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A remark on Schottky's theorem.

MASATSUGU TSUJI.

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Let $f(z)=a_0+a_1 z+\ldots\ldots$ be regular for $|z|<1$ and $f(z)\neq 0, \neq 1$, then by the precise form of Schottky's theorem due to Bohr and Landau,[1)]

$$|f(z)|\leqq \exp \frac{D\log(|a_0|+2)}{1-r} \quad (|z|=r<1), \quad (1)$$

where D is a numerical constant.
Since from (1), $\{f(z)\}$ forms a normal family, we can easily prove that if $a_0\to 0$, then $f(z)\to 0$ uniformly in the wider sense in $|z|<1$. But the right hand side of (1) does not tend to zero for $a_0\to 0$. Hence it is desirable to obtain a majorant of $|f(z)|$, which tends to zero for $a_0\to 0$. We will now prove the following

Theorem. $|f(z)|\leqq \exp\left(A \log|a_0|\cdot(1-r)+B\frac{\log(|a_0|+2)}{1-r}\right)$,

where A, B are positive numerical constants.
Proof. Let D_0 be the domain on $\zeta=x+iy$-plane, such that $o<x<1$, $y>0$, $|\zeta-\frac{1}{2}|>\frac{1}{2}$ and we map D_0 on the upper half w-plane by $w=\lambda(\zeta)$, such that $\lambda(o)=o$, $\lambda(1)=1$, $\lambda(\infty)=\infty$.

Then $\lambda(\zeta)$ is automorphic with respect to a modular group G, whose fundamental domain is $\varDelta=D_0+D_0'$, where D_0' is the image of D_0 with respect to the imaginary axis.
As well known, $w=\lambda(\zeta)$ has a simple pole at $t=o$, where $t=e^{\pi i\zeta}$, so that if $y\geqq\eta$ (>1),

$$e^{\frac{\pi}{2}y}\leqq|w|\leqq e^{2\pi y} \quad (y\geqq\eta), \quad \text{or} \qquad (2)$$

1) H. Bohr und E. Landau: Über das Verhalten von $\zeta(s)$ und $\zeta_\kappa(s)$ in der Nähe der Geraden $\sigma=1$. (Göttinger Nachr. 1910).

$$\frac{\pi}{2}\, y \leqq \log |w| \leqq 2\pi\, y, \tag{3}$$

hence the line $y=\eta$ is mapped on a curve, which lies in a ring donain:

$$e^{\frac{\pi}{2}\eta} \leqq |w| \leqq e^{2\pi\eta}. \tag{4}$$

Let $\zeta=\nu(w)$ be the inverse function of $w=\lambda(\zeta)$, which is infinitely many valued. We consider the branch of $\nu(w)$, which lies in $\varDelta$ and let $\zeta_0=x_0+iy_0=\nu(a_0)$ lie in $\varDelta$.

We assume that $|a_0|$ is so large that

$$|a_0| \geqq e^{8\pi\eta}, \tag{5}$$

then a_0 lies outside the circle $|w|=e^{2\pi\eta}$, so that by (4), $y_0 \geqq \eta$ and hence by (3),

$$\frac{\pi}{2}\, y_0 \leqq \log |a_0| \leqq 2\pi\, y_0. \tag{6}$$

Since

$$\zeta = x+iy=\nu(f(z))=\nu(a_0)+b_1 z+\ldots\ldots=\zeta_0+b_1 z+\cdots$$

is regular for $|z|<1$ and $y=\Im(\nu f(z))>0$, we have

$$y_0\frac{1+r}{1-r} \geqq y \geqq y_0\frac{1-r}{1+r} \quad (|z|=r). \tag{7}$$

Since by (6)

$$y \geqq y_0\frac{1-r}{1+r} \geqq y_0\frac{(1-r)}{2} \geqq \frac{\log|a_0|}{4\pi}(1-r), \tag{8}$$

we have for $0 \leqq r \leqq \frac{1}{2}$,

$$y \geqq \frac{\log|a_0|}{8\pi} \geqq \eta,$$

so that by (2), (8),

$$|f(z)| \geqq e^{\frac{\pi}{2}\cdot y} \geqq \exp\frac{\log|a_0|(1-r)}{8} \quad \left(o \leqq r \leqq \frac{1}{2}\right). \tag{9}$$

Since

$$\frac{1}{f(z)}=\frac{1}{a_0}+c_1 z+\ldots\ldots$$

satisfies the same condition as $f(z)$, we have from (9), if $\frac{1}{|a_0|} \geqq e^{8\pi\eta}$, or $|a_0| \leqq e^{-8\pi\eta}$,

$$\left|\frac{1}{f(z)}\right| \geqq \exp \frac{\log\left|\frac{1}{a_0}\right|(1-r)}{8}, \text{ or}$$

$$|f(z)| \leqq \exp \frac{\log|a_0|(1-r)}{8} \quad (o \leqq r \leqq \frac{1}{2}). \quad (10)$$

