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On the invariant differential forms of local Lie groups.

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Let $\mathbf{G}$ be an n-dimensional connected compact Lie group and let p_i $(i=0,1,\ldots, n)$ be its Betti numbers. The polynomial $P_G(t)=\sum_{i=0}^{n} p_{n-i}t^i$ is called the *Poincaré polynomial* of $\mathbf{G}$. H. Hopf [1] has proved in 1941 the following remarkable results:

(i) $P_G(t)=(1+t^{m_1})(1+t^{m_2})\cdots(1+t^{m_l})$,

(ii) $m_i\equiv 1 \pmod{2} \quad (i=1,2,\ldots, l)$,

(iii) $n=m_1+\cdots\cdots+m_l$.

On the other hand H. Cartan [1] has called $P_G(t)=\sum_{i=0}^{n} q_{n-i}t^i$ the Poincaré polynomial of $\mathbf{G}$ where q_i is the dimension of the module of all the i-th invariant differntial forms of $\mathbf{G}$ over the field of real numbers. Then $P_G(t)$ is defined also for a local Lie group. By the results of de Rham [1] and Cartan [1] $p_i=q_i$ $(i=0,1,\ldots,n)$ hold for connected compact Lie groups.

In this note we shall prove that the Hopf's results (i), (ii) are also true for the Poincaré polynomials in Cartan's sense if we consider local Lie groups with the property $\mathbf{P}$, formulated in Theorem 1. This property $\mathbf{P}$ concerns with the complete reducibility of some representations of $\mathbf{G}$. Hence every compact Lie groups and every real seml-simple Lie groups have the property $\mathbf{P}$, and we can apply our results for these Lie groups. Our proof rests entirely on the Cartan's local method of differential forms and does not use any topological methods in the large.

1. Let $\mathbf{G}=\{S_a\}$ be a real n-dimensional local Lie group with parameter $a=(a_1,\ldots,a_n)$ in a neighbourhood U^n of $(0,\ldots,0)$ in n-dimensional Euclid space R^n. Denote by $c_i=\varphi^i(a, b)$ $(i=1,\ldots,n)$ the composition function of $\mathbf{G}$: $S_c=S_aS_b$. Let

$$(1)\qquad \omega^p=\sum_{i(1)<\cdots<i(p)} A_{i(1)\ldots i(p)}(x)\, dx_{i(1)}\cdots dx_{i(p)}$$

be a Grassmann-Cartan's differential form of degree p defined on some neighbourhood of $(0,\ldots\ldots,0)$ in R^n. The left (right) translations $x_i\to\bar{x}_i=\varphi_i(a, x)\cdot$ $(x_i\to\bar{\bar{x}}_i=\varphi_i(x, a))$ $(i=1,\ldots,n)$ with parameter a induce the left (right) transformations

(2) $$\omega^p \to \omega^p = S_a\omega_p \qquad (\omega_p \to \omega^p = \omega^p S_a)$$

These transformations satisfy

(3) $$S_a(a\omega_1^p + \beta\omega_2^p) = aS_a\omega_1^a + \beta S_a\omega_2^p, \quad S_a(\omega^p\omega^q) = (S_a\omega^p)(S_a\omega^q),$$
$$S_a(d\omega^p) = d(S_a\omega^p),$$

and the same relations hold for the right transformations. We call a differential form ω^p a *left* (*right*) *invariant differential form* (in short L.I.D.F. or R.I.D.F) if

$$S_a\omega^p = \omega^p \quad (\omega^p S_a = \omega_p)$$

hold for all $S_a \in \mathbf{G}$. A differential form which is both left and right invariant is called an *invariant differential form* (in short I.D.F.). Put

$$a_{ij}(x) = \left(\frac{\partial\varphi_i(x, c)}{\partial c_j}\right)_{c=0}, \quad (\beta_{ij}(x))_{ij=1\cdots n} = (a_{ij}(x))^{-1}_{ij=1\cdots n}$$

and

(4) $$\omega_i = \sum_{j=1}^{n} \beta_{ij}(x)dx_j \qquad (i=1,\ldots n).$$

Then $\omega_1,\ldots\ldots,\omega_n$ are left invariant. The differential form of degree p

$$\omega^p = \sum A_{i(1)\cdots i(p)}\omega_{i(1)}\cdots\cdots\omega_{i(p)} \qquad (A_{i(1)\cdots\cdots i(p)} \text{ are constants})$$

is also left invariant, and conversly any L.I.D.F. is of this type. We can define the structure constants c_i^{jk} $(i, j, k=1,\ldots, n)$ of $\mathbf{G}$ by

