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On the structure and representations of Clifford algebras.

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The structure and representations of Clifford algebras over the complex number field were studied by many authors.[1] The purpose of this note is to investigate them over any ground field K with $\chi(K)\neq 2$. Moreover, to apply the results to the problems of Eddington on sets of anticommuting matrices,[2] we shall consider slightly generalized Clifford algebras. In Appendix we shall give irreducible representations of such algebras in their explicite form.

1. Let K be any field with the characteristic $\chi(K)\neq 2$, and n, g two integers such that $0\leqq g\leqq n$, $n>0$. The *Clifford algebra of type* (n, g) $C(n,g)/K$ over K is defined as an algebra with generators.

$$u_0, u_1, \dots, u_n$$

and with fundamental relations

$$(1)\quad u_0^2=u_0,\ u_0u_i=u_iu_0=u_i,\ u_i^2=u_0\ (1\leqq i\leqq g),\ u_i^2=-u_0\ (g+1\leqq i\leqq n),$$
$$u_iu_j+u_ju_i=0\ \ (i\neq j,\ i>0,\ j>0).$$

$C(n,g)$ has rank 2^n and

$$u_0, u_i\ (1\leqq i\leqq n),\ u_iu_j\ (1\leqq i<j\leqq n), \dots\dots, u_1u_2\dots n_n$$

form a basis of $C(n,g)/K$.[3] $C(n,0)/K$ is the ordinary Clifford algebra.[3]

We distinguish now three cases according to the properties of K:

Case I. There is an element $\lambda\in K$ with $1+\lambda^2=0$.

Case II. There is no solution $\lambda\in K$ of $1+\lambda^2=0$, but there are elements

$$\alpha, \beta\in K \text{ with } 1+\alpha^2+\beta^2=0.$$

Case III. There are no solutions $\alpha, \beta\in K$ of $1+\alpha^2+\beta^2=0$.

All three cases may arise, when $\chi(K)=0$. Of course we have Case I when K is the complex number field, and Case III when K is the real number field. If $\chi(K)=p\neq 0$, then we have either Case I or Case II.[4] Case I occurs when $p\equiv 1$ (mod. 4), and Case II when $p\equiv 3$ (mod. 4) for prime field K.[5]

Now we consider three algebras. The one is the quarternion algebra $Q/K=C(2,0)/K$:

$$Q/K = K + iK + jK + kK,$$
$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j,$$

In Cases I, II we have

$$Q/K \simeq K(2)$$

(We mean by $K(r)$ the full matric algebra over K with degree r for the convention of printing). In Case III, Q is a normal division algebra over K.[6]

Lemma 1. *Let* $Q_1 = K + iK + jK + kK$ *and* $Q_2 = K + pK + qK + rK$ *be two quarternion algebras. Then their direct product* $Q_2 \times Q_2$ *is isomorphic to* $K(4)$.[7]

The other two are $C(2,1)/K$, $C(2,2)/K$.

Lemma 2. $C(2,1)/K \simeq C(2,2)/K \simeq K(2)$.[8]

2. *The structure of* $G(n,g)/K$ *for* $n=2m$, $g=2d+\delta$ $(\delta=0,1)$. We denote $u_0, u_1, \ldots, u_{2m}$ by u_0, v_1, w_1, v_2, $w_2, \ldots, v_m$, w_m, in this order, so that

$$(2) \begin{cases} v_k^2 = w_k^2 = u_0 \ (1 \leqq k \leqq d),\ v_{d+1}^2 = (-1)^{1+\delta} u_0,\ w_{d+1}^2 = -u_0, \\ v_k^2 = w_k^2 = -u_0 \ (d+2 \leqq k \leqq m). \end{cases}$$

We put then

$$(3) \begin{cases} p_1 = v,\ q_1 = w_1,\ r_1 = v_1 w_1, \\ p_k = v_1 w_1 v_2 w_2 \ldots v_{k-1} w_{k-1} v_k,\ q_k = v_1 w_1 v_2 w_2 \ldots v_{k-1} w_{k-1} w_k,\ r_k = v_k w_k \\ (\leqq k \leqq m). \end{cases}$$

