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On the Cluster Sets of Analytic Functions in a Jordan Domain.

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I. Cluster Sets defined by the convergence set.

1. Let D be a Jordan domain, C its boundary, E any set on $D+C$[1] and z_0, z_0' two points on C. Divide C into two parts C_1 and C_2 by z_0 and z_0'. We denote the part of D, C, E, C_1 and C_2 in $|z-z_0|\leqq r$ by D_r, C_r, E_r, $C_r^{(1)}$ and $C_r^{(2)}$ respectively and the part of $|z-z_0|=r$ in D by θ_r. Let $w=f(z)$ be a meromorphic fnnction in D and $\mathfrak{D}_r$ the set of values taken by $f(z)$ in D_r. Then the intersection $\bigcap_{r>0}\overline{\mathfrak{D}_r}=S_{z_0}^{(D)}$[2] is called the *cluster set* of $f(z)$ in D at z_0 and the intersection $\bigcap_{r>0}\mathfrak{D}_r=R_{z_0}^{(D)}$ the *range of values* of $f(z)$ in D at z_0. The intersection $\bigcap_{r>0}\overline{M_r^{(E)}}=S_{z_0}^{(E)}$, where $M_r^{(E)}$ is the union $\bigcup S_{z'}^{(D)}$, for $z_0\neq z'\in E$, $S_{z'}^{(D)}$ consisting of the single value $f(z')$ for $z'\in D$, is called the cluster set of $f(z)$ on E at z_0. For example, $S_{z_0}^{(C)}$, $S_{z_0}^{(C_1)}$, $S_{z_0}^{(C_2)}$ and $S_{z_0}^{(L)}$, where L is a Jordan curve in D terminating at z_0, are thus defined. If $S_{z_0}^{(L)}$ consists of only one value α, we call α the asymptotic value, L the asymptotic path and we denote the set of all the asymptotic values at z_0 by $\Gamma_{z_0}^{(D)}$, and call it the *convergence set* of $f(z)$ at z_0. When $f(z)$ omits at least three values in the neighbourhood of z_0[3], $\Gamma_{z_0}^{(D)}$ consists of at most one value [4]. Then we call the value of non-empty $\Gamma_{z_0}^{(D)}$ the *boundary value* at z_0, and denote it by $f(z_0)$. Furthermore the intersection $\bigcap_{r>0}\overline{Y_r^{(E)}}=\Gamma_{z_0}^{(E)}$ for $E\subset C$, $Y_r^{(E)}$ being the union $\bigcup\Gamma_{z'}^{(D)}$ for $z_0\neq z'\in E_r$, is called the cluster set of the convergence set of $f(z)$ on E at z_0.

$S_{z_0}^{(D)}$ includes all the other cluster sets and $S_{z_0}^{(E)}$ includes $\Gamma_{z_0}^{(E)}$. $S_{z_0}^{(D)}$, $S_{z_0}^{(C_1)}$, $S_{z_0}^{(C_2)}$ and $S_{z_0}^{(L)}$ are continuums but not necessarily $\Gamma_{z_0}^{(C)}$, $\Gamma_{z_0}^{(C_1)}$ and $\Gamma_{z_0}^{(C_2)}$ are [5].

2. Let $f(z)$ be bounded in the neighbourhood of z_0. Then it is known that [6]

$$\overline{\lim_{z\to z_0}}\,|f(z)|=\overline{\lim_{C\ni z'\to z_0}}\,\Big(\overline{\lim_{z\to z'\neq z_0}}\,|f(z)|\Big),$$

and that this is equivalent to $B(S_{z_0}^{(D)})\subset B(S_{z_0}^{(C)})$, $B(S)$ being the boundary set of S[7]. Also it is known that $B(S_{z_0}^{(D)})\subset B(\Gamma_{z_0}^{(C)})$ holds in the case where D is a circle [8]; then it holds also in the general case where D is a Jordan domain, by means of a one-to-one continuous correspondence between them, with their boundaries included. By the same reason we may, and shall, assume that D is a circle $|z|<1$ and $z_0=1$ in proofs of our theorems 1.1 to 1.3.

Theorem 1.1. *Let D be a Jordan domain, C its boundary, z_0 a point on C and $f(z)$ a bounded regular function in D. Then*

$$B(S_{z_0}^{(D)}) \subset B(\Gamma_{z_0}^{(C)}).$$

Proof. Transform the circle $|z|<1$ onto $|\zeta|<1$ by the transformation $\zeta=\dfrac{z-z_1}{1-\bar{z}_1 z}$ $(|z_1|<1)$ and put $z_1=1+x, f(z(\zeta))=F(\zeta)$ and $\zeta=\rho e^{i\varphi}$. Then

$$|\zeta+1|=\left|\frac{x+\bar{x}z}{1-z-\bar{x}z}\right|\leqq\frac{2|x|}{|1-z|-|x|}.$$

Hence for $|1-z|\geqq\delta$ and $|z|\leqq 1$, $\zeta+1$ tends to 0 uniformly as $x\to 0$. Put $\overline{\lim\limits_{\theta\to\pm 0}}|f(e^{i\theta})|=m$ and suppose $|f(e^{i\theta})|\leqq m+\varepsilon$ when $|\theta|\leqq\delta_1$, for any given positive ε. Let this arc be transformed into the arc $\widehat{\alpha\beta}$ by $\zeta=\zeta(z)$ and suppose the length of $\widehat{\alpha\beta}\geqq 2\pi-\varepsilon$ on taking $|x|$ sufficiently small. This is possible, because the both end-points of $\widehat{\alpha\beta}$ tend to -1 as $x\to 0$. Put $|F(\zeta)|=|f(z)|<M$ and let E be the set of points on $\widehat{\alpha\beta}$ where $F(e^{i\varphi})$ exists, and $\widehat{\alpha\beta}'$ the complementary set of $\widehat{\alpha\beta}$ with respect to $|\zeta|=1$. Then by Cauchy's formula and Lebesgue's theorem

$$|f(z_1)|=|F(0)|\leqq\overline{\lim_{\rho\to 1}}\frac{1}{2\pi}\int_0^{2\pi}|F(\rho e^{i\varphi})|d\varphi=\overline{\lim_{\rho\to 1}}\frac{1}{2\pi}\int_E|F(\rho e^{i\varphi})|d\varphi$$
$$+\overline{\lim_{\rho\to 1}}\frac{1}{2\pi}\int_{\widehat{\alpha\beta}'}|F(\rho e^{i\varphi})|\,d\varphi\geqq\frac{1}{2\pi}\int_E|F(e^{i\varphi})|\,d\varphi+\frac{M\varepsilon}{2\pi}\leqq\frac{m+\varepsilon}{2\pi}(2\pi-\varepsilon)$$
$$+\frac{M\varepsilon}{2\pi}=m+\frac{\varepsilon}{2\pi}(2\pi-\varepsilon+M-m).$$

