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An Asymptotic Series for the Number of Three-Line Latin Rectangles

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Introduction

An (n,k)-latin rectangle is a (k,n)-matrix having k permutations of degree n as its k rows and admitting no coincidences of letters in each of letters in each of its n columns. For the number $f(n,k)$ of such latin rectangles, P. Erdös and I. Kaplansky[1)] recently proved an asymptotic relation

$$f(n,k)\sim e^{-k(k-1)/2}(n!)^k.$$

And for the special ease of $f(n, 3)$, numerous results are reported to be obtained by authors of the United States and other countries thongh we have access to only a few of them.[2)]

In this paper we shall give some formulas for the number $f(n, 3)$. Explicit formulas are given in **1.** They would require heavy computations. Our principal aim is an asymptotic series for $f(n, 3)$.

$$f(n, 3)\sim$$
$$e^{-3}(n!)^3\left\{1-\frac{1}{n}-\frac{1}{2}\cdot\frac{1}{n(n-1)}+\frac{5}{6}\cdot\frac{1}{n(n-1)(n-2)}+\frac{1}{24}\cdot\frac{1}{n(n-1)(n-2)(n-3)}-\cdots\right\}$$

given in **2.** The close-up to the coefficients M_s of this series will be found in **3,** and finally in **4** numerical values of $N_s=s!\,M_s$ and $\psi_n=f(n, 3)/n!$ are given for $n\leqq 20$. Our series seems to give far better approximations than we can prove, at least as far as $n\leqq 20$.

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1. Explicit Formulas

We shall first modify the numbers $f(n, 2)$ and $f(n, 3)$ slightly:

$$\varphi_n=f(n, 2)/n!,\quad \psi_n=f(n, 3)/n!,$$

and use them exclusively. These are the numbers of *reduced* latin rectangles, i.e. of those latin rectangles, whose first rows consist of natural permutations. For φ_n a well-known theorem (*problème des rencontres*) states that

(1) $$\varphi_n = n!\left\{1-\frac{1}{1!}+\frac{1}{2!}-+\dots+(-)^n\frac{1}{n!}\right\}.$$

Moreover I make usc of the gereral *partial discordance numbers*

(2) $$\zeta_{r,k}=\varphi_r+\binom{k}{1}\varphi_{r-1}+\dots+\varphi_{r-k},$$

or

(2*) $$\zeta_{r,r-h}=r!-\binom{h}{1}(r-1)!+-\dots+(-)^h(r-h)!.$$

The numbers φ_r and $r!$ are the both extremities of $\zeta_{r,k}$.

$$\varphi_r=\zeta_{r,0}=\bar{\Delta}^r r!=\Delta^r 0!, \qquad r!=\zeta_{r,r}=\bar{\Delta}^0 r!=\Delta^0 r!.$$

Originally $\zeta_{r,k}$ is defined as the number of permutations of degree r, which leave at most k *preassigned*, e.g. the first k, letters unchanged (which change at least all of the $r-k$ *preassigned*, e.g. all of the last $r-k$ letters). But we prefer to consider $\zeta_{r,k}$ as the number of discordant *arrangements*

$$a=\begin{pmatrix}1 & 2 \dots\dots & r\\ a_1 & a_2\dots\dots & a_r\end{pmatrix} \quad (1\neq a_1,\ 2\neq a_2,\dots\dots,r\neq a_r)$$

of a prescribed set $\{a_1, a_2,\dots\dots,a_r\}$ of r letters, where there are just k letters in this set not appearing in $\{1,2,\dots\dots,r\}$. We shall call these k letters *heterogeneous particles* of the second row of our arrangement a. Then the recurrence relation

$$\zeta_{r,k}=\zeta_{r,k-1}+\zeta_{r-1,\,k-1} \quad (r,\ k\geq 1),\ \zeta_{r,0}=\varphi_r$$

is easily verified to hold, which is immediately extended to integral forms (2) and (2*).

Now we proceed to obtain the number $\psi_n=f(n,\,3)/n!$.

Lemma 1. *The number ψ_n is expressible by the partial discordance numbers $\zeta_{r,k}$ as follows:*

$$\psi_n=\sum_{r=0}^{n}(-)^r\binom{n}{r}\sum_{k=0}^{r_0}\binom{r}{k}\binom{n-r}{k}\zeta_{r,k}\,\zeta^2_{n-r,k}.$$

The inner summation extends from $k=0$ to $k=r_0=$ min. $(r,\ n-r)$.

