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On the decomposition of an (L)-group

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The purpose of the present note is to give a decomposition theorem of an (L)-group[1] which is a generalization of the well known theorem of Levi[2] in the theory of Lie groups.

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§1. A locally compact group G is called an *(L)-group*, if G contains a system of closed normal subgroups $\{N_\alpha\}$ such that

1) G/N_α are all Lie groups and
2) $\bigcap_\alpha N_\alpha = e$,

where e denotes the unit element of G. If an (L)-group G is connected, G contains a system of compact normal subgroups $\{K_\alpha\}$ such that G/K_α are all Lie groups and $\bigcap_\alpha K_\alpha = e$. Moreover we may assume that all K_α are contained in a compact normal subgroup and that the intersection of any finite number of K_α is contained in the system $\{K_\alpha\}$[3]. Such a system $\{K_\alpha\}$ is denoted in the following as a *canonical system* of G.

A subgroup L of an (L)-group G is called a Lie subgroup, if it is generated by a local Lie group L_l which is contained in a neighbourhood of the unit element of G. If we take as the neighbourhoods of the unit element in the group L those of the local group L_l we may introduce a new topology in L, which we shall call the inner topology of L.

Now let G be an arbitrary topological group and let H_1 and H_2 be the subgroups of G. We denote by $[H_1, H_2]$ the subgroup of G generated by the elements of the form $[h_1, h_2]=h_1 h_2 h_1^{-1} h_2^{-1}$. The closure of $[H_1, H_2]$ will be denoted by $C(H_1, H_2)$ and is called the topological commutator group of H_1 and H_2. In particular $C(G, G)$ is called the topological commutator group of G and is denoted by $D(G)$. We define inductively the groups $D_n(G)$ by the relations $D_0(G)=G$, $D_n(G)=D(D_{n-1}(G))$. Analogously the subgroups $N_n(G)$ are defined by $N_0(G)=G$. $N_n(G)=C(G, N_{n-1}(G))$. A connected locally compact group G is *solvable* (*nilpotent*), if $D_n(G)=e$ $(N_n(G)=e)$ for some integer n. In the case of the Lie groups these definitions of the solvability and the nilpotency coincide with the usual ones.

As K. Iwasawa has shown, there exists in any connected locally compact group G a unique maximal solvable connected closed normal subgroup which is called the *radical* of G[3].

Definition 1. A connected (L)-group is said to be *semi-simple* if its radical contains only the unit element.

By a result of M. Gotô[4] any connected semi-simple (L)-group G is the product of its maximal connected compact normal subgroup C and a closed connected normal subgroup L, which is a semi-simpl Lie group containing no compact connected normal subgroup other than the group (e) consisting only of the unit element, such that $[L, C]=e$ and $L\cap G$ is a finite group[5]. We call such a decomposition of a connected semi-simple (L)-group a *canonical decomposition.*

§**2.** In this section we prove some lemmas which are necessary in the following sections.

Lemma 1. Let G be a connected Lie group and R its radical and let N be a closed normal subgroup. Then RN is closed and RN/N is the radical of G/N.

We omit the proof.

Lemma 2. Let G be a connected (L)-group and R its radical. If N is a closed normal subgroup of G such that G/N is a Lie group, then RN is closed and RN/N is the radical of G/N.

Proof. Let K be a compact normal subgroup of G such that G/K is a Lie group. Then $G/N\cap K$ is also a Lie group and since $K\cap N$ is compact, we see that $R(K\cap N)/K\cap N$ is the raical of $G/K\cap N$[6]. By the homomorphic mapping of $G/K\cap N$ on G/N, $R(K\cap N)/K\cap N$ is mapped on RN/N. Then by Lemma 3 we see that RN is closed and RN/N is the radical of G/N.

Lemma 3. Let G be a connected semi-simple (L)-group and N a closed o-dimensional normal subgroup. If G/N is a Lie group, then G itself is a Lie group and N is discrete.

Proof. Since N is o-dimensional, N is contained in the center of G. Let K be a compact open subgroup of N. As N is central, K is a compact o-dimensional normal subgroup of G. Since N/K is discrete, we see that G/K is a Lie group, whence G is a Lie group[6]. Since each o-dimesional closed subgroup of a Lie group is discrete, N is discrete.