Hence

$$\overline{\lim_{z\to 1}}\,|f(z)|\leqq m,$$

that is

$$\overline{\lim_{z\to 1}}\,|f(z)|\leqq\overline{\lim_{\theta\to\pm 0}}\,|f(e^{i\theta})|.$$

From this relation it follows easily $B(S_{z_0}^{(D)}) \subset B(\Gamma_{z_0}^{(C)})$ (7).

Now we divide C into C_1 and C_2.

Lemma 1 (9). *Under the same conditions as in theorem* 1.1, *there exists a domain G bounded by a part of C_1 and a curve L in D terminating at z_0 such that $S_{z_0}^{(C_1)} = S_{z_0}^{(G)}$.*

Proof. Take a sequence of points $Q_1 \supset Q_2 \supset \ldots$ (10), $Q_n \to z_0$, on C_1, and a neighbourhood N in D at every point P, $Q_k \supseteqq P \supset Q_{k+1}$, such that every point of the image of N_P in the w-plane has a distance $<\frac{1}{k}$ from $S_P^{(D)}$. Then the arc $Q_k \supseteqq P \supseteqq Q_{k+1}$ can be covered by a finite number of N_P, which we denote by $N_1^{(k)}, \ldots\ldots, N_{n_k}^{(k)}$. Put $\bigcup_{k=1}^{\infty} \bigcup_{\nu=1}^{n_k} N_\nu^{(k)} = G$. Then G satisfies the conditions required.

Theorem 1. 2. *Under the same conditions as in lemma* 1,

$$B(S_{z_0}^{(Ci)}) \subset B(\Gamma_{z_0}^{(Ci)}), (i=1,\ 2) \text{ and } B(S_{z_0}^{(C)}) \subset B(\Gamma_{z_0}^{(C)}).$$

Proof. Put $\overline{\lim}_{\theta \to +0} |f(e^{i\theta})| = m$ and $\overline{\lim}_{\theta \to +0} (\overline{\lim}_{z \to e^{i\theta}} |f(z)|) = M$, and assume $m < M$. For any given positive ε, there exists $r_0 > 0$ such that $\overline{Y_{r_0}^{(C_1)}}$ is included in the circle $|w| < m + \varepsilon$, and $\overline{M_{r_0}^{(G)}}$ in $|w| < M + \varepsilon$, G being the domain in lemma 1. Map conformally the domain, bounded by C_{r_0} and parts of L in lemma 1 and θ_{r_0}, on the unit circle in the ζ-plane so that $C_{r_0}^{(1)}$ corresponds to the upper semicircle and z_0 to $\zeta = 1$, and put $f(z(\zeta)) = F(\zeta)$ and $\zeta = \rho e^{i\varphi}$. Then $|F(\zeta)| < M + \varepsilon$. The boundary values $F(e^{i\varphi})$ exist at almost all points $e^{i\varphi}$, $0 \leqq \varphi \leqq 2\pi$, by Fatou's theorem and $|F(e^{i\varphi})| < m + \varepsilon$ for $o < \varphi \leqq \pi$, since $\overline{Y_{r_0}^{(C_1)}}$ is included in $|w| < m + \varepsilon$. Put $F(\zeta) \cdot \overline{F(\bar{\zeta})} = G(\zeta)$, $\bar{\zeta}$ and $\bar{F}$ designating the conjugate values of ζ and F. Then for almost all $e^{i\varphi}$, $0 \leqq \varphi \leqq 2\pi$, $|G(e^{i\varphi})| = |F(e^{i\varphi})| \cdot |F(e^{-i\varphi})| < (M+\varepsilon)(m+\varepsilon) = m_1$. Similarly as in theorem 1.1 $|G(\zeta)| < m_1$ holds for all ζ in the unit circle. Especially for each real value $\zeta = t$, $|G(t)| = |F(t)|^2 < m_1 < M^2$ holds for sufficiently small ε. Applying theorem 1.1 to the upper semicircular disc, the cluster set of $F(\zeta)$ at $\zeta = 1$, consequently the cluster set on the upper semicircle, which is nothing but the set $S_{z_0}^{(C_1)}$, is included in $|w| < \sqrt{m_1} < M$. According to the definition of M, there exists, however, a point of $S_{z_0}^{(C_1)}$ on $|w| = M$. This is a contradiction, and we get $m \geqq M$. But obviously $m \leqq M$ and so $m = M$, i.e. $\overline{\lim}_{\theta \to +0} |f(e^{i\theta})| = \overline{\lim}_{\theta \to +0} (\overline{\lim}_{z \to e^{i\theta}} |f(z)|)$. The equivalence of this with the proposition $B(S_{z_0}^{(C_1)}) \subset B(\Gamma_{z_0}^{(C_1)})$ can be shown as usual (7).

Similarly $B(S_{z_0}^{(C_2)}) \subset B(\Gamma_{z_0}^{(C_2)})$ and from both relations it follows $B(S_{z_0}^{(C)}) \subset B(\Gamma_{z_0}^{(C)})$.