Proof. We can pick up from a given reduced three-line latin rectangle

$$\begin{pmatrix}1 & 1 & \dots\dots n\\ a_1 & a_2 & \dots\dots a_n\\ b_1 & b_2 & \dots\dots b_n\end{pmatrix}$$

the two discordant permutations

$$\begin{pmatrix} 1 & 2 & \ldots\ldots & n \\ a_1 & a_2 & \ldots\ldots & a_n \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & \ldots\ldots & n \\ b_1 & b_2 & \ldots\ldots & b_n \end{pmatrix},$$

which are discordant to each other. Conversely any such an ordered pair of discordant permutations determines a three-line latin rectangle in the reduced form. Hence ψ_n is also the number of such pairs. If we apply Poincare's *inclusion and exclusion method*[3] to obtain this number, the mutual discordance reflects itself in the following manner:

$$\psi_n=\varphi\frac{2}{n}-\binom{n}{1}\mu_1+\binom{n}{2}\mu_2-+\cdots+(-)^n\mu_n. \tag{3}$$

Herein μ_r denotes the number of pairs of discordant permutations having coincidences in all of the r preassigned, e.g. in all of the first r, *positions.* This number μ_r, however, can be immediately calculated if we know the number $\lambda_{a_1\cdots a_r}$ of discordant permutations

$$\pi=\begin{pmatrix} 1 & \ldots\ldots & r & r+1 & \ldots\ldots & n \\ a_1 & \ldots\ldots & a_r & * & \ldots\ldots & * \end{pmatrix}$$

of degree n, such that $1,\ldots\ldots,r$ are matched against the prescribed letters $a_1,\ldots\ldots,a_r$ respectively. Indeed,

$$\mu_r=\sum\lambda^2_{a_1\cdots a_r}, \tag{4}$$

summation extending over all discordant partial arrangements

$$\alpha_1=\begin{pmatrix} 1 & \ldots\ldots & r \\ a_1 & \ldots\ldots & a_r \end{pmatrix}$$

($a_1,\ldots\ldots,a_r$ being arbitrararily selected from among $1,\ldots\ldots,n$ under the condition of discordance). But as is easily seen, the number $\lambda_{a_1\ldots\ldots a_r}$ depends only upon the distribntion of elements of our prescribed set $\{a_1,\ldots\ldots,a_r\}$ between $\{1,\ldots\ldots,r\}$ and $\{r+1,\ldots\ldots,n\}$. Indeed, if there are just $r-k$ elements common to $\{a_1,\ldots\ldots,a_r\}$ and $\{1,\ldots\ldots,r\}$, then since the remaining k elements of $\{1,\ldots\ldots,r\}$ should appear in the second row of the second partial arrangement

$$\alpha_2=\begin{pmatrix} r+1 & \ldots\ldots & n \\ * & \ldots\ldots & * \end{pmatrix}$$

as k heterogeneous particles, $\lambda_{a_1\cdots a_r}$ reduces to $\zeta_{n-r,k}$. If we fix the value k, there are $\binom{r}{r-k}\binom{n-r}{k}$ ways to choose the set $\{a_1,\ldots\ldots,a_r\}$ which shares just $r-k$ elements common with $\{1,\ldots\ldots,r\}$. And after choosing this set there are $\zeta_{r,k}$ ways of arranging this set so as to be discordant to $1,\ldots\ldots,r$. Hence there are

$$\binom{r}{k}\binom{n-r}{k}\zeta_{r,k}$$

terms in the sum (4), for which $\lambda_{a_1\ldots\ldots a_r}$ turns to $\zeta_{n-r,k}$. Thus (4) may be written :

$$\mu_r=\sum_{k=0}^{r_0}\binom{r}{k}\binom{n-r}{k}\zeta_{r,k}\,\zeta^2_{n-r,k}\,, \tag{5}$$

and this together with (3) furnishes the proof of Lemma 1.

In the course of the proof above we eventually classified discordant permutations of degree n with regard to the first r positions. Hence if we replace $\zeta^2_{n-r,k}$ in (5) by $\zeta_{n-r,k}$ itself we would restore φ_n (for any value of r) :

$$\varphi_n=\sum_{k=0}^{r_0}\binom{r}{k}\binom{n-r}{k}\zeta_{r,k}\,\zeta_{n-r,k}.$$

We can generalize (6) to the following Lemma, which is essential in our further discussions.

LEMMA 2. *For* $p\leqq r_0=\min.(r,\ n-r)$ *there holds the identity :*

$$\sum_{k=p}^{r_0}\binom{r}{k}\binom{n-r-p}{k-p}\zeta_{r,k}\zeta_{n-r,k}=\frac{r\,!}{(r-p)!}\zeta_{n-p,\,p}.$$

PROOF. The proof goes analogusly to that of Lemma 1. Suppose there are given a set of $n+p$ symbols

$$\{1,\ldots\ldots,\ n-p\ ;\ n-p+1,\ldots\ldots,\ n,\ \overline{n-p+1},\ldots\ldots,\overline{n}\}$$

and consider of various discordant arrangements

$$a=\binom{1\ \ldots\ldots n}{a_1\ldots\ldots a_n},$$

where $\{a_1\ldots\ldots,a_n\}$ consists of $1,\ldots\ldots,n-p,\overline{n-p+1},\ldots\ldots,\overline{n}$. Thus in a. $n-p+n\ldots\ldots n$ and $\overline{n-p+1},\ldots\ldots,\overline{n}$ are the respective heterogeneous particles of the first and the second rows. Let us determine the number γ of discordant arrangements a with the following property :

(C) all the p heterogeneneous particles of the second row are matched against those *homogeneous* particles of the first row *contained in the part* $\{1,\ldots\ldots,\ r\}$.