Lemma 4. Let G be a connected (L)-group and L be a closed connected normal subgroup which is a semi-simple Lie group. Suppose that there exists a closed normal subgroup M such that $G=L\cdot M$, $[L,M]=e$.

Then $G=L\cdot M_0$, where M_0 denotes the component of the unit element in M.

Proof. As G/M is a semi-simple Lie group, M contains the radical R of G. Hence $M_0 \supseteqq R$ and G/M_0 is a connected semi-simple (L)-group. But since the factor group of G/M_0 with respect to M/M_0 is a Lie group and since M/M_0 is o-dimensional, G/M_0 is a Lie group and M/M_0 is discrete by Lemma 3. We have $G/M_0=LM_0/M_0\cdot M/M_0$, where LM_0/M_0 is a Lie subgroup of G/M_0. As we may easily see, the Lie groups G/M_0 and LM_0/M_0 are locally isomorphic, whence $G/M_0=L\cdot M_0/M_0$. Thus $G=L\cdot M_0$.

Lemma 5. Let G be a connected (L)-group and R its radical and let $D(G)$ be the topological commutator group. Then $G=D(G)\cdot R$.

Proof. As $G/D(G)$ is abelian, there exist closed normal subgroups K and H such that $K\cdot H=G$, $K\cap H=D(G)$, $K/D(G)$ is compact and $H/D(G)$ is a vector group. Since G/K is a vector group we see by Lemma 4 that RK/K is the radical of G/K. But since G/K is abelian, we have $G=R\cdot K$. Next let R_1 be the radical of $D(G)$ and put $K/R_1=\tilde{K}$, $D(G)/R_1=\tilde{D}$. Then $\tilde{D}$ is semi-simple and $\tilde{K}/\tilde{D}$ is a compact abelian group. Let $\tilde{N}$ be the centraliser of $\tilde{D}$ in $\tilde{K}$. Then $\tilde{K}=\tilde{D}\cdot\tilde{N}$[7]. Now let R_2 be the radical of K. Then $\tilde{R}_2=R_2/R_1$ is the radical of $\tilde{K}$ and as we may easily see $\tilde{R}_2$ is contained in $\tilde{N}$. Since $\tilde{N}/\tilde{N}\cap\tilde{D}$ ($\simeq \tilde{K}/\tilde{D}$) and $\tilde{N}\cap\tilde{D}$ are abelian, $\tilde{N}$ is solvable, whence the component of the unit element of $\tilde{N}$ coincides with $\tilde{R}_2$. Let $\tilde{D}=\tilde{L}\cdot\tilde{C}$ be the canonical decomposition of $\tilde{D}$. The $\tilde{K}=\tilde{L}\cdot\tilde{C}\cdot\tilde{N}$ and $\tilde{M}=\tilde{C}\tilde{N}$ is the closed normal subgroup of $\tilde{K}$ such that $[\tilde{L}, \tilde{M}]=e$. Therefore we get by Lemma 4 $\tilde{K}=\tilde{L}\cdot\tilde{M}_0$, where $\tilde{M}_0$ denotes the component of the unit element of $\tilde{M}$. But we see easily that $\tilde{M}_0=\tilde{C}\cdot\tilde{R}_2$, whence $\tilde{K}=\tilde{L}\cdot\tilde{C}\cdot\tilde{R}_2$. Hence $K=D\cdot R_2$. Therefore $G=K\cdot R=D(G)\cdot R$.

Lemma 6. Let G be a connected (L)-group such that the radical R is a Lie group. If the factor group G/K with respect to a compact o-dimensional subgroup K is a Lie group, then G itself is a Lie group.

Proof. It is sufficient to prove that G/R is a Lie group. But since G/KR is a Lie group and KR/R is a compact o-dimensional normal subgroup of G/R, G/R is a Lie group.

§3. In this section we prove our decomposition theorem and the uniqueness of such decomposition up to inner automorphismus.

