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On Projectively Connected Spaces whose Groups of Holonomy Fix a Hyperquadric

Tominosuke Otsuki

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Introduction. This paper deals with n-dimensional projectively connected spaces P_n whose groups of holonomy fix a hyperquadric. For the spaces with normal connexion, similar results as ours have been obtained independently by S. Sasaki and K. Yano.[1]

In § 1, 2, we shall arrive at the fundamental equations (22) and (24) in the case of the spaces with no torsion, following the general method of E. Cartan in his famous paper.[2] In § 3, we shall consider the Riemannian space R^*_{n+1} which is associated with the space P_n and obtain a relation between these two spaces which shows that the condition in order that the connexion of P_n be normal is equivalent to that of R^*_{n+1} being an Einstein space. In § 4, we shall investigate the relations between the space P_n and the Riemannian space $R_n{}^*$ which is a hypersurface in R^*_{n+1}. Then, in § 5, we shall show that there exist a Riemannian space R_n which is projective to P_n and that, if the connexion of P_n is normal, R_n is an Einstein space. Lastly, in § 6, we shall show that R_n is the space treated by E. Cartan.[3]

§ 1. According to E. Cartan, let $R: (A,\ A_i)$ $(i=1,2,\ldots\ldots n)$ be a frame of an n-dimensional space P_n with projective connexion. Then the connexion is given by a system of Pfaffians $\omega^\lambda_\mu(\lambda,\ \mu=0,1,2,\ldots\ldots,n)$ such that

$$(1)\qquad dA=\omega^0_0A+\omega^iA_i, \qquad dA_i=\omega^0_iA+\omega^j_i\ A_j$$

where $\omega^i=\omega^i_0$. The equations of structure of P_n are

$$(2)\qquad (\omega^\mu_\lambda)'=[\omega^\rho_\lambda\ \omega^\mu_\rho]-\Omega^\mu_\lambda,$$

$$\Omega^\mu_\lambda-\delta^\mu_\lambda\Omega^0_0=\frac{1}{2}A_\lambda{}^\mu{}_{ij}\ [\omega^i\omega^j]$$

where $A_\lambda{}^\mu{}_{ij}$ are the components of the tensor of curvature and torsion of the space. In a coordinate neighborhood (y^i), we can use natural frames such that the following relations hold:

$$(3)\qquad \omega^i=dy^i, \qquad \omega^i_i-n\omega^0_0=0.$$

Let us now represent a non degenerate hyperquadric Q_{n-1}, which the group of holonomy of P_n at a given point A_0 fixes, by

(4) $$G^0_{\lambda\mu}x^\lambda x^\mu=0, \qquad |G^0_{\lambda\mu}|\neq 0$$

in reference of the frame R^0 : (A^0_λ) at the point A^0. Suppose that the point does not lie on Q_{n-1} and that the coordinates of A^0 are $y^i=0$ ($i=1,2,\ldots\ldots,n$). Now, if we consider a system of curves which pass through the point A^0

(5) $$y^i=a^it,$$

where a^i is a constant and $(a^i)\neq(0)$, and integrate the differential equations

(6) $$\begin{cases} \dfrac{d}{dt}b^\alpha_\mu+b^\beta_\mu\dfrac{\omega^\alpha_\beta(at.\ a\ dt)-\delta^\alpha_\beta\omega^0_0(at,\ a\ dt)}{dt}=0, \\ \dfrac{d}{dt}c^\lambda_\alpha-c^\lambda_\beta\dfrac{\omega^\beta_\alpha(at,\ a\ dt)-\delta^\beta_\alpha\omega^0_0(at,\ a\ dt)}{dt}=0 \end{cases}$$

under the initial conditions $b^\alpha_\mu(O)=d^\alpha_\mu$, $c^\lambda_\alpha(O)=\delta^\lambda_\alpha$, then we see that the solutions can be written as $b^\alpha_\mu=b^\alpha_\mu(y)$, $c^\lambda_\alpha=c^\lambda_\alpha(y)$ and that $b^\alpha_\lambda(y)c^\mu_\beta(y)=\delta^\alpha_\beta$. Making use of these solutions, we transform our frame such that

$$B_0=b^\lambda_0A_\lambda, \qquad B_i=b^\lambda_iA_\lambda$$

where $A_0=A$. If we define $\tilde\omega^\mu_\lambda$ by the relations

$$dB_\lambda=\tilde\omega^\mu_\lambda\ B_\mu,$$

we get

(7) $$\tilde\omega^\lambda_\mu b^\alpha_\lambda=db^\alpha_\mu+b^\beta_\mu\ \omega^\alpha_\beta$$

or

$$\tilde\omega^\lambda_\mu=c^\lambda_\alpha\ db^\alpha_\mu+b^\beta_\mu\ \omega^\alpha_\beta\ c^\lambda_\beta,$$

because

$$dB_\mu=(db^\alpha_\mu+b^\beta_\mu\ \omega^\alpha_\beta)A_\alpha=\tilde\omega^\lambda_\mu\ B_\lambda=\tilde\omega^\lambda_\mu\ b^\alpha_\lambda A_\alpha.$$

Since we have $\tilde\omega^\lambda_\mu=0$ along the curves (5) by virtue of (6), $\tilde\omega^\lambda_\mu$ are components of infinitesimal transformations of the group of holonomy according to a theorem of E. Cartan.[3] Hence $\tilde\omega^\lambda_\mu$ must satisfy the equations

(8) $$G^0_{\lambda\rho}\ \tilde\omega^\rho_\lambda+G^0_{\rho\mu}\ \tilde\omega^\rho_\lambda=G^0_{\lambda\mu}\quad\cdot$$

where π is a Pfaffian form. If we represent the equations of structure of the space with respect to the frame R : (B_λ) by