Conversely we can represent $v_1, w_1, \ldots, v_m$, w_m by p_k, q_k, r_k:

$$(4) \begin{cases} v_1 = p_1,\ w_1 = q_1, \\ v_k = (-1)^{k-1} r_1 \ldots r_{k-1} p_k,\ w_k = (-1)^{k-1} r_1 \ldots r_{k-1} q_k.\ (2 \leqq k \leqq d+1), \\ v_k = (-1)^{k+\delta-1} r_1 \ldots r_{k-1} p_k,\ w_k = (-1)^{k+\delta-1} r_1 \ldots r_{k-1} q_k\ (d+2 \leqq k \leqq m). \end{cases}$$

Hence p_k, q_k, r_k $(1 \leqq k \leqq m)$ generate the algebra $C(2m, g)/K$. Let us put

$$(5) \qquad R^{(k)} = u_0 K + p_k K + q_k K + r_k K \quad (1 \leqq k \leqq m).$$

Since

$$(6) \begin{cases} p^2 = (-1)^{k-1} u_0,\ q_k^2 = (-1)^{k-1} u_0,\ p_k q_k = -q_k p_k = (-1)^{k-1} r_k (1 \leqq k \leqq d), \\ p_{d+1}^2 = (-1)^{d+\delta+1} u_0,\ q_{d+1}^2 = (-1)^{d+1} u_0,\ p_{d+1} q_{d+1} = (-1)^d r_{d+1} \\ p_k^2 = (-1)^{k+\delta} u_0,\ q_k^2 = (-1)^{k+\delta} u,\ p_k q_k = -q_k p_k = (-1)^{k-1} r_k \\ \qquad (d+2 \leqq k \leqq m) \end{cases}$$

hold, we have

$$(7)\begin{cases} R^{(k)}\cong Q \text{ for } 1\leqq k\leqq d,\ k\equiv 0 \pmod{2} \text{ or for } d+2\leqq k\leqq m, \\ \qquad\qquad \delta+k\equiv 1 \pmod{2}, \\ R^{(k)}\cong C(2,2) \text{ for } 1\leqq k\leqq d,\ k\equiv 1 \pmod{2} \text{ or for } d+2\leqq k\leqq m, \\ \qquad\qquad \delta+k\equiv 0 \pmod{2}, \\ R^{(d+1)}\cong\begin{cases} C(2,0)=Q & \text{for } \delta=0,\ d\equiv 0 \pmod{2} \\ C(2,2) & \text{for } \delta=0,\ d\equiv 1 \pmod{2} \\ C(2,1) & \text{for } \delta=1. \end{cases} \end{cases}$$

We can also easily verify that $a_k a_h = a_h a_k$ for $a_h \in R$, $a_k \in R^{(k)} (h \neq k)$. From this and from the fact $C(2m, g) = R^{(1)}R^{(2)}\dots R^{(m)}$, and comparing the rank over K, we have a decomposition of $C(2m, g)$ in a direct product of subalgebras $R^{(k)}$:

$$(8) \qquad C(2m,g) \cong R^{(1)} \times R^{(2)} \times \dots \times R^{(m)}.$$

From Lemma 1, 2 and (7),(8) we can give the structure of $C(2m,g)$ by simple enumeration as follows:

Theorem 1. *$C(2m,g)$ is a normal simple algebra over K, and its Wedderburn's representation as a direct product of a normal division algebra and full matric algebra is given as follows:*

Case I, II:

$$(9) \qquad C(2m,g) \cong K(2^m),$$

Case III:

$$(10) \qquad C(2m,g) \cong K(2^m) \qquad \textit{for } g-m\equiv 0,\ 1 \ (\textit{mod. } 4)$$

$$(11) \qquad C(2m, g) \cong Q \times K(2^m) \quad \textit{for } g-m\equiv 2,\ 3 \ (\text{mod. } 4).$$

$C(2m, g)$ has only one (equivalent class of) irreducible representation D in K, and deg $D=2^m$ for (9), (10), and deg $D=2^{m+1}$ for (11) respectively. It is not difficult to give them explicitely since (8) is known.[9]

3. *The structure of $C(n, g)$ for $n=2m+1$, $g=2d+\delta$* $(\delta=0,1)$. We permute $u_0,u_1,\dots,u_n$ in appropriate order and denote them by $u_0,v_1,\dots,v_m$, $w_1,\dots,w_m$, x, so that

$$(12)\begin{cases} v_k^2=w_k^2=u_0 \ (1\leqq k\leqq d),\ v_k^2=w_{k0}=-u_0 \ (d+1\leqq k\leqq m), \\ x^2=(-1)^{\delta+1}u_0. \end{cases}$$