Theorem 1. *Let G be a connected (L)-group. Then G decomposes into the form $G=L\cdot C\cdot R$, where*

1) *R is the radical of G, C is a compact connected semi-simple subgroup and L is a semi-simple Lie subgroup which contains no compact connected normal subgroup different from (e),*

2) *$[L, C]=e$ and $L\cap C$ is a finite group and*

3) *LR and CR are the closed normal subgroups such that $G/R=LR/R\cdot CR/R$ is the canonical decomposition of the connected semi-simple (L)-group G/R.*

Proof. We first consider the case where G is a Lie group. Let $G=S\cdot R$ be a Levi decomposition of G, where S is a semi-simple Lie subgroup. Now let $S=L\cdot C$ be the canonical decomposition of the semi-simple Lie group S. Then since the inner topology of S is stronger than the relative topology of S as a subgroup of G, C is also a compact connected semi-simple subgroup of G. Since S and G/R are locally isomorphic and the compactness is the local property in the case of the semi-simple Lie groups by Weyl's theorem,[8] the image CR/R of C by the locally isomorphic mapping of S on G/R is the maximal compact connected normal subgroup of G/R. Moreover since L is the component of the unit element of the centraliser of C, LR/R coincides locally with the component of the centraliser of CR/R. Hence LR/R is closed and $G/R=LR/R\cdot CR/R$. is the canonical decomposition of G/R.

Next we consider the case of an (L)-group. But we proceed stepwise as follows: First we consider the case where R is a simply connected Lie group, then the case where R is nilpotent, and finally the general case.

1) R is a simply connected Lie group.

Let $G/R=L_1/R\cdot C_1/R$ be the canonical decomposition of G/R and K_0 be the maximal compact connected normal subgroup of C_1. Clearly K_0 is a normal subgroup of G. We first show that G/K_0 is a Lie group. Since L_1 is a Lie group and $G/K_0=L_1K_0/K_0\cdot C_1/K_0$, it is sufficient to show that C_1/K_0 is a Lie group. Let K be the maximal compact normal subgroup of C_1. Then C_1/K is a Lie group[9] and since K/K_0 is compact and o-dimensional and the radical RK_0/K_0 of C_1/K_0 is a Lie group, we see by Lemma 6 that C_1/K_0 is a Lie group. Now since the kernel K_0R/R of the homomorphism $G/R\sim G/K_0R$ is compact and connected, C_1/KR is the maximal compact connected normal subgroup of G/K_0R and $G/K_0R=L_1K_0/K_0R\cdot C_1/K_0R$ is the canonical decomposition of the semi-simple Lie group G/K_0R. Put $G^*=G/K_0$, $R^*=RK_0/K_0$, $L^*_1=L_1K_0/K_0$ and $C_1^*=C_1/K_0$. Then by what has been already proved there exists a semi-simple Lie subgroup S^* of the Lie group G^* such that $L^*_1=L^*R^*$ and $C_1^*=C^*R^*$, where $S^*=$

L^*C^* is the canonical decomposition of S^*. Now let $L_1=L\cdot R$ be a Levi decomposition of the Lie group L_1, where L is a semi-simple Lie subgroup. Then $L^*_1=L_1K_0/K_0=L\cdot K_0/K_0\cdot RK_0/K_0$ is a Levi decomposition of L_1^* and since $L^*_1=L^*R^*$ is also such a decomposition, there exists an element r in R such that $rLr^{-1}K_0/K_0=L^*$[10]. Now let C be the complete inverse image of C^* under the homomorphism $G/G\sim K_0$. Clearly C is contained in C_1 and since C^* and K_0 are both compact, connected and semi-simple, the same holds for C. Since $G=L_1\cdot C_1$, $L_1=rLr^{-1}\cdot R$, $C_1=CR$, we have $G=rLr^{-1}\cdot C\cdot R$. Now since $[L^*, C^*]=e$, *i.e.* $[rLr^{-1}K_0/K_0, C/K_0]=e$ we have $[rLr^{-1}, C]\subseteqq K_0$. On the other hand since $[L_1, C_1]\subseteqq R$, we have $[rLr^{-1}, C]\subseteqq R$, whence $[L, C]\subseteqq R\cap K_0$. But as R is simply connected, R contains no compact subgroup, whence $R\cap K_0=e$. Thus $[L, C]=e$. That $rLr^{-1}\cap C$ is a finite group will be shown afterwards.