(5) $$d\omega_i = \sum_{j<k} c_i^{jk}\,\omega_j\omega_k \qquad (i=1,\ldots, n).$$

Denote by $\mathbf{L}$ the Grassmann-algebra of all L.I.D.F with dimension 2^n over the field k of real numbers, and by $\mathbf{L}^p$ the module of all the L.I.D.F. of degree p with dimension $\binom{p}{n}$ over k. We have $\mathbf{L}=\mathbf{L}^0+\ldots+\mathbf{L}^n$. Put

(6) $$\mathbf{Z}^p = \{\omega^p\,;\ \omega^p \in \mathbf{L}^k,\ d\omega^p=0\}, \quad \mathbf{Z}=\mathbf{Z}^0+\ldots+\mathbf{Z}^n$$

and

(7) $$\mathbf{H}^p = \{\omega^p;\ \omega^p = d\omega_{p-1},\ \omega_{p-1} \in \mathbf{L}_{k-1}\}, \quad \mathbf{H}=\mathbf{H}^0+\ldots+\mathbf{H}^n.$$

By the relations

(8) $$\omega^p\omega^q = (-1)^{pq}\omega^q\omega^p, \quad d\,(a\omega_1^p + \beta\omega_2^p) = a(d\omega_1^p) + \beta(d\omega_2^p),$$
$$d(\omega^p\omega^q) = (d\omega^p)\omega^q + (-1)^p\omega^p(d\omega^q), \quad d(d\omega^p)=0.$$

$\mathbf{Z}$ is a subalgebra of $\mathbf{L}$ and $\mathbf{H}$ is an ideal of $\mathbf{Z}$. The factor algebra

(9) $$\mathbf{B} = \mathbf{Z}/\mathbf{H}$$

is called the *cohomology ring* of $\mathbf{G}$ and the factor modules

(10) $$\mathbf{B}^p = \mathbf{Z}^p/\mathbf{H}^p \qquad (p=0,1,\ldots\ldots, n)$$

are called the p-th *cohomology groups* of **G**.

Now we consider the right transformation by S_a for $\omega^p \in \mathbf{L}^p (p=1,\dots, n)$. It is easy to see that

(11) $$\omega_i S_a = \sum^n d_{ij}(a)\omega_j \qquad (i=1,\dots n)$$

holds for $\omega_1,\dots\dots,\omega_n$ defined by (4), where $d_{ij}(a)$ is defined by

$$(d_{ij}(a)) = (\beta^*_{ij}(a))(\alpha_{ij}(a)), \quad (\beta^*_{ij}(a)) = (\alpha^*_{ij}(a))^{-1}, \quad \alpha^*_{ij} = \left(\frac{\partial\varphi_i(c,a)}{\partial c_j}\right)_{c=0}$$

The correspondence $S_a \to D^{(1)}(a) = (d_{ij}(a))$ satisfies the relation

(12) $$D^{(1)}(c) = D^{(1)}(a)D^{(1)}(b) \qquad \text{for } S_c = S_a S_b.$$

$D^{(1)}(a)$ is called the *adjoint representation* of **G**. Now put

$$u_{i(1)\dots i(p)} = \omega_{i(1)}\dots\dots\omega_{i(p)} \qquad (i(1) < \dots\dots < i(p)).$$

These are the basis of $\mathbf{L}^p$. If we apply the right transformation S_a on $u_{i(1)\dots i(p)}$ we have the representation $D^{(p)}(a)$:

(13) $$u_{i(1)\dots i(p)} S_a = \sum_{j(1)<\dots<j(p)} \begin{vmatrix} d_{i(1)j(1)}(a)\dots\dots d_{i(1)j(p)}(a) \\ d_{i(p)j(1)}(a)\dots\dots d_{i(p)j(p)}(a) \end{vmatrix} u_{j(1)\dots j(p)}$$

with degree $\binom{n}{p}$ $(p=1,\dots\dots,n)$. Especially

(14) $$D^{(n)}(a) = \det |d_{ij}(a)| .$$

Denote by **I** and $\mathbf{I}^p$ $(p=1,\dots, n)$ the modules af all the I.D.F. and the I.D.F. of degree p respectively. $\mathbf{I}^p$ is the submodule of all the L.I.D. F. of $\mathbf{L}^p$ which are also right invariant. **I** is also an algebra.

2. From now on we consider a canonical parameter of **G**. Put

(15) $$\gamma_{ij}(a) = \left(\frac{\partial\, d_{ij}(\lambda a)}{\partial\lambda}\right)_{\lambda=0}$$

Let the expansion of φ_i be $\varphi_i(a,b) = a_i + b_i + \sum_{jk} d_i^{kj} a_k b_j + \dots\dots$ Then we have

$$\alpha_{ij}(a) = \delta_{ij} + \sum_k d_i^{kj} a_k + \dots\dots, \quad \beta^*_{ij}(a) = \delta_{ij} - \sum_k d_i^{jk} a_k + \dots\dots$$

and

$$\gamma_{ij}(a) = \sum_k (d_i^{kj} - d_i^{jk}) a_k = \sum_k a_k c_i^{jk} \qquad (i, j=1,\dots\dots, n).$$

Now we define the infinitesimnl transformation T_a for $\omega_1\dots\dots,\omega_n$ defined in (4) by

(19) $$\omega_i T_a = \sum_{j=1}^n \gamma_{ij}(a)\omega_j = \sum_{kj} a_k c_i^{kj} \omega_j \qquad (i=1,\dots\dots, n)$$

and for $\omega^p = \sum A_{i(1)\dots i(p)}\, \omega_{i(p)}\dots\dots\omega_{i(p)} \in \mathbf{L}_p$ $(A_{i(1)\dots i(p)}$ are constants)

$$\omega^p T_a = \sum A_{i(1),\dots,i(p)} \left(\sum_{j=1}^{p} \omega_{i(1)} \cdots\cdots (\omega_{i(j)} T_a) \cdots\cdots \omega_{i(p)}\right). \tag{17}$$

This infinitesimal transformation T_a may be defined by

$$\omega^p T_a = \lim_{\lambda \to 0} \frac{1}{\lambda}\, \omega^p (S_{\lambda a} - S_0). \tag{18}$$

It is easy to see

$$(\alpha\omega_1^p + \beta\omega_2^p) T_a = \alpha(\omega_1^p T_a) + \beta(\omega_2^p T_a),\ (\omega^p \omega^q) T_a = (\omega^q T_a)\omega^q + \omega^p(\omega^q T_a),$$
$$(d\omega^p) T_a = d(\omega^p T_a) \tag{19}$$

$$\omega^p T_a = \sum_i \lambda_i\, (\omega^p T_{a(i)}) \qquad \text{for } a = \sum_i \lambda_i a(i). \tag{20}$$

It follows from (18)

Lemma 1. *A L. I. D. F. ω^p is invariant if and only if $\omega^p T_a = 0$ for all $S_a \in$ **G**.*

From the relations (5), (8), (16), (17), (18) we can verify

$$d\omega^p = \frac{1}{2} \sum_{j=1}^{n} \omega_j (\omega^p \ T_{e(j)}) \tag{21}$$

for $e(j) = (0,\dots\dots,0,1,0,\dots\dots,0)$. Hence we have the following Cartan's theorem:

Lemma 2. $\qquad \mathbf{I}^p \subset \mathbf{Z}^p \qquad (p, = 0,1,\dots\dots, n).$