We define p_k, q_k, r_k, $R^{(k)}$ $(1\leqq k\leqq m)$ as above by (3) and (5), and put

$$(13) \qquad z=r_1r_2\dots r_mx, \text{ then } z^2=(-1)^{m+\delta+1}u_0$$

(14) $Z=u_0K+zK.$

Then z belongs to the center of $C(2m+1,g)$ and we have conversely

(15) $$x=(-1)^m r_1 \dots r_m z$$

(16) $$\begin{cases} v_1=p_1,\ w_1=q_1, \\ v_k=(-1)^{k-1}r_1\dots r_{k-1}\ p_k,\ w_k=(-1)^{k-1}r_1\dots r_{k-1}\ q_k\ (2\leqq k\leqq m). \end{cases}$$

Therefore, by the similar reasoning as in § 2, it holds that

(17) $$C(2m+1,g)\cong R^{(1)}\times\dots\times R^{(m)}\times Z.$$

Here the elements p_k, q_k, r_k satisfies the following relations:

(18) $$\begin{cases} p_k^2=q_k^2=(-1)^{k-1}\ u_0,\ p_kq_k=-q_kp_k=(-1)^{k-1}r_k\ (1\leqq k\leqq d) \\ p_k^2=q_k^2=(-1)^k u_0,\ p_kq_k=-q_k p_k=(-1)^{k-1}r_k\ (d+1\leqq k\leqq m), \end{cases}$$

so that we have

(19) $$\begin{cases} \overset{(k)}{R}\cong Q, & \text{for } 1\leqq k\leqq d,\ k\equiv 0 \pmod 2 \text{ or for } d+1\leqq k\leqq m,\ k\equiv 1 \pmod 2 \\ \overset{(k)}{R}\cong C(2,2), & \text{for } 1\leqq k\leqq d,\ k\equiv 1 \pmod 2 \text{ or for } d+1\leqq k\leqq m,\ k\equiv 0 \pmod 2 \end{cases}$$

and Z is the center of $C(2m+1,g)$.

Now we distinguish three cases:

(i) $m+g\equiv 1$ (mod. 2). We put

(20) $$e_1=(u_0+z)/2,\ e_2=(u_0-z)/2.$$

(ii) $m+g\equiv 0$ (mod. 2), Case I. We put

(21) $$e_1=(u_0+\lambda z)/2,\ e_2=(u_0-\lambda z)/2,\quad (\lambda^2=-1).$$

In these cases Z is a direct sum of two fields:

$$Z=e_1K+e_2K,\ u_0=e_1+e_2,\ e_1^2=e_1,\ e_2^2=e_2\ e_1e_2=e_2e_1=0.$$

(iii) $m+g\equiv 0$ (mod. 2), Case II, III. Then Z is a quadratic field over K: $Z=K(\sqrt{-1})$. From these data, we can give the structure of $C(2m+1,g)$ by simple enumeration as follows:

Theorem 2. *$C(2m+1,g)$ is a semi-simple algeb a over K and its structure is:*

Case I:

(22) $$C(2m+1,g)\cong K(2^m)+K(2^m).$$

Case II:

(23) $C(2m+1,g)\cong K(2^m)+K(2^m)$ *for* $m+g\equiv 1$ (mod. 2)

(24) $C(2m+1,g)\cong Z\times K(2^m)$ *for* $m+g\equiv 0$ (mod. 2).

Case III:

(25) $C(2m+1,g) \cong Z \times K(2^m)$ *for* $g-m \equiv 0, 2 \pmod 4$

(26) $C(2m+1,g) \cong Q(2^{m-1}) + Q(2^{m-1})$ *for* $g-m \equiv 3 \pmod 4$

(27) $C(2m+1,g) \cong K(2^m) + K(2^m)$ *for* $g-m \equiv 1 \pmod 4$,

where Z *is a quadratic field over* K *given by* $Z = K(\sqrt{-1})$.