Theorem 1. 3. *If there exists a value a such that $a \in S_{z_0}^{(D)} - \Gamma_{z_0}^{(C)}$ and $a \bar{\in} R_{z_0}^{(D)}$, under the same conditions as in theorem 1.1, then $a=f(z_0)$.*

Proof. We may suppose that $a=0$. For sufficiently small $r_0>0$, $0 \bar{\in} \mathfrak{D}_{r_0}$ and the distance ρ_1 from 0 to the set $\overline{Y_{r_0}^{(C)}}$ is positive. We may suppose by taking r_0 suitably that at the two end-points of θ_{r_0} the boundary values exist. Then $|f(z)|>\rho_2>0$ for $z \in \theta_{r_0}$. Put $Min\ (\rho_1, \rho_2)=\rho>0$. Since $0 \in S_{z_0}^{(D)}$, there is a point z_1 in D_{r_0}, whose image $w_1=f(z_1)$ lies in $|w|<\rho$. Take an inverse element e_{z_1} and continue it analytically (with algebraic characters) in any way along the radius from w_1 to $w=0$. Since $0 \bar{\in} \mathfrak{D}_{r_0}$ the continuation up to 0 is impossible: it must end at a point β on the radius $\overline{0w_1}$. There corresponds a curve L in D_{r_0} such that $f(z)\to\beta$ when z approaches to C_{r_0} on L. If L oscillates, $f(z)$ reduces to a constant by Koebe's theorem, so that L terminates at a point on C_{r_0} and β is a boundary value at this point. But $\overline{Y_{r_0}^{(C)}}$ has no point in $|w|<\rho$ and so L terminates at $z_0=1$ and $f(z_0)=\beta$. However, if we take another element e_{z_2} corresponding to $z_2 \in D_{r_0}$ at a point $w_2=f(z_2)$ in $|w|<\rho$ which is near w_1, but not on $\overline{0w_1}$, then follows similarly $f(z_0)=\gamma$, γ being a point on the radius $\overline{0w}$. Accordingly $f(z_0)=\beta=\gamma=0$.

The following theorem is an immediate consequence of theorem 1.3.

Theorem 1. 4. *Under the same conditions as in theorem 1.1, every value belonging to $S_{z_0}^{(D)} - \Gamma_{z_0}^{(C)}$ belongs to $R_{z_0}^{(D)}$ except at most one value.*

3. Formerly we have defined $\Gamma_{z_0}^{(C)}$, $\Gamma_{z_0}^{(C_1)}$ and $\Gamma_{z_0}^{(C'_2)}$ by considering all the boundary values on the general Jordan domain D. But we shall consider hereafter only the case when D is the unit circle $|z|<1$. Let e be any set of points of Lebesgue measure zero on $|z|=1$, put $C-e=C'$, $C_1-e=C_1'$ and $C_2-e=C_2'$ and consider $\Gamma_{z_0}^{(C')}$, $\Gamma_{z_0}^{(C_1')}$ and $\Gamma_{z_0}^{(C'_2)}$. Then a theorem similar to theorem 1.1 is obtained: we shall call it theorem 1.1′. Furthermore, using the same method as in theorem 1.2, we can prove $B(S_{z_0}^{(C_i)}) \subset B(S_{z_0}^{(C_i')}) \subset B(\Gamma_{z_0}^{(C_i')})$ $(i=1,2)$ and $B(S_{z_0}^{(C)}) \subset B(S_{z_0}^{(C')}) \subset B(\Gamma_{z_0}^{(C')})$, which we shall call theorem 1.2′. However, theorems corresponding to theorems 1.3 and 1.4 must be stated in somewhat different forms. Namely:

Theorem 1.3′. *If there exists a value a such that $a \in S_{z_0}^{(D)} - \Gamma_{z_0}^{(C')}$ and and $a \bar{\in} R_{z_0}^{(D)}$ under the same conditions as in theorem 1.1 (with $D=$unit circle), then $a=f(z_0)$ or there is a sequence $z_1, z_2, \ldots, z_n \to z_0$ of points on $|z|=1$, such that $a=f(z_n)$.*

Proof. To prove this theorem we have to employ a method different from that used in the proof of theorem 1.3. We may suppose that $a=0$, and we determine r_0 and ρ as in theorem 1.3, provided that the two end-points of θ_{r_0} do not belong to the exceptional set e. Since $0 \in S_{z_0}^{(D)}$, there is a point z_1 in D_{r_0} such that $w_1=f(z_1)$ is in $|w|<\rho$ and consequently there exists a domain $\varDelta_1$ in D_{r_0}, in which $f(z)$ takes the values in $|w|<\rho$ and on whose boundary $|f(z)|=\rho$ in $|z|<1$. Hence $\varDelta_1$ has no common point with θ_{r_0} and is a simply connected domain because $f(z)$ is regular in $|z|<1$. Now we shall prove that $1/f(z)$ is not bounded in $\varDelta_1$. Map $\varDelta_1$ conformally on $|\zeta|<1$ and put $f(z(\zeta))=F(\zeta)$. Then by Fatou's theorem there exist boundary values of both $F(\zeta)$ and $z(\zeta)$ at almost all points on $|\zeta|=1$. Now, let E be the set of points on $|\zeta|=1$ at which both $F(\zeta)$ and $z(\zeta)$ exist and the relation : $|z(\zeta)|=1$ holds, and E' be the image of E by $z(\zeta)$. By Kametani-Ugaheri's theorem ([11]) $m_*E \leq m^*E'$. Then we have $E' \subset e$, because $\lim f(z)$ exists along a curve terminating at every point of E'. Therefore $mE'=0$ and $m_*E=0$. By Tsuji ([12]) the set of all points on $|\zeta|=1$ at which boundary values $z(\zeta)$ exist and the relation: $|z(\zeta)|=1$ holds is measurable. This set consists of E and a set of measure zero where boundary values of $F(\zeta)$ do not exist, so that E is also measurable and $mE=0$. Consequently both $F(\zeta)$ and $z(\zeta)$ exist on $|\zeta|=1$, $|z(\zeta)|<1$ and hence $|F(\zeta)|=\rho$ holds almost everywhere. If $1/F(\zeta)$ were bounded, we would have as in lemma 1, $1/|F(\zeta)|\leq 1/\rho$. Hence $|F(\zeta)|\leq\rho$ and this is a contradiction. Therefore $1/F(\zeta)$ is unbounded and there exists a point z_2 in $\varDelta_1$ such that $|f(z_2)|<\rho/2$. Let $\varDelta_2$ be the component of the image of $|w|<\rho/2$ which contains z_2. Similarly as in the proof of Iversen's theorem ([13]) there exists a curve L in D_{r_0} along which $f(z)\to 0$. However small r_0 may be taken, there exists such a curve L in D_{r_0} and the theorem is proved.

Theorem 1.4′. *Under the same conditions as in theorem* 1.3′, $S_{z_0}^{(D)}-\Gamma_{z_0}^{(C)}$ *is contained in* $R_{z_0}^{(D)}$ *except at most a set of capacity zero*([14]).