(Note that $p\leqq r\leqq n-p$.) We shall show

$$\gamma=\sum_{k=p}^{r_0}\binom{r}{k}\binom{n-r-p}{k-p}\zeta_{r,k}\zeta_{n-r,\,k}. \tag{7}$$

In fact, let us devide a into two parts

$$\alpha_1=\begin{pmatrix}1 \ldots\ldots r \\ a_1\ldots\ldots a_r\end{pmatrix} \text{ and } \alpha_2=\begin{pmatrix}r+1\ldots\ldots n \\ a_{r+1} \ldots\ldots a_n\end{pmatrix}$$

and suppose in α_1 there are

u elements common to $\{a_1,\ldots\ldots,a_r\}$ and $\{1,\ldots\ldots, r\}$, then there are

$r-p-u$ elements common to $\{a_1,\ldots\ldots,a_r\}$ and $\{r+1,\ldots\ldots, n-p\}$, because there should be

p elemements common to $\{a_1,\ldots\ldots,a_r\}$ and $\{\overline{n-p+1},\ldots\ldots, \bar{n}\}$ according to (C). Hence in α_2 there are

$r-u$ elements common to $\{a_{r+1},\ldots\ldots,a_n\}$ and $\{1,\ldots\ldots, r\}$,

$n-r-(r-u)$ elements common to $\{a_{r+1},\ldots\ldots, a_n\}$ and $\{r+1,\ldots\ldots n-p\}$,

and no element common to $\{a_{r+1},\ldots\ldots a_n\}$ and $\{\overline{n-p+1},\ldots\ldots, \bar{n}\}$.
The two partial arrangements α_1 and α_2 have the same number $k=r-u$ of (relative) heterogeneous particles. If we fix α_1 there are $\zeta_{n-r,\,k}$ ways to obtain α_2's or to enlarge α_1 to an α.
And if we fix the number k, there are

$$\binom{r}{u}\binom{n-r-p}{r-p-u} \zeta_{k,\,k}=\binom{r}{k}\binom{n-r-p}{k-p}\zeta_{r,k}$$

ways to obtain α_1's. This proves (7).

In order to give another form to γ, we loose the condition (C) imposed upon α and require only that

(C^*) all the p heterogeneous particles of the second row are matched against homogeneous particles of the first row.
The number γ^* of arrangements α satisfying the new condition (C^*) is easily calculated:

$$(7^*) \qquad \gamma^*=\binom{n-p}{p} p' \zeta_{n-p,\,p}.$$

Indeed, if we devide α into the two parts

$$\beta_1=\begin{pmatrix}1 \ldots\ldots n-p \\ a_1\ldots\ldots a_{n-p}\end{pmatrix} \text{ and } \beta_2=\begin{pmatrix}n-p+1\ldots\ldots n \\ a_{n-p+1} \ldots\ldots a_n\end{pmatrix},$$

β_1 and β_2 have p (relative) heterogeneous paiticles. For a fixed β_1 there are p' ways to obtain β'_2s or to enlarge β_1 to an α.
And there are $\binom{n-p}{p}\zeta_{n-p,\,p}$ ways to obtain β_1's, hence (7*) is true. Moreover the ratio $\gamma^*:\gamma$ is equal to $\binom{n-p}{p}:\binom{r}{p}$. This is obvious because γ^* can be devided into $\binom{n-p}{p}$ "classes", each containing the same num-

ber of a's, as are characterized by the p homogeneous elements of the first row matching against heterogeneous ones of the second row, while γ has only $\binom{r}{p}$ of such "classes". Hence

$$\gamma=\frac{\binom{r}{p}}{\binom{n-p}{p}}\cdot\gamma^*.$$

Combining this with (7) and (7*) we obtain the desired proof.

By the help of this Lemma we can simplify the formula for ψ_n given in Lemma 1.

LEMMA 3. *The number ψ_n is expressible in terms of rencontre numbers φ_m as follows:*

$$\psi_n=\sum_{r=0}^{n}\sum_{p=0}^{r_0}\sum_{q=0}^{p}(-)^r\frac{\varphi_{n-r-p}\ \varphi_{n-p-q}}{(n-r-p)!(r-p)!(p-q)!q!}.$$

PROOF.