In cases (22),(23),(27), $C(2m+1,g)$ has two (equivalent classes) of irreducible representations in K, and both are of degree 2^m [10]. In cases (24) (25), $C(2m+1,g)$ has only one irreducible representation in K of degree 2^{m+1} [10]. In case (26), $C(2m+1,g)$ has two irreducible representations in K of degree 2^{m+1} [10].

For later applications (§ 4), we give here the table of the degrees of irreducible representations of $C(n,0)$ for various values of n and for Cases I, II, III (cf. Theorem 1,2) :

Case \ n	$8h$	$8h+1$	$8h+2$	$8h+3$	$8h+4$	$8h+5$	$8h+6$	$8h+7$
I	2^{4h}	* 2^{4h}	2^{4h+1}	* 2^{4h+1}	2^{4h+2}	* 2^{4h+2}	2^{4h+3}	* 2^{4h+3}
II	2^{4h}	2^{4h+1}	2^{4h+1}	* 2^{4h+1}	2^{4h+2}	2^{4h+3}	2^{4h+3}	* 2^{4h+3}
III	2^{4h}	2^{4h+1}	2^{4h+2}	* 2^{4h+2}	2^{4h+3}	2^{4h+3}	2^{4h+3}	* 2^{4h+3}

(For the cases * there are two inequivalent representations.)

4. Now we consider two problems of A. S. Eddington in a somewhat generalized form. Let $E_k (k=1,2,\dots,N)$ be matrices with degree l, whose coefficients belong to K, satisfying the relations

$$(28) \qquad E_k^2 = -1, \quad E_k E_j = -E_j E_k \quad (j \neq k).$$

The first problem is to find the maximal value of N for a given l. Clearly the matrices $\{E_k\}$ form representations of $\{u_k\}$ of $C(N,0)$ with degree l. Since $C(N,0)/K$ is semi-simple we can decompose them as a direct sum of irreducible representations. Let $l = 2^m q$, $(2,q)=1$. It is easy to see that N depends only on the factor 2. The maximal value of N for $l=2^m$ is given as the largest value n in the above table such that $C(n,0)$ has an irreducible representation with degree 2^m. Hence we have

Theorem 3. *The maximal number* N *of the anticommuting matrices* E_k $(1 \leq k \leq N)$ *in* K *with* (28) *is given for various values* $l = 2^m$ *of their degrees and for Cases* I, II, III *as follows*:

Case \ l	2^{4h}	2^{4h+1}	2^{4h+2}	2^{4h+3}
I	* $8h+1$	* $8h+3$	* $8h+5$	* $8h+7$
II	$8h$	* $8h+3$	$8h+4$	* $8h+7$
III	$8h$	$8h+1$	* $8h+3$	* $8h+7$

(For the cases* there are two inequivalent solutions.)

The second problem concerns the Cases II, III. Let Λ be a field of Case II, III, and K be a quadratic field over Λ given by $K=\Lambda\ (\lambda)$, $\lambda^2=-1$. Then from the second table there exist $2m+1$ matrices E_k $(1\leqq k\leqq 2m+1)$ in K with degree 2^m satisfying the relations (28). We call E_k real, if all its coefficients belong to Λ and purely imaginary if $E_k=\lambda\ F_k$ (F_k : real). If we set a further condition that g matrices E_k among them are purely imaginary and the rest $2m+1-g$ are real, what values of g will be possible ?

As can be seen easily, this problem is equivalent to the one as follows : "Λ being a field of Case II, III, find the condition that $C(2m+1,g)/\Lambda$ has a representation of degree 2^m in Λ ! "

We can answer immediately by Theorem 2, that is :

Theorem 4. *A necessary and sufficient condition that among* $2m+1$ *matrices* $\{E_k\}$ *with degree* 2^m *and satisfying* (28), g *are purely imaginary and* $2m+1-g$ *are real, is*

(i) *Case* II *for* Λ, $g+m\equiv 1$ (mod. 2)

(ii) *Case* III *for* Λ, $g-m\equiv 1$ (mod. 4).

Remark. Theorem 1 gives us the condition that $C(2m,g)/\Lambda$ has a representation of degree 2^m in Λ as follows :

Case II : no restriction, Case III : $g-m\equiv 0,1$ (mod. 4).

If there exists a representation of $C(2m+\delta,\ g)/\Lambda$ $(\delta=0,1)$ in Λ of degree 2^m, it is necessarily irreducible.

