Journal of the Mathematical Society of Japan Vol. 2, Nos. 1-2, Sept, 1950.

On the dimension of normal spaces. II.

Kiiti MORITA

(Received Sep. 1, 1949)

In the present paper we shall generalize some of the results obtained in a previous paper [5].[1]

As is well known, a normal space R is called to be of dimension not greater than n, $\dim R \leqq n$, in case for any finite open covering of R there exists an open refinement of order not greater than $n+1$. Our main theorem reads as follows: Let $\{G_\alpha;\ \alpha \in \Omega\}$ be a locally finite system of open sets in a normal space R and $\{F_\alpha;\ \alpha \in \Omega\}$ a system of closed sets such that $F_\alpha \subset G_\alpha$, $\alpha \in \Omega$. If the dimension of a closed set A of R is not greater than n, then there exists a system $\{U_\alpha;\ \alpha \in \Omega\}$ of open sets in R such that (1) $F_\alpha \subset U_\alpha \subset G_\alpha$ $\alpha \in \Omega$ and (2) the order of the system $\{A \cdot (\bar{U}_\alpha - U_\alpha);\ \alpha \in \Omega\}$ is not greater than n.

As an application of this theorem, we can prove a theorem that for a metrizable space R the relation $\dim R \leqq n$ implies the relation $\dim^* R \leqq n$, where we mean by $\dim^* R$ the dimension of R in the sense of Menger-Urysohn. In particular, for the case that R is a metric space with the star-finite property, the relation $\dim R \leqq n$ is shown to be equivalent to $\dim^* R \leqq n$. This theorem may be considered as a generalization of a well-known theorem for separable metric spaces, since such spaces have necessarily the star-finite property (Cf. [6]).

Besides the results mentioned above some other theorems will also be obtained.

§ 1. Locally finite systems.[2]

A system $\mathfrak{W}$ of subsets in a topological space R is called to be locally finite, if for each point p of R there exists a neighbourhood $U(p)$ such that $U(p)$ intersects a finite number of sets of $\mathfrak{W}$.

Theorem 1.1. *Let* $\{G_\alpha;\ \alpha \in \Omega\}$ *be a locally finite open covering of a*

1) Numbers in brackets refer to the Bibliography at the end of the paper.

2) The results of §§ 1,2 and 3 were published in [7] except Theorems 2.4 and 3.2.

normal space R. Then there exists a closed covering $\{F_\alpha;\ \alpha \in \Omega\}$ such that $F_\alpha \subset G_\alpha$, $\alpha \in \Omega$.

This is a known theorem. (Cf. [4] or [6]).

Theorem 1.2. *Let $\mathfrak{G}=\{G_\alpha;\ \alpha \in \Omega\}$ be a locally finite system of open sets in a topological space R and let $\{F_\alpha;\ \alpha \in \Omega\}$ be a system of closed sets such that $F_\alpha \subset G_\alpha$, $\alpha \in \Omega$. If we denote by $\mathfrak{G}'$ the intersection of all the binary coverings*

$$\{G_\alpha,\ R-F_\alpha\},\quad \alpha \in \Omega,$$

then the covering $\mathfrak{G}'$ is a locally finite open covering of R, and it holds that $S(F_\alpha, \mathfrak{G}') \subset G_\alpha$. Here we mean by $S(F_\alpha, \mathfrak{G}')$ the sum of the sets of $\mathfrak{G}'$ which intersect F_α.

Proof. A non-empty set G' of $\mathfrak{G}'$ is expressed in the form

$$G'=\prod_{\alpha \in \Gamma} G_\alpha \cdot \prod_{\beta \bar{\in} \Gamma}(R-F_\beta) \tag{1}$$

where Γ is a subset of Ω. Since $\mathfrak{G}$ is locally finite, Γ is a finite set. We shall show that $\mathfrak{G}'$ is locally finite. For a point p of R there exists a neighbourhood $U(p)$ such that the set $\Gamma_0(p)$ of indices α for which $U(p)\cdot G_\alpha \neq 0$ is a finite subset of Ω. If a set G' of $\mathfrak{G}'$ expressed in the form (1) intersects $U(p)$, then $\Gamma \subset \Gamma_0(p)$. Hence the number of sets of $\mathfrak{G}'$ intersecting $U(p)$ is finite. This shows that $\mathfrak{G}'$ is locally finite.

Next we shall prove that $\mathfrak{G}'$ is an open covering. Take a point p of $G' \in \mathfrak{G}'$. If $\beta \bar{\in} \Gamma_0(p)$, then $U(p)G_\beta=0$, and hence $U(p) \subset R-F_\beta$. Therefore, if we put

$$V(p)=\prod_{\alpha \in \Gamma} G_\alpha \cdot \prod_{\beta \in \Gamma_0(p)-\Gamma}(R-F_\beta)\cdot U(p),$$

$V(p)$ is a neighbourhood of p and $V(p) \subset G'$. Thus G' is an open set. Finally, if $G' \cdot F_\alpha \neq 0$, then we have $\alpha \in \Gamma$, that is, $G' \subset G_\alpha$. This shows that $S(F_\alpha, \mathfrak{G}') \subset G_\alpha$.

Corollary. *A locally finite open covering of a normal space admits an open Δ-refinement which is locally finite.*

This corollary is an immediate consequence of Theorems 1.1 and 1.2. (Cf. [6], [8])

Theorem 1.3. *Let $\mathfrak{G}=\{G_\alpha;\ \alpha \in \Omega\}$ be a locally finite system of open sets in a normal space R and $\mathfrak{F}=\{F_\alpha;\ \alpha \in \Omega\}$ a system of closed sets of R*

such that $F_\alpha \subset G_\alpha$, $\alpha \in \Omega$. *Then there exists a system of open sets* U_α, $\alpha \in \Omega$ *such that*

1° $F_\alpha \subset U_\alpha$, $\bar{U}_\alpha \subset G_\alpha$, $\alpha \in \Omega$;

2° *the system* $\{\bar{U}_\alpha;\ \alpha \in \Omega\}$ *is similar to the system* $\mathfrak{F}$.

Proof. Let us assume that the set of indices α consists of all (transfinite) ordinal numbers which are less than a fixed ordinal Ω_0. If we denote by Φ_1 the system of sets which are expressible as finite intersections $F_{\alpha_1} \ldots\ldots F_{\alpha_r}$ of sets of $\mathfrak{F}$ and are disjoint to F_1, then Φ_1 is locally finite, since $\mathfrak{F}$ is locally finite. Hence, if we denote the sum of the sets of Φ_1 by S_1, S_1 is closed in R, and $F_1 \cdot S_1 = 0$. Therefore there is an open set U_1 such that

$$F_1 \subset U_1, \quad \bar{U}_1 \subset G_1, \quad \bar{U}_1 \subset R - S_1.$$

If we construct a system $\mathfrak{U}_1 = \{\bar{U}_1, F_2, \ldots\}$ by replacing F_1 in $\mathfrak{F}$ by $\bar{U}_1$, it follows that the system $\mathfrak{U}_1$ is similar to $\mathfrak{F}$.