Proof. Since $S_{z_0}^{(D)}-\Gamma_{z_0}^{(C)}$ is an open set by theorem 1.1, it consists of an at most enumerably infinite number of connected domains and it suffices to prove the theorem for a component Ω chosen arbitrarily. The intersection of Ω and the complement of $R_{z_0}^{(D)}$, namely the exceptional set, is a Borel set. Assume that its capacity is positive. Take a sequence $r_1>r_2>\dots$, $r_n\to 0$ and let E_n be the set of values in Ω not belonging to $\mathfrak{D}_{r_n}$. Since $E_1 \subset E_2 \subset \dots$ and $\bigcup_{n=1}^{\infty} E_n$ is the exceptional set, there exists n_0 such that $E_n(n\geq$

n_0) is of positive capacity. We may suppose that in D_{r_n} $f(z)$ takes no value of a closed set E of positive capacity in Ω, which is then of positive distance from the boundary of Ω. By Frostman's theorem (15) there exists a positive mass-distribution $\mu(w)$ on E such that $u(w)=\int_E \log\frac{1}{|w-\omega|} d\mu(\omega)$ is bounded: $u(w)\leqq k$, $u(w)=k$ holds on E except a set of capacity zero and $u(w)$ is harmonic outside E. Let $v(w)$ be the conjugate function of $u(w)$ and put $g(w)=e^{u(w)+iv(w)}$. Then $|g(w)|\leqq e^k$. Take r_n sufficiently small and let the distance between E and $\overline{Y_{r_n}{}^{(C')}}$ be positive. Put $\chi=F(z)=g(f(z))$ by selecting a branch of $g(w)$. Then $F(z)$ is a one-valued bounded regular function in D_{r_n} and $|F(e^{i\theta})|\leqq e^{k'}$, where $F(e^{i\theta})$ is the boundary value on C' and $k'=\max u(w)$ for $w\in\overline{Y_{r_n}{}^{(C')}}$. Applying theorem 1.1′ to $F(z)$ and D, we have $\overline{\lim\limits_{z\to z_0}}\,|F(z)|\leqq e^{k'}$. Since $E\subset S_{z_0}^{(D)}$, there exists a sequence $z_1, z_2,\dots,z_n\to z_0$ such that $f(z_n)\to w_0\in E$, where $u(w_0)=k$. Therefore $|F(z_n)|\to e^k$. Since $k'<k$, this is a contradiction. Hence the exceptional set of values in Ω must be of capacity zero.

Example. Exclude a non-empty closed set E of capacity zero from a circle $|w|<1$; map conformally the remaining domain on a circle D: $|z|<1$ and let z_0 be a singular point of $w(z)$ on C: $|z|=1$. Then $S_{z_0}^{(D)}=S_{z_0}^{(C)}$ is the closed circle $|w|\leqq 1$ and $\Gamma_{z_0}^{(C)}$ is the sum of E and the circumference $|w|=1$. If we exclude the image of E from C, which is of measure zero, $\Gamma_{z_0}^{(C')}$ is $|w|=1$ for remaining C' and $S_{z_0}^{(D)}-\Gamma_{z_0}^{(C')}$ is $|w|<1$ and is included in $R_{z_0}^{(D)}$ except a set of capacity zero, which is just the excluded set E.

4. Now we remove the restriction of boundedness of $f(z)$. If $S_{z_0}^{(D)}$ is not the whole plane, it is easily reduced by a linear transformation to the case where $f(z)$ is bounded. If $S_{z_0}^{(D)}$ is the whole plane, theorem 1.1 is trivial. If both $S_{z_0}^{(C_1)}$ and $S_{z_0}^{(C_2)}$ are the whole planes, theorem 1.2 is trivial, but if $S_{z_0}^{(C_1)}$, for example, is not the whole plane although $S_{z_0}^{(C_2)}$ is, lemma 1 and hence the relation: $B(S_{z_0}^{(C_1)})\subset B(\Gamma_{z_0}^{(C_1)})$ holds good still. When $f(z)$ is of class a near z_0, theorems 1.3 and 1.3′ hold and are proved in fact by generalized Koebe's theorem (16) and by the following theorem, to which we shall give a simple proof.

Theorem (*Cartwright*) (17). *Let $f(z)$ be meromorphic in a circle $|z|<1$. If $f(z)$ is of class a near z_0, then boundary values of $f(z)$ exist at points which are dense on $|z|=1$ near z_0.*

Proof. It is sufficient to prove that in any neighbourhood on $|z|=1$

of z_0, there exists a point at which a boundary value exists. Suppose that $f(z)$ omits three values α, β, γ in D_r. If $S_{z_0}^{(D)}$ is not the whole plane, we can prove the theorem by reducing to the case where $f(z)$ is bounded. Hence we may suppose $\alpha \in S_{z_0}^{(D)}$ and there exists a sequence $z_1, z_2, \dots, z_n \to z_0$ such that $w_n = f(z_n) \to \alpha$. Continue the inverse element e_{z_n} from w_n toward α along $\overline{w_n\alpha}$. Since $f(z) \neq \alpha$ in D_r, the continuation up to α is impossible and must stop at a point on $\overline{w_n\alpha}$. The z-image L_n does not oscillate by generalized Koebe's theorem. Therefore each L_n terminates at a point on C_r or θ_r. But if there exists an infinite number of L_n terminating on θ_r, $f(z) \to \alpha$ on these curves which accumulate on $C_r^{(1)}$ or $C_r^{(2)}$ and $f(z)$ reduces to a constant α by generalized Koebe's theorem. Hence every $L_n (n \geqq n_0)$ terminates at some point on C_r and the theorem is proved, because we can take r arbitrarily small and any point near z_0, instead of z_0.

In the proof of theorem 1.3 ([18]), we take a curve L in D_r, whose two end-points terminate at two points on $C_r^{(1)}$ and $C_r^{(2)}$ respectively where boundary values exist, instead of θ_r.

For theorem 1.3′ ([18]), it may happen that there exists no such point belonging to C'. But to prove the theorem for α we take instead of θ_r a curve whose two end-points on $C_r^{(1)}$ and $C_r^{(2)}$ have boundary values different from α. The existence of such points is shown as in the proof of Cartwright's theorem. Next we shall consider theroems 1.4 and 1.4′. Theorem 1.4 ([18]) is deduced directly from theorem 1.3 ([18]) and it can be stated in the following form.