(9) $$(\tilde\omega^\lambda_\mu)'-[\tilde\omega^\rho_\mu\ \tilde\omega^\lambda_\rho]=-\tilde\Omega^\lambda_\mu,$$

$\tilde\Omega^\lambda_\mu$'s satisfy the relations

(10) $$\tilde\Omega^\lambda_\mu=c^\lambda_\alpha\ \Omega^\alpha_\beta\ b^\beta_\mu$$

because these quantities are components of a projective tensor. Differentiating (8) exteriorly and making use of (9) we get the following relations

$$(\pi)' G^0_{\lambda\mu} = G^0_{\lambda\rho}\,(\tilde{\omega}^\rho_\mu)' + G^0_{\rho\lambda}\,(\tilde{\omega}^\rho_\lambda)'$$
$$= G^0_{\lambda\rho}\,\{[\tilde{\omega}^\nu_\mu\ \tilde{\omega}^\rho_\nu] - \tilde{\Omega}^\rho_\mu\} + G^0_{\rho\mu}\,\{[\tilde{\omega}^\nu_\lambda\ \tilde{\omega}^\rho_\nu] - \tilde{\Omega}^\rho_\lambda\}$$
$$= -G^0_{\lambda\rho}\tilde{\Omega}^\rho_\mu - G^0_{\rho\mu}\tilde{\Omega}^\rho_\lambda + [\tilde{\omega}^\nu_\mu(\pi G^0_{\lambda\nu} - G^0_{\rho\nu}\,\tilde{\omega}^\rho_\lambda)] + [\tilde{\omega}^\nu_\lambda(\pi G^0_{\mu\nu} - G^0_{\rho\nu}\tilde{\omega}^\rho_\mu])$$
$$= -G^0_{\nu\rho}\ \tilde{\Omega}^\rho_\mu - G^0_{\rho\mu}\ \tilde{\Omega}^\rho_\lambda,$$

that is,

$$G^0_{\lambda\rho}\tilde{\Omega}^\rho_\mu + G^0_{\rho\mu}\tilde{\Omega}^\rho_\lambda = -(\pi)'\ G^0_{\lambda\mu}. \tag{11}$$

If we put

$$G_{\alpha\beta} = G^0_{\lambda\mu}\ c^\lambda_\alpha\ c^\mu_\beta, \tag{12}$$

we get from (11)

$$G_{\alpha\gamma}\Omega^\gamma_\beta + G_{\gamma\beta}\Omega^\gamma_\alpha = -(\pi)'\ G_{\alpha\beta}. \tag{13}$$

Let us suppose that the space P_n has no torsion, that is, the conditions

$$\Omega^i_0 \equiv \Omega^i = o \tag{14}$$

are satisfied. Then, if we put $\lambda=\mu=0$ in (13), we have $2G_{00}\Omega^0_0 = -(\pi)'$ G_{00}. At the point A^0 we have $G_{00} = G^0_{\lambda\mu}c^\lambda_0\ c^\mu_0 \neq 0$ by (6′) and by our assumption, hence we can suppose $G_{00} \neq 0$ in a neighborhood of A^0. Accordingly, we get from the above relation

$$\Omega^0_0 = -(\pi)'. \tag{15}$$

Thus we obtain

$$G_{\alpha\gamma}(\Omega^\gamma_\beta - \delta^\gamma_\beta\Omega^0_0) + G_{\gamma\beta}(\Omega^\gamma_\alpha - \delta^\gamma_\alpha\Omega^0_0) = 0 \tag{13′}$$

from (13) and (15).

Substituting (14), (15) in the following equations which are derived from (2) by exterior defferentiation:

$$(\Omega^\lambda_\mu)' = -[\Omega^\rho_\mu\ \omega^\lambda_\rho] + [\omega^\rho_\mu\ \Omega^\lambda_\rho]$$

we get

$$[\omega^k(\Omega^i_k - \delta^i_k\Omega^0_0)] = o, \qquad [\omega^k\ \Omega^0_k] = o. \tag{16}$$

Hence for the case $n \geqq 3$ we get from (16) and (2)

$$A_i{}^\lambda{}_{jk} + A_j{}^\lambda{}_{ki} + A_k{}^\lambda{}_{ij} = 0. \tag{16′}$$

Contracting (13′) with $G^{\alpha\beta}$; we have $\Omega^i_i - n\Omega^0_0 = 0$. Hence we see that *the infinitesimal transformations associated with infinitesimal closed circuits fix the point A and their dual transformations are unimodular affine.* By (2),

(3) we have $\Omega_i^i - n\Omega_0^0 = -[\omega_i^0\ \omega^i]$. Accordingly, putting

(17) $$\omega_\mu^\lambda - \delta_\mu^\lambda \omega_0^0 = \Gamma_{\mu i}^\lambda \omega^i = \Gamma_{\mu i}^\lambda\ dy^i,$$

we get

(18) $$\Gamma_{ij}^\lambda = \Gamma_{ji}^\lambda$$

from the above equation and (14).