2) R is nilpotent.

Let K be the maximal compact subgroup of R. Then K is connected and is contained in the center of R[11]. Therefore K is the unique maximal compact subgroup and a central subgronp of G and R/K is a simply connected Lie group. Let $G/R=L_1/R\cdot C_1/R$ be the canonical decomposition of G/R and put $G'=G/K$, $R'=R/K$, $L_1'=L_1/K$ and $C_1'=C_1/K$. Then $G'/R'=L_1'/R'\cdot C_1'/R'$ is the canonical decomposition of G'/R' and since the radical R' of G' is simply connected, we have by 1) a decomposition of G such that $G'=L'\cdot C'\ R'$, $L'_1=L'R'$, $C_1'=C'R'$, $[L', C']=e'$, where e' denotes the unit element of G'. Now since $L'_1=L_1/K$ is a Lie group and K is compact and abelian, there exists in L_1 a semi-simple Lie subgroup L such that $L_1=L\cdot R$ and $L'=LK/K$.[13] Next let $\tilde{C}$ be the complete inverse image of C' under the homomorphism $C_1\sim C_1'$. Then $\tilde{C}$ is compact and connected and K is the radical of $\tilde{C}$. As we may see from the structure theory of compact groups,[14] there exists a connected compact semi-simple subgroup C such that $\tilde{C}=C\cdot K$. Then $C_1=\tilde{C}\cdot R=C\cdot R$. Thus $G=L\cdot C\cdot R$, $L_1=L\cdot R$, $C_1=C\cdot R$. Since $[L', C']=e'$, we have $[L, C]\subseteqq K$. As K is central, $(lcl^{-1}c^{-1})c=c(lcl^{-1}c^{-1})$, *i. e.* $lcl^{-1}=clcl^{-1}c^{-1}$ for $c\in C$, $l\in L$. Multiplying c^{-1} from the left, we obtain $c^{-1}(lcl^{-1})=(lcl^{-1})c^{-1}$. Then $(lcl^{-1}c^{-1})\cdot(ldl^{-1}d^{-1})=c^{-1}(lcl^{-1})\ (ldl^{-1}d^{-1})=c^{-1}\ (lcdl^{-1}d^{-1}c^{-1})\ c=lcdl^{-1}(c\ d)^{-1}$. Hence, for fixed l, the correspondence $c\rightarrow lcl^{-1}c^{-1}$ is a continuous (not necessarily open) homomorphism of the group C into the group K. Denoting by N the kernel of this homomorphism, we see that C/N is an abelian group. But since C is semi-simple, we have $C=N$. Therefore every element in C commutes with $l\in L$. Thus we obtain $[L, C]=e$. Moreover since L and L_1/R are locally isomorphic and L_1/R contains no compact connected normal subgroup except (e), the same holds for L by Weyl's theorem. Now we prove that $L\cap C$ is a finite group. Let $\{K_\alpha\}$ be a canonical system of the connected (L)-group G. Then G is the limit group of the system of groups $\{G/K_\alpha\}$ and hence $L\cap C$ is the limit group of the system of groups $\{(L\cap C)\ K_\alpha/K_\alpha\}$. We have $G/K_\alpha=LK_\alpha/K_\alpha\cdot CK_\alpha/K_\alpha\ RK_\alpha/K_\alpha$ and $LK_\alpha/K_\alpha\cdot CK_\alpha/K_\alpha$ are the maximal semi-simple Lie subgroups S_α of the Lie groups G/K_α. Since $L\sim LK_\alpha/K_\alpha$ and L contains no compact connected normal subgroup except (e), the same hold for LK_α/K_α. Hence CK_α/K_α are the maximal compact connected normal subgroups of S_α and $S_\alpha=LK_\alpha/K_\alpha\cdot CK_\alpha/K_\alpha$ are the canonical decompositions of S_α. Therefore $(L\cap C)K_\beta/K_\alpha$ are the finite groups. Thus $L\cap C$ is a limit group of a system of finite groups, whence $L\cap C$ is compact. On the other hand $L\cap C$ is contained in the center of the semi-simple Lie group L and hence it is enumerable. Therefore $L\cap C$ must be a finite group.[15]

3) General case.