Theorem 1.4″. *Let $f(z)$ be meromorphic in a Jordan domain. Then $S_{z_0}^{(D)} - \Gamma_{z_0}^{(C)} \subset R_{z_0}^{(D)}$ holds except at most two values. Especially if $f(z)$ omits just two values near z_0, $R_{z_0}^{(D)}$ contains all values except these two values.*

In theorem 1.4′ ([18]) we may suppose that E is a bounded closed set and boundary values exist almost everywhere near z_0, because $f(z)$ is of bounded type near z_0 on account of the assumption that $f(z)$ omits values of positive capacity ([19]). Therefore the theorem is proved similarly as before.

5. Seidel ([20]) has proved that if $f(z)$ is regular in $|z|<1$, $|f(z)|<1$ and $|f(e^{i\theta})|=1$ on an arc A almost everywhere, then an inner point of A is a regular point of $f(z)$ or $S_{z_0}^{(D)}$ at any singular point $z_0 \in A$ is a closed circular disc $|w| \leqq 1$, by the same method as in the proof of Schwarz's theorem. We shall call such function a function of class U'. From this and theorem 1.3′ we have

Theorem (*Seidel*) ([20]). *Let $f(z)$ be a function of class U' and be not regular on A. If $f(z) \neq a$ ($|a| < 1$) in $|z| < 1$, $f(z)$ has boundary value a at any singular point or at points on A accumulating on this singular point.*

From theorem 1.4′ we have

Theorem (*Extension of Seidel's theorem*) ([20]). *Let $f(z)$ be a function of class U' and not regular on A. Then $R_{z_0}^{(D)}$ at any singular point contains every value except at most values of capacity zero.*

From theorem 1.4″ the next theorem is easily proved.

Theorem (*Cartwright*) ([17]). *Let $f(z)$ be meromorphic in $|z| < 1$ and $w_0 \in \Gamma_{z_0}^{(D)}$. If each $\Gamma_{z'}^{(D)}$, for $z' \in C$, has no value in d: $0 < |w - w_0| < \eta$ for some η, then $f(z) \equiv w_0$ or $R_{z_0}^{(D)}$ contains d': $0 < |w - w_0| < \eta'$ for some η'.*

II. On Hössjer's theorems.

1. We add to $S_{z_0}^{(C_1)}$ all the possible bounded domains limited by $S_{z_0}^{(C_1)}$, which we will call holes of $S_{z_0}^{(C_1)}$, and denote the continuum by Ω_1. Similarly we get Ω_2. G. Hössjer proved([21])

Theorem I (*Hössjer*). *Under the same conditions as in theorem 1.2, Ω_1 and Ω_2 have at least one common point and $S_{z_0}^{(D)} \subset \Omega_1 \cup \Omega_2 \cup \Delta$ holds, where Δ denotes the set of bounded domains limited by $\Omega_1 \cup \Omega_2$.*

This theorem is a consequence of the theorem that for any component Δ_i of the complementary set of $S_{z_0}^{(C)}$ with respect to w-plane either $\Delta_i \subset S_{z_0}^{(D)}$ or $\Delta_i \cap S_{z_0}^{(D)} = \phi$ holds([22]), and this latter theorem is easily proved from $B(S_{z_0}^{(D)}) \subset B(S_{z_0}^{(C)})$ ([23]).

Corollary. *Every value of $S_{z_0}^{(D)}$ which belongs to some hole of $S_{z_0}^{(C_1)}$ but not to Ω_2, or to some hole of $S_{z_0}^{(C_2)}$ but not to Ω_1, or to Δ, belongs to $R_{z_0}^{(D)}$ without exception.*

Proof. If one such value a does not belong to $R_{z_0}^{(D)}$, then by theorem 1.3 there exists a curve L in D terminating at z_0 such that the cluster set on L consists of one value a and this value does not belong to Ω_2 or not to Ω_1 (or not to both). Applying Hössjer's theorem to the domain lying between L and C_2 or C_1, a contradiction is obtained.

Moreover Δ is unnecessary in theorem I; we have namely $S_{z_0}^{(D)} \subset \Omega_1 \cup \Omega_2$ or $S_{z_0}^{(D)} \cap \Delta = \phi$ ([24]). To prove this assertion, the following lemma is useful.

Lemma 2 (*Gross*) ([9]). *Under the same conditions as in lemma 1, there exists a curve L_1 in D terminatng at z_0 such that $S_{z_0}^{(L_1)} = S_{z_0}^{(C_1)}$.*

Proof. Consider the domain G in lemma 1. Let $a_1, a_2, \ldots$ be a sequence of points which are dense in $S_{z_0}^{(G)}$. Put $D_{\frac{1}{n}} \cap G = G_n$. Since $a_n \in S_{z_0}^{(G)}$, there exists a point $Q_n \in G_n$ such that $\overline{Q_n a_n} < \frac{1}{n}$ for each n. By connecting $Q_1, Q_2, \ldots$ and removing the superfluous parts we gain L_1.

Remark. Since we may suppose that two domains G for C_1 and C_2 are disjoint, we can take L_1 and L_2 disjoint in D.