Let $\omega^p = \sum_{i(1)<\dots<i(p)} A_{i(1)\dots i(p)} \omega_{i(1)} \cdots \omega_{i(p)} \in \mathbf{Z}^p$. Taking terms in (21) which contain ω_1 we have the relation

$$\omega^p T_{e(1)} = d\omega_1^{p-1}$$

for $\omega_1^{p-1} = \sum_{1<i(2)<\dots<i(p)} A_{1\, i(2)\dots i(p)} \omega_{i(2)} \cdots\cdots \omega_{i(p)}$. Using (20) we have

Lemma 3. *If $\omega^p \in \mathbf{Z}^p$ then $\omega^p T_a \in \mathbf{H}^p$ holds for any $S_a \in$ **G**.*

By these Lemmas we can prove the following theorem:

Theorem 1. *Suppose that an n-dimensional local Lie group **G** satisfies the following property **P**:*

P: *The represemaions $S_a \to D^{(p)}(a)$ $(p = 1,\dots\dots, n)$ of degree $\binom{n}{p}$ of **G** defined by (13) are completely reducible.*

Then $\qquad \mathbf{B}^p \cong \mathbf{L}^p \qquad (p = 1,\dots\dots, n)$

and $\qquad \mathbf{B} \cong \mathbf{L} \qquad$ *(ring-isomorphism)*

hold. Moreover $\mathbf{Z}^p = \mathbf{I}^p + \mathbf{H}^p$ $(p = 1,\dots\dots, n)$ *and* $\mathbf{Z} = \mathbf{I} + \mathbf{H}$ *hold.*

Proof. We can prove this theorem just as in the proof of Theorem 19.1 in Chevalley-Eilenberg [1]. Let $\tilde{\mathbf{L}}^p, \tilde{\mathbf{Z}}^p, \tilde{\mathbf{H}}^p, \tilde{\mathbf{I}}^p$ be the modules derived from $\mathbf{L}^p$, $\mathbf{Z}^p$, $\mathbf{H}^p$, $\mathbf{I}^p$ respectively by the extension of the coefficient field k to the field K of complex numbers. Since $\Omega^p \in \tilde{\mathbf{L}}^p$ is representable as $\Omega^p = \omega_1^p + i\omega_2^p$ ($\omega_1^p, \omega_2^p \in \mathbf{L}^p$) the relations (19), (20), (21) etc. hold also for L. I. D. F. with complex coefficients. By the relations (3) $\tilde{\mathbf{I}}^p, \tilde{\mathbf{Z}}^p, \tilde{\mathbf{H}}^p$ are representation modules for the right transformations $S_a \in \mathbf{G}$. Hence by our assumption **P** we can decompose

$$\tilde{\mathbf{L}}^p = \tilde{\mathbf{N}}^p + \tilde{\mathbf{Z}}^p \quad , \qquad \tilde{\mathbf{Z}}_p = \tilde{\mathbf{M}}^p + \tilde{\mathbf{H}}^p$$

as the direct sum of representation modules.

Since $\tilde{\mathbf{M}}^p$ is a representaion module we have $\Omega^p T_a \in \tilde{\mathbf{M}}_p$ for $\Omega^p \in \tilde{\mathbf{M}}^p$ from the relation (18). On the other hand it follows from Lemma 3 that $\Omega^p T_a = d\Omega^{p-1} \in \tilde{\mathbf{H}}^p$ for $\Omega^p \in \tilde{\mathbf{Z}}^p$. Hence $\Omega^p\, T_a \in \tilde{\mathbf{M}}^p \cap \tilde{\mathbf{H}}^p = 0$ for $\Omega^p \in \tilde{\mathbf{M}}^p$. Thus we have $\tilde{\mathbf{M}}^p \subseteq \tilde{\mathbf{I}}^p$.