We shall prove the theorem by transfinite induction. For this purpose let us suppose that for any β less than some fixed ordinal $\alpha < \Omega_0$ there exists an open set U_β such that $F_\beta \subset U_\beta$, $\bar{U}_\beta \subset G_\beta$ and the system $\mathfrak{U}_\beta = \{\bar{U}_\gamma;\ \gamma \leqq \beta,\ F_\gamma;\ \beta < \gamma < \Omega_0\}$ is similar to $\mathfrak{F}$. Then the system $\{\bar{U}_\gamma;\ \gamma < \alpha,\ F_\gamma;\ \alpha \leqq \gamma < \Omega_0\}$ is also locally finite and similar to $\mathfrak{F}$, as is easily shown. Hence by the method described above we can construct an open set U_α such that $F_\alpha \subset U_\alpha$, $\bar{U}_\alpha \subset G_\alpha$ and the system $\mathfrak{U}_\alpha = \{\bar{U}_\gamma;\ \gamma \leqq \alpha,\ F_\gamma;\ \alpha < \gamma < \Omega_0\}$ is similar to $\mathfrak{F}$. The system $\mathfrak{U} = \{\bar{U}_\alpha;\ \alpha \in \Omega\}$ of open sets U_α constructed in such a way is shown to be similar to $\mathfrak{F}$. This proves the theorem.

§ 2. Locally finite coverings and the dimension.

Theorem 2.1. *Let R be a normal space of dimension $\leqq n$, and let $\mathfrak{G} = \{G_\alpha;\ \alpha \in \Omega\}$ be a locally finite open covering of R. Then there exists an open covering $\mathfrak{U} = \{U_\alpha;\ \alpha \in \Omega\}$ of order $\leqq n+1$ such that $U_\alpha \subset G_\alpha$, $\alpha \in \Omega$.*

This theorem is proved by C.H. Dowker [1][3]. We shall give here our proof based on the same idea as in a previous paper [5].

Proof. In case the cardinal number of the set Ω is finite the theorem reduces to the definition dim $R \leqq n$. To prove the theorem by transfinite induction, we shall prove the theorem for the case that the cardinal number of Ω is $\aleph_\nu$, under the assumption that the theorem holds in case the car-

3) Dowker's proof seems to be contained in his paper "Mapping theorems for non-compact spaces" in Amer. Jour. Math. 69, but his paper is not yet available to us.

dinal number of Ω is less than $\aleph_\nu$ $(\nu \geqq 0)$. For this purpose let us suppose that the set Ω of indices α consists of all transfinite ordinals α less than an "Anfangszahl" ω_ν belonging to $\aleph_\nu$. For the sake of simplicity we write $\Omega = \omega_\nu$, that is, $\mathfrak{G} = \{G_\alpha\,;\ \alpha < \Omega\}$.

Since $\mathfrak{G}$ is locally finite there exists a closed covering $\{F_\alpha\,;\ \alpha < \Omega\}$ such that $F_\alpha \subset G_\alpha$. There exists an open set L_α such that $F_\alpha \subset L_\alpha$, $\bar{L}_\alpha \subset G_\alpha$. Let us construct an open covering $\{U_\alpha\}$ by transfinite induction. Suppose that for any ordinal β less than some fixed α we have constructed open sets $U_{\beta\gamma}(\gamma < \Omega)$ such that

$$(C_\beta) \qquad \begin{cases} U_{\beta\gamma} = 0, \quad \text{for} \quad \beta < \gamma < \Omega\,; \\ \bar{U}_{\beta\gamma} \subset L_\beta L_\gamma, \quad \text{for} \quad \gamma \leqq \beta\,; \\ F_\beta \subset \sum\limits_{\gamma \leqq \beta} U_{\beta\gamma}\,; \\ \text{the order of } \{\sum\limits_{\sigma \leqq \beta} \bar{U}_{\sigma\gamma}\,;\ \gamma \leqq \beta\} \leqq n+1. \end{cases}$$

Then we shall show the existence of open sets $U_{\alpha\gamma}(\gamma < \Omega)$ satisfying the condition (C_α). If we put

$$(2) \qquad U_\gamma' = \sum_{\beta < \alpha} U_{\beta\gamma}, \text{ for } \gamma < \alpha,$$

then we have

$$(3) \qquad \bar{U}_\gamma' = \sum_{\alpha > \alpha} \bar{U}_{\beta\gamma} \subset L_\gamma,$$

since $\bar{U}_{\beta\gamma} \subset L_\beta$ and hence the system $\{\bar{U}_{\beta\gamma}\,;\ \beta < \alpha\}$ is locally finite. Then we have

(4) the order of the system $\{\bar{U}_\gamma'\,;\ \gamma < \alpha\} \leqq n+1$.

Because, if there is a point p such that $p \in \bar{U}_{\gamma i}'$, $i = 1, 2, \ldots, n+2$, there exist $\beta_i < \alpha$ such that $p \in \bar{U}_{\beta_i \gamma_i}$ and since there exists a β such that $\beta_i \leqq \beta < \alpha$, $\gamma_i \leqq \beta$ for $i = 1, 2, \ldots, n+2$, we have

$$p \in \sum_{\sigma \leqq \beta} \bar{U}_{\sigma\gamma_i}, \quad i = 1, 2, \ldots, n+2,$$

which contradicts the last condition of (C_β).

According to Theorem 1.3 there exist open sets V_γ $(\gamma < \alpha)$ such that

$$(5) \qquad \bar{U}'_\gamma \subset V_\gamma, \quad \bar{V}_\gamma \subset L_\gamma\,;$$

$$(6) \qquad \{V_\gamma\,;\ \gamma < \alpha\} \text{ is similar to } \{\bar{U}'_\gamma\,;\ \gamma < \alpha\}.$$

If we construct the intersection $\mathfrak{V}$ of all the binary coverings $\{V_\gamma, R-\bar{U}'_\gamma\}$, $\gamma<\alpha$, $\mathfrak{V}$ is a locally finite open covering of R and satisfies the condition

$$(7) \qquad S(\bar{U}'_\gamma, \mathfrak{V}) \subset V_\gamma, \quad \gamma<\alpha,$$

by virtue of Theorem 1.2. Here the cardinal number of the family of sets of $\mathfrak{V}$ is not greater than the cardinal number of the set $\{\gamma;\ \gamma<\alpha\}$, and hence less than $\aleph_\nu$. Hence, by the assumption of induction and the relation $\dim F_\alpha \leqq \dim R \leqq n$, there is a system of open sets $H_\lambda(\lambda \in \Lambda)$ such that

$$(8) \qquad \begin{cases} F_\alpha \subset \sum_\lambda H_\lambda, \quad \bar{H}_\lambda \subset L_\alpha, \\ \{\bar{H}_\lambda;\ \lambda \in \Lambda\} \text{ is a refinement of } \mathfrak{V} \text{ and has order} \leqq n+1. \end{cases}$$

Here we may assume that $\{\bar{H}_\lambda;\ \lambda \in \Lambda\}$ is also locally finite, since $\mathfrak{V}$ is locally finite.