As the radical R_1 of the topological commutor group $D(G)$ is nilpotent;[16] we have by 2) a decomposition of $D(G)$ such that $D(G)=L\cdot C\cdot R_1$, $[L, C]=e$ and $L\cap C$ is a finite group. Then since $G=D(G)\cdot R$ by Lemma 7, we have $G=L\cdot C\cdot R$. We must prove that LR and CR are the closed normal subgroups such that $G/R=LR/R\cdot CR/R$ is the canonical decomposition of G/R. For this purpose let $G/R=L_1/R\cdot C_1/R$ be the canonical decomposition of G/R. Since CR_1 is a characteristic subgroup of $D(G)$, $CR=CR_1\cdot R$ is a closed normal subgroup of G and clearly $CR\supseteqq C_1$. Now we have $G/R_1=R/R_1\cdot D/R_1$ and the radical R/R_1 of G/R_1 is central. Since R/R_1 is also the radical of C_1/R_1 and $C_1/R_1/R/R_1$ compact, C_1/R_1 is by 2) the product of R/R_1 and a compact semi-simple connected subgroup. Hence $D(C_1/R_1)$ is compact and since $D(C_1/R_1)\subseteqq D(G/R_1)=D(G)/R_1$, we have $D(C_1/R_1)\subseteqq CR_1/R_1$. But $D(C_1/R_1)=\overline{D(C_1)R_1/R_1}$, whence $D(C_1)R_1\subseteqq CR_1$, Therefore $\overline{D(C_1)R}\subseteqq CR$. On the other hand since C_1/R is semi-simple, we have $D(C_1/R)=C_1/R$ and hence $\overline{D(C_1)R}=C_1$. Thus we obtain $C_1=CR$. Now $G/R=C_1/R\cdot LR/R$ and clearly $L_1/R\supseteqq LR/R$. But as $G/R/C_1/R$ is locally isomorphic with LR/R and also with L_1/R, the Lie groups L_1/R and LR/R must be of the same dimensions. Therefore $L_1/R\doteq LR/R$, whence $L_1=LR$. q. e. d.

Theorem 2. *Let $G=L\cdot C\cdot R$ and $G=L'C'R$ be two decompositions of a connected (L)-group G which satisfy the conditions 1) and 2) in Theorem*

2. *Then there exists an element r in R such that $rLr^{-1}=L'$, $rCr^{-1}=C'$.*

Proof. We may assume that the one of these decompositions, for example $G=L\cdot C\cdot R$, is the one obtained in the proof of Theorem 1. First we prove that $LR=L'R$ and $CR=C'R$. Let $\{K_\alpha\}$ be a canonical system of G. Then $G/K_\alpha=LK_\alpha/K_\alpha\cdot CK_\alpha/K_\alpha RK_\alpha/K_\alpha$ and $S_\alpha=LK_\alpha/K_\alpha\cdot CK_\alpha/K_\alpha$ is a maximal semi-simple Lie subgroup of G/K_α. As was shown before, CK_α/K_α is the maximal compact connected normal subgroup of S_α. By the same reason, also $C'K_\alpha/K_\alpha$ is the maximal compact normal subgroup of the maximal semi-simple Lie subgroup $S_\alpha'=L'K_\alpha/K_\alpha\cdot C'K_\alpha/K_\alpha$. Hence from what has been already noticed in the proof of Theorem 1 we obtain $CK_\alpha/K_\alpha\cdot RK_\alpha/K_\alpha=C'K_\alpha/K_\alpha RK/K_\alpha$ i.e. $CRK_\alpha=C'RK_\alpha$. Considering the intersections of each sides for all α, we get $CR=C'R$. But since $G/R=LR/R\cdot CR/R=L'R/R\cdot CR/R$ and the former is the canonical decomposition of G/R, we have $LR/R\supseteq L'R/R$. But as $G/R/CR/R$ is locally isomorphic with LR/R and also with $L'R/R$, we obtain $LR=L'R$ as before.