§ 2. Let us now consider the qantity

(19) $$DG_{\lambda\mu} = dG_{\lambda\mu} - \omega_\lambda^\rho G_{\rho\mu} - \omega_\mu^\rho G_{\lambda\rho} + 2\omega_0^0 G_{\lambda\mu}$$

and its exterior derivative. Since

$$-(DG_{\lambda\mu})' = -\{\Omega_\lambda^\rho G_{\rho\mu} + \Omega_\mu^\rho G_{\lambda\rho} - 2\Omega_0^0 G_{\lambda\mu}\}$$
$$-\{[\omega_\lambda^\rho\ DG_{\rho\mu}] + [\omega_\mu^\rho DG_{\lambda\rho}] - 2[\omega_0^0\ DG_{\lambda\mu}] - 2[\omega^i\omega_i^0]G_{\lambda\mu},$$

we get by (13′) and (18)

(20) $$(DG_{\lambda\mu})' = [\omega_\lambda^\rho\ DG_{\rho\mu}] + [\omega_\mu^\rho\ DG_{\lambda\rho}] - 2[\omega_0^0\ DG_{\lambda\mu}].$$

On the other hand, by (6), we have along the curves (5) $y^i = a^i t$

$$DG_{\alpha\beta} = G_{\lambda\mu}^0 \{dc_\alpha^\lambda - \omega_\alpha^\tau c_\tau^\lambda + \omega_0^0 c_\alpha^\lambda\}\ c_\beta^\mu + G_{\lambda\mu}^0\ c_\alpha^\lambda \{dc_\beta^\mu - \omega_\beta^\tau c_\tau^\mu + \omega_0^0 c_\beta^\mu\} = 0$$

Hence, if we denote the variations of t by δ and those of a^i by d, we have from (20)

$$\delta\ DG_{\lambda\mu} = e_\lambda^\rho\ DG_{\rho\mu} + e_\mu^\rho\ DG_{\lambda\rho} - 2e_0^0\ DG_{\lambda\mu}$$

where $e_\lambda^\mu = \omega_\lambda^\mu (a^i\ t,\ a^i\ \delta t)$, or

(21) $$\frac{\delta}{\delta t} DG_{\lambda\mu} = \frac{e_\lambda^\rho}{\delta t} DG_{\lambda\rho} + \frac{e_\mu^\rho}{\delta t} DG_{\lambda\rho} - 2\frac{e_0^0}{\delta t} DG_{\lambda\mu}.$$

Since $DG_{\lambda\mu}$'s satisfy the linear differential equations as above and are equal to zero at the point A^0, it follows that at any point and for any variation

(22) $$DG_{\lambda\mu} = 0.$$

Thus we see that the hyperquadric $G_{\lambda\mu} x^\lambda x^\mu = 0$ in the tangent projective space at each point A overlaps one upon another in the developement of our space.

Let us now write (22) in another form by means of (17):

(22′) $$\frac{\partial}{\partial y^k} G_{\lambda\mu} - \Gamma_{\lambda k}^\rho\ G_{\rho\mu} - \Gamma_{\mu k}^\rho\ G_{\lambda\rho} = 0$$

or

(23$_1$) $$\frac{\partial}{\partial y^k} G_{00} - 2G_{k0} = 0,$$

$$(23_2)\qquad \frac{\partial}{\partial y^k}G_{i0}-\Gamma^{\rho}_{ik}G_{\rho 0}-G_{ik}=0,$$

$$(23_3)\qquad \frac{\partial}{\partial y_k}G_{ij}-\Gamma^{\rho}_{ik}\,G_{\rho j}-\Gamma^{\rho}_{jk}\,G_{i\rho}=0.$$

Contracting (22′) with $G^{\lambda\mu}$, we see by means of (3) that

$$G^{\lambda\mu}\frac{\partial}{\partial y^k}G_{\lambda\mu}=\frac{\partial}{\partial y^k}\log G=2\Gamma^{\rho}_{\rho k}=2\Gamma^{i}_{ik}=0,$$

that is, $G=|G_{\nu\mu}|=$constant. G_{00} is evidently a projective scalar. Let us denote it by 2φ; then we have

$$G_{\lambda 0}=\frac{\partial}{\partial y_k}\varphi=\varphi_k,\qquad \varphi_0=\varphi.$$

$\varphi_0, \varphi_1, \ldots\ldots, \varphi_n$ are obviously components of an analytic covariant projective vector.

From (23_3) we have

$$\frac{1}{2}\left\{\frac{\partial G_{jk}}{\partial y^i}+\frac{\partial G_{ki}}{\partial y^i}-\frac{\partial G_{ij}}{\partial y^k}\right\}=\Gamma^{\rho}_{ij}\,G_{\rho k}$$

and from (23_2)

$$\frac{1}{2}\left\{\frac{\partial G_{jo}}{\partial y^i}+\frac{\partial G_{io}}{\partial y^j}-2G_{ij}\right\}=\Gamma^{\rho}_{ij}\,G_{\rho o}.$$

Making use of the above relations, we get

$$(24)\qquad \Gamma^{\lambda}_{ij}=\frac{1}{2}G^{\lambda k}\left(\frac{\partial G_{kj}}{\partial y^i}+\frac{\partial G_{ik}}{\partial y^j}-\frac{\partial G_{ij}}{\partial y^k}\right)+G_{\lambda o}\left(\frac{\partial^2\varphi}{\partial y^i\partial y^j}-G_{ij}\right)$$

Thus we obtain the following result:

The parameters of connexion of our space are given by (24) *in terms of* $G_{\lambda\mu}$; *conversely, in such spaces that* Γ^{λ}_{ij} *are given by* (24) *the groups of holonomy fix a hyperquadric.*

§ 3. Let us now consider such an $(n+1)$-dimensional Riemannian space R^*_{n+1} that its fundamental tensor is given by

$$(25)\qquad G^*_{\lambda\mu}=e^{2y^0}G_{\lambda\mu},\qquad G^{*\lambda\mu}=e^{-2y^0}G^{\lambda\mu}.$$

As the quadratic form $G^*_{\lambda\mu}x^{\lambda}x^{\mu}$ is not always positive definite, this space is a Riemannian space *in a general sense*.