Let us denote by $U_{\alpha 1}$ the sum of the sets H_λ such that $\bar{H}_\lambda \cdot \bar{U}_1' \neq 0$ and by $U_{\alpha 2}$ the sum of the sets H_λ such that $\bar{H}_\lambda \cdot \bar{U}_1'=0$, but $\bar{H}_\lambda \cdot \bar{U}_2' \neq 0$, and so on. Further we shall denote by $U_{\alpha\alpha}$ the sum of the sets H_λ such that $\bar{H}_\lambda \cdot \bar{U}'_\gamma=0$ for each $\gamma<\alpha$. For $\gamma>\alpha$, let us put $U_{\alpha\gamma}=0$. Then these open sets $U_{\alpha\gamma}(\gamma<\Omega)$ satisfies the condition (C_α). To prove this we have only to prove

(9) the order of $\{\bar{U}'_\gamma+\bar{U}_{\alpha\gamma};\ \gamma<\alpha,\ \bar{U}_{\alpha\alpha}\} \leqq n+1$,

(10) $\bar{U}_{\alpha\gamma} \subset L_\alpha L_\gamma, \quad \gamma<\alpha.$

By the construction of H_λ we have

$$(11) \qquad U_{\alpha\gamma} \subset S(\bar{U}'_\gamma, \mathfrak{V}) \subset V_\gamma, \quad \bar{V}_\gamma \subset L_\gamma, \text{ for } \gamma<\alpha,$$

and hence we have (10). To prove (9) let us suppose that

$$X=\prod_{i=1}^{r}(\bar{U}'_{\gamma_i}+\bar{U}_{\alpha\gamma_i}) \cdot \bar{U}_{\alpha\alpha} \neq 0, \ \gamma_1<\gamma_2<\ldots<\gamma_r<\alpha.$$

Then, since $\bar{U}'_{\gamma_i} \cdot \bar{U}_{\alpha\alpha}=0$, we have

$$X=\prod_{i=1}^{r} \bar{U}_{\alpha\gamma_i} \cdot \bar{U}_{\alpha\alpha} \neq 0$$

and hence $r+1 \leqq n+1$ by (8). If

$$X=\prod_{i=1}^{r}(\bar{U}'_{\gamma_i}+\bar{U}_{\alpha\gamma_i}) \neq 0,$$

then we have, by (11), $X \subset \prod_{i=1} V_{\gamma i}$ and hence the inequality $r \leqq n+1$ is established by (6), (4).

Thus we have proved for any ordinal β less than Ω, the existence of open sets $U_{\beta\gamma}$ satisfying the condition (C_β). If we put

$$U_\alpha = \sum_\sigma U_{\sigma\alpha},\ \alpha < \Omega,$$

then we have

$$U_\alpha \subset L_\alpha \subset G_\alpha,$$
$$R = \sum_\alpha U_\alpha.$$

Hence $\mathfrak{U} = \{U_\alpha;\ \alpha < \Omega\}$ is an open covering of R and its order is not greater than $n+1$, since, if $p \in U_{\alpha_i}$, $i=1,\ldots,n+2$, we have $p \in U_{\sigma_i \alpha_i}$ for some $\sigma_i < \Omega$ and hence $p \in \sum_{\sigma > \beta} U_{\sigma\alpha_i}$, $i=1,\ldots,n+2$ for some β such that $\sigma_i \leqq \beta$, $\sigma_i \leqq \beta$, $\alpha_i \leqq \beta < \Omega$ contradicting the condition (C_β). Thus Theorem 2.1 is completely proved.

In the above proof the following theorem is essentially proved.

Theorem 2.2. *Let $\mathfrak{G} = \{G_\alpha;\ \alpha \in \Omega\}$ be a regular*[4] *open covering of a normal space R such that every subsystem of $\mathfrak{G}$ whose cardinal number is less than that of $\mathfrak{G}$ is locally finite. Then, if* dim $R \leqq n$, *there exists an open refinement of $\mathfrak{G}$ of order $\leqq n+1$.*

Corollary. *If $\mathfrak{G}$ is a countable regular open covering of a normal space of dimension $\leqq n$, then there exists an open refinment of order $\leqq n+1$.*

The following theorem is an immediate consequence of Theorems 2.1 and 1.3.

Theorem 2.3. *Let $\mathfrak{G} = \{G_\alpha;\ \alpha \in \Omega\}$ be a locally finite open covering of a normal space R. If the dimension of a closed subset A of R is not greater than n, then there exists an open covering $\{U_\alpha;\ \alpha \in \Omega\}$ such that $\bar{U}_\alpha \subset G_\alpha$, $\alpha \in \Omega$ and the order of $\{A \cdot \bar{U}_\alpha;\ \alpha \in \Omega\}$ is not greater than $n+1$.*

Theorem 2.4. *Let R be a fully normal space. In order that* dim $R \leqq n$ *it is necessary and sufficient that for any open covering of R there exists an open refinement of order $\leqq n+1$.*

This theorem follows readily from Theorem 2.1 and a theorem of A. H. Stone [8].

§ 3. The sum theorems.

Proceeding analogously as in the proof of Theorem 2.1 we can estab-

4) Cf. [5].

lish the following theorem.

Theorem 3.1. *Let* $\mathfrak{F}=\{F_\alpha\,;\ \alpha\in\Omega\}$ *be a closed covering of a normal space* R *and let the dimension of each set* F_α *be not greater than* n. *If there exists an open covering* $\mathfrak{G}=\{G_\alpha\,;\ \alpha\in\Omega\}$ *such that every subsystem of* $\mathfrak{G}$ *whose cardinal number is less than that of* $\mathfrak{G}$ *is locally finite and* $F_\alpha\subset G_\alpha$, $\alpha\in\Omega$, *then we have* $\dim R\leqq n$.

The sum theorem in the usual sense is a special case of this theorem.

Corollary. *A normal space which is the sum of a countable number of closed subsets of dimension* $\leqq n$ *has dimension* $\leqq n$.

Theorem 3.2. *Let* $\{F_\alpha\,;\ \alpha\in\Omega\}$ *be a locally finite closed covering of a fully normal space* R. *If for each* α $\dim F_\alpha\leqq n$, *then we have* $\dim R\leqq n$.

For the proof of Theorem 3.2 it is sufficient, in view of Theorem 3.1., to prove the following lemma.

Lemma. *Let* $\mathfrak{F}=\{F_\alpha\,;\ \alpha\in\Omega\}$ *be a locally finite system of closed subsets in a fully normal space* R. *Then there exists a locally finite system of open sets* G_α, $\alpha\in\Omega$ *such that* $F_\alpha\subset G_\alpha$.