$$\psi_n=\sum_{r=0}^{n}(-)^r\binom{n}{r}\sum_{k=0}^{r_0}\binom{r}{k}\binom{n-r}{k}\ \zeta_{r,k}\zeta^2_{n-r,\,k} \qquad \text{(Lemma 1)}$$

$$=\sum_{r=0}^{n}\sum_{k=0}^{r_0}(-)^r\binom{n}{r}\binom{r}{k}\binom{n-r}{k}\zeta_{r,k}\zeta_{n-r,\,k}\sum_{p=0}^{k}\binom{k}{p}\varphi_{n-r-p} \qquad [(2)]$$

$$=\sum_{r=0}^{n}\sum_{p=0}^{r_0}(-)^r\binom{n}{r}\binom{n-r}{p}\varphi_{n-r-p}\sum_{k=0}^{r_0}\binom{r}{k}\binom{n-r-p}{k-p}\zeta_{r,k}\zeta_{n-r,\,k}$$

$$=\sum_{r=0}^{n}\sum_{p=0}^{r_0}(-)^r\binom{n}{r}\binom{n-r}{p}\varphi_{n-r-p}\ \frac{r\,!}{(r-p)\,!}\ \zeta_{n-p},$$

$$=\sum_{r=0}^{n}\sum_{p=0}^{r_0}(-)^r\binom{n}{r}\binom{n-p}{p}\frac{r\,!}{(r-p)\,!}\varphi_{n-r-p}\ \sum_{q=0}^{p}\binom{p}{q}\varphi_{n-p-q} \qquad [(2)]$$

$$=n!\sum_{r=0}^{n}\sum_{p=0}^{r_0}\sum_{q=0}^{p}(-)^r\frac{\varphi_{n-r-p}\ \varphi_{n-p-q}}{(n-r-p)\,!\,(r-p)\,!\,(p-q)\,!q!}, \qquad \text{q.e.d.}$$

This formula is not convenient for the actual computation. We state here another formula though it is also dtfficultly.

LEMMA 4. *The number ψ_n is given by*

$$\psi_n=n'\sum_{s=0}^{n}C_{n-s}\ \varphi_{n-s}\ ,$$

where the number C_{n-s} is the coefficient of $\theta^{n-s}\xi^s$ in the Taylor expansion of an analytic function

$$(1-\theta)^{-1}\cdot e^{-\xi^2-\xi\theta-2\theta}.$$

PROOF. If we change the arguments r, p, q in the fotmula of Lemma 3 to

$$w=r-q,\ s=p+q \text{ and } q,$$

the summation domain

$$0 \leqq q \geqq p \leqq r,\ n-r \leqq n \tag{8}$$

is changed to

$$n \geqq w+s,\ w,s \geqq 0,\ \frac{s}{2} \geqq q \geqq 0, \frac{s-w}{2}.$$

Thus

$$\psi_n = n! \sum_{s=0}^{n} \sum_{w=0}^{n-s} (-)^w \frac{\varphi_{n-s}\, \varphi_{n-s-w}}{w!\,(n-s-w)!} \sum_{q=\max.([\frac{s-w}{2}],0)}^{[\frac{s}{2}]} (-)^q \frac{1}{q!} \binom{w}{s-2q}$$

$$= n!\sum_{s=0}^{n} \varphi_{n-s} \sum_{w=0}^{n-s} \frac{\varphi_{n-s-w}}{(n-s-w)!} \cdot \frac{(-1)^w}{w!}\, C_{w,s},$$

where $C_{w,s} = \sum_{q=\max.([\frac{s-w}{2}],0)}^{[\frac{s}{2}]} (-)^q \frac{1}{q!}\binom{w}{s-2q}$ is the coefficient of ξ^s in the expansion of $(1+\xi)^w e^{-\xi^2}$ and hence $(-)^w \frac{C_{w,s}}{w!}$ is the coefficient of $\xi^s\theta^w$ of $e^{-(1+\xi)\theta-\xi^2}$. Thus

$$\sum_{w=0}^{n-s} \frac{\varphi_{n-s-w}}{(n-s-w)!} \frac{(-1)^w}{w!} C_{w,s}$$

is the coefficient of $\xi^s\theta^{n-s}$ in $\frac{e^{-\theta}}{1-\theta} e^{-(1+\xi)\theta-\xi^2}$, because the generating function $\sum_{m=0} \frac{\varphi_m}{m!} \theta^m$ for the *rencontre* numbers is $\frac{e^{-\theta}}{1-\theta}$.

2. Asymptotic Series

Our next step and chief aim is to obtain an asymptotic series which simplifies our former results in the sense of actual calculation. This is done by a simple idea to substitute $e^{-1} m!$ for φ_m in the formula of Lemma 3. If we substitute

$$\varphi_m = m! \left(e^{-1} + \frac{\varepsilon_m}{(m+1)!}\right) \quad (|\varepsilon_m| < 1) \tag{9}$$

in that formula, we have

$$\frac{\psi_n}{n!}\sum_{r,p,q}(-)^r\frac{(n-p-q)!}{(r-p)!(p-q)!q!}\left(e^{-1}+\frac{\varepsilon_{n-r-p}}{(n-r-p+1)!}\right)\left(e^{-1}+\frac{\varepsilon_{n-p-q}}{(n-p-q-1)!}\right)$$

$$=e^{-2}\sum_{r,p,q}(-)^r\frac{(n-p-q)!}{(r-p)!\ (p-q)!\ q!}$$

$$+e^{-1}\sum_{r,p,q}(-)^r\frac{\varepsilon_{n-r-p}}{(n-r-p+1)!(r-p)!p-q)!\ q!}$$

$$+e^{-1}\sum_{r,p,q}(-)^r\frac{\varepsilon_{n-p-q}}{(r-p)!\ (p-q)!\ q!\ (n-p-q+1)}$$

$$+e^{-1}\sum_{r,p,q}(-)^r\frac{\varepsilon_{n-r-p}\ \varepsilon_{n-p-q}}{(r-p)!\ (p-q)!\ q!\ (n-r-p+1)!\ (n-p-q+1)}\ .$$

