Journal of the Mathematical Society of Japan Vol. 1, No. 2, July, 1949

Notes on Fourier Analysis (X).
On the summability of Fourier series.

Noboru MATSUYAMA.

(Received Dec. 10, 1947)

1. Let $\phi(x)$ be an L-integrable and periodic function with period 2π. For any $k \geqq 0$ and any $\alpha \geqq 0$ we define $\Psi_k(x)$ and $\Psi_k^\alpha(x)$ the formula:

$$\Psi_k(x) = \frac{1}{\Gamma(k)}\int_x^\pi \left(\log\frac{u}{x}\right)^{k-1}\phi(u)\frac{du}{u},$$

$$\Psi_0(x) = \phi(x)$$

and

$$\Psi_k^\alpha(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-v)^{\alpha-1}\Psi_k(v)\,dv,$$

$$\Psi_k^0(x) = \Psi_k(x).$$

If $\Psi_k^\alpha(x)/x^\alpha(\log 1/x)^k = o(1)$ as $x\to 0$, we say that $\phi(x)$ is (α, k)-continuous at $x=0$.

Let $\sum a_n$ be a given series and $A(u) \equiv \sum_{\nu<u} a_\nu$ be its partial sum. For any $k \geqq 0$ we define Riesz's sum $R_k(\omega)$ of order k by

$$R_k(\omega) = \sum_{n<\omega}\left(\log\frac{\omega}{n}\right)^k a_n = \frac{1}{\Gamma(k)}\int_1^\omega \left(\log\frac{\omega}{u}\right)^{k-1}\frac{A(u)}{n}du,$$

$$R_0(\omega) = A(\omega)$$

and for any $\alpha \geqq 0$

$$R_k^\alpha(\omega) = \frac{1}{\Gamma(\alpha)}\int_0^\omega (\omega - v)^{\alpha-1}R_k(v)\,dv,$$

$$R_k^0(\omega) = R_k(\omega).$$

If

$$R_k^\alpha(\omega)/\omega^\alpha(\log\omega)^k \to s \text{ as } \omega\to\infty,$$

then we say that $\sum a_n$ be (α, k)-summable to the sum s and denote it by

$$\sum a_n = s(\alpha, k).$$

Let $\phi(x)$ be an even periodic function with period 2π, and its Fourier series be

$$\phi \sim \frac{1}{2} a_0 + \sum_{n=1}^{n} a_n \cos nx.$$

Let us consider the (α, k)-summability of Fourier series of $\phi(t)$ at x. For the sake of simplicity, we suppose that $x=0$, $s=0$ and $\phi(0)=0$. This summability has been already treated by Hardy[1], Kawata-Wang[2] and Bosanquet-Offord[3] Especially Bosanquet and Offord proved the following theorem :

If
$$|\phi(t)| = O\left(\log \frac{1}{t}\right) \quad (C,\ 1)$$

as $t \to 0$, then a necessary and sufficient condition that for any $\delta > -1$ $\mathfrak{S}[\phi]$ is $(\delta, 1)$-summable, is that for some β; $\phi(x)$ is $(\beta, 1)$-continuous.

As an extension of this theorem, we prove the following

Theorem. *If for $k \geqq 1$*

$$\Psi_{k-1}(t) = O\left(\log \frac{1}{t}\right)^k \quad (C,\ 1)$$

as $t \to 0$, then the necessary and sufficient condition that $\mathfrak{S}[\phi]$ is (α, k)-summable for any $\alpha > -1$, is that for some β, $\phi(x)$ is (β, k)-continuous.

2. Let S_n^α be the n-th Cesàro sum of the series $\sum a_n$ of order α. Concerning the relation between S_n^α and $R_k^\beta(\omega)$ we have

Lemma 1. *For any positive integer k and α such as $\alpha - k > -1$,*

$$S_n^\alpha = A_0 R_k^\alpha(n) + A_1 n R_k^{\alpha-1}(n) + \dots\dots + A_k n^k R_k^{\alpha-k}(n),$$

where $A_i (i=0, 1, \dots\dots, k)$ depends only on α and k.

Proof. By the definition

$$S_n^\alpha = \frac{1}{\Gamma(\alpha)} \int_0^n (n-v)^{\alpha-1} s(v)\, dv$$

where $s(v) = \sum_{n<v} a_n$. Consequently

$$\Gamma(\alpha) S_n^\alpha = \Big[R_1(v)\, v(n-v)^{\alpha-1} \Big]_0^n - \int_0^n R_1(v) \frac{d}{dv} \{ v(n-v)^{\alpha-1} \} dv$$

$$= -\alpha \int_0^n R_1(v)(n-v)^{\alpha-1} dv + (\alpha-1) n \int_0^n R_1(v)(n-v)^{\alpha-2} dv$$

$$= -\alpha \Gamma(\alpha) R_1^\alpha(n) + (\alpha-1) \Gamma(\alpha-1) n R_1^{\alpha-1}(n),$$

