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On the jump of a function and its Fourier series. Notes on Fourier Analysis (XXXIII)

Noboru Matsuyama.

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§ **1.** Let $f(x)$ be an integrable and periodic function with period 2π and its Fourier series be

$$\mathfrak{S}[f]=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n \text{ con } nx+b_n \sin nx).$$

Fejér has proved that, if there is an s such that

$$\int_0^1|\psi(u)-s|du=o(t),$$

$$\psi(u)=(f(u)-f(-u))/2,$$

then the sequence (nb_n) is $(R, \log n, 1)$-summable to $2s/\pi$.

Recently, O. Szàsz has proved that if

$$\int_0^t(\psi(u)-s)\,du=o(t) \tag{1}$$

and

$$\int_0^t|\psi(u)-s|du=O(t), \tag{2}$$

then the sequence (nb_n) is $(C, 2)$-summable to $2s/\pi$.

We shall now consider the $(R, \log n, \alpha)$-summability of the sequence (nb_n). In fact we shall prove the following theorems:

Theorem 1. *If for any* $\alpha\geqq 0$

$$\lim_{t\to 0}\psi(t)=s \quad (R, \log n, \alpha),$$

then the sequence (nb_n) *is* $(R, \log n, 1+\alpha+\delta)$*-summable to* $2s/\pi$*, where* δ *is any positive number.*

Theorem. 2. *If for any* $\alpha\geqq 1$, (bn) *is* $(R, \log n, \alpha)$*—summable to* $2s/\pi$*, then*

$$\lim_{t\to 0}\psi(t)=s \quad (R, \log n, \alpha+1+\delta),$$

δ *being any positive number.*

§ **2. Lemmas.** Let us put

$$l_\alpha(t)=\frac{1}{t}\int_0^t(\log (t/u))^\alpha \sin u\, du$$

for $\alpha > -1$.

Lemma 1.

(3) $$\frac{d}{dt}(tl_\alpha(t)) = \alpha l_{\alpha-1}(t) \quad (\alpha > 0),$$

(4) $$l_{\alpha+\beta+1}(t) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_0^t (\log\frac{t}{u})^\alpha l_\beta(u)\,du \quad (\alpha,\beta > -1),$$

(5) $$\frac{2}{\pi}\int_0^\infty l_\alpha(u)\sin tu\,du = (\log\frac{1}{t})^\alpha \text{ for } 0 < t < 1,$$
$$= 0 \text{ for } 1 \leq t \quad (\alpha > -1),$$

(6) *if* $t > 0$, *then* $l_\alpha(t) = O(1)$ $(\alpha > -1)$, $l'_\alpha(t) = O(1/t)$ $(\alpha > 0)$, $l'_0(t) = O(1)$, $l_\alpha(t) = O(1/t^2)$ $(\alpha \geq 1)$. *And if* $t \geq 2$, *then* $l_\alpha(t) = O((\log t)^\alpha / t)$ $(\alpha \geq 0)$, $l_\alpha(t) = O(1/t^{\alpha+1})$ $(\alpha \leq 0)$, $l'_\alpha(t) = O((\log t)^\alpha/t^2)$ $(\alpha \geq 1)$, *and* $l'_\alpha(t) = O(1/t^{\alpha+1})$ $(0 \leq \alpha < 1)$, $l''_\alpha(t) = O((\log t)^\alpha/t^3)$ $(\alpha \geq 1)$.

(7) $l_\alpha(0) = 0$ $(\alpha > -1)$.

(8) $l_0(t) = (1-\cos t)/t$.

Proof of this lemma is easy.

Let $D_\alpha(\omega)$ be the $(R, \log n, \alpha)$-mean of (nb_n). By definition we have

(9) $$D_\alpha(\omega) = \frac{\alpha}{(\log \omega)^\alpha}\sum_{n<\omega}\left(\log\frac{\omega}{n}\right)^{\alpha-1} b_n$$

for $\alpha > 0$.

If $\alpha > 1$, then $l'_{\alpha-1}(t)$ is integrable in $(0, \infty)$ and $l_{\alpha-1}(t) = o(1)$ as $t \to \infty$. After S. Pollard we have

$$\int_0^\infty \psi(t) l_{\alpha-1}(\omega t)\,dt = \sum_{n=1}^\infty b_n \int_0^\infty l_{\alpha-1}(\omega t)\sin nt\,dt$$
$$= \sum_{n=1}^\infty b_n \frac{1}{\omega}\int_0^\infty l_{\alpha-1}(t)\sin\frac{nt}{\omega}\,dt$$
$$= \frac{\pi}{2}\,\frac{1}{\omega}\sum_{n<\omega} b_n (\log\frac{\omega}{n})^{\alpha-1}.$$

Consequently we have

(10) $$D_\alpha(\omega) = \frac{2\alpha}{\pi}\,\frac{\omega}{(\log\omega)^\alpha}\int_0^\infty \psi(t) l_{\alpha-1}(\omega t)\,dt$$

for $\alpha > 1$.