The summation is over the domain (8). Let us denote the four sums by S_1, S_2, S_3 and S_4 respectively.

Before we can estimate these sums effectively it is convenient to prove a simple lemma.

LEMMA 5. *Let the numbers σ_m be defined by*

$$\sigma_m=\sigma_m(x)=\sum_{u=0}^{\left[\frac{m}{2}\right]}\binom{m-u}{u}x^u.$$

Then their generating function $G(\theta)=\sum_{m=0}^{\infty}\sigma_m\,\theta^m$ *is*

$$G(\theta)=\frac{1}{1-\theta-x\theta^2}\ .$$

PROOF. σ_m's satisfy the recurrent relation

$$\sigma_{m+2}-\sigma_{m+1}-x\sigma_m=0,\qquad \sigma_0=\sigma_1=1.$$

This leads to the following equation for $G(\theta)$:

$$G(\theta)=\sigma_0+\sigma_1\theta+\sum_{m=0}^{\infty}(\sigma_{m+1}+x\sigma_m)\ \theta^{m+2}$$

$$=1+\theta+\theta(G(\theta)-1)+x\theta^2G(\theta)=1+(\theta+x\theta^2)G(\theta),$$

$$(1-\theta-x\theta^2)G(\theta)=1,\quad \text{q.e.d.}$$

We need in the sequel only the two special cases: $x=1$ and $x=\dfrac{1}{4}$.

In these cases $G(\theta)$ and σ_m become respectively:

$$G(\theta)=\frac{1}{2\sqrt{5}}\left(\frac{\sqrt{5}+1}{1-\frac{\sqrt{5}+1}{2}\theta}+\frac{\sqrt{5}-1}{1+\frac{\sqrt{5}-1}{2}\theta}\right),$$

$$(10)\quad \sigma_m \sum_{u=0}^{\left[\frac{m}{2}\right]} \binom{m-u}{u} = \frac{1}{\sqrt{5}} \left\{ \left(\frac{\sqrt{5}+1}{2} \right)^{m+1} - \left(\frac{-\sqrt{5}+1}{2} \right)^{m+1} \right\},$$

and

$$G(\theta) = \frac{1}{2\sqrt{2}} \left(\frac{\sqrt{2}+1}{1-\frac{\sqrt{2}+1}{2}\theta} + \frac{\sqrt{2}-1}{1+\frac{\sqrt{2}-1}{2}\theta} \right),$$

$$(10^*)\quad \sigma_m = \sum_{u=0}^{\left[\frac{m}{2}\right]} \binom{m-u}{u} 2^{-2u} = \frac{1}{\sqrt{2}} \left\{ \left(\frac{\sqrt{2}+1}{2} \right)^{m+1} - \left(\frac{-\sqrt{2}+1}{2} \right)^{m+1} \right\}.$$

Now let us proceed to estimate the sums S_1, S_2, S_3 and S_4 succesively. In all of them we shall change the arguments r, p, q to

$$s=p+q,\ t=r-p \text{ and } q.$$

The summation is then to extend over all integral triples (s, t, q) in the domain equivalent to (8), or in the domain defined by

$$0 \leqq t \leqq n-2(s-q),\quad 0 \leqq q \leqq \frac{s}{2} \leqq \frac{n}{2}.$$

First S_1 becomes (using (9))

$$S_1 = e^{-2} \sum_{s=0}^{n} \sum_{q=\max.\left(s-\left[\frac{n}{2}\right],\,0\right)}^{\left[\frac{s}{2}\right]} (-)^{s+q} \frac{(n-s)!}{q!\,(s-2q)!} \sum_{t=0}^{n-2(s-q)} (-)^t \frac{1}{t!}$$

$$= e^{-2} \sum_{s=0}^{n} \sum_{q=\max.\left(s-\left[\frac{n}{2}\right],\,0\right)}^{\left[\frac{s}{2}\right]} (-)^{s+q} \frac{(n-s)!}{q!\,(s-2q)!} \left(e^{-1} + \frac{\varepsilon_{n-2s+2q}}{(n-2s+2q+1)!} \right)$$

$$= e^{-3} \sum_{s=0}^{n} \sum_{q=0}^{\left[\frac{s}{2}\right]} (-)^{s+q} \frac{(n-s)!}{q!\,(s-2q)!} + e^{-3} \sum_{s=\left[\frac{n}{2}\right]-1}^{n} \sum_{q=0}^{s-\left[\frac{n}{2}\right]-1} (-)^{1+s+q} \frac{(n-s)!}{q!\,(s-2q)!}$$

$$+ e^{-2} \sum_{s=0}^{n} \sum_{q=\max.\left(s-\left[\frac{n}{2}\right],0\right)}^{\left[\frac{s}{2}\right]} \frac{(n-s)!\ \varepsilon_{n-2s+2q}}{q!\,(s-q)!\ (n-2s+2q-1)!}.$$