Appendix.

We shall give here an explicite form of irreducible representations of Clifford algebras. For its sake we shall consider first subalgebras of them.

I. Quarternion $Q/K=C(2,0)/K$ in Cases I, II. Then $Q/K\cong K(2)$ and a system of matric units is given by

$$e_{11}=(1+\alpha i+\beta j)/2, \qquad e_{12}=(-k-\beta i+\alpha j)/2,$$

$$e_{21}=(k-\beta i+\alpha j)/2, \qquad e_{22}=(1-\alpha i-\beta j)/2,$$

where $1+\alpha^2+\beta^2=0$. An irreducible representation of Q is given by

$$1\rightarrow\begin{pmatrix} 1 & \\ & 1\end{pmatrix}, \; i\rightarrow A_1=\begin{pmatrix} -\alpha & \beta \\ \beta & \alpha\end{pmatrix}, \; j\rightarrow A_2=\begin{pmatrix} -\beta & -\alpha \\ -\alpha & \beta\end{pmatrix}, \; k\rightarrow A_3=\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

II. Direct product $Q_1\times Q_2$ of two quarternion algebras in Case I, II, III. Then $Q_1\times Q_2\cong K(4)$, and a system of matric units $\{e_{ij}\}$ (i, $j=1,2,3,4$) is given by

$$(4e_{ij})=\begin{pmatrix} 1+pi+qj+rk & -i+p+qk-rj & -j-pk+q+ri & -k+pj-qi+r \\ i-p+qk-rj & 1+pi-qj-rk & -k+pj+qi-r & j+pk+p+ri \\ j-pk-q+ri & k+pj+qi+r & 1-pi+qj-rk & -i-p+qk+rj \\ k+pj-qi-r & -j+pk-q+ri & i+p+qk+rj & 1-pi-qj+rk \end{pmatrix}$$

An irreducible representation of $Q_1\times Q_2$ is given by

$$i\rightarrow B_1=\begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}, \; j\rightarrow B_2=\begin{pmatrix} & & -1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{pmatrix}, \; k\rightarrow B_3=\begin{pmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix},$$

$$p\rightarrow C_1=\begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}, \; q\rightarrow C_2=\begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}, \; r\rightarrow C_3=\begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix},$$

We use later also the matrices :

$$kp\rightarrow D_1=\begin{pmatrix} & & -1 & \\ & & & 1 \\ -1 & & & \\ & 1 & & \end{pmatrix}, \; kq\rightarrow D_2=\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}, \; kr\rightarrow D_3=\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$

III Algebras $C(2,1)$ and $C(2,2)$.

$C(2,1)$: matric units system is given by

$$e_{11}=(u_0+u_1)/2, \; e_{12}=(u_2+u_1u_2)/2, \; e_{21}=(-u_2+u_1u_2)/2, \; e_{22}=(u_0-u_1)/2$$

and an irr. rep. of $C(2,1)$ is given by

$$u_1 \to G_1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},\quad u_2 \to G_1 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix},\quad (-u_1u_2) \to G_2 = \begin{pmatrix} & -1 \\ -1 & \end{pmatrix}$$

$C(2,2)$: matric units system is given by

$$e_{11} = (u_0 + u_1)/2,\quad e_{12} = (n_2 + u_1u_2)/2,\quad e_{21} = (u_2 - u_1u_2)/2,\quad e_{22} = (u_0 - u_1)/2$$

and an irr. rep. of $C(2,2)$ is given by

$$u_1 \to F_1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},\quad u_2 \to F_2 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix},\quad (-u_1u_2) \to F_3 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