On the other hand if we put

$$\mu(t)=1 \text{ in } (0,\pi) \text{ and } =-1 \text{ in } (-\pi,0),$$

then

$$\mu(t)\sim\frac{2}{\pi}\sum_{1}^{\infty}\frac{1-(-1)^n}{n}\cos nt.$$

If we replace $D_\alpha(\omega)$ and $\psi(u)$ by $\chi_\alpha(\omega)$ and $\mu(u)$ in (10), then we have

$$\text{(11)}\qquad \chi_\alpha(\omega)=\frac{2\alpha}{\pi}\frac{\omega}{(\log\omega)^\alpha}\int_0^\infty\mu(t)l_{\alpha-1}(\omega t)\,dt.$$

Since the sequence $((1-(-1)^n)2/\pi)$ is (C,δ)-summable to $2/\pi$, it is also $(R,\log n,\delta)$-summable to $2/\pi$. Hence (10) and (11) give us

$$D_\alpha(\omega)-s\chi_\alpha(\omega)=\frac{2\alpha}{\pi}\frac{\omega}{(\log\omega)^\alpha}\int_0^\infty(\psi(t)-s\mu(t))l_{\alpha-1}(\omega t)\,dt.$$

Thus we get

Lemma 2. *For any $\alpha>1$, the necessary and sufficient condition that the sevuence (nb_n) is $(R,\log n,\alpha)$-summable to $2s/\pi$, is*

$$\text{(12)}\qquad I_\alpha\equiv\omega(\frac{\omega}{(\log\omega)^\alpha}\int_0^\infty g(t)l_{\alpha-1}(\omega t)\,dt=o(1)$$

as $\omega\to\infty$, where

$$\text{(13)}\qquad g(t)=\psi(t)-\mu(t)s.$$

§3. **Proof of Theorem 1.** Let us put

$$G(u)=\int_0^t g(u)\,du,$$

and

$$G^*(u)=\int_9^t|g(u)|\,du,$$

then $G(u)=O(1)$ and $G^*(u)=O(u)$ as $u\to\infty$. If $\alpha\geqq 2$,

$$\frac{\omega}{(\log\omega)^\alpha}\int_\pi^\infty g(t)l_{\alpha-1}(\omega t)\,dt=\frac{\omega}{(\log\omega)^\alpha}[G(t)l_{\alpha-1}(\omega t)]_\pi^\infty$$

$$-\frac{\omega^2}{(\log\omega)^\alpha}\int_\pi^\infty G(t)l'_{\alpha-1}(\omega t)\,dt$$

$$=O\Big(\frac{1}{(\log\omega)}\Big)+O\Big(\frac{\omega^2}{(\log\omega)^\alpha}\int_\pi^\infty\frac{(\log\omega t)^{\alpha-1}}{\omega^2t^2}dt\Big)$$

$$=O\Big(\frac{1}{\log\omega}\Big)=o(1).$$

On the other hand if $1<\alpha<2$, then

$$\frac{\omega}{(\log\omega)^\alpha}\int_\pi^\infty g(t)l_{\alpha-1}(\omega t)\,dt=\frac{\omega}{(\log\omega)^\alpha}\Big(\int_\pi^{\lambda/\omega}+\int_{\lambda/\omega}^\infty\Big)\equiv P+Q,$$

say. We have

$$|P|\leqq\frac{\omega}{(\log\omega)^{\alpha}}\int_{\pi}^{\lambda w}|g(t)|\ \frac{(\log\omega t)^{\alpha-1}}{\omega t}\ dt$$

$$=\frac{1}{(\log\omega)^{\alpha}}\left[G^{*}(t)\ \frac{(\log\omega t)^{\alpha-1}}{t}\right]_{\pi}^{\lambda w}+\frac{1}{(\log\omega)^{\alpha}}\int_{\pi}^{\lambda/\omega}G^{*}(t)$$

$$\frac{(\alpha-1)(\log\omega t)^{\alpha-2}+(\log\omega t)^{\alpha-1}}{t^{2}}dt$$

$$=O\left(\frac{(\log\lambda)^{\alpha-1}}{(\log\omega)^{\alpha}}\right)+O\left(\frac{1}{\log\omega}\right)+O\left(\frac{(\log\lambda)^{\alpha}}{(\log\omega)^{\alpha}}\right)+O(1)$$