Proof of Lemma. For each point p of R there exists a neighbourhood $U(p)$ such that $U(p)$ intersects only a finite number of sets of $\mathfrak{F}$. Let us put $\mathfrak{U}=\{U(p)\ ;\ p\in R\}$ and construct a Δ-refinement $\mathfrak{V}$ of some Δ-refinement of $\mathfrak{U}$, $\mathfrak{V}^{\Delta\Delta}<\mathfrak{U}$[5]. If we denote by G_α the set $S(F_\alpha,\ \mathfrak{V})$, then $\{G_\alpha\}$ satisfies the condition of the lemma. Because, if $S(x,\ \mathfrak{V})\cdot G_\alpha\neq 0$, we have $S(x,\ \mathfrak{V}^{\Delta})F_\alpha\neq 0$, and since $S(x,\ \mathfrak{V}^{\Delta})$ is contained in some $U(p)$, $S(x,\ \mathfrak{V})$ intersects only a finite number of sets G_α.

Finally we state the following theorem, which can be proved similarly as in the previous paper [5], with the aid of Therem 2.1 and theorems of § 1. For a detailed proof, Cf. [7]. For another proof, Cf. § 4.

Theorem 3.3. *Let* $\mathfrak{G}$ *be a locally finite open covering of a normal space* R *and let* A, B *be two closed subsets of* R. *If it holds that*

$$(\mathfrak{G})\text{-}\dim A\leqq n,\quad (\mathfrak{G})\text{-}\dim B\leqq n,\quad \dim A\cdot B\leqq n-1,$$

then we have

$$(\mathfrak{G})\text{-}\dim\,[A+B]\leqq n.$$

4. Main theorem.

Now we shall proceed to the proof of our main theorem.

Theorem 4.1. *Let* $\{G_\alpha\,;\ \alpha\in\Omega\}$ *be a locally finite system of open sets in a normal space* R *and* $\{F_\alpha\,;\ \alpha\in\Omega\}$ *a system of closed sets such that* F_α

5) Cf. [9], p. 44.

$\subset G_\alpha$, $\alpha \in \Omega$. *If the dimension of a closed subset* A *of* R *is not greater than* n, *then there exists a system of open sets* U_α, $\alpha \in \Omega$ *such that*

(12) $\quad F_\alpha \subset U_\alpha \subset G_\alpha, \quad \alpha \in \Omega,$

(13) $\quad$ *the order of the system* $\{A(\bar{U}_\alpha - U_\alpha);\ \alpha \in \Omega\} \leqq n$.

In case the set Ω of indices α is a finite set this theorem has already been proved in [5].

Proof. Let us construct the intersection $\mathfrak{H}$ of all the binary coverings $\{G_\alpha, R-F_\alpha\}$, $\alpha \in \Omega$ and put $\mathfrak{H}=\{H_\lambda;\ \lambda \in \Lambda\}$ where

$$H_\lambda = \prod_{\alpha \in \lambda} G_\alpha \cdot \prod_{\gamma \bar{\in} \lambda} (R - F_\gamma), \tag{14}$$

and λ means a finite subset of Ω (including the empty set) and Λ means the family of finite subsets of Ω. Since $\mathfrak{H}$ is a locally finite open covering of R by Theorem 1.2., there exists an open covering $\mathfrak{L}=\{L_\lambda;\ \lambda \in \Lambda\}$ such that

(15) $\quad L_\lambda \subset H_\lambda, \quad \lambda \in \Lambda,$

(16) $\quad$ the order of $\{A \cdot L_\lambda;\ \lambda \in \Lambda\} \leqq n+1$.

According to Theorem 1.1., since $\mathfrak{L}$ is locally finite, there is a closed covering $\mathfrak{N}=\{N_\lambda;\ \lambda \in \Lambda\}$ such that $N_\lambda \subset L_\lambda$.

Let us construct for each $\lambda \in \Lambda$ a continuous function $f_\lambda(x)$ such that $0 \leqq f_\lambda(x) \leqq 1$ and

$$f_\lambda(x) = \begin{cases} 0, & x \in N_\lambda \\ 1, & x \bar{\in} L_\lambda, \end{cases} \tag{17}$$

and put, for $0<\theta<1$,

$$M_\lambda(\theta) = \{x;\ f_\lambda(x) < \theta\}. \tag{18}$$

Then $M_\lambda(\theta)$ is clearly an open set and

(19) $\quad N_\lambda \subset M_\lambda(\theta) \subset L_\lambda, \quad 0<\theta<1, \quad \lambda \in \Lambda,$

(20) $\quad \overline{M_\lambda(\theta_1)} \subset M_\lambda(\theta_2), \quad$ for $\theta_1 < \theta_2$.

If we set

(21) $\quad \Omega(\lambda) = \{\alpha;\ F_\alpha \cdot N_\lambda \neq 0, \ \alpha \in \Omega\},$

then we have

(22) $\quad \Omega(\lambda) \subset \lambda.$

Because, if $\alpha \in \Omega(\lambda)$, then $F_\alpha \cdot N_\lambda \neq 0$ and hence $F_\alpha \cdot H_\lambda = F_\alpha \cdot \prod_{\beta \in \lambda} G_\beta \cdot \prod_{\gamma \bar{\in} \lambda} (R - F_\gamma) \neq 0$, so we have $\alpha \in \lambda$.

Since λ is a finite subset of Ω, $\Omega(\lambda)$ is also finite. We can, therefore, correspond to each $\alpha \in \Omega(\lambda)$ a real number $\theta_\lambda(\alpha)$ such that $0 < \theta_\lambda(\alpha) < 1$ and

$$(23) \qquad \theta_\lambda(\alpha) \neq \theta_\lambda(\beta), \quad \text{for} \quad \alpha \neq \beta, \quad \alpha, \beta \in \Omega(\lambda).$$

Now let us put

$$(24) \qquad M_\lambda(\alpha) = M_\lambda(\theta_\lambda(\alpha)), \quad \alpha \in \Omega(\lambda),$$

and

$$(25) \qquad U_\alpha = \sum_\lambda M_\lambda(\alpha), \quad \alpha \in \Omega(\lambda),$$

where $\sum$ means the sum extending over all λ such that $\alpha \in \Omega(\lambda)$. We shall prove that these U_α satisfy the condition of the theorem.

Since $\{N_\lambda ; \lambda \in \Lambda\}$ is a closed covering of R we have

$$F_\alpha \subset S(F_\alpha, \mathfrak{N}) = \sum_{\Omega(\lambda) \ni \alpha} N_\lambda \subset U_\alpha,$$

and it follows from Theorem 1.2 that

$$U_\alpha \subset S(F_\alpha, \mathfrak{L}) \subset S(F_\alpha, \mathfrak{H}) \subset G_\alpha.$$

Thus U_α satisfies the condition (12) of the theorem.