The Christoffel symbols $\{^{\nu}_{\lambda\mu}\}^*$ formed by $G^*_{\lambda\mu}$ are

$$\{^{\nu}_{\lambda\mu}\}^*=\frac{1}{2}G^{\nu\rho}\left(\frac{\partial G_{\lambda\rho}}{\partial y^h}\delta^h_\mu+\frac{\partial G_{\rho\mu}}{\partial y^h}\delta^h_\lambda-\frac{\partial G_{\lambda\mu}}{\partial y^h}\delta^h_\rho\right)+\delta^{\nu}_{\lambda}\delta^0_\mu+\delta^{\nu}_{\mu}\delta^0_\lambda-G^{\nu 0}G_{\lambda\mu}.$$

From these we get

$$\left\{ {\nu \atop ij} \right\}^* = \frac{1}{2} G^{\nu k} \left(\frac{\partial G_{kj}}{\partial y^i} + \frac{\partial G_{ik}}{\partial y^j} - \frac{\partial G_{ij}}{\partial y^k} \right) + G^{\nu o} \left(\frac{\partial^2 \varphi}{\partial y^i \partial y^j} - G_{ij} \right),$$

$$\left\{ {\nu \atop oj} \right\}^* = \frac{1}{2} G^{\nu k} \left(\frac{\partial G_{ko}}{\partial y^j} - \frac{\partial G_{jo}}{\partial y^k} \right) + G^{\nu o} \left(\frac{1}{2} \frac{\partial G_{00}}{\partial y^j} - G_{jo} \right) + \delta^\nu_j,$$

$$\left\{ {\nu \atop 00} \right\}^* = -G^{\nu h} G_{h0} - G^{\nu 0} G_{00} + 2\delta^\nu_o.$$

$$\left\{ {\lambda \atop ij} \right\}^* = \Gamma'^{\lambda}_{ij}, \quad \left\{ {\nu \atop o\mu} \right\}^* = \delta^\nu_\mu. \tag{26}$$

Let us now consider the Pfaffians $\omega^{*\lambda}_\mu$ of R^*_{n+1} with respect to its natural frame

$$\omega^{*\lambda}_\mu = \left\{ {\lambda \atop \mu\rho} \right\}^* dy^\rho ; \tag{27}$$

then by (26) we have

$$\omega^{*\lambda}_i = \omega^\lambda_i + \delta^\lambda_i \, dy^0, \qquad \omega^{*\lambda}_o = dy^\lambda. \tag{28}$$

Accordingly we have for the frame (A, e_λ) the relations

$$\begin{aligned} &dA = dy^\lambda e_\lambda, \\ &de_0 = dy^\lambda e_\lambda, \\ &de_i = \omega^0_i \, e_0 + \omega^k_i \, e_k + dy^0 e_i, \end{aligned} \tag{29}$$

for which we obtain $d(A-e_0)=0$. Thus we see that the group of holonomy of R^*_{n+1} fixes a point. Moreover, for the curves $y^i=$const. we have $dA=dy^0 e_0$, $de_0=dy^0 e_0$. Hence, these curves constitute a family of geodesics each of which converges to a point corresponding to this fixed point 0, because we may consider from the first that $G_{00}>0$.

Now we consider an n-dimensional hypersurface $y^0=$const., on which we have $dA=dy^i e_i$, $de_0=dy^i e_i$, $de_i=\omega^0_i \, e_0+\omega^j_i \, e_j$. Then, if we put $e_\lambda=A_\lambda$, we get (1). Thus we see that, if we consider at each point in R^*_{n+1} the tangent $(n+1)$-dimensional Euclidean space $E_{n+1}(A)$, any two neighbouring spaces $E_{n+1}(A)$ and $E_{n+1}(A+dA)$ are situated such that they have the point 0 in common. Moreover, the above relations show that *we have the connexion of our space P_n if we project the tangent hyperplane $E_n(A+dA)$ at $A+dA$ onto the tangent hyperplane $E_n(A)$ at A from the point* 0.

Let us now determine the curvature tensor $\Omega^{*\lambda}_\mu$ of R^*_{n-1}. By (28) we have

$$\begin{aligned} -\Omega^{*\lambda}_i &= (\omega^{*\lambda}_i)' - [\omega^{*\rho}_i \; \omega^{\lambda *}_\rho] \\ &= (\omega^\lambda_i)' - [\omega^k_i \; \omega^\lambda_k] - [\omega^0_i \; dy^\lambda] - [dy^0 \; \omega^\lambda_i] + \delta^\lambda_k [dy^0 \; \omega^k_i] = -\Omega^\lambda_i, \end{aligned}$$

where we put $\omega^0_0=0$ as this may be generally admissible. And we have by (18)

$$\Omega^{*\lambda}_0 = [dy^k (\omega^\lambda_k + \delta^\lambda_k \, dy^0)] + [dy^0 \; dy^\lambda] = 0,$$

$$\Omega_0^0 = [\omega^i \ \omega_i^0] = 0.$$

Putting

(30) $$\Omega^{*\lambda}_{\mu} = \frac{1}{2} R^{*\lambda}_{\mu\rho\nu} \, [dy^\rho \ dy^\nu,]$$

we obtain from the above relations

(31) $$R^{*}{}_{i}{}^{\lambda}{}_{hk} = A_i{}^{\lambda}{}_{kh}, \qquad R^{*\lambda}_{\mu\nu o} = R^{*\lambda}_{o\mu\nu} = 0.$$

Accordingly for the Ricci tensor $R^*_{\lambda\mu} = R^*{}_{\lambda}{}^{\rho}{}_{\mu\rho}$ of R^*_{n+1} we have the relations

(32) $$R^*_{ik} = A_{ik}, \qquad R^*_{io} = R^*_{00} = 0.$$

If the connexion of our space P_n is normal, we have

(33) $$A_{ik} = 0.$$

Thus we see that:

The necessary and sufficient condition *that the connexion of the space P^n may be normal is that the Riemannian space R^*_{r+1} associated to P_n is an Einstein space whose scalar curvature is equal to* 0.