Let us denote the three sum by A, S_1'[4] and S_1'' respectively. Above all

$$A = e^{-3} \sum_{s=0}^{n} M_s (n-s)!,$$

where

$$M_s = (-)^s \sum_{q=0}^{\left[\frac{s}{2}\right]} (-)^q \frac{1}{q!\ (s-2q)!}$$

is a well-defined function of s, *independent of n*. This term A virtually furnishes the principal part of our asymptotic series. Next we shall show the remaining sums S_1', S_1'', S_2, S_3 and S_4 are all relatively small.

$$|S_1'| < e^{-3}\sum_{s=\left[\frac{n}{2}\right]}^{n}\sum_{q=0}^{s-\left[\frac{n}{2}\right]-1}\frac{1}{2}\cdot\frac{1}{q!}\geqq\frac{1}{2}e^{-3}\sum_{s=\left[\frac{n}{2}\right]-1}^{n}\frac{e}{\left(s-\left[\frac{n}{2}\right]\right)!}$$

$$(11) \qquad < \frac{e^{-2}(e-1)}{2}.$$

Here we make use of the relation

$$\frac{(n-s)!}{(s-2q)!}\leqq\frac{1}{2}.$$

(This inequality follows from

$$0\leqq q\leqq s-\left[\tfrac{n}{2}\right]-1.$$

Indeed, $(s-2q)-(n-s)\geqq 2$, $\quad \dfrac{(s-2q)!}{(n-s)!}\geqq 2$.)

$$|S_1''| < e^{-2}\sum_{q=0}^{\left[\frac{n}{2}\right]}\sum_{s=2q}^{q+\left[\frac{n}{2}\right]}\frac{(n-s)!}{(n-2s+2q)!\,q!\,(s-2q)!}$$

$$= e^{-2}\sum_{q=0}^{\left[\frac{n}{2}\right]}\frac{1}{q!}\sum_{w=0}^{\left[\frac{n}{2}\right]-q}\binom{n-2q-w}{w} \qquad (w=s-2q)$$

$$= e^{-2}\sum_{q=0}^{\left[\frac{n}{2}\right]}\frac{1}{q!}\left\{\left(\frac{\sqrt{5}+1}{2}\right)^{n+1-2q}-\left(\frac{-\sqrt{5}+1}{2}\right)^{n+1-2q}\right\} \qquad (10)$$

$$< \frac{e^{-2}}{\sqrt{5}}\left\{e^{\frac{3-\sqrt{5}}{2}}\left(\frac{\sqrt{5}+1}{2}\right)^{n+1}+e^{\frac{3+\sqrt{5}}{2}}\left(\frac{\sqrt{5}-1}{2}\right)^{n+1}\right\}$$

$$(12) \qquad < e^{-2}\left(\frac{\sqrt{5}+1}{2}\right)^{n} \qquad (n\geqq 4).$$

$$|S_2| < e^{-1}\sum_{q=0}^{\left[\frac{n}{2}\right]}\sum_{s=q}^{\left[\frac{n}{2}\right]+q}\sum_{t=0}^{n-2(s-q)}\frac{(n-s)!}{(n-t-2s+2q)!\,t!\,q!\,(s-2q)!}$$

$$= e^{-1}\sum_{q}\sum_{s}\frac{1}{q!}\binom{n-s}{s-2q}\sum_{t=0}^{n-2(s-q)}\binom{n-2(s-q)}{t}$$

$$= e^{-1}\sum_{q}\sum_{s}\frac{1}{q!}\binom{n-s}{s-2q}2^{n-2(s-q)}$$

$$=2^n e^{-1}\sum_{q=0}^{\left[\frac{n}{2}\right]}\frac{2^{-2q}}{q\,!}\sum_{w=0}^{\left[\frac{n}{2}\right]-q}\binom{n-2q-w}{w}2^{-2w} \qquad (w=s-2q)$$

$$=-\frac{2^n e^{-1}}{\sqrt{2}}\sum_{q=0}^{\left[\frac{n}{2}\right]}\left\{\left(\frac{\sqrt{2}+1}{2}\right)^{n-2q+1}-\left(\frac{-\sqrt{2}+1}{2}\right)^{n-2q+1}\right\} \qquad [(10^*)]$$

$$<\frac{e^{-1}}{2\cdot\sqrt{2}}\left\{e^{3-2\sqrt{2}}\ (\sqrt{2}+1)^{n+1}+e^{3+2\sqrt{2}}(\sqrt{2}-1)^{n+1}\right\}$$

$$(13) \qquad <\frac{e^{-1}}{2}\ (\sqrt{2}+1)^{n+1} \qquad (n\geqq 7).$$

Similarly :

$$(14) \qquad |S_3|<e^2-e \quad |S_4|>e^3-e^2-1.$$

Combining (11), (12), (13), (14) we now conclude:

$$\left|\frac{\psi_n}{n\,!}-A\right|<|S_1'|+|S_1''|+|S_2|+|S_3|+|S_4|<(\sqrt{2}+1)^n \quad (n\geqq 7).$$