To prove (13) we take $n+1$ different indices α_i $(i=1,2,\dots,n+1)$ belonging to Ω and consider the set

$$(26) \qquad P = A \cdot \prod_{i=1}^{n+1} (\bar{U}_{\alpha_i} - U_{\alpha_i}).$$

Then our aim is to prove that P is empty. Since $\{L_\lambda ; \lambda \in \Lambda\}$ is locally finite, we have

$$\bar{U}_{\alpha_i} = \sum_{\Omega(\lambda) \ni \alpha_i} \overline{M_\lambda(\alpha_i)}, \quad i=1,2,\dots, n+1$$

and hence

$$P = A \cdot \prod_{i=1}^{n+1} \Big[\sum_{\Omega(\lambda) \ni \alpha_i} \overline{M_\lambda(\alpha_i)} \prod_{\Omega(\lambda) \ni \alpha_i} \{R - M_\lambda(\alpha_i)\}\Big]$$
$$= \sum_{\Omega(\lambda_1) \ni \alpha_1} \cdots \sum_{\Omega(\lambda_{n+1}) \ni \alpha_{n+1}} Q(\lambda_1, \dots, \lambda_{n+1}),$$

where

$$Q(\lambda_1, \dots, \lambda_{n+1}) = A . \prod_{i=1}^{n+1} \Big[\overline{M_{\lambda_i}(\alpha_i)} \prod_{\Omega(\lambda) \ni \alpha_i} \{R - M_\lambda(\alpha_i)\}\Big].$$

Consequently it is sufficient to prove that $Q(\lambda_1, \lambda_2, \dots, \lambda_{n+1}) = 0$. For this purpose we distinguish two cases; the first is the case that among $\lambda_1,\dots,\lambda_{n+1}$ there are at least two indices λ_i, λ_j such that $\lambda_i=\lambda_j$ and the second is the case that $\lambda_1,\dots,\lambda_{n+1}$ are different from one another.

Case 1). Let us put $\lambda_i=\lambda_j=\lambda$. Then we have $a_i \in \Omega(\lambda)$, $a_j \in \Omega(\lambda)$, $a_i \neq a_j$. By (23), $\theta_\lambda(a_i) \neq \theta_\lambda(a_j)$ and so we may assume that $\theta_\lambda(a_i) < \theta_\lambda(a_j)$, Then it follows from (20) and (24) that $\overline{M_{\lambda_i}(a_i)} \subset M_{\lambda_j}(a_j)$. Hence we have

$$Q(\lambda_1,\dots,\lambda_{n+1}) \subset \overline{M_{\lambda_i}(a_i)}\{R-M_{\lambda_j}(a_j)\}=0.$$

Case 2). First we shall show that

$$Q(\lambda_1,\dots,\lambda_{n+1}) \cdot N_\lambda=0, \text{ for every } \lambda \in \Lambda. \tag{27}$$

a). In case λ is equal to some λ_i, we have

$$N_\lambda \subset M_\lambda(a_i)=M_{\lambda_i}(a_i),$$

and hence

$$Q(\lambda_1,\dots,\lambda_{n+1}) \cdot N_\lambda \subset \{R-M_{\lambda_i}(a_i)\} \cdot N_\lambda=0.$$

b). In case $\lambda \neq \lambda_i$ for $i=1,2,\dots,n+1$, we have

$$Q(\lambda_1,\dots,\lambda_{n+1}) \cdot N_\lambda \subset A\{\prod_{i=1}^{n+1} \overline{M_{\lambda_i}(a_i)}\}N_\lambda$$

$$\subset A \cdot \{\prod_{i=1}^{n+1} L_{\lambda_i}\} \cdot L_\lambda=0,$$

in view of (16).

Hence the relation (27) is established. Since $\{N_\lambda;\ \lambda \in \Lambda\}$ is a covering of R it follows from (27) that $Q(\lambda_1,\dots,\lambda_{n+1})=0$.

Thus we have proved that $P=0$. This shows that the order of the system $\{A(\bar{U}_\alpha-U_\alpha)\ ;\ \alpha \in \Omega\}$ is not greater than n, and the theorem is completely proved.

According to Theorem 1.3 we can easily deduce the following theorem from Theorem 4.1. (Cf. [5]).

Theorem 4.2. *Under the same assumption as in Theorem* 4.1 *there exist two systems of open sets* U_α, $\alpha \in \Omega$; V_α, $\alpha \in \Omega$ *such that*

$$F_\alpha \subset V_\alpha, \quad \bar{V}_\alpha \subset U_\alpha \subset G_\alpha, \quad \alpha \in \Omega, \tag{28}$$

(29) *the order of* $\{A(\bar{U}_\alpha-V_\alpha)\ ;\ \alpha \in \Omega\} \leqq n.$

As an application of Theorem 4.1 we give here a proof of Theorem 3.3.

Proof of Theorem 3.3. Let $\mathfrak{G}=\{G_\alpha\ ;\ \alpha<\Omega\}$ be a locally finite open covering of R. By the hypothesis that $(\mathfrak{S})$-dim $A \leqq n$, $(\mathfrak{S})$-dim $B \leqq n$, there exist closed sets A_α, B_α and open sets L_α, M_α $(\alpha < \Omega)$ such that

$$A=\sum A_\alpha, \quad A_\alpha \subset L_\alpha, \quad \bar{L}_\alpha \subset G_\alpha,$$
$$B=\sum B_\alpha, \quad B_\alpha \subset M_\alpha, \quad \bar{M}_\alpha \subset G_\alpha,$$

and the orders of the systems $\{\bar{L}_\alpha;\ \alpha<\Omega\}$, $\{\bar{M}_\alpha;\ \alpha<\Omega\}$ do not exceed $n+1$. According to Theorem 4.1 we can find open sets U_α, V_α such that

$$A_\alpha \subset U_\alpha \subset L_\alpha, \qquad B_\alpha \subset V_\alpha \subset M_\alpha,$$

and the order of $\{AB\ (\bar{U}_\alpha - U_\alpha),\ AB(\bar{V}_\beta - V_\beta);\ \alpha, \beta < \Omega\} \leqq n-1$. If we put $P_\alpha = U_\alpha - \sum_{\beta<\alpha} \bar{U}_\beta$, $Q_\alpha = V_\alpha - \sum_{\beta<\alpha} \bar{V}_\beta$, then we have $A=\sum A\cdot\bar{P}_\alpha$, $B=\sum B\cdot\overline{Q}_\alpha$, and the order of $\{A\cdot\bar{P}_\alpha,\ B\cdot\overline{Q}_\beta;\ \alpha, \beta<\Omega\}$ does not exceed $n+1$. Because, if it holds that for $\alpha_1 < \ldots < \alpha_r$, $\beta_1 < \ldots < \beta_s$,

$$X=\prod_{\nu=1}^{r} A\cdot\bar{P}_{\alpha_\nu} \prod_{\nu=1}^{s} B\cdot\overline{Q}_{\beta_\nu} \neq 0,$$

then we can show that $r+s \leqq n+1$:

Case 1). In case $r=0$ or $s=0$, we have clearly $r+s \leqq n+1$.