§ 4. The equations of geodesics in R^*_{n+1}, as well known, are

(34) $$\frac{d^2 y^\lambda}{dt^2} + \{ {}^{\lambda}_{\mu\nu} \} \frac{dy^\mu}{dt} \frac{dy^\nu}{dt} = F\left(y, \frac{dy}{dt}\right) \frac{dy^\lambda}{dt}$$

where F is a suitable function. Making use of (26), we have from (34)

(34) $$\frac{d^2 y^0}{dt^2} + \Gamma^0{}_{ij} \frac{dy^i}{dt^i} \frac{dy^j}{dt} = \left(F - \frac{dy^0}{dt} \right) \frac{dy^0}{dt},$$

(34) $$\frac{d^2 y^i}{dt^2} + \Gamma^i_{jk} \frac{dy^j}{dt} \frac{dy^k}{dt} = \left(F - 2\frac{dy^0}{dt} \right) \frac{dy^i}{dt}.$$

On the other hand, the equations of projective geodesics of P_n are

$$\frac{d^2 y^i}{dt^2} + \Gamma^i_{jk} \frac{dy^j}{dt} \frac{dy^k}{dt} = H\left(y, \frac{dy}{dy} \right) \frac{dy^i}{dt}.$$

From these equations we see that:

The geodesics of P_u are those curves on the surface $y^0 = const.$ which we obtain by projecting the geodesics of R^*_{n+1} along the geodesics through the point corresponding to the point 0.

Let us now consider the surface $y^0 =$ const. as an n-dimensional Riemannian space $R_n{}^*$ and denote its fundamental tensor by γ_{ij}:

(35) $$\gamma_{ij} = G_{ij}$$

(here we suppose that $|G_{ij}| \neq 0$). Then we can see that

$$\gamma^{ij}=G^{ij}-\frac{G^{ij}\ G^{jo}}{G^{00}}.$$

Let us denote the Christoffel symbols of $R_n{}^*$ by Λ_{ij}^h, then we have

$$(36) \qquad \Lambda_{ij}^h=\Gamma'^h_{ij}-\frac{G^{ho}}{2\ G_{00}}\ \Gamma_{ij}^0,$$

because

$$\Lambda_{ij}^h=\frac{1}{2}\gamma^{hk}\Big(\frac{\partial\ \gamma_{kj}}{\partial\ y^i}+\frac{\partial\ \gamma_{ik}}{\partial\ y^i}-\frac{\partial\ \gamma_{ij}}{\partial\ y^k}\Big)$$

$$=\frac{1}{2}\Big(G^{hk}-\frac{G^{ho}\ G^{ko}}{G^{00}}\Big)\Big(\frac{\partial\ G_{kj}}{\partial\ y^i}+\frac{\partial\ G_{ik}}{\partial\ y^k}-\frac{\partial\ G_{ij}}{\partial\ y^k}\Big)$$

$$=\Gamma_{ij}^h-\frac{G^{ho}}{2}\Big\{\frac{G^{ko}}{G^{00}}\Big(\frac{\partial\ G_{kj}}{\partial\ y^i}+\frac{\partial\ G_{ik}}{\partial\ y^i}-\frac{\partial\ G_{ij}}{\partial\ y^k}\Big)$$

$$+2\Big(\frac{\partial^2\ \varphi}{\partial y^i\ \partial y^j}-G_{ij}\Big)\Big\}$$

$$=\Gamma'^h_{ij}-\frac{G^{ho}}{2G^{00}}\ \Gamma_{ij}^0.$$

Hence we find by (36) the equations of geodesics of $R_n{}^*$ as

$$(37) \qquad \frac{d^2\ y^i}{dt^2}+\Gamma_{jk}^i\ \frac{dy^j}{dt}\ \frac{dy^k}{dt}=M\frac{dy^i}{dt}+\frac{G^{io}}{2G^{00}}\Gamma_{jk}^0\frac{dy^i}{dt}\ \frac{dy^k}{dt}$$

where M is a suitable function of y^i and $\dfrac{dy^i}{dt}$. From this it follows:

The necessary and sufficient condition that *the geodesics of P_n coincide with those of $R_n{}^*$ is $\Gamma_{jk}^0=0$ or $G^{io}=0$.* In the first case: by (29) *$R_n{}^*$ becomes a totally geodesic surface in R_{n+1}^* and P_n can be considered as a Riemannian space with the group of holonomy which fixes a point.* Because the connexion of P_n fixes the plane at infinity $[A_1,\ldots\ldots,A_n]$ and is accordingly affine, and furthermore, since the group of holonomy fixes the hyperquadric $G_{\alpha\beta}\ x^\alpha\ x^\beta=0$ ($|G_{ij}|\neq 0$), P_n can be considered as a Riemannian space. In the second case: the condition $G^{io}=0$ is equivalent to the condition $G_{io}\gamma^{ik}=0$ or, by virtue of our assumption $|\gamma_{ij}|\neq 0$, $G_{io}=\dfrac{\partial\varphi}{\partial y^i}=0$.