THEOREM 1. *For the number ψ_n of reduced three-line latin rectangles with n columns, there holds an asymptotic expression*

$$\psi_n\sim n!\ A=e^{-3}(n\,!)^2\sum_{s=0}^{n}M_s\frac{(n-s)!}{n\,!}\ ,$$

where the coefficient M_s are given by

$$M_s=(-)^s\sum_{q=0}^{\left[\frac{s}{2}\right]}(-)^q\frac{1}{q!\,(s-2q)!}.$$

and the approximation error is

$$|\psi_n-n\,!A|<n\,!\ \ (\sqrt{2}+1)^n. \quad ^{5)}$$

REMARK. The last inequality was proved for $n\geqq 7$. But we can verify this for $n<7$ by actual calculation (see 4).

3. The Coefficients M_s

Now let us make clear some properties of the numbers M_s, which will enable us to compute them actually. The following Lemma is also important.

LEMMA 6. *For any value of $s\geqq 2$,*

$$|M_s|<1.$$

PROOF. Let us define

$$M^*_s = \sum_{q=0}^{[\frac{s}{2}]} \frac{1}{q!\ (s-2q)!}.$$

Then we can show the sequences $\{M_{2r}^*\}$ and $\{M_{2r+1}^*\}$ are decreasing but for the first terms. In fact, if $s \geqq 2$, $M_s^* - M_{s+2}^* = -\frac{1}{0!(s+2)!} + \left(\frac{1}{0!\ s!} - \frac{1}{1!\ s!}\right) + \left(\frac{1}{1!(s-2)!} - \frac{1}{2!(s-2)!}\right) + \sum_{q=2}^{[\frac{s}{2}]}\left(\frac{1}{q!(s-2q)!} - \frac{1}{(q+1)!(s-2q)!}\right)$

$< \frac{1}{2(s-2)!} - \frac{1}{(s+2)!} > 0.$

We know on the other hand

$$M_4^* = \frac{25}{24},\ M_4^* = \frac{81}{120'} M_6^* = \frac{331}{720}.$$

Therefore

$$|M_s| < M_s^* < 1 \qquad \text{for} \quad s \geqq 5.$$

For $s \leqq 4$, M_s has the values

$$M_0 = 1,\quad M_1 = -1,\quad M_2 = -\frac{1}{2},\quad M_3 = \frac{5}{6},\quad M_4 = \frac{1}{24}.$$

Thus Lemma 6 is proved.

LEMMA 7. *The generating function* $K(\theta) = \sum_{s=0}^{\infty} M_s \theta^s$ *for the numbers* M_s *is :*

$$K(\theta) = e^{-(\theta+\theta^2)} = e^{\frac{1}{4}} \cdot e^{-(\theta+\frac{1}{2})^2}.$$

PROOF.

$$e^{-\theta} \cdot e^{-\theta^2} = \sum_{p=0}^{\infty} (-)^p \frac{\theta^p}{q!} \cdot \sum_{q=0}^{\infty} (-)^q \frac{\theta^{2q}}{q!}$$

$$= \sum_{s=0}^{\infty} \left\{ \sum_{q=0}^{[\frac{s}{2}]} (-)^{s+q} \frac{1}{q!\ (s-2q)!} \right\} \theta^s = K(\theta), \text{ q.e.d.}$$

This Lemma reveals an interesting fact that the numerator

$$N_s = s!\ M_s$$

of M_s becomes

$$N_s = \left.\frac{d^s K(\theta)}{d\theta^s}\right|_{\theta=0} = e^{\frac{1}{4}} \cdot \left.\frac{d^s e^{-x^2}}{dx^s}\right|_{x=\frac{1}{2}} = e^{x^2} \left.\frac{d^s e^{-x^2}}{dx^s}\right|_{x=\frac{1}{2}}$$

$$=(-)^s H_s(\frac{1}{2})=H_s(-\frac{1}{2}),$$

where $H_s(x)=(-)^s e^{x^2}\frac{d^s e^{-x^2}}{dx^s}$ is the *Hermite polynomial*[6] of degree s.

LEMMA 8. *The numbers M_s and N_s satisfy the following recurrence relations :*

$$(s+2)M_{s+2}+M_{s+1}+2M_s=0, \quad M_0=1, \quad M_1=-1;$$
$$N_{s+2}+N_{s+1}+2(s+1)N_s=0, \quad N_0=1, \quad N_1=-1.$$

PROOF. The Lemma is nothing but the recurrence relation for Hermite polynomials.