Case 2). In case $r>0$, $s>0$, we have

$$X \subset A\cdot B \prod_{\nu=1}^{r-1} (\bar{U}_{\alpha_\nu} - U_{\alpha_\nu}) \prod_{\nu=1}^{s-1} (\bar{V}_{\beta_\nu} - V_{\beta_\nu}),$$

and hence $(r-1)+(s-1) \leqq n-1$, that is, $r+s \leqq n+1$. *Q.E.D.*

By virtue of Theorem 3.3, as in the case of separable metric spaces, we can prove the following theorem, if we utilize Zorn's lemma instead of Brouwer's reduction theorem.

Theorem 4.3. *Any n-dimensional bicompact normal space contains a subset which is an n-dimensional Cantor-manifold.*

§ 5. Metric spaces with the star-finite property.

Let us define, after K. Menger, the dimension dim* R of a topological space R by induction as follows: (1) If R is empty, dim* $R=-1$, (2) If for each point p of R and its any neighbourhood U there exists a neighbourhood V such that $p \subset V \subset U$, dim* $(\bar{V}-V) \leqq n-1$, then we define dim* $R \leqq n$. We shall first prove

Theorem 5.1. *For any metric space R the relation* dim $R \leqq n$ *implies the relation* dim* $R \leqq n$.

Proof. According to a theorem of A.H. Stone [8], there exists a countable collection $\mathfrak{U}_1, \mathfrak{U}_2, \ldots$ of locally finite open coverings such that for each point p of R $\{S(p, \mathfrak{U}_i);\ i=1,2,\ldots\}$ is a basis of neighbourhoods at p. Let us assume that $\mathfrak{U}_i$ consists of open sets $U_{i\alpha}$, $\alpha \in \Omega$. By Theorem 1.1 there is a closed covering $\{F_{i\alpha};\ \alpha \in \Omega\}$ such that $F_{i\alpha} \subset U_{i\alpha}$, $\alpha \in \Omega$. Since $\dim R \leqq n$ there exist, by Theorem 4.2, open sets $U_{1\alpha}^1$, $V_{1\alpha}^1$, $\alpha \in \Omega$ such that

(30) $$F_{1\alpha} \subset V_{1\alpha}^1, \quad \bar{V}_{1\alpha}^1 \subset U_{1\alpha}^1 \subset U_{1\alpha}, \quad \alpha \in \Omega,$$

(31) the order of $\{\bar{U}_{1\alpha}^1 - V_{1\alpha}^1;\ \alpha \in \Omega\} \leqq n$.

By an inductive process we can construct successively open sets $U_{k\alpha}^i$ $V_{k\alpha}^i$, $1 \leqq k \leqq i$ such that

(32) $$F_{i\alpha} \subset V_{i\alpha}^i, \quad \bar{V}_{i\alpha}^i \subset U_{i\alpha}^i \subset U_{i\alpha}, \quad \alpha \in \Omega,$$

(33) $$\bar{V}_{k\alpha}^{i-1} \subset V_{k\alpha}^i, \quad \bar{V}_{k\alpha}^i \subset U_{k\alpha}^i \subset U_{k\alpha}^{i-1}, \quad 1 \leqq k < i,$$

(34) the order of $\{\bar{U}_{k\alpha}^i - V_{k\alpha}^i;\ k=1,2,\ldots,i,\ \alpha \in \Omega\} \leqq n$.

The existence of such open sets is assured by Theorem 4.2, since the system $\{U_{k\alpha}^{i-1};\ k=1,2,\ldots, i-1,\ \alpha \in \Omega,\ U_{i\alpha};\ \alpha \in \Omega\}$ is locally finite. Now let us put

$$V_{k\alpha} = \sum_{i=k}^{\infty} V_{k\alpha}^i, \quad \alpha \in \Omega.$$

Then we have

$$V_{k\alpha}^i \subset V_{k\alpha} \subset U_{k\alpha}^i, \quad \alpha \in \Omega.$$

If $n+1$ pairs of (k_ν, α_ν), $\nu=1,2,\ldots, n+1$ are different from one another, we have, for any integer k such that $k > k_\nu$ for $\nu=1,\ldots, n+1$,

$$V_{k_\nu \alpha_\nu}^k \subset V_{k_\nu \alpha_\nu} \subset U_{k_\nu \alpha_\nu}^k$$

and hence by (34)

$$\prod_{\nu=1}^{n+1} (\bar{V}_{k_\nu \alpha_\nu} - V_{k_\nu \alpha_\nu}) \subset \prod_{\nu=}^{n+1} (\bar{V}_{k_\nu \alpha_\nu}^k - V_{k_\nu \alpha_\nu}^k) = 0.$$

The system of open sets $V_{i\alpha}$, $i=1,2,\ldots;\ \alpha \in \Omega$ is clearly an open basis of R, and the order of the system $\{\bar{V}_{i\alpha} - V_{i\alpha};\ \alpha \in \Omega,\ i=1,2,\ldots\}$ is not greater than n, as is shown above. Therefore our theorem is established if the following lemma is proved.

Lemma. *If for a topological space R there exists an open basis $\{V_\alpha;\ \alpha \in \Omega\}$ such that the order of $\{\bar{V}_\alpha - V_\alpha;\ \alpha \in \Omega\}$ does not exceed n, then we* *have* $\dim^* R \leqq n$.

Proof of Lemma is the same as in [5]. If $n=0$, then $\bar{V}_\alpha = V_\alpha$ and the lemma is evident by the definition of the dimension $\dim^* R$. Suppose that the lemma holds for $n-1$. Then for each point p and its arbitrary neighbourhood U there is some open set V_α such that $p \subset V_\alpha \subset U$. If we denote by A the subspace $\bar{V}_\alpha - V_\alpha$, then $\{A \cdot V_\gamma;\ \gamma \in \Omega,\ \gamma \neq \alpha\}$ is an open basis of the space A and the order of the system $\{A \cdot \overline{A \cdot V_\gamma} - A \cdot V_\gamma;\ \gamma \neq \alpha\}$ does not exceed $n-1$ by the assumption of the lemma. Hence, by the assumption of induction we have $\dim^* A \leqq n-1$. Therefore it holds that $\dim^* R \leqq n$.

Theorem 5.2. *Let R be a normal space with the star-finite property. If* $\dim^* R \leqq n$, *then we have* $\dim R \leqq n$.