Thus we see that geodesics of R_{n-1}^* through the point corresponding to O intersect the surface corresponding to the $R_n{}^*$ at those points which are apart from the fixed point by a constant distance, hence that *this surface must be totally umbilical.*

§ 5. It is evident that we can introduce a Riemannian metric into our space P_n by means of the hyperquadric (4) which is fixed by the group of holonomy of P_n. The non-Euclidean distance with respect to the absolute hyperquadric (4) between A and $A+dA$ is equal to the distance with respect to the absolute hyperquadric $G_{\alpha\beta}\, x^\alpha x^\beta=0$ at the point A which is projective to (4).

The homogeneous coordinates of A and $A+dA$ with respect to the natural frame (A_λ) are respectively $(1, 0, \ldots\ldots, 0)$, $(1, dy^1, \ldots\ldots, dy^n)$. Then let us consider the straight line passing through A and $A+dA$ in the tangent projective space at A, and suppose that this line intersects the hyperquadric $G_{\alpha\beta}\, x^\alpha x^\beta=0$ at the points M and N. If we put the coordinates of these points of intersection $x^0=\lambda+\mu$, $x^i=\mu\, dy^i$ and substitute these in $G_{\alpha\beta}\, x^\alpha\, x^\beta=0$, we have

$$\lambda^2 G_{00}+2\lambda\mu(G_{00}+G_{i0}\, dy^i)+\mu^2(G_{00}+2G_{i0}\, dy^i+G_{ij}\, dy^i\, dy^j)=0$$

or

$$\frac{\lambda}{\mu}=-1-\frac{G_{i0}\, dy^i}{G_{00}}\pm\sqrt{-1}\sqrt{\left(\frac{G_{ij}}{G_{00}}-\frac{G_{i0}}{G_{00}}\,\frac{G_{j0}}{G_{00}}\right)dy^i\, dy^j}.$$

Hence we have .

$$\text{Double ratio } (A,\ A+dA,\ L,\ M)$$
$$=\frac{\left\{1+\frac{G_{i0}}{G_{00}}dy^i-\sqrt{-1}\sqrt{\left(\frac{G_{ij}}{G_{00}}-\frac{G_{i0}}{G_{00}}\,\frac{G_{j0}}{G_{00}}\right)dy^i\, dy^j}\right\}^2}{1+2\frac{G_{j0}}{G_{00}}dy^j+\frac{G_{jk}}{G_{00}}dy^j\, dy^k}.$$

Accordingly we have

$$\frac{1}{2\sqrt{-1}}\log(A,\ A+dA,\ L,\ M)=\sqrt{\left(\frac{G_{ik}}{G_{00}}-\frac{G_{j0}}{G_{00}}\,\frac{G_{k0}}{G_{00}}\right)dx^j\, dx^k}+\ldots$$

From the above relation we now define a Riemannian metric in P_n by the equation

$$ds^2=\epsilon\, k^2\left(\frac{G_{ij}}{G_{00}}-\frac{G_{i0}}{G_{00}}\,\frac{G_{j0}}{G_{00}}\right)dy^i\, dy^j \tag{38}$$

where ϵ is 1 or -1 and k is a proper constant, and denote by R_n the corresponding Riemannian space. Since $G_{\alpha\beta}$ is an projective tensor, the quantities

$$g_{ij}=\epsilon\, k^2\left(\frac{G_{ij}}{G_{00}}-\frac{G_{i0}}{G_{00}}\,\frac{G_{j0}}{G_{00}}\right) \tag{39}$$

are evidently the components of an affine tensor in ordinary sense with respect to coordinate transformations. The components g^{ij} of the contravariant tensor conjugate to the covariant tensor g_{ij} are obviously

$$g^{ij}=\epsilon\frac{G_{00}}{K^2}G_{ij}. \tag{40}$$

Then let us form the Christoffel symbols $\{{}^{h}_{ij}\}$ of R_n. From (39) we have

$$\frac{\partial g_{ij}}{\partial y^k}=\epsilon k^2\Big\{\frac{1}{2\varphi}\frac{\partial G_{ij}}{\partial y^k}-\frac{G_{ij}}{2\varphi^2}-\frac{\partial^2\log\sqrt{\varphi}}{\partial y^i\,\partial y^k}\frac{\partial\log\sqrt{\varphi}}{\partial y^i}$$
$$-\frac{\partial\log\sqrt{\varphi}}{\partial y^i}\frac{\partial^2\log\sqrt{\varphi}}{\partial y^j\,\partial y^k}\Big\},$$

hence

$$\{{}^{h}_{ij}\}=\frac{1}{2}G^{hk}\Big(\frac{\partial G_{kj}}{\partial y^i}+\frac{\partial G_{ik}}{\partial y^j}-\frac{\partial G_{ij}}{\partial y^k}\Big)-\frac{\partial\log\sqrt{\varphi}}{\partial y^i}(\delta^h_j-G^{ho}\varphi_j)$$
$$-\frac{\partial\log\sqrt{\varphi}}{\partial y^i}(\delta^h_i-G^{ho}\varphi_i)+\Big(G_{ij}-2\varphi\frac{\partial^2\log\sqrt{\varphi}}{\partial y^i\,\partial y^j}\Big)G^{hk}\frac{\partial\log\sqrt{\varphi}}{\partial y^k}.$$

On the other hand, we have

$$G^{hk}\varphi_k=G^{hk}G_{ko}=-G^{ho}G_{00}=-2\varphi G^{ho}$$
$$2\varphi\frac{\partial^2\log\sqrt{\varphi}}{\partial y^i\,\partial y^j}=\frac{\partial^2\varphi}{\partial y^i\,\partial y^j}-\frac{1}{\varphi}\varphi_i\,\varphi_j,$$

hence we get by (24) the following relations:

$$\Gamma^h_{ij}=\{{}^{h}_{ij}\}+\delta^h_i\frac{\partial\log\sqrt{\varphi}}{\partial y^j}+\delta^h_j\frac{\partial\log\sqrt{\varphi}}{\partial y^i}. \tag{41}$$

From (41) we see that *P_n and R_n are mutually projective with respect to the parameters of connexions.* But this is obvious from the fact that the geodesics of R_n coincide with those of P_n by our projective definition of the metric of R_n.