Theorem 2.1. *Under the same conditions as in theorem* 1.2

$$S_{z_0}^{(D)} \subset \Omega_1 \cup \Omega_2.$$

Proof. Without loss of generality we may suppose that D is a circle $|z| < 1$, $z_0 = 1$ and $f(z)$ is regular on $|z| = 1$ except at z_0 since L_1 and L_2 may be taken instead of C_1 and C_2, by lemma 2. Assume that there exists a hole Δ_{i_0} which is included in $S_{z_0}^{(D)}$, whence in $R_{z_0}^{(D)}$ by the corollary. In it we take a point w_{i_0}, which is not an image of a double point of $f(z)$ ([25]). We cover Ω_1 and Ω_2 by bounded simply connected domains Φ_1 and Φ_2 with boundaries Γ_1 and Γ_2 of analytic closed curves, having w_{i_0} as their outer point. Connect w_{i_0} with infinity outside $\overline{\Phi}_2$ by an analytic curve L which passes no branch point. Because of the analyticity of Γ_1 and Γ_2 the number of holes of $\overline{\Phi}_1 \cup \overline{\Phi}_2$, each of which is contained in some hole of $\Omega_1 \cup \Omega_2$, is finite and we denote these holes by $\delta_i (i = 1, 2, \ldots, p)$. According to the definition of Φ_1 and Φ_2, w_{i_0} belongs to some hole δ_n. We enumerate δ_i such that L meets $\delta_1, \delta_2, \ldots, \delta_n$, and only those, in this order coming from infinity; so in particular $\infty \in \delta_1$ and $w_{i_0} \in \delta_n$. And we assume $\overline{\delta}_m \cap S_{z_0}^{(D)} = \phi$ but $\overline{\delta}_{m+1} \subset S_{z_0}^{(D)}$. Then $\overline{\delta}_{m+1} \subset R_{z_0}^{(D)}$ by corollary. If it is shown that this is impossible, we have $\overline{\delta}_n \cap S_{z_0}^{(D)} = \phi$ by induction, hence $w_{i_0} \overline{\in} S_{z_0}^{(D)}$ which is a contradiction. We take a point w_1 which is the first intersection of L with $\overline{\delta}_{m+1}$ counting from infinity, and denote by L_1 the part of L between w_1 and the point w_2, which L meets for the first time counting from w_1 toward infinity. Then $L_1 \subset \overline{\Phi}_1$. Connect w_1 with infinity by a curve L_2, lying outside $\overline{\Phi}_1$ except w_1, and which divides δ_{m+1} into two domains and passes no branch point.

Let us turn to the z-plane. For sufficiently small $r_0 > 0$, $\mathfrak{D}_{r_0} \cap \overline{\delta}_m = \phi$, $\overline{M}_{r_0}^{(C_1)} \subset \Phi_1$ $\overline{M}_{r_0}^{(C_2)} \subset \Phi_2$. Since $w_1 \in R_{z_0}^{(D)}$, there exists a point z_1 in D_{r_0} such that $f(z_1) = w_1$. Let $l_1^{(1)}$ and $l_1^{(2)}$ be the curves through z_1 corresponding to L_1 and L_2 respectively and put $l_1^{(1)}+l_1^{(2)}=l_1$. $l_1^{(1)}$ and $l_1^{(2)}$ terminate at points on the boundary of D_{r_0}, and the end-points of $l_1^{(1)}$ and $l_1^{(2)}$ are not on C_2 and C_1 respectively except for z_0, because the boundary values at that end-points are outside $\overline{\Phi}_2$ and $\overline{\Phi}_1$ respectively and $\overline{M}_{r_0}^{(C_1)}\subset\Phi_1$ and $\overline{M}_{r_0}^{(C_2)}\subset\Phi_2$. Moreover each end-point is different from z_0, because according to Hössjer's theorem applied to the domain lying between $l_1^{(1)}$ and C_2 or $l_1^{(2)}$ and C_1 it is impossible that the cluster set on $l_1^{(1)}$ or $l_1^{(2)}$, which consists of that boundary value only, is outside Ω_2 or Ω_1.

Therefore l_1 is a cross-cut of D_{r_0} and hence D_{r_0} is divided into two domains by it, only one of which has z_0 on its boundary and will be denoted by G_1. Since $w_1\in R_{z_0}^{(D)}$, there is a point z_2 in G_1 such that $f(z_2)=w_1$.

Similarly we get $l_2^{(1)}$, $l_2^{(2)}$, l_2 and G_2. There exists a sequence $z_\nu(\nu=1,2,\ldots)$ of points such that $z_\nu\to 1$ as $\nu\to\infty$ and $f(z_\nu)=w_1$, and we get $l_\nu^{(1)}$, $l_\nu^{(2)}$, l_ν and $G_\nu(\nu=1,2\ldots)$ such that l_ν and $l_{\nu+1}$ have no common point in D_{r_0} and $G_{\nu+1}\subset G_\nu$. Since $f(z)$ is regular on C except at z_0, l_ν and G_ν converge to z_0 as $\nu\to\infty$ and there exists a number ν_0 such that end-points of $l_\nu^{(1)}$, $l_\nu^{(2)}$, for $\nu\geqq\nu_0$ terminate on $C_{r_0}^{(1)}$, $C_{r_0}^{(2)}$ except at z_0 respectively. We take a point w_3 in δ_{m+1} but not on L_2. Since $w_3\in R_{z_0}^{(D)}$ by corollary, there exists a domain G_0, which is enclosed by l_{ν_1}, l_{ν_1+1} $(\nu_1\geqq\nu_0)$ and parts of $C_{r_0}^{(1)}$, $C_{r_0}^{(2)}$ and which contains a point z' such that $f(z')=w_3$. Denote the part of the boundary of G_0 composed of $l_{\nu_1}^{(1)}$, $l_{\nu_1+1}^{(1)}$ and a part of $C_{r_0}^{(1)}$ by k_1 and the part composed of $l_{\nu_1}^{(2)}$, $l_{\nu_1+1}^{(2)}$ and a part of $C_{r_0}^{(2)}$ by k_2.

By the principle of argument the number of zero points of $f(z)-w_3$ in G_0,

$$\frac{1}{2\pi}\int_{k_1+k_2} d\arg\ (f(z)-w_3)>0.$$

Now it is possible by using L_2 to connect w_3 with infinity by a curve having no common point with the image of k_1 which is a closed curve on $L_1\cup\Phi_1$, therefore

$$\int_{k_1} d\arg\ (f(z)-w_3)=0.$$

Since $w_2\in\overline{\delta_m}$, there holds $w_2\bar{\in}\mathfrak{D}_{r_0}$ and hence

$$\int_{k_1+k_2} d\arg\ (f(z)-w_2)=0,$$

furthermore

$$\int_{k_1} d\arg\ (f(z)-w_2)=0,$$

because we can connect w_2 with infinity with a curve having no common point with the set $L_1 \cup \Phi_1$.
Consequently

$$\int_{k_2} d \arg (f(z)-w_2)=0.$$

But by using L_1 it is also possible to connect w_2 with w_3 by a curve without having common point with $L_2 \cup \Phi_2$, on which the image of k_2 lies. Accordingly

$$\int_{k_2} d \arg (f(z)-w_3)=0.$$

whence

$$\int_{k_1+k_2} d \arg (f(z)-w_3)=0.$$

This is a contradiction and the theorem is proved.