Remark. In case for any open covering of a topological space R there exists a refinement which is a star-finite open covering, R is said to have the star-finite property. It is to be noted that for a regular space the Lindelöf property implies the star-finite pryperty. (Cf. [6])

Proof of Theorem 5.2. We shall carry out the proof by induction. For this purpose we shall assume the theorem for $n-1$ and prove the theorem for n. Let us first prove that if $\dim^* R \leqq n$, then for an open set G and a closed set F such that $F \subset G$ there exists an open set H satisfying the conditions

(35) $$F \subset H \subset G,$$

(36) $$\dim\ (\bar{H} - H) \leqq n-1.$$

Now take an open set L such that $F \subset L$, $\bar{L} \subset G$ and construct for each point p of R a neighbourhood $U(p)$ such that

(37) $$\begin{cases} \overline{U(p)} \subset G, & \textit{in case } p \in \bar{L}, \\ \overline{U(p)}\bar{L} = 0, & \textit{in case } p \bar{\in} \bar{L} \end{cases}$$

(38) $$\dim^* (\overline{U(p)} - U(p)) \leqq n-1.$$

Then from the star-finite property of R it follows that there exists a star-finite open covering $\mathfrak{U}$ which is a refinement of the covering $\{U(p);\ p \in R\}$. We may assume that $\mathfrak{U}$ consists of open sets U^i_γ, $\gamma \in \Gamma$, $i=1,2,\ldots$ such that

(39) $$U^i_\gamma \cdot U^j_\delta = 0, \quad \textit{for } \gamma \neq \delta.$$

If we put

(40) $$U_\gamma=\sum_{i=1}^{\infty} U_\gamma^i,$$

U_γ is at the same time closed and open. By the construction, for any U_γ^i there is a neighbourhood $U(p_\gamma^i)$ of some point p_γ^i of R satisfying the conditions (37), (38) and containing U_γ^i. Now let us put

$$V_\gamma^i=U_\gamma\cdot U(p_\gamma^i).$$

Then we have $\bar{V}_\gamma^i-V_\gamma^i\subset U_\gamma\cdot\{\overline{U(p_\gamma^i)}-U(p_\gamma^i)\}$ and hence by (38) $\dim^*(\bar{V}_\gamma^i-V_\gamma^i)\leqq n-1$, and consequently by the assumption of induction $\dim(\bar{V}_\gamma^i-V_\gamma^i)\leqq n-1$. Therefore

(41) $$U_\gamma=\sum_{i=1}^{\infty} V_\gamma^i,\quad \dim(\bar{V}_\gamma^i-V_\gamma^i)\leqq n-1.$$

If we construct open sets

$$W_\gamma^1=V_\gamma^1,\ W_\gamma^2=V_\gamma^2-\bar{V}_\gamma^1,\dots,\ W_\gamma^i=V_\gamma^i-(\bar{V}_\gamma^1+\dots+\bar{V}_\gamma^{i-1}),\dots$$

and denote the W_γ^i whose closures intersect $\bar{L}$ by X_γ^1, X_γ^2,...and denote the other W_γ^i by Y_γ^1, Y_γ^1,..., and put

$$Y=\sum_{\gamma\in\Gamma} Y_\gamma,\qquad Y_\gamma=\sum_{i=1}^{\infty} Y_\gamma^i,$$

then we have

(42) $$\bar{Y}-Y=\sum(\bar{Y}_\gamma-Y_\gamma),\qquad \bar{Y}_\gamma-Y_\gamma\subset\sum_{i=1}^{\infty}(\bar{V}_\gamma^i-V_\gamma^i).$$

By virtue of the sum theorem, $\dim(\bar{Y}_\gamma-Y_\gamma)\leqq n-1$ and so it follows from Theorem 3.2 that

(43) $$\dim(\bar{Y}-Y)\leqq n-1,$$

since any normal space with the star-finite property is fully normal. (But in this case (43) can be proved directly without appealing to Theorem 3.2.) Hence, if we put $H=R-\bar{Y}$, it is easily seen that

$$L\subset H\subset G,\qquad \dim(\bar{H}-H)\leqq n-1.$$

Thus the existence of an open set H satisfying the condtions (35), (36) is proved. According to Theorem 3.3 in [5], this shows that $\dim R\leqq n$.

By Theorem 5.1 and 5.2 we obtain the following theorems.

Theorem 5.3. *Let R be a metric space with the star-finite property. Then the relation $R\leqq n$ is equivalent to the relation* $\dim^*R\leqq n$.

Theorem 5.4. *Let R be a metric space with the star-finite property. In order that* dim $R \leqq n$ *it is necessary and sufficient that for each pair of a closed set F and an open set G such that $F \subset G$ there exists an open set U satisfying the conditions*

$$F \subset U \subset G, \qquad \dim (\bar{U} - U) \leqq n-1.$$

§ 6. Some theorems on mappings in spheres.

In a previous paper [5] we have proved the theorem: Let $\mathfrak{G}$ be a finite open covering of a normal space R and let A be a closed set of R. If $(\mathfrak{G})$-dim $A \leqq n$, dim $(R-A) \leqq n$, it holds that $(\mathfrak{G})$-dim $R \leqq n$. The validity of this theorem is also seen from the proof of Theorem 2.1. Corresponding to this theorem and Theorem 3.3 we have the following theorems concerning mappings in spheres (Cf. [3]).

Theorem 6.1. *Let A be closed subset of a normal space R and f a continuous mapping of A into an n-dimensional sphere S^n. If* $\dim(R-A) \leqq n$, *then f can be extended to a continuous mapping of R in S^n.*[6]

Theorem 6.2. *Let a normal space R be the sum of two closed subsets A and B and let f and g be continuous mappings of A and B into S^n. If* dim $A \cdot B \leqq n-1$, *then f can be extended over R.*

Proof. of Theorem 6.1. Let $a_0, a_1, \dots, a_{n+1}$ be linearly independent points situated in an $(n+1)$-dimensional Euclidean space. If we denote by T_i the closed cell determined by $a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_{n+1}$, then the sum $T_0 + T_1 + \dots + T_{n+1}$ is homeomorphic to an n-sphere. Hence we may set $S^n = T_0 + T_1 + \dots + T_{n+1}$. It is to be noted that $T_0 T_1 \dots T_{n+1} = 0$.

If we put

$$A_i = f^{-1}(T_i), \quad i = 0, 1, \dots, n+1 \tag{44}$$

and construct an open covering $\mathfrak{G} = \{G_0, \dots, G_{n+1}\}$ of R such that $A_i \subset G_i$, then we have $(\mathfrak{G})$-dim $A \leqq n$, since $A_0 A_1 \dots A_{n+1} = 0$. Therefore by the theorem referred to above we have $(\mathfrak{G})$-dim $R \leqq n$, in particular there is a closed covering $\{C_0, C_1 \dots C_{n+1}\}$ such that

$$C_0 C_1 \dots C_{n+1} = 0; \quad C_i \supset A_i, \quad i = 0, 1, \dots n+1. \tag{45}$$

---

6) It has already been known that a normal space R has dimension $\leqq n$ if and only if for each closed set C and each continuous mapping f of R in S^n there is an extension of f over R (Cf. [1], [2], [5]). The "only if" part of this theorem follows immediately from Theorem 6.1. In particular, in this case the existence of a closed covering $\{C_0, \dots, C_{n+1}\}$ satisfying (45) is proved simply as follows. Since $\{R-A_0, \dots, R-A_{n+1}\}$ is an open covering of R and dim $R \leqq n$, there is an open covering $\{U_0, \dots, U_{n+1}\}$ such that $U_i \subset R-A_i$ and $U_0 \cdot U_1 \dots U_{n+1} = 0$. If we put $C_i = R - U_i$, $\{C_0, \dots, C_{n+1}\}$ is a closed covering satisfying (45). For a simple proof of the "if" part, cf. [2], [5].