Now we investigate the properties of R_n. If we put

$$\tilde{\omega}^h_i=\{{}^{h}_{ik}\}dy^k=\omega^h_i-\delta^h_i\,d\rho-\rho_i\,dy^h$$

where $\rho=\log\sqrt{\varphi}$, $\varphi_i=\dfrac{\partial\rho}{\partial y^i}$, we have

$$(\tilde{\omega}^h_i)'=(\omega^h_i)'-[d\rho_i\;dy^h],$$
$$[\tilde{\omega}^k_i\;\tilde{\omega}^h_k]=[\omega^k_i\;\omega^h_k]-\rho_k[\omega^k_i\;dy^h]+\rho_i[d\rho\;dy^h].$$

Denoting by $\tilde{\Omega}^h_i$ the curvature of R_n, we have from the above equations

$$-\tilde{\Omega}_i^h=(\tilde{\omega}_i^h)'-[\tilde{\omega}_i^k\ \tilde{\omega}_k^h]$$
$$=(\omega_i^h)'-[\omega_i^k\ \omega_k^h]-[d\rho_i-\omega_i^k\ \rho_k+\rho_i\ d\rho,\ dy^h].$$

On the other hand, we get from (23_2)

$$DG_{io}=d\varphi_i-\omega_i^k\ \varphi_k-G_{ik}\ dy^k-2\varphi\omega_i^o=0,$$

hence by (40)

$$d\rho_i-\omega_i^k\ \rho_k+\rho_i\ d\rho=\frac{1}{2\varphi}\ (d\varphi_i-\omega_i^k\ \varphi_k)-\frac{\varphi_i\ d\varphi}{4\varphi^2}$$
$$=\frac{1}{2\varphi}(G_{ik}\ dy^k+2\varphi\omega_i^o)-\frac{\varphi_i\ d\varphi}{4\varphi^2}=\omega_i^o+\frac{\varepsilon}{k^2}\ g_{ik}\ dy^k.$$

Substituting these equations on the right of $\tilde{\Omega}_i^h$, we obtain

$$-\tilde{\Omega}_i^h=(\omega_i^h)'-[\omega_i^k\ \omega_k^h]-[\omega_i^o\ dy^h]-\frac{\epsilon}{k^2}g_{ij}[dy^j\ dy^h] \tag{42}$$

or

$$\tilde{\Omega}_i^h=\Omega_i^h+\frac{\epsilon}{k^2}g_{ij}\ [dy^j\ dy^h].$$

If we denote the components of the Riemann tensor of R_n by $\tilde{\Omega}_i^h=\frac{1}{2}\tilde{R}_i{}^h{}_{jk}\ [dy^j\ dy^k]$, we get from (42)

$$\tilde{R}_i{}^h{}_{jk}=A_i{}^h{}_{jk}+\frac{\epsilon}{k^2}\ (g_{ij}\ \delta_k^h-g_{ik}\ \delta_j^h) \tag{43}$$

and

$$\tilde{R}_{ij}=A_{ij}+\frac{\epsilon\,(n-1)}{k^2}g_{ij} \tag{43}$$

where $\tilde{R}_{ij}=\tilde{R}_i{}^h{}_{jh}$ are the components of the Ricci tensor of R_n. When the connexion of P_n is normal, that is, the condition $A_{ij}=0$ is satisfied, we have from (44)

$$\tilde{R}_{ij}=\frac{\epsilon}{k^2}\ (n-1)g_{ij}. \tag{45}$$

This shows that the space R_n is an Einstein space. Our result can be stated in the following

THEOREM. *If a normal projectively connected space P_n $(n>2)$ has the group of holonomy which fixes a hyperquadric, the space is projective to an Einstein space.*

From (43) we have a well known result: The necessary and sufficient condition that P_n *may be flat is that* R_n *be a space with constant curvature.*

Next we shall consider the inverse problem. From (23_2) we get

$$-\Gamma^o_{ij}=\frac{1}{2\varphi}G_{ij}+\frac{1}{2\varphi}\Gamma^h_{ij}\varphi_h-\frac{1}{2\varphi}\frac{\partial^2\varphi}{\partial y^i\,\partial y^j}$$
$$=\frac{\epsilon}{k^2}g_{ij}-\Big(\frac{\partial\rho_i}{\partial y^j}-\Gamma^h_{ij}\rho_h+\rho_i\rho_j\Big)=\frac{\epsilon}{k^2}g_{ij}-\varphi_{i\,;\,j}+\rho_i\rho_j$$

where $\rho_{i\,;\,j}$ is the covariant derivative of ρ_i with respect to y^j in R_n. Hence we have the following equations

$$(46)\qquad \begin{cases}\Gamma^h_{ij}=\{{}^{\,h}_{ij}\}+\delta^h_i\,\rho_j+\delta^h_j\,\rho_i,\\ \Gamma^o_{ij}=-\dfrac{\epsilon}{k^2}g_{ij}+\rho_{i\,;\,j}-\rho_i\,\rho_j.\end{cases}$$

Thus we obtain the result:

For a given Einstein space R_n $(n>2)$, if we consider a space with connexion given by (46), this space has the group of holonomy which fixes a hyperquadric.