Then we shall prove $\tilde{\mathbf{I}}^p \cap \tilde{\mathbf{H}}^p = 0$. Take $\Omega^p \in \tilde{\mathbf{I}}^p \cap \tilde{\mathbf{H}}^p$. By the mapping $\Omega^{p-1} \rightarrow d\Omega^{p-1}$ from $\tilde{\mathbf{L}}^{p-1}$ into $\tilde{\mathbf{L}}^p$ we have $\tilde{\mathbf{N}}^{p-1} \cong \tilde{\mathbf{H}}^p$. Since $\Omega^p = d\,\Omega^{p-1}$, $\Omega^{p-1} \in \tilde{\mathbf{N}}^{p-1}$ for $\Omega^p \in \tilde{\mathbf{H}}^p$, it follows from $\Omega^p \in \tilde{\mathbf{I}}^p$ that $0 = \Omega^p T_a = (d\Omega^{p-1})\, T_a = d(\Omega^{p-1}\, T_a)$, that is, $\Omega^{p-1}\, T_a \in \tilde{\mathbf{Z}}^{p-1}$. On the other hand we have $\Omega^{p-1} T_a \in \tilde{\mathbf{N}}^{p-1}$ and so $\Omega^{p-1}\, T_a \in \tilde{\mathbf{Z}}^{p-1} \cap \tilde{\mathbf{N}}^{p-1} = 0$. Hence $\Omega^{p-1} \in \tilde{\mathbf{I}}^{p-1}$ and $\Omega^p = d\Omega^{p-1} = 0$. Thus we have $\tilde{\mathbf{I}}^p \cap \tilde{\mathbf{H}}^p = 0$.

From $\tilde{\mathbf{M}}^p \subset \tilde{\mathbf{I}}^p$ and $\tilde{\mathbf{I}}^p \cap \tilde{\mathbf{H}}^p = 0$ we have $\tilde{\mathbf{M}}^p = \tilde{\mathbf{I}}^p$ and $\tilde{\mathbf{Z}}^p = \tilde{\mathbf{I}}^p + \tilde{\mathbf{H}}^p$. From this follows also $\mathbf{Z}^p = \mathbf{I}^p + \mathbf{H}^p$ and $\mathbf{Z} = \mathbf{I} + \mathbf{H}$, Q.E.D,

Corollary. *Every residue class of* $\mathbf{Z}^p$ mod. $\mathbf{H}^p$ *contains exactly one I. D.F. of* $\mathbf{I}^p$. We denote this representative I.D.F. by $I\omega^p \in \mathbf{I}^p$. We have then

$$(22) \qquad \begin{aligned} &I(\alpha\omega_1^p + \beta\omega_2^p) = \alpha(I\omega_1^p) + \beta(I\omega_2^p) \quad \text{for } \omega_1^p,\ \omega_2^p \in \mathbf{Z}^p\ . \\ &I\ (\omega^p\, \omega^q) = (I\omega^p)(I\omega^q) \quad \text{for } \omega^p \in \mathbf{Z}^p,\ \omega^q \in \mathbf{Z}^q. \end{aligned}$$

3. Let $\mathbf{G}_1$ and $\mathbf{G}_2$ be two local Lie groups. Let $\mathbf{I}_1$ and $\mathbf{I}_2$ be their algebra of all I.D.F. and

$$\mathbf{I}_1 = \sum_{i=1}^{M} k\ \omega_i \quad , \qquad \mathbf{I}_2 = \sum_{j=1}^{N} k\ \theta_j$$

where ω_i, θ_j are homogeneous differential forms. Now we define formal product of $\mathbf{I}_1$ and $\mathbf{I}_2$ by

$$\mathbf{I}_1 \times \mathbf{I}_2 = \sum_i \sum_j \ k(\omega_i \cdot \theta_j).$$

We assume here that $\omega_i \cdot \theta_j$ $(i=1,\ldots\ldots, M; j=1,\ldots,N)$ are linearly independent,

$$(23) \qquad (\omega_i \cdot \theta_j)(\omega_k \cdot \theta_l) = (-1)^{\alpha\beta}(\omega_i\ \omega_k)\cdot(\theta_i\ \theta_l),\ \alpha = \deg\,\omega_k, \beta = \deg\,\theta_j,$$

and the distributive law for multiplication. We can easily prove the following Lemma;

Lemma 4. *The algebra* **I.** *of all the I.D.F. of the direct product local Lie groups* $\mathbf{G}=\mathbf{G}_1\times\mathbf{G}_2$ *is isomorphic with* $\mathbf{I}_1\times\mathbf{I}_2$.