$$Q=\frac{\omega}{(\log\omega)^{\alpha}}\left[G(t)l_{\alpha-1}(\omega t)\right]_{\lambda/\omega}^{\infty}-\frac{\omega^{2}}{(\log\omega)^{\alpha}}\int_{\lambda/\omega}^{\infty}G(t)l'_{\alpha-1}(\omega t)\,dt$$

$$=O\left(\frac{(\log\lambda)^{\alpha-1}}{(\log\omega)^{\alpha}}\cdot\frac{\omega}{\lambda}\right)+O\left(\frac{\omega}{(\log\omega)^{\alpha}}\right)\cdot\left(\int_{\lambda/\omega}^{\infty}\frac{dt}{\omega^{\alpha}\ t^{\alpha}}\right)$$

$$=O\left(\frac{(\log\lambda)^{\alpha=1}}{(\log\omega)^{\alpha}}\cdot\frac{\omega}{\lambda}\right)+O\left(\frac{\lambda^{1-\alpha}}{(\log\omega)^{\alpha}}\right).$$

If we put $\lambda=\omega^{1/(\alpha-1)}$, then $P=O(1)$ and $Q=o(1)$. Thus we have proved the formula,

$$I_{\alpha}=\frac{\omega}{(\log\omega)^{\alpha}}\int_{0}^{\pi}g(t)l_{\alpha-1}(\omega t)\,dt+o(1),\quad for\ 2\geqq\alpha, \tag{14}$$

$$I_{\alpha}=\frac{\omega}{(\log\omega)^{\alpha}}\int_{0}^{\pi}g(t)l_{\alpha-1}(\omega t)\,dt+O(1),\quad for\ 1<\alpha<2. \tag{15}$$

In the case $\alpha=1$, we have

$$D_{1}(\omega)-s\chi_{1}(\omega)=\frac{2}{\pi}\ \frac{\omega}{\log\omega}\int_{0}^{\pi}g(t)l_{0}(\omega t)\,dt+o(1),$$

by the direct calculation. Consequently

$$I_{1}=\frac{\omega}{\log\omega}\int_{0}^{\pi}g(t)l_{0}(\omega t)\,dt+o(1). \tag{16}$$

By the hypothesis

$$g_{\alpha}(t)=\frac{1}{\Gamma(\alpha)}\int_{t}^{\pi}\left(\log\frac{u}{t}\right)^{\alpha-1}\frac{g(u)}{u}\ du=o\left(\log\frac{1}{t}\right)^{\alpha} \tag{17}$$

for $\alpha\geqq 0$. Now

$$J_{\alpha+1}=\frac{\omega}{(\log\omega)^{\alpha+1}}\int_{0}^{\pi}g(t)l_{\alpha}(\omega t)\,dt$$

$$=C\ \frac{\omega}{(\log\omega)^{\alpha+1}}\int_{0}^{\pi}g(t)l_{0}(\omega t)\,dt$$

$$=C\frac{\omega}{(\log\omega)^{\alpha-1}}\left(\int_0^{2/\omega}+\int_{2/\omega}^{\pi}\right)$$
$$=P+Q,$$

say. We have

$$P=\frac{C}{(\log\omega)^{\alpha-1}}\int_0^{2/\omega}g_\alpha(t)\,1\frac{(1-\cos\omega t)}{t}\,dt$$
$$=o\left(\frac{\omega^2}{(\log\omega)^{\alpha+1}}\right)\left(\int_0^{2/\omega}\left(\log\frac{1}{t}\right)^\alpha t\,dt\right)=o\left(\frac{\omega^2}{(\log\omega)^{\alpha+1}}\right.$$
$$\left.\frac{1}{\omega^2}(\log\omega)^\alpha\right)=o(1),$$

$$Q=\frac{C}{(\log\omega)^{\alpha+1}}\int_{2/\omega}^{\pi}g_\alpha(t)\frac{1-\cos\omega t}{t}\,dt$$
$$=o\left(\frac{1}{(\log\omega)^{\alpha+1}}\right)\left(\int_{2/\omega}^{\pi}\left(\log\frac{1}{t}\right)^\alpha\frac{dt}{t}\right)$$
$$=o\left(\frac{1}{(\log\omega)^{\alpha+1}}(\log\omega)^{\alpha+1}\right)=o(1).$$

Consequently if $\alpha\geqq 1$ or $\alpha=0$, then

$$I_{\alpha+1}(\omega)=J_{\alpha+1}(\omega)+o(1)=o(1),$$

and if $1>\alpha>0$ then

$$I_{\alpha+1}(\omega)=J_{\alpha+1}(\omega)+O(1)=O(1),$$

which proves the theorem.