To obtain the majorant for $\frac{1}{2} < r < 1$, let

$$M(r) = \underset{|z|=r}{\text{Max.}} \ |f(z)|,$$

$$r_1 = \frac{1}{2}, \ r_2 = r \ , \ r_3 = \frac{1+r}{2} \ , \quad (11)$$

then by Hadamard's three clrcles theorem,

$$\log M(r) = \log M(r_2) \leqq$$

$$\frac{\log r_3 - \log r_2}{\log r_3 - \log r_1} \log M(r_1) + \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1} \log M(r_3)$$

$$= \frac{\log \frac{1+r}{2r}}{\log (1+r)} \log M(r_1) + \frac{\log 2r}{\log (1+r)} \log M(r_3) \ . \quad (12)$$

By (10), (1),

$$M(r_1) \leqq \exp \frac{\log |a_0|}{16}, \ M(r_3) \leqq \exp \frac{D \log (|a_0|+2)}{1-\frac{1+r}{2}}$$

$$= \exp \frac{2D \log (|a_0|+2)}{1-r} \quad (13)$$

and there exists a constant $o < a < 1$, such that

$$\frac{\log \frac{1+r}{2r}}{\log (1+r)} \geqq a(1-r), \ \frac{\log 2r}{\log (1+r)} \leqq 1 \ (\frac{1}{2} \leqq r < 1).$$

Since $\log |a_0| < 0$, we have from (12), (13),

$$M(r) \leqq \exp \left\{ \frac{a \log |a_0|}{16} (1-r) + \frac{2D \log (|a_0|+2)}{1-r} \right\}$$

$$= \exp \left\{ a_1 \log |a_0| (1-r) + \beta_1 \frac{\log (|a_0|+2)}{1-r} \right\} \ (\frac{1}{2} \leqq r < 1) \quad (14)$$

$$a_1 = \frac{a}{16} \ , \qquad \beta_1 = 2D.$$

Since $\alpha_1 < \frac{1}{8}$, we have from (10), (14), if $|a_0| \leqq e^{-8\pi\eta}$,

$$|f(z)| \leqq \exp \left\{ \alpha_1 \log |a_0| (1-r) + \frac{\beta_1 \log (|a_0|+2)}{1-r} \right\} (o \leqq r < 1). \quad (15)$$

If $|a_0| \geqq e^{-8\pi\eta}$, let $\gamma = \frac{8\pi\ \eta\alpha_1}{\log 2}$, then $-8\pi\eta\alpha_1 + \gamma \log 2 = 0$, so that

$$\alpha_1 \log |a_0| + \gamma \log (|a_0|+2) > 0,$$

hence

$$\alpha_1 \log |a_0|(1-r) + \gamma \frac{\log (|a_0|+2)}{1-r} >$$

$$(1-r)\ (\alpha_1 \log |a_0| + \gamma \log (|a_0|+2) > 0. \quad (16)$$

Since by (1),

$$|f(z)| \leqq \exp \frac{D \log (|a_0|+2)}{1-r} \quad (o \leqq r < 1),$$

we have from (16),

$$|f(z)| \leqq \exp \left\{ \alpha_1 \log |a_0|(1-r) + \frac{(D+\gamma) \log (|a_0|+2)}{1-r} \right\}$$

$$(o \leqq r < 1). \quad (17)$$

From (15), (17), if we put $A = \alpha_1$, $B = D + \gamma + \beta_1$,

$$|f(z)| \leqq \exp \left\{ A \log |a_0| (1-r) + B \frac{\log (|a_0|+2)}{1-r} \right\} (o \leqq r < 1),$$

which proves the theorem.

Remark. If we apply our theorem on $\frac{1}{f(z)}$, we have a minorant of $f(z)$:

$$|f(z)| \geqq \exp\left(A \log |a_o| (1-r) - B \frac{\log (\frac{1}{|a_o|}+2)}{1-r} \right).$$

If we change slightly the reasoning, which we have obtained (9), we have the following Valiron's theorem:[2] If $|a_o| \geqq a_o(r, \varepsilon)$, which depends on r and ε $(o < \varepsilon < 1)$, then

$$\log |f(z)| \geqq (1-\varepsilon) \frac{1-r}{1+r} \log |a_o|.$$

Mathematical Institute, Tokyo University.

2) G. Valiron: Le théorème de Picard et le complément de M. Julia. Jour. de Math. (1928).