§ 6. In this last section we compare our method with that of E. Cartan who has introduced a Riemannian metric into the space P_n.

Since the group of holonomy of P_n is transitive, we can transform the frame R : (A_λ) to the other $\bar{R}$: $(\bar{A}_\lambda)$

$$\bar{A}=A,\qquad \bar{A}_i=p^o_i\,A+p^j_i\,A_j$$

such that the new system of Pfaffians $\bar{\omega}^\lambda_\mu$ satisfies the equations (8)

$$G^o_{\lambda\rho}\,\bar{\omega}^\rho_\mu+G^o_{\rho\mu}\,\bar{\omega}^\rho_\lambda=\pi G^o_{\lambda\mu}$$

where $\bar{\omega}^\lambda_\mu$ are defined by $\bar{\omega}^\lambda_\mu\,p^\alpha_\lambda=dp^\alpha_\mu+p^\beta_\mu\,\omega^\alpha_\beta$. From these we get

$$\Big(\frac{G^o_{i\rho}}{G_{00}}-2\frac{G^o_{io}}{G^o_{oo}}\,\frac{G^o_{\rho o}}{G^o_{oo}}\Big)\bar{\omega}^\rho_o+\frac{G^o_{\rho o}}{G^o_{oo}}\bar{\omega}^\rho_i=0,$$

$$\frac{G^o_{i\rho}}{G^o_{oo}}\bar{\omega}^\rho_j+\frac{G^o_{\rho j}}{G^o_{oo}}\,\bar{\omega}^\rho_i-2\frac{G^o_{ij}}{G^o_{oo}}\,\bar{\omega}^\rho_o=0$$

or

$$\bar{\omega}^o_i=\Big(\frac{G^o_{ij}}{G^o_{oo}}-2\frac{G^o_{io}}{G^o_{oo}}\,\frac{G^o_{jo}}{G^o_{oo}}\Big)\bar{\omega}^j+\frac{G^o_{oo}}{G^o_{oo}}\,(\bar{\omega}^j_i-\delta^j_i\,\bar{\omega}^o_o),$$

$$\frac{G^o_{ok}}{G^o_{oo}}(\bar{\omega}^k_j-\delta^k_j\bar{\omega}^o_o)+\frac{G^o_{jo}}{G^o_{oo}}(\bar{\omega}^k_i-\delta^k_i\,\bar{\omega}^o_o)+\frac{G^o_{io}}{G^o_{oo}}\bar{\omega}^o_j+\frac{G^o_{jo}}{G^o_{oo}}\,\bar{\omega}^o_i=0.$$

If we put

$$(47)\qquad \frac{G^o_{ij}}{G^o_{oo}}-\frac{G^o_{jo}}{G^o_{oo}}\,\frac{G^o_{io}}{G^o_{oo}}=a_{ij},\qquad \frac{G^o_{io}}{G^o_{oo}}=a_i$$

and substitute these in the above equations, we have

$$(48) \qquad a_{ik}(\bar{\omega}_j^k - \delta_j^k \bar{\omega}_o^o) + a_{kj}(\bar{\omega}_j^k - \delta_i^k \ \bar{\omega}_o^o) - (a_i \ a_{jk} + a_j \ a_{ik} - 2a_i a_j a_k)\bar{\omega}^k = 0.$$

Denoting $\bar{\omega}_\mu^\lambda$ for the variations only secondary parameters by e_μ^λ, we get from (48)

$$a_{ik}(e_j^k - \delta_j^k \ e_o^o) + a_{kj}(e_{ki} - \delta_i^k \ e_o^o) = 0.$$

On the other hand, as we assume P_n has no torsion, that is,

$$(\bar{\omega}^i)' - [\bar{\omega}^k(\bar{\omega}_k^i - \delta_k^i \ \bar{\omega}_o^o)] = 0,$$

it follows that

$$\delta\bar{\omega}^j = -\bar{\omega}^k \ (e_k^j - \delta_k^j \ e_o^o).$$

Accordingly we have

$\delta(a_{ij} \ \bar{\omega}^i \ \bar{\omega}^j) = 2a_{ij} \ \bar{\omega}^i \ \delta\bar{\omega}^j = -2a_{ij} \ \bar{\omega}^i\bar{\omega}^k \ (e_k^j - \delta_k^j e_o^o) = 0$, which shows that the form

$$d\bar{s}^2 = a_{ij} \ \bar{\omega}^i\bar{\omega}^j = \left(\frac{G_{ij}^o}{G_{oo}^o} - \frac{G_{io}^o}{G_{oo}^o} \ \frac{G_{jo}^o}{G_{oo}^o}\right)\bar{\omega}^i \ \bar{\omega}^j = \left(\frac{G_{ij}}{G_{oo}} - \frac{G_{io}}{G_{oo}} \ \frac{G_{jo}}{G_{oo}}\right)\bar{\omega}^i\bar{\omega}^j.$$

$$= \frac{\epsilon}{k^2} g_{ij} \ \omega^i \ \omega^j = \frac{\epsilon}{k^2} ds^2$$

determines *a Riemannian metric defined by E. Cartan in another form* and equivalent to ours (39) but a constant factor.

Mathematical Institute
Kyushu University

Notes

1) S. Sasaki and K. Yano, On the structure of spaces with normal projective connexions whose holonomy groups fix a hyperquadric or an (n-2) dimensional quadric in a hyperplane, Tohoku Math. J, (2) 1 no. 1.

2) E. Cartan, Les groupes d'holonomie des espaces généralisés, Acta Mathematica 48 (1926) pp. 1—42.

3) E. Cartan, Leçon sur la théorie des espaces à connexion projective, (1937).