Then there are open sets H_i such that $C_i \subset H_i$, $H_0 H_1 \dots H_{n+1} = 0$.

Define real-valued continuous functions $w_i(x)$ such that $0 \leqq w_i(x) \leqq 1$. and

$$w_i(x) = \begin{cases} 0, & \text{for } x \in C_i, \\ 1, & \text{for } x \bar{\in} H_i. \end{cases}$$

Then the continuous mapping φ defined by

$$\varphi(x) = (\sum_{i=0}^{n+1} w_i(x) a_i) / \sum_{i=0}^{n+1} w_i(x).$$

carries R into S^n, since for each point x we have $w_i(x) > 0$ for some i and $w_j(x) = 0$ for some j.

Take any point x of A and assume that $f(x) \in T_i$. Then $x \in A_i$ and hence $w_i(x) = 0$. This shows that $\varphi(x)$ belongs to T_i. Thus for any point x of A the points $f(x)$ and $\varphi(x)$ are contained in the same T_i. Hence, if we denote by $\varphi|A$ the partial mapping φ operating on the set A, f and $\varphi|A$ are homotopic. By Borsuk's theorem f can be extended to a continuous mapping of R into S^n.

Proof of Theorem 6.2. We use the same notation as above. If we put

$$A_i = f^{-1}(T_i), \quad B_i = g^{-1}(T_i), \quad i = 0, 1, \dots, n+1$$

and construct an open covering $\mathfrak{G} = \{G_0, \dots, G_{n+1}\}$ such that $A_i + B_i \subset G_i$. Then we have $(\mathfrak{G})$-dim $A \leqq n$, $(\mathfrak{G})$-dim $B \leqq n$ and hence by the proof given in [5] of Theorem 3.3 for the finite case there is a closed covering $\{C_0, C_1, \dots, C_{n+1}\}$ such that

$$C_0 C_1 \dots C_{n+1} = 0; \quad A_i \subset C_i, \quad i = 0, 1, \dots, n+1.$$

From now on we may proceed in exactly the same way as in the above proof of Theorem 6.1. This proves Theorem 6.2.

Theorem 6.2 is also deduced, as in [3], from the following theorem.

Theorem 6.3. *Let f and g be continuous mappings of a normal space R into an n-dimensional sphere S^n such that the set D of the points for which $f(x)$ is not equal to $g(x)$ has dimension $\leqq n$-1. Then f and g are homotopic.*

Remark. Here as well as in the proof of Theorem 6.1 we understand the notion of homotopy in the sense of P. Alexandroff and H. Hopf, Topologie, I p. 319. If two continuous mappings of R in S^n are homotopic in this sense, they are also homotopic in the sense of Hurewicz and Wallman [3]. Hence our theorem states much more than the theorem in [3] p. 87. In case R is bicompact these definitions are, however, equivalent.

If two continuous mappings f and g of R in S^n are homotopic, their extensions over Cêch's bicompactification $\beta(R)$ of R are also homotopic. Since Dowker's proof stated in [3] p. 86 remains true for bicompact spaces, we obtain, returning to the original space R, Borsuk's theorem in exactly the same form as given in [3].

Proof of Theorem 6.3. Let us consider Cêch's bicompactification $\beta(R)$ of R and the extensions of f and g over $\beta(R)$ which shall be denoted by $\tilde{f}$ and $\tilde{g}$ respectively. If we put for $\varepsilon > o$

$$U = \{x\,;\ \rho[f(x),\ g(x)] < \varepsilon,\ x \in R\}$$

then we have

$$\eta(\bar{U}) \subset \{x\,;\ \rho[\tilde{f}(x),\ \tilde{g}(x)] \leqq \varepsilon,\ x \in \beta(R)\},$$

where $\eta(X)$ means the closure of a subset X of R in the space $\beta(R)$. If ε is sufficiently small, then the partial mappings $\tilde{g}|\eta(\bar{U})$ and $\tilde{f}|\eta(\bar{U})$ are homotopic, and hence, by Borsuk's theorem, the partial mapping $\tilde{g}|\eta(\bar{U})$ admits an extension G defined over $\beta(R)$ such that G is homotopic to $\tilde{f}$.

We shall show that two mappings G and $\tilde{g}$ of R in S^n are homotopic. For this purpose let us note that

$$\beta(R) = \eta(\bar{U}) + \eta(R-U),$$
$$\dim\ \eta(R-U) = \dim\ (R-U) \leqq \dim\ D \leqq n-1$$

and proceed as in [3]. Namely we define a continuous mapping $F(x,\ t)$ of C in S^n as follows:

$$F(x,\ t) = G(x) = \tilde{g}(x), \qquad \text{for } x \in \eta(\bar{U}),\ o \leqq t \leqq 1.$$
$$F(x,\ o) = G(x),\ F(x,\ 1) = \tilde{g}(x) \qquad \text{for } x \in \beta(R),$$

where we denote by C the set consisting of points $(x,\ t)$ for $x \in \eta(\bar{U})$, $o \leqq t \leqq 1$ and $(x,\ o)$, $(x,\ 1)$ for $x \in \beta(R)$. C is clearly closed in the product space $\beta(R) \times I$, where I is the unit segment $o \leqq t \leqq 1$. Since the complement of C is contained in $\eta(R-U) \times I$ which has dimension $\leqq n$ (Cf. [2]), the mapping $F(x,\ t)$ is shown to be extensible over $\beta(R) \times I$.[7] Hence G and $\tilde{g}$ are homotopic. As G is homotopic to $\tilde{f}$, $\tilde{f}$ is homotopic to $\tilde{g}$. Therefore f and g are homotopic. This proves the theorem.

Institute of Mathematics,
Tôkyô University of Education.

7) By Theorem 6. 1.

Bibliography.

[1] C. H. Dowker, Lebesgue dimension of a normal space, Bull. Amer. Math. Soc., 52 (1946), p. 243 (Abstract).

[2] E. Hemmingsen, Some theorems in dimension theorey for normal Hausdorff spaces, Duke Math. Jour., 13, (1946), pp. 495–404.

[3] W. Hurewicz and H. Wallman, Dimension theory, 1941.

[4] S. Lefschetz, Alebraic topology, 1942.

[5] K. Morita, On the dimension of normal spaces, to appear in Jap. Jour. of Math.

[6] K. Morita, Star-finite coverings and the star-finite property, Mathematicae, 1 (1948), pp. 60–68.

[7] K. Morita, On the dimension theory I, Shijô Sûgaku Danwa-kwai' 2nd ser., 10, July 1948. (in Japanese).

[8] A. H. Stone, Paracompactness and product spaces, Bull. Amer. Math. Soc., 54 (1948), pp. 677–688.

[9] J. W. Tukey, Convergence and unitormity in topology, 1941.