Now take $\omega_1=\theta_1=1$ and we write

$$\text{(24)}\qquad \mathbf{I}\ni\Omega=\sum_{ij} a_{ij}\ \omega_i\cdot\theta_j=\Pi(\Omega)+\Lambda(\Omega)+\sum_{i>1}\sum_{j>1}a_{ij}\ \omega_i\cdot\theta_j\ -a_{11}$$

where

$$\text{(25)}\qquad \Pi(\Omega)=\sum_{i=1}^{M} a_{i1}\ \omega_i,\quad \Lambda\ (\Omega)=\sum_{j=1}^{N} a_{1j}\ \theta_j$$

It is easy to see that the mappings $\Omega\to\Pi(\Omega)$ and $\Omega\to\Lambda(\Omega)$ from **I.** into $\mathbf{I}_1$ and $\mathbf{I}_2$ respectively are ring-homomorhic. We note here the following fact. If Ω is a homogeneous differential form of degree k, then $a_{ij}\neq 0$ in (24) only for $k=\deg\ \omega_i+\deg\ \theta_j$, so that $0<\deg\ \omega_i<k$ and $0<\deg\ \theta_i<k$ for $(i>1, j>1)$.

Denote $b_i=\mu_i(a)$ $(i=1,\ldots,n)$ for $S_a^{-1}=S_b$ and put

$$\text{(26)}\qquad \psi_i(a,b)=\varphi_i(a,\ \mu(b))\quad(i=1,\ldots,n).$$

Now we consider the mapping $S_z=S_xS_y^{-1}$, namely

$$(x_1,\ldots\ldots,x_n,y_1,\ldots\ldots,y_n)\longrightarrow(\psi_1(x,\ y),\ldots\ldots,\psi_n(x,\ y))$$

from $\mathbf{G}\times\mathbf{G}$ into $\mathbf{G}$. This mapping ψ induces the inverse transformation for the differential form $\omega_p=\sum A_{i(1)\ldots i(1)}(z)dz_{i(1)}\ldots\ldots dz_{i(p)}$ on **G** to the differential form $\Psi(\omega^p)$ on $\mathbf{G}\times\mathbf{G}$:

$$\text{(27)}\qquad \Psi(\omega^p)=\sum A_{i(1)\ldots i(p)}(\psi(x,\ y))\Big(\sum\frac{\partial\psi_{i(1)}}{\partial x_{j(1)}}dx_{j(1)}+\sum\frac{\partial\psi_{i(1)}}{\partial y_{k(1)}}\ dy_{k(1)}\Big)\cdots$$
$$\Big(\sum\frac{\partial\psi_{i(p)}}{\partial x_{j(p)}}\ dx_{j(p)}+\sum\frac{\partial\psi_{i(p)}}{\partial y_{k(p)}}\ dy_{k(p)}\Big)$$

Denote by $S_a^{(1)}$ and $S_b^{(2)}$ the left transformations to the first and second component of $S_x\times S_y\in\mathbf{G}\times\mathbf{G}$, namely

$$\text{(28)}\qquad S_a^{(1)}\cdot(S_x\times S_y)=(S_aS_x)\times S_y,\quad S_b^{(2)}\cdot(S_x\times S_y)=S_x\times(S_bS_y).$$

Then it holds that

$$\text{(29)}\qquad S_a^{(1)}\Psi(\omega^p)=\Psi(S_a\omega_p),\quad S_b^{(2)}\Psi(\omega^p)=\Psi(\omega^pS_b^{-1})$$

Hence $\Psi(\omega^p)$ is a L.I.D F. on $\mathbf{G}\times\mathbf{G}$ if ω^p is a I.D.F. on **G**. Moreover, from $d\omega^p=0$ follows $d\Psi(\omega^p)=0$.