Remark. We denote holes of $S_{z_0}^{(C_1)}$ and $S_{z_0}^{(C_2)}$ by $\{\omega_i^{(1)}\}$ and $\{\omega_j^{(2)}\}$ respectively and call also the complements of Ω_1 and Ω_2 holes. Then for each of $\{\omega_i^{(1)}\}$ and $\{\omega_j^{(2)}\}$, we can decide whether it belongs to $S_{z_0}^{(D)}$ or not in the following sense. When it belongs to $S_{z_0}^{(D)}$, it does to $R_{z_0}^{(D)}$ with one possible exception. When $\omega_n^{(1)}$ for example, does not, then $\{\omega_n^{(1)}-(S_{z_0}^{(C_2)}+\sum'\omega_j^{(2)})\} \cap S_{z_0}^{(D)}=\phi$, where Σ' means the summation for $\omega_j^{(2)}$ which belongs to $S_{z_0}^{(D)}$. And the one possible exception cannot lie in the hole, be it of $S_{z_0}^{(C_1)}$ or $S_{z_0}^{(C_2)}$, which does not belong to $S_{z_0}^{(D)}$. These facts, which contain theorem 2.1, are shown by the same method as the one used in this theorem.

2. In the same paper G. Hössjer proved

Theorem II. (*Hössjer*). *Under the same conditions as in theorem I and under the hypothesis that $f(z)$ is continuous on $D+C$ except at z_0, there exists a Jordan curve L on $D+C$ terminating at z_0 such that*

$$S_{z_0}^{(L)} \subset \Omega_1 \cap \Omega_2 = \Omega.$$

But his proof seems to be imperfect in some point([26]) and unless theorem 2.1 is proved, we can say only $S_{z_0}^{(L)} \subset \Omega \cup \Delta$ when Δ exists. We state the theorem in the following form.

Theorem 2.2. *Under the same conditions as in theorem* 1.2, *there exists a Jordan curve L in D terminating at z_0 such that*

$$S_{z_0}^{(L)} \subset \Omega.$$

To prove this theorem the following lemma is to be mentioned.

Lemma 3. *Let D be a Jordan domain, z_0 be on its boundary, $\Omega_i (i=1, 2,\ldots)$ be the sequence of cross-cuts in D, disjoint of each other, not terminating at z_0 and not accumlating in D (27). D being divided by Q_i into two domains, let D_i be the one which has z_0 on its boundary and let the area of each $D_i \geqq k > 0$ (28). Then $D_0 = \bigcap_{i=1}^{\infty} D_i$ is a domain.*

Proof. Take an arbitrary sequence of domains $G_n (n=1,2,\ldots)$, such that $\overline{G_n} \subset G_{n+1} \rightarrow D$. If there is a sequence of domains $D_{i_n} (n=1,2,\ldots)$ such that $D_{i_n} \cap G_n = \phi$, then the area of $D_{i_n} \rightarrow 0$. Consequently there exists a number n_0 such that for each $n \geqq n_0$, $G_n \cap D_i \neq \phi$ $(i=1,2,\ldots)$. Since only a finite number of cross-cuts $Q_{i_1}, Q_{i_2},\ldots,Q_{i_p}$ has common points with G_n and for other cross-cuts Q_i, $D_i \supset G_n$, so $D_0 \cap G_n = (\bigcap_{j=1}^{p} D_{i_j}) \cap G_n$ is a non-empty open set(29). Since $D_0 = D_0 \cap (\bigcup_{n=1}^{\infty} G_n) = \bigcup_{n=1}^{\infty} (D_0 \cap G_n)$, D_0 is a non-empty open set and consists of components of domains.

Assuming that there are at least two components of D_0, connect a point z_1 in one component H_1 with a point z_2 in other component H_2 by a polygonal curve in D. Let z_3 be the point at which the curve has a point in common with the boundary of H_1 finally counting from z_1 and Q_{i_0} be the cross-cut on which z_3 lies. Since the one side of Q_{i_0} belongs to H_1, the curve does not enter into H_1 across Q_{i_0} after z_3 and hence z_2 can not belong to D_{i_0} because the another side of Q_{i_0} does not belong to D_{i_0}. This contradicts the definition of D_0. Therefore D_0 is a domain.

Proof of theorem 2.2. Without loss of generality, we may suppose that D is a circle $|z| < 1$, $z_0 = 1$ and $f(z)$ is regular on C except at z_0 by lemma 2. We shall first consider the case where one of Ω_1, Ω_2 does not contain the other. Approximate Ω_1 and Ω_2 by two sequences of simply connected domains $\Phi_n^{(1)}$, $\Phi_n^{(2)}$ $(n=1,2,\ldots)$ respectively so that $\Phi_n^{(i)} \supset \Omega_i$, $\Phi_n^{(i)} \supset \overline{\Phi_{n+1}^{(i)}}$ $(i=1, 2)$ and the boundary $\Gamma_n^{(i)}$ of $\Phi_n^{(i)}$ $(i=1, 2)$ is an analytic curve and passes no branch point.

For fixed n, there exists a positive number r_n such that $\overline{\mathfrak{D}_{r_n}} \subset \Phi_n^{(1)} \cup \Phi_n^{(2)}$ by theorem 2.1 and $\overline{M_{r_n}^{(C_i)}} \cup \Phi_n^{(i)}$ $(i=1, 2)$. Then there is no point of D_{r_n} which corresponds to the point on $\Gamma_n^{(1)}$ outside $\Phi_n^{(2)}$ or on $\Gamma_n^{(2)}$ outside $\Phi_n^{(1)}$, because these points are not in $\Phi_n^{(1)} \cup \Phi_n^{(2)}$.

Consider the domains in D_{r_n} in which $f(z)$ takes the values belonging to $\Phi_n^{(1)}$ and let $D_n^{(1)}$ be a component which is in contact with $C_{r_n}^{(1)}$. The values, which $f(z)$ takes on $C_{r_n}^{(1)}$ except at z_0, belong to $\Phi_n^{(1)}$, and hence some part of D_{r_n} near $C_{r_n}^{(1)}$, is contained in $D_n^{(1)}$.

Next we shall investigate the boundary curves of $D_n^{(1)}$ inside D_{r_n}. These curves are images of an analytic $\Gamma_n^{(1)}$, and hence consist of at most an enumerably infinite number of cross-cuts having no common point with each other, not accumulating in D_{r_n} and not terminating on $C_{r_n}^{(1)}$, including z_0. For if a cross-cut terminates at z_0, the cluster set on that curve consists of one point on $\Gamma_n^{(1)}$ and $\Omega_1 \subset \Phi_n^{(1)}$, and they are disjoint, but it is impossible by Hössjer's theorem. And further $D_n^{(1)}$ is a simply connected domain.