§4. **Proof of Theorem 2.** From the hypothesis $I_\alpha=o(1)$ and then

$$\int_0^\pi g_{\alpha-1}(t)l_0(\omega t)\,dt=\frac{1}{\omega}\Big[g_\alpha(t)(1-\cos\omega t)\Big]_0^\pi-\frac{\omega}{\omega}\int_0^\pi g_\alpha(t)\sin\omega t\,dt$$
$$=-\int_0^\pi g_\alpha(t)\sin\omega t\,dt$$
$$=o((\log\omega)^\alpha/\omega)\quad\text{for } \alpha=1 \text{ or } \alpha\geqq 2,$$
$$=O((\log\omega)^\alpha/\omega)\quad\text{for } 1<\alpha<2.$$

If we put

$$g_\alpha(t)\sim\sum_1^\infty c_n\sin nt,$$

then we have $c_n=o((\log n)^\alpha/n)$ for $\alpha=1$, or $\alpha\geqq 2$, and $c_n=O((\log n)^\alpha/n)$ for $1<\alpha<2$. Hence

$$\frac{1}{t}\int_0^t g_\alpha(u)\,du=\sum_1^\infty c_n\frac{1-\cos nt}{nt}$$

$$=\sum_{nt<1} o\left(\frac{(\log n)^{\alpha}}{n}\right) nt+\sum_{nt\geq 1} o\left(\frac{(\log n)^{\alpha}}{n}\right)\frac{1}{nt}$$

$$=o\left(\log\frac{1}{t}\right)^{\alpha},$$

for $\alpha=1$ or $2\leqq\alpha$. Smilarly we have

$$\frac{1}{t}\int_0^t g_\alpha(u)\ du=O(\log 1/t)^{\alpha},$$

for $1<\alpha<2$. Consequently we have

$$g_{\alpha+1}(t)=o(\log 1/t)^{\alpha+1} \text{ for } \alpha=1 \text{ or } \alpha\geqq 2,$$

$$g_{\alpha+1}(t)=O(\log 1/t)^{\alpha+1} \text{ for } 1<\alpha<2.$$

Thus the theorem is proved.

§5. We conclude this paper by the following theorem:

Theorem 3. *If*

$$g_\alpha(t)=o(\log 1/t)^{\alpha}$$

and

$$\int_t^\pi |g_{\alpha-1}(t)|/t\ dt=O\left(\log\frac{1}{t}\right)^{\alpha}$$

then (nb_n) *is* $(R, \log n, \alpha+\delta)$*-summable to* $2s/\pi$, α *being* $\geqq 1$.

Proof. We have

$$J_\alpha=\frac{\omega}{(\log\omega)^{\alpha}}\int_0^\pi g(t)l_{\alpha-1}(wt)dt=\frac{\omega}{(\log\omega)^{\alpha}}\int_0^\pi g(t)l_0(\omega t)dt$$

$$=\frac{\omega}{(\log\omega)^{\alpha}}\left(\int_0^{2/\omega}+\int_{2/\omega}^{\pi}\right)\equiv P+Q,$$

say. By integration by parts

$$P=\frac{C}{(\log\omega)^{\alpha}}[g_\alpha(t)(1-\omega t)]_0^{2/\omega}-C\frac{\omega}{(\log\omega)^{\alpha}}\int_0^{2/\omega} g_\alpha(t)\sin\omega t\ dt$$

$$=o\left(\frac{1}{(\log\omega)^{\alpha}}\cdot(\log\omega)^{\alpha}\right)+o\left(\frac{\omega^2}{(\log\omega)^{\alpha}}\int_0^{2/w}\left(\left(\log\frac{1}{t}\right)^{\alpha} t\ dt\right)\right)$$

$$=o(1)+o\left(\frac{\omega^2}{(\log\omega)^{\alpha}}\ \frac{1}{\omega^2}(\log\omega)^{\alpha}\right)=o(1).$$

and

$$|Q|\leqq C\frac{\omega}{(\log\omega)^{\alpha}}\int_{2/\omega}^{\pi}|g_{\alpha-1}(t)|\frac{dt}{\omega t}=O\left(\frac{1}{(\log\omega)^{\alpha}}\cdot(\log\omega)^{\alpha}\right)=O(1).$$

Thus we have $I_\alpha=O(1)$ for $\alpha\geqq 1$. By the first condition of the theorem

3 and Theorem 1 we have $I_{\alpha+1+\delta}(\omega)=o(1)$. Hence we have $I_{\alpha+\delta}(\omega)=o(1)$ for any $\delta>0$, which is the reqired.

Corollary. *If $g(t)$ satisfies*

$$\int_0^t g(u)\ du=o(t) \text{ and } \int_0^t |g(u)|du=O(t)$$

then

$$\lim\ nb=2s/\pi \quad (R,\ \log\ n,\ 1+\delta).$$

Mathematical Institute
Tôhoku University

Notes

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