IV. We give here as an example the explicite form of irr. rep. of $C(2m,0)/K$.

(i) $m = 4h$. The decomposition (8) becomes here (cf. (7))

$$C(2m,0) = Q^{(1)} \times Q^{(2)} \times \dots \times Q^{(2h)} \times P^{(1)} \times P^{(2)} \times \dots \times P^{(2h)},$$

where $Q^{(i)} \cong R^{(2i-1)} \cong C(2,0) = Q$, $P^{(i)} = R^{(2i)} \cong C(2,2)$. Correspondingly we have (cf. (4))

$$\begin{cases} v_{4k+1} = (r_1r_3)(r_5r_7)\dots(r_{4k-3}r_{4k-1})p_{4k+1}(r_2r_4\dots r_{4k}) \\ v_{4k+2} = -(r_1r_3)(r_5r_7)\dots(r_{4k-3}\ r_{4k-1})r_{4k+1}(r_2r_4\dots r_{4k}p_{4k+2}) \\ v_{4k+3} = (r_1r_3)(r_5r_7)\dots(r_{4k-3}r_{4k-1})(r_{4k+1}r_{4k+3})(r_2r_4\dots r_{4k+2}) \\ v_{4k+4} = -(r_1r_3)(r_5r_7)\dots(r_{4k-3}r_{4k-1})(r_{4k+1}r_{4k+3})(r_2r_4\dots r_{4k+2}r_{4k+4}) \end{cases}$$

and analogous formulas for the w_k. Hence we have an irr. rep. D with degree 2^{4h} (cf. Appendix II, III) :

$$\begin{cases} v_{4k+1} \to V_{4k+1} = \\ \quad \overbrace{D_3 \times \dots \times D_3}^{k} \times \overbrace{B_1 \times E_4 \times \dots \times E_4}^{h-k} \times \overbrace{F_3 \times \dots \times F_3}^{2k} \times \overbrace{E_2 \times E_2 \times \dots \times E_2}^{2(h-k)} \\ v_{4k+1} \to V_{4k+2} = \\ \quad -D_3 \times \dots \times D_3 \times B_3 \times E_4 \times \dots \times E_4 \times F_3 \times \dots \times F_3 \times F_1 \times E_2 \times \dots \times E_2 \\ v_{4k+3} \to V_{4k+3} = \\ \quad D_3 \times \dots \times D_3 \times D_1 \times E_4 \times \dots \times E_4 \times F_3 \times \dots \times F_3 \times F_3 \times E_2 \times \dots \times E_2 \\ v_{4k+4} \to V_{4k+4} = \\ \quad -D_3 \times \dots \times D_3 \times D_3 \times E_4 \times \dots \times E_4 \times F_3 \times \dots \times F_3 \times F_3 \times F_1 \times E_2 \times \dots \times E_2 \end{cases}$$

and analogous formulas for w_k. (Here E_i denotes unit matrix with degree i)

Analogously we can give an irr. rep. for the cases

(ii) $m = 4h+1$, Case I, II, deg $D = 2^{4h+1}$.

(iii) $m=4h+2$, Case I, II, deg $D=2^{4h+2}$; Case III. deg $D=2^{4h+3}$

(iv) $m=4h+3$, deg $D=2^{4h+3}$

V. The Clifford algebras $C(n,g)/K$ can be treated almost similarly. Here we outline only $C(2m+1,0)/K$. As can be seen easily, $C(2m+1,0) \simeq C(2m,0)\times Z$.

In cases (22), (23), (26), (27) : $C(2m+1,0)\simeq e_1C(2m,0)+e_2C(2m,0)$, where e_i are given by (20) or (21). $C(2m+1,0)$ has two irr. rep. D_1, D_2. Let an irr. rep. of $C(2m,0)$ be D'. Since

$$ze_1=e_1,\ ze_2=-e_2 \quad \text{or} \quad ze_1=\lambda e_1,\ ze_2=\lambda e_2$$

according to (20) or (21) respectively, we have

$$D_i(v_k)=D'(v_k),\quad D_i(w_k)=D'(w_k) \qquad (1\leqq k\leqq m)$$

$$D_i(x)=(-1)^m\ \xi_i\ D'(r_1r_2\dots r_m) \quad (i=1,2),$$

where $\xi_1=1$, $\xi_2=-1$ for (20) and $\xi_1=-\lambda$, $\xi_2=\lambda$ for (21). As D' is known by IV, these are the desired irr. rep. (deg $D_1=$deg $D_2=$deg D').

In cases (24), (25) :

(i) $m\equiv 0$ (mod. 2), $K=$Case II (Case (24))

$$C(2m+1,0)=C(2m,0)\times Z,\ C(2m,0)\simeq K(2^m).$$

Hence we can easily give its irr. rep. in K explicitely from IV and by using Kronecker product of irr. rep. of $C(2m,0)$ and of Z.