Considering $\Phi_n^{(2)}$, we get another domain $D_n^{(2)}$ with the same character. The boundary curves of both domains inside D_{r_n} are cross-cuts not accumulating in D_{r_n}, not terminating at z_0 and free from each other, because the common point corresponds to the point of intersection of $\Gamma_n^{(1)}$ and $\Gamma_n^{(2)}$, and this is outside $\mathfrak{D}_{r_n}$ by selecting r_n sufficiently small. Considering that any cross-cut is the boundary curve of non-empty $D_n^{(1)}$ or $D_n^{(2)}$, the further assumption of lemma 3 is satisfied and the intersection $D^n = D_n^{(1)} \cap D_n^{(2)}$ is a domain.

For each n we get domains $D_n^{(1)}$, $D_n^{(2)}$ and D^n such that $D_{n+1}^{(i)} \subset D_n^{(i)}$ $(i=1,2)$ and hence $D^{n+1} \subset D^n$ holds. If we take $r_n \to 0$, then $D^n \to z_0$. Let z_n be a point in D^n, connect z_n with z_{n+1} in D^n by a polygonal curve, combine them and make it a simple curve by removing the superfluous parts from it. Then it is easily seen that $S_{z_0}^{(L)} \subset \Omega$.

Now in the case where the one contains the other, for instance $\Omega_1 \subset \Omega_2$, we get L by lemma 2.

Remark. When Ω consists of many continuums, $S_{z_0}^{(L)}$ belongs to a component of Ω since $S_{z_0}^{(L)}$ is a continuum, and there is no more such a curve on which the cluster set belongs to the other component of Ω, because of Hössjer's theorem.

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Notes.

(1) We use + for sums of disjoint sets.

(2) $\mathfrak{D}_r$ denotes the closure of $\mathfrak{D}_r$: ditto concerning $\bar{M}_r^{(E)}$ $\vec{Y}_r^{(E)}$ etc.

(3) We will call this a function of class a.

(4) *Cf.* W. Gross: Zum Verhalten der konformen Abbildung am Rande. Math. Zeit. **3** (1919).

(5) An example will be shown at the end of n°3.

(6) F. Iversen: Sur quelques propriétés des fonctions monogènes au voisinage d'un point singulier. Öfv. af Finska Vet-Soc. Förh. **58** (1916).

W. Seidel: On the cluster values of analytic functions Trans. Amer. Math. Soc. **34** (1932).

(7) J. L. Doob: On a theorem of Gross and Iversen. Ann. of Math. **33** (1932).

K. Noshiro: On the singularities of analytic functions. Jap. Jour. Math. **17** (1940).

(8) *Cf.* S. Ishikawa: On the cluster sets of analytic functions. Nippon Sugaku-Butsurigaku Kaishi. **13** (1939) (in Japanese).

(9) W. Gross: Zum Verhalten analytischer Funktionen in der Umgebung singulärer Stellen. Math. Zeit. **2** (1918).

(10) $Q_1 \supset Q_2$ represents that Q_2 is nearer to z_0 than Q_1.

(11) S. Kametani and T. Ugaheri: A remark on Kawakami's extension of Löwner's lemma. Proc. Imp. Acad. Tokyo. **18** (1942).

(12) M. Tsuji: On an extension of Löwner's theorem. Proc. Imp. Acad. Tokyo. **18** (1942).

(13) F. Iversen: Recherches sur les fonctions inverses des fonctions méromorphes. Thèse de Helsingfors. 1914.

K. Noshiro: loc. cit. (7).

(14) Capacity means logarithmic capacity.

(15) O. Frostman: Potentiel d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions. Meddel. Lunds Univ. Mat. Sem. **3** (1935).

(16) We denote Koebe's theorem for a function of class a by generalized Koebe's theorem. *Cf.* W. Gross: Über die Singularitäten analytischer Funktionen. Mh. Math. u. Physik. **29** (1918).

(17) M. L. Cartwright: On the behaviour of analytic functions in the neighbourhood of its essential singularities. Math. Ann. **112** (1936).

(18) Here, theorems for a function of class a are considered.

(19) *Cf.* R. Nevanlinna: Eindeutige analytische Funktionen. Berlin. 1936.

(20) W. Seidel: On the distribution of values of bounded analytic function. Trans. Amer. Math. **36** (1934).

(21) G. Hössjer: Bemerkung über einen Satz von E. Lindelöf. Fysiogr. Sällsk. Lunds. Förh. **6** (1937). G. Hössjer assumed the continuity of $f(z)$ on the closed Jordan domain except for z_0, but here it is unnecessary.

(22) ϕ represents an empty set.

(23) K. Noshiro: loc. cit. (7).

(24) W. Gross has obtained already some similar results. But our results are different from his in several points. W. Gross: loc. cit. (9).

(25) We will call it briefly the branch point (in the w-plane).

(26) Giving two sequences of points $\{z_n\}$ and $\{z_n'\}$ which converge to z_0 on C_1 and C_2 respectively and proving that any curve in D connecting two points z_n and z_n' meets at least

one of given domains in D, he concluded the existence of a domain having z_0 on its boundary among these domains. But it seems hasty to conclude so.

(27) That is, there runs only a finite number of curves near any point in D.

(28) Area means the inner extent in Jordan's sense.

(29) Since Q_{ij} ($j=1, 2, \ldots\ldots, p$) don't pass z_0, some neighbourhood of z_0 in D is included in $\bigcap_{j=1}^{p} D_{ij}$. Connect z_0 with a point z_1 in G_n by a curve in D. If this curve does not meet Q_{ij} ($j=1, 2, \ldots\ldots, p_i$), z_1 will belong to $\bigcap_{j=1}^{p} D_{ij}$, otherwise there will exist a cross-cut Q_{io} which the curve intersects at the first time counting from z_0. Since one side of Q_{i_0} belongs to $\bigcap_{j=1}^{p} D_{ij}$ and some part of Q_{io} lies in G_n, it is possible to enter into G_n staying inside $\bigcap_{j=1}^{p} D_{ij}$. Accordingly $(\bigcap_{j=1}^{p} D_{ij}) \cap G_n$ is a non empty open set.