Let $\omega^p\to\rho(\omega^p)$ of **I** onto itself be the isomorphic transformation induced on **I** by the mapping $x\to x^{-1}$ of **G**. Then we have

$$\sum A_{i(1)\dots i(p)}(\psi(x,y))\Big(\sum\frac{\partial\psi_{i(1)}}{\partial x_{j(1)}}dx_{j(1)}\Big)\cdots\Big(\sum\frac{\partial\psi_{i(p)}}{\partial x_{j(p)}}dx_{j(p)}\Big)=\omega^p(x,dx)S_y^{-1}$$
$$=\omega^p(x,dx)$$
$$\sum A_{i(1)\dots i(p)}(\psi(x,y))\Big(\sum\frac{\partial\psi_{i(1)}}{\partial y_{k(1)}}dy_{k(1)}\Big)\cdots\Big(\sum\frac{\partial\psi_{i(p)}}{\partial y_{k(p)}}dy_{k(p)}\Big)=S_x\rho(\omega^p(y,dy))$$
$$=\rho(\omega^p(y,dy)S_x^{-1})=\rho(\omega^p(y,dy)).$$

Applying these relations it follows from (27) that

$$\Psi(\omega^p)=\omega^p(x,dx)+\rho(\omega^p(y,dy))+\sum_i\omega_{1i}(x,dx)\cdot\omega_{2i}(y,dy)$$

where $\omega_{1i}(x, kx)$, $\omega_{2i}(y, dy)$ are homogeneous differential forms with dimension $<p$. We consider here the mapping I defined in the Corollary of Theorem 1. Then we see that

$$I(\Psi(\omega^p))=I\omega^p(x,dx)+I\rho(\omega^p(y,dy))+\sum_i I\omega_{1i}(x,dx)\cdot I\omega_{2i}(y,dy)$$
$$=\omega^p(x,dx)+\rho(\omega^p(y,dy))+\sum_i\Omega_{1i}(y,dy)\cdot\Omega_{2i}(y,dy)$$

where Ω_{1i}, Ω_{2i} are homogeneous I.D.F. of degree $<p$. We state these facts in the following Lemma:

Lemma 5. *Let* **I** *and* **I*** *be the algebra of all the I.D.F. of* **G** *and* **G** × **G** *respectively. Then there exists a ring-homomorphic mapping* $\omega^p\to\Phi(\omega^p)=I\Psi(\omega^p)$ *from* **I** *into* **I*** *such that*

(30) $$\Phi(\omega^p)=\omega^p(x,dx)+\rho(\omega^p(y,dy))+\sum\Omega_{1i}(x,dx)\cdot\Omega_{2i}(y,dy)$$

where Ω_{1i}, Ω_{2i} *are homogeneous I.D.F. of degree* $<p$.

By means of this Lemma we can prove the following theorem just as in the proof of the corresponding theorem in Hopf [1].

Theorem 2. *Let* **G** *be an n-dimensional local Lie group with the property* **P** *in Theorem* 1. *Let* q_i *be the dimension of the module* $\mathbf{I}^i$ *of all the I.D.F. of degree i of* **G** *over k, and put*

(31) $$P(t)=\sum_{i=0}^{n}q_{n-i}\,t^i.$$

Then the following relations hold:

(i) $$P_G(t)=(1+t^{m_1})(1+t^{m_2})\dots\dots(1+t^{m_l})$$

(ii) $$m_i\equiv 1 \pmod 2 \qquad (i=1,\dots\dots,l).$$

Moreover, the algebra **I** *of all the I.D.F. of* **G** *has l homogeneous I.D.F.* $\theta_1, \theta_2,\dots\dots,\theta_l$ *with degree* $m_1,\dots\dots, m_l$ *respectively such that*

$$1,\ \theta_i,\ \theta_i\theta_j(i<j),\ \ \theta_i\theta_j\theta_k\ \ (i<j<k),\ldots\ldots,\ \theta_1\theta_2\ldots\theta_l$$

are the basis of **I** *over k.*

If the adjoint representation of **G** *satisfies the condition:*

$$\text{(32)} \qquad \det|d_{ij}(a)|=1$$

for every element $S_a \in$ **G**, *then*

$$\text{(iii)} \qquad n=m_1+\cdots\cdots+m_l$$

holds.