Next we consider the case where the radical R is nilpotent. Let K be the maximal compact subgroup of R. As we have already remarked, K is connected and is contained in the center of G. Then CK is a maximal compact subgroup of $C_1=CR=C'R$. For, since $C_1/K=CK/K\cdot R/K$ and R/K is a simply connected Lie group, CK/K is a maximal compact subgroup of C_1/K and hence CK is maximally compact in C_1. By the same way $C'K$ is also a maximal compact subgroup of C_1. Hence there exists an element $a=c\cdot r(c\in C,\ r\in R)$ in C_1 such that $aCKa^{-1}=C'K$. Since K is central, we have $rCr^{-1}K=C'K$. Let $C'K=M$. Then K is the radical of M and $rCr^{-1}\cdot K$ and $C'K$ are two decompositions of the compact group M as the products of the semi-simple connected compact normal subgroups and the radical. As such a decomposition of the compact group is unique, we have $rCr^{-1}=C$. Hence it is sufficient to show that, if $G=L\cdot C\cdot R=L'\cdot C\cdot R$ are the decompositions of G, then there exists an element r in R such that $rLr^{-1}=L'$ and $rCr^{-1}=C$. If G is a Lie group, this is easy to verify. First let R be simply connected. Let K_0 be the maximal compact connected normal subgroup of $C_1=CR$. Then as we have already shown G/K_0 is a Lie group and hence there exists an element r in R such that $rLr^{-1}\cdot K_0=L'K_0$ and $rCr^{-1}=C$. Since $K_0\subseteq C$, we have $[rLr^{-1}, K_0]=e$, $[L, K_0]=e$ and $rLr^{-1}\cap K_0$ and $L'\cap K_0$ are finite groups. Hence $L'K_0$ is algebraically isomorphic with $L'\times K_0/D$, where D is a finite group. If we introduce a topology in $L'K_0$ as the factor group of $L'\times K_0$, then $L'K_0$ becomes a connected semi-simple (L)-group and L' is a closed subgroup.

Since L' contains no compact connected normal subgroup, K_0 is the maximal compact connected normal subgroup of $L'\cdot K_0$. Let $L'K_0=L_0K_0$ be the canonical decomposition of $L'K_0$, then L' is contained in L_0 and since L_0 and L' are locally isomorphic Lie groups we get $L_0=L'$. rLr^{-1} is also contained in L_0 and L' and rLr^{-1} are also locally isomorphic. Hence $L' =rLr^{-1}$. If R is not simply connected, consider the group G/K. Then we have $rLr^{-1}\cdot K=L'K$ and $rCr^{-1}K=CK$. Then considering the commutator group in the former and the topological commutator group in the latter, we obtain $rLr=L'$ and $rCr=C$. Finally we consider the general case. Let R_1 be the radical of $D(G)$. Then since the decomposition $G=L\cdot C\cdot R$ is the one obtained in the proof of Theorem 1, $D(G)/R_1=LR_1/R_1\cdot CR_1/R_1$ is the canonical decomposition of $D(G)/R_1$. Now the component of the unit element of the group $CR\cap D\ (G)$ is CR_1. For, since $C\subseteq D(G)$, we have $CR\cap D(G)=C(R\cap D(G))$ and as tha component of the unit element of $R\cap D(G)$ is R_1, we may easily verify the above proposition. Since $C'R=CR$ and C' is also contained in $D(G)$, $C'(R\cap D(G))=C(R\cap D(G))$ and considering the components of the both sides, we have $CR_1=C'R_1$. As L' is also contained in $D(G)$ and $[L', C']=e$, we have $LR_1/R_1\supseteqq L'R_1/R_1$ and by the same argument as before we obtain $LR_1=L'R_1$. Therefore $D(G)=L\cdot C\cdot R_1=L'C'R_1$. Since R_1 is nilpotent there exists an element r in R_1 such that $rLr^{-1}=L'$ and $rCr^{-1}=C$. q.e.d.

