial. Therefore let us assume that R is not totally bounded; then, for some n_0, the open covering $\{S_{1/n}(x) \mid x \in R\}$ $(n \geqq n_0)$ has no finite subcovering, where we mean by $S_\varepsilon(x)$ the set of all points with the distance less than ε from x.
Therefore $\mathfrak{S}_n = \{S^c_{1/n}(x) \mid x \in R\}$ $(n \geqq n_0)$ is a vanishing filter, and the cardinal number of $\{\mathfrak{S}_n \mid n \geqq n_0\}$ is countable.

Let $\mathfrak{F} = \{F_\alpha\}$ be an arbitrary maximum vanishing filter of R. Since R is complete, $\mathfrak{F}$ can not be a Cauchy filter, that is, there exists n' such that if $n \geqq n'$, for every $F_\alpha \in \mathfrak{F}$ and $x \in R$, $F_\alpha \not\subset S_{1/n}(x)$ holds. Let $n \geqq n_0, n'$; then

$$F_\alpha \cap S^c_{1/n}(x) \neq \phi \quad \text{for all } F_\alpha \in \mathfrak{F} \text{ and } x \in R.$$

Since $\mathfrak{F}$ is maximum, it must be

$$S^c_{1/n}(x) \in \mathfrak{F} \quad \text{for all } x \in R;$$

hence $\qquad \mathfrak{S}_n \subset \mathfrak{F}.$

Therefore the collection $\{\mathfrak{S}_n \mid n \geqq n_0\}$ has the property of the collection of vanishing filters in Shanin's theorem, and R is accordingly a G_δ-set of $W(R)$, *i. e.* R is topologically complete.

From the method of this proof, we see that the follwing corollary holds.

Corollary. *Let R be a fully normal topological space, in which a uniformity with the cardinal number at most $\mathfrak{n}$ can be introduced. In order that R can be represented as an intersection of at most $\mathfrak{n}$ open sets in some bicompact T_2-space, it is necessary and sufficient that a complete uniformity with a cardinal number at most $\mathfrak{n}$ can be introduced in R.*

Mathematical Institute
Ôsaka University

Notes.

1) E. Čech: On bicompact spaces. Annals of Math, 38, (1937.)

2) N. A. Shanin: On the theory of bicompact extensions of topological spaces. Comptes Rendus (Doklady) USSR, 38, (1943). This theorem is stated in the more general form.

3) We denote by $F^c{}_{n,\alpha}$ the complement of $F_{n,\alpha}$.

4) Cf. Tukey: Convergence and uniformity in topology. 1940.

5) $$S(a\ \mathfrak{P}) = \sum_{a \in P \in \mathfrak{P}} P.$$