For the completeness we shall sketch the proof. Let a system of homogenous I.D.F. $\{\theta_1,\ldots\ldots,\theta_l\}$ be a generating system of the algebra **I.** We assume that this system is irreducible, namely any proper subest of it is not a generating system of **I.** For the proof of (i), (ii) it is sufficient to see that

$$\text{(a)} \qquad \theta_2\theta_2\cdots\cdots\theta_l\neq 0$$

$$\text{(b)} \qquad m_i=\deg\ \theta_i\equiv 1\ (\text{mod. } 2) \quad (i=1,\ldots\ldots,\ l).$$

We shall prove (a) by induction. Namaly we shall assume that $\theta_2\theta_3\cdots\cdots\theta_k\neq 0$ $(k\leq 1)$ and $\deg\ \theta_1\geq\deg\ \theta_i$ $(i=2,\ldots\ldots,k)$. Let $\mathbf{U}=(\theta_2,\ldots\ldots,\theta_l)$ be the ideal of **I** generated by $\theta_2,\ldots\ldots,\theta_l$. Then $\mathbf{U}^*=\{\sum_h\omega_h(x,dx)\cdot\varphi_h(y,dy)\ ;\ \omega_h\in\mathbf{U}\}$ is an ideal of $\mathbf{I}^*=\mathbf{I}\times\mathbf{I}$. From (30) follows.

$$\Phi(\theta_1)\equiv\theta_1(x,\ dy)+\rho\theta_1(y,\ dy) \quad (\text{mod. } \mathbf{U}^*)$$

$$\Phi(\theta_i)\equiv \qquad \rho\theta_i(y,\ dy) \quad (\text{mod. } \mathbf{U}^*) \quad (i=2,\ldots\ldots,k).$$

Since Φ is a ring homomorphism of **I** into **I** we have

$$\Phi(\theta_1\ldots\theta_k)\equiv\theta_1(x,dx)\cdot\rho(\theta_2(y,dy)\ldots\theta_k(y,dy))$$
$$+\rho(\theta_1(y,dy)\cdots\cdots\theta_k(y,dy)) \qquad (\text{mod. } \mathbf{U}^*).$$

If $\theta_1\cdots\cdots\theta_k=0$, then we would have

$$\theta_1(x,dx)\cdot\rho(\theta_2(y,dy)\cdots\cdots\theta_k(y,dy)) \in \mathbf{U}^*$$

and so $\theta_1(x,dx)\in\mathbf{U}=(\theta_2,\ldots\ldots,\theta_l)$; which is a contradiction. Hence we have $\theta_1\cdots\cdots\theta_l\neq 0$.

We can prove (b) analogously by means of Lemma 5.

If the condition (32) holds, then **I** contains an I.D.F. $\Omega=\omega_1\cdots\cdots\omega_n$ with degree n. Since the term of **I** with the highest degree is $\theta_1\cdots\cdots\theta_l$, we have $\Omega=\theta_1\cdots\cdots\theta_l$. Therefore,

$$n=\deg\ \Omega=\deg\ \theta_1+\cdots\cdots+\deg\ \theta_l=m_1+\cdots\cdots+m_l\ ,\ \text{q.e.d.}$$

Corollary 1. *Let* **G** *be an n-dimensional real semi-simple local Lie group.*

Then the Poincaré polynomial (31) *of* **G** *satisfies the relations* (i), (ii), (iii) *in Theorem* 2.

From the well known results of de Rham and Cartan we have alao

Corollary 2. *Let* **G** *be an n-dimensional connected compact Lie group. Then the Poincaré polynomial* $P_G(t)=\sum_{i=0}^{n} p_{n-i}\, t^i$ *of* **G** *satisfies the relations* (i), (ii), (iii).

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Literature

E. Cartan [1] Sur les invariants intégraux de certains espaces homogènes clos et propriétés topologiques de ces espaces, Annales de la polonaise de math., **8** (1929).

C. Chevalley and S. Eilenberg [1] Cohomology theory of Lie groups and Lie algebras, Trans. Amer. Math. Soc., **63** (1948).

H. Hopf [1] Ueber die Topologie der Gruppenmannigfaltigkeiten und ihre Verallgemeinerungen, Annals of Math., 42 (1941).

G. de Rham [1] Sur l'analysis situs des variétés á *n* dimensions, Jour. de Math. pures et appl. **10** (1931).