§**4.** Now we may give somewhat different formulation to Theorems 1 and 2.

Definition 2. Let G be a connected (L)-group. A subgroup H of G is called an *semi-simple (L)-subgroup*, if, for each closed normal subgroup N such that G/N is a Lie group, HN/N is a semi-simple Lie subgroup of G/N.

As we may easily see, each closed semi-simple subgroup is a semi-simple (L)-subgroup.

Theorem 3. *Let G be a connected (L)-group and R its radical. Then there exists a semi-simple (L)-subgroup S such that $G=S\cdot R$. If S' is another semi-simple (L)-subgroup such that $G=S'\cdot R$, then there exists an element r in R such that $rSr^{-1}=S'$.*

Proof. Let $G=L\cdot C\cdot R$ be a decomposition in Theorem 1. Then since L and C are semi-simple (L)-subgroups and $[L, C]=e$, we see that the subgoup $S=L\cdot C$ is a semi-simple (L)-subgroup such that $S\cdot R=G$. Now let $\{K_\alpha\}$ be a canonical system of G and put $G_\alpha=G/K_\alpha$, $S'_\alpha=S'K_\alpha/K_\alpha$. Then S'_α are maximal semi-simple Lie subgroups of the Lie groups

G_α and G is the limit group of the system $\{G_\alpha\}$ of Lie groups. Let α and β be an arbitrary pair of indices such that $K_\beta \subset K_\alpha$. Then G_β is homomorphic to G_α. If $\varphi_{\alpha\beta}$ denotes the homomorphic mapping of G_β on G_α, then there corresponds to every element x of G the system $\{x_\alpha\}$, where $x_\alpha \in G_\alpha$ and $x_\alpha = \varphi_{\alpha\beta}(x_\beta)$ for any pair α, β such that $K_\beta \subset K_\alpha$. In particular the elements of S' ars determined by the systems $\{s'_\alpha\}$, $s'_\alpha \in S'_\alpha$. Now let $S'_\alpha = L'_\alpha \cdot C'_\alpha$ and $S'_\beta = L'_\beta \cdot C'_\beta$ be the canonical decompositions of the semi-simple Lie groups S'_α and S'_β. Then $\varphi_{\alpha\beta}(S'_\beta) = S'^\alpha$, $\varphi_{\alpha\beta}(L'_\beta) = L'_\alpha$ and $\varphi_{\alpha\beta}(C'_\beta) = C'_\alpha$. Clearly the systems $\{c'_\alpha\}$ such that $c'_\alpha \in C'_\alpha$ determine a compact subgroup $C' \subseteqq S'$. Further since the kernels $K_{\alpha\beta}$ of the homomorphisms $\varphi_{\alpha\beta}$ are compact, we see that $\varphi_{\alpha\beta}$ are locally isomorphic mappings of L'_β on L'_α. In fact, let $\mathfrak{K}_{\alpha\beta}$ be the ideals of $\mathfrak{G}_\beta$ which correspond to the group $K_{\alpha\beta}$, where $\mathfrak{G}_\beta$ denotes the Lie algebra of the Lie group G_β. Then since $K_{\alpha\beta}$ is compact, $\mathfrak{K}_{\alpha\beta} = \mathfrak{S}_{\alpha\beta} + \mathfrak{Z}_{\alpha\beta}$, where $\mathfrak{S}_{\alpha\beta}$ is the semi-simple ideal and $\mathfrak{Z}_{\alpha\beta}$ is the center. Since $\mathfrak{K}_{\alpha\beta}$ is the ideal of $\mathfrak{G}_\beta$ $\mathfrak{S}_{\alpha\beta}$ is contained in the Lie algebra $\mathfrak{S}'_\beta$ of the maximal semi-simple Lie subgroup S_β and $\mathfrak{Z}_{\alpha\beta}$ is contained in the radical of $\mathfrak{G}_\beta$ Moreover since $\mathfrak{S}_{\alpha\beta}$ generates the compact group, $\mathfrak{S}_{\alpha\beta}$ is contained in the Lie algebra of the group C'_β. Hence the intersection of $\mathfrak{K}_{\alpha\beta}$ and the Lie algebra of the Lie subgroups L'_β contains only zero. This proves the above assertion Therefore we may choose a sufficiently small neighbourhood L^0_α of the unit element of each L'_α such that $\varphi_{\alpha\beta}$ are the one-to-one mappings of L^0_β on L^0_α for all pairs (α, β) such that $K_\beta \subset K_\alpha$. Further let ψ_α be the homomorphic mappings of G on G_α and put $L^0 = \cap_\alpha \psi_\alpha^{-1}(L^0_\alpha)$. There correspond to the elements of L^0 the systems $\{x_\alpha\}$ such that $x_\alpha \in L^0_\alpha$ and, for fixed α, every element x_α in L^0_α appears in such a system. Hence L^0 is a local Lie group isomorphic with the local Lie groups L^0_α. Let L' be the Lie subgroup generated by L^0. Then $L' \subset S'$ and $\psi_\alpha(L') = L_\alpha$, *i.e.* $L'K_\alpha/K_\alpha = L_\alpha$. We may easily see that $S' = L' \cdot C'$ and $[L', C] = e$. Further we may prove by the same argument as in the proof of Theorem 1 that $L' \cap C'$ is a finite group. Hence we obtain a decomposition $G = L' \cdot C'R$ which satisfies the conditions 1) and 2) in Theorem 2 such that $S' = L'C'$. Then by Theorem 3 there exists an element r in R such that $rLr^{-1} = L'$ and $rCr^{-1} = C'$. Hence $rSr^{-1} = S'$. q.e.d.