(ii) $m\equiv 0$ (mod. 2), $K=$Case III (Case (25)).

$$C(2m+1,0)=C(2m,\ 0)\times Z,\ C(2m,0)\simeq\begin{cases} K(2^m) & \text{for } m\equiv 0 \text{ (mod. 4)} \\ Q\times K(2^{m-1}) & \text{for } m\equiv 2 \text{ (mod. 4)}. \end{cases}$$

If $C(2m,0)\simeq K(2^m)$, we proceed as in (i).
If $C(2m,0)\simeq Q\times K(2^{m-1})$, then $Q\times Z=K(2)\times Z$ holds and we can give its system of matric units by

$$e_{11}=(1+\lambda i)/2,\ e_{12}=(\lambda j-k)/2,\ e_{21}=(\lambda j+k)/2,\ e_{22}=(1-\lambda i)/2$$

and an irr. rep. of $Q\times Z$ by

$$i\to\begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix},\ j\to\begin{pmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{pmatrix},\ k\to\begin{pmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix},\ \lambda\to\begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}$$

where $\lambda=z$, $\lambda^2=-1$.

We can easily give an irr. rep. of $C(2m+1,0)$ from these matrices and the decomposition $(m=4h+2)$

$$C(2m+1,0)=Q^{(1)}\times\ldots\times Q^{(2h)}\times P^{(1)}\times\times P^{(2h+1)}\times(Q^{(2h+1)}Z)$$

and from the results in IV.

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1) W. K. Clifford, Amer. J. Math., 1 (1878). A. S. Eddington, J. London Math. Soc., 7 (1932), 8(1933). M. H. A. Newmann, Ibid. 7(1932). J. E. Littlewood, Ibid. 9(1934). R. Brauer, H. Weyl, Amer. J. Math., 57(1935), H. Weyl, Classical groups, (1939). D. Wajnsztejn, Studia. Math., 9(1940). C. Chevalley, Theory of Lie groups, (1946). H. C. Lee, Ann. of Math., 49(1948). For the representations by real orthogonal matrices cf. A. Hurwitz (Werke II), J. Radon, Abh. Math. Sem. Hamburg, 1(1923), B. Eckmann, Comment. Math. Helv., 15(1943).

2). Cf. A. S. Eddington and M. H. A. Newmann, loc. cit. 1). Theorem 2 of Newmann's paper is incomplete, cf. Theorem 4 of this note.

3) Cf. C. Chevalley, loc. cit. 1).

4) In fact for a prime field K with $\chi(K)=p\neq 2$ there are α, β such that $1+\alpha^2+\beta^2=0$. Let $p=2m+1$ and let $a_1,\ldots a_m$ be a system of representatives of quadratic rests mod. p. If $\alpha^2+\beta^2+1$ were $\not\equiv 0 (\text{mod.}\ p)$ for any α, β then 0, $a_1,\ldots,a_m$, $-1-a_1,\ldots,-1-a_m$ (mod. p) would be a complete system of representatatives mod p. If we add them all, then $-m\equiv 1+2+\ldots+(p-1)\equiv 0$ (mod. p), which is a contradiction.

5) In fact $1+\lambda^2\equiv 0$ (mod. p) has a solution if and only if $p\equiv 1$ (mod. 4).

6) Cf. L. E. Dickson, Algebren und ihre Zahlentheorie (1927). or R. Brauer-E. Noether, S. B. Preuss. Akad., (1927). See also Appendix I. Cf also A. A. Albert, Structure of algebras, (1939), p. 147, Theorem 27.

7) This follows immediately from a well-known theorem of algebra (for example M. Deuring, Algebren (1934), p. 58, Satz 1). See also Appendix II.

8) See Appendix, III. Cf. also A.A. Albert, loc. cit 6).

9) See Appendix IV.

10) See Appendix V.

11) For the case when K is the complex number field this result is given by M. H. A. Newmann, loc. cit. 1).

12) For the case when Λ is the real number fleld, this result is given by A. S. Eddington for $m=2$, and by M. H. A. Newmann for $m=3$. For $m>3$ the result of Newmann (Theorem 2) is incomplete. He misses the possibility of the case where all matrices are purely imaginary.

Added in proof

Cf. also E. Witt, Crellés Journ., 176 (1937), 31–44.