The two equations in Lemma 8 are known as *Poincaré's equation*[7] in the theory of finite differences. Hence we observe first

$$\lim\frac{M_{s+1}}{M_s}=0,$$

which is another flank of fact stated in Lemma 6. More precisely, Poincaré himself proved that[8]

$$N_s'=2^{-\frac{s}{2}}(s!)^{-\frac{s}{2}}N_s$$

is the sum of two numbers

$$N_s'=P_s+Q_s$$

such that

$$(15) \qquad \lim\frac{P_{s+1}}{P_s}=i, \quad \lim\frac{Q_{s+1}}{Q_s}=-i \qquad (i=\sqrt{-1}),$$

although the sequence

$$\left\{\frac{N'_{s+1}}{N'_s}\right\}$$

itself is not convergent. From (15) follows immediately

$$\lim\frac{N'_{s+2}}{N'_s}=-1$$

or

$$N_{s+2}\sim -2sN_s \qquad (\text{as } s\to\infty.)$$

THEOREM 2. *The coefficient M_s of the asymptotic series in Theorem 1 is $\frac{H_s(-\frac{1}{2})}{s!}$, where $H_s(x)$ is the Hermite polynomial of degree s. The sequence*

$\{M_s\}$ converges to 0, and

$$M_{s+2} \sim -\frac{2}{s} M_s \quad (as\ s \to \infty.).$$

COROLLARY.[9] *For an arbitrary, but fixed non-negetive integral value of r, there holds an asymptotic relation :*

$$\psi_n = e^{-3}(n!)^2 \left\{ M_0 + M_1 \cdot \frac{1}{n} + M_2 \cdot \frac{1}{n(n-1)} + \cdots\cdots \right.$$
$$\left. + M_r \frac{1}{n(n-1)\ldots(n-r+1)} + 0\left(\frac{1}{n^{r+1}}\right) \right\}.$$

PROOF. The propsition follows immediately from Theorem 1 and the boundedness of the sequence $\{M_s\}$, established in Theorem 2 (or Lemma 6).

4. Numerical Tables

The following is a short table for the numbers N_s, calculated through the recurrence relation. Lemma 8.

s	N_s	s	N_s
0	+1	11	+107 029
1	−1	12	−604 031
2	−1	13	−1 964 665
3	+5	14	+17 669 471
4	+1	15	+37 341 149
5	−41	16	−567 425 279
6	+31	17	−627 491 489
7	+461	18	+19 919 950 975
8	−895	19	+2 669 742 629
9	−6 481	20	−759 627 879 679
10	+22 591	21	+652 838 174 519

The following is a beginning of the table for the numbers ψ_n : in each pair of values, the upper are the true,[10] the lower are approximated ones $n!\ A = e^{-3} n! \sum_{s=0}^{n} M_s (n-s)!$.

n	ψ_n and $n!\,A$
3	2
	1.3
4	24
	21.4
5	552
	564
6	21 280
	21 249
7	1 073 760
	1 073 89
8	70 299 264
	70 297 8
9	5 792 853 248
	5 792 866
10	587 159 944 704
	587 159 83
11	71 822 743 499 520
	71 822 744 6
12	10 435 273 503 677 440
	10 435 273 482
13	1 776 780 700 509 416 448
	1 776 780 700 65
14	350 461 958 856 515 690 496
	350 461 958 854 4
15	79 284 041 282 622 163 140 608
	79 284 041 282 652
16	20 392 765 404 792 755 583 221 760
	20 392 765 404 792 31
17	5 917 934 230 798 104 348 783 083 520
	5 917 934 230 798 115
18	1 924 427 226 324 694 427 836 833 857 536
	1 924 427 226 324 694 58
19	696 979 289 286 274 520 909 680 184 328 192
	696 979 289 286 274 523 1
20	279 603 955 400 790 511 301 713 870 268 399 616
	279 603 355 400 790 511 59

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References

1) The author has not yet access to this paper.

2) In this country, M. Takasaki obtained a recursive formula for $f(n, 3)$, involving three other rows of numbers. But it was sterile for asymptotic behaviors.

4) The presence of this term was pointed out by Prof. Kawada.

5) This estimation is undoubtedly too crude. I believe it should be reduced to $\leqq Cn!$ with some constant C, but I cannot prove.

6) See e.g. N. Wiener: The Fourier integral and certain of its applications, Cambridge 1933, p. 54.

7) H. Poincaré: Sur les equations linéaires aux differetielles ordinaires et aux differences finies. Amer. J. Math. **7** (1885), pp. 1—56=Oeuvres, t. I, Paris 1928, pp. 226—289.

8) H. Poincaré: loc. cit., paragraph VII.

9) I owe this form of the proposition to Prof. Kawada.

10) I have computed these values through Takasaki's formula, and verified by Riordan's formula ([2] and [3]).