Now we consider the case where G is a connected (l)-group[17] and show that in this case the semi-simple (L)-subgroup S is closed.

Theorem 4. *If G is a connected (l)-group, then every semi-simple (L)-subgroup S such that $G = S \cdot R$ is a closed subgroup.*

Proof. If K is a compact normal subgroup of G such that G/K is a Lie group, then G/K is faithfully representable (*f.r.*)[18]. Let $\bar{S}$ be the closure of S. Then $\bar{S}K/K$ is the closure of SK/K. But since $\bar{S}K/K$ is a semi-simple Lie subgroup of the *f. r.* Lie group G/K, it is closed.[19] Hence $\bar{S}K/K=SK/K$. Thus $\bar{S}$ is a semi-simple (L)-subgroup such that $G=\bar{S}\cdot R$. But since S is conjugate with $\bar{S}$, S is also closed.

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Notes

1) For (L)-groups, see K. Iwasawa. On some types of topological groups, Annals of Math. Vol. 50, No. 3 (1949). We shall refer to this paper as I.
2) For Levi's theorem, see J.H.C. Whitehead. On the decomposition of an infinitesimal group, Proc. Cambr. Phil. Soc. v. **32**. (1932), A. Malcev. On the representation of an algebra as a direct sum of the radical and a semi-simple algebra, C. R. DRSS, **36** (1942) and M. Gotô, On a theorem of E.E. Levi, Mathematica Japonicae. Vol. 1. No. 3 (1949).
3) See, I.
4) M. Gotô, Linear representations of topological groups, to appear shortly. We shall refer to this paper as G.
5) L is the component of the unit element in the centraliser of C.
6) See, G.
7) See, G.
8) H. Weyl, Theorie der Darstellung kontinuierlicher halbeinfacher Gruppen durch lineare Transformationen, I-III, Math. Zeit. Bd. **23-24** (1924-25).
9) See, I.
10) See, A. Malcev, loc. cit.
11) See, G.
12) For, by I, maximal compact subgroups are conjugate with each other and any compact abelian normal subgroup is contained in the center of G.
13) See, I. Lemma 4.8

14) See, H. Freudenthal, Topologische Gruppen mit genügend vielen fastperiodischen Funktionen, Ann. of Math. **37** (1936).

15) See, G.

16) See, G.

17) For the definition and properties of (l)-groups, see, G.

18) A Lie group is said to be faithfully representable, if it admits an isomorphic continuous representaion by matrices.

19) K. Yosida, A theorem concerning the semi-simple Lie groups, Tohoku Math. Journ. v. **43** (1937).