Dividing by $\Gamma(\alpha)$, we get

$$S_n^\alpha = -\alpha R_1^\alpha(n) + nR_1^{\alpha-1}(n).$$

Repeating this process we get

$$S_n^\alpha = A_0 R_k^\alpha(n) + A_1 n R_k^{\alpha-1}(n) + \dots\dots + A_{k-1} n^{k-1} R_k^{\alpha-k+1}(n)$$
$$+ A_k' n^k \int_0^n R_k(v)(n-v)^{\alpha-k-1} dv,$$

where the last term is equal to

$$\frac{A_k'}{\alpha-k} n^k \frac{d}{dn} \int_0^n R_k(v)(n-v)^{\alpha-k} dv$$

$$= \frac{A_k'}{\alpha-k} \Gamma(\alpha-k+1) n^k \frac{d}{dn} R_k^{\alpha-k+1}(n) = A_k n^k R_k^{\alpha-k}(n).$$

Consequently, for $\alpha-k > -1$,

$$S_n^\alpha = A_0 R_k^\alpha(n) + A_1 n R_k^{\alpha-1}(n) + \dots\dots + A_k n^k R_k^{\alpha-k}(n).$$

Thus the lemma is proved.

As the immediate corollary of Lemma 1 we get

Lemma 2. *For any positive integer k and* $\alpha > -1$, $\sum a_n = 0(\alpha, k)$ *implies* $\sum a_n = o(\log n)^k(\alpha+k, 0)$.

On the other hand the following theorem is known:

Theorem (Kawata-Wang[2]). *For any* $\alpha, \beta > 0$, $\sum a_n = 0(\alpha, \beta)$ *implies* $\sum a_n = 0(0, \alpha+\beta)$.

3. For any $k \geqq 1$ the k-th Riesz sum $R_k(\omega)$ of Fourier series $\mathfrak{S}[\phi]$ is represented by

$$R_k(\phi, \omega) = R_k(\omega) = \frac{2}{\pi} \int_0^\pi \Psi_k(t) \frac{\sin \omega t}{t} dt + o(\log \omega)^k.$$

That is

$$R_k(\omega) = \frac{2}{\pi} \int_0^\pi \frac{\sin\left(\omega + \frac{1}{2}\right)t}{2 \sin t/2} \Psi_k(t) dt + o(\log \omega)^k + o(1)$$
$$= S_\omega(\Psi_k) + o(\log \omega)^k.$$

In particular,

$$R_1(\Psi_k, \omega) = S_\omega((\Psi_k)_1) + o(\log \omega) = S_\omega(\Psi_{k+1}) + o(\log \omega),$$

(1) $$R_1(\Psi_k, \omega) = R_{k+1}(\Phi, \omega) + o(\log \omega)^{k+1}.$$

As the consequence of Lemma 1 and (1) we have

$$\begin{aligned} S^{\alpha}_{\omega}(\Psi_{k-1}) &= A_0 R^{\alpha}_1(\Psi_{k-1}, \omega) + A_1 \omega R^{\alpha-1}_1(\Psi_{k-1}, \omega) \\ &= A_0\{R^{\alpha}_k(\phi, \omega) + o(\omega^{\alpha} \log^k \omega)\} + A_1\omega\{R^{\alpha-1}_k(\phi, \omega) \\ &\qquad + o(\omega^{\alpha-1} \log^k \omega)\} \\ &= A_0 R^{\alpha}_k(\phi, \omega) + A_1 \omega R^{\alpha-1}_k(\phi, \omega) + o(\omega^{\alpha} \log^k \omega). \end{aligned}$$

Thus we get

$$S^{\alpha}_{\omega}(\Psi_{k-1}) = A_0 R^{\alpha}_k(\phi, \omega) + A_1 \omega R^{\alpha-1}_k(\phi, n) + o(\omega^{\alpha} \log^k \omega),$$

provided that $\alpha > 0$ and $k \geqq 1$.

Lemma 3. *If $\alpha > 0$ and $k \geqq 1$, then the necessary and sufficient condition that $\mathfrak{S}[\phi]$ is $(\alpha-1, k)$-summable, is that*

1°. $S^{\alpha}_{\omega}(\Psi_{k-1}) = o(\omega^{\alpha}(\log \omega)^k)$.

2°. *There exists β such that, $\mathfrak{S}[\phi]$ is (β, k)-summable.*

Proof. Necessity is trivial. If $\beta > \alpha$, then from (2)

$$\frac{S^{\beta}_{\omega}(\Psi_{k-1})}{\omega^{\beta}(\log \omega)^k} = A_0 \frac{R^{\beta}_k(\phi, \omega)}{\omega^{\beta}(\log \omega)^k} + A_1 \frac{R^{\beta-1}_k(\phi, \omega)}{\omega^{\beta-1}(\log \omega)^k} + o(\omega^{\alpha-\beta})$$

and Lemma 1 $\mathfrak{S}[\phi]$ is $(\beta-1, k)$-summable. If $h = [\beta-\alpha]+1$, we have easily $(\beta-h, k)$-summability of $\mathfrak{S}[\phi]$ and then (α, k)-summability of $\mathfrak{S}[\phi]$. On the other hand if $\beta \leqq \alpha$, then the lemma is evident.

Lemma 4. *For any $\alpha > 0$*

$$\Psi^{\alpha+1}_{k-1}(t) = B_0 \Psi^{\alpha+1}_k(t) + B_1 t \Psi^{\alpha}_k(t),$$

where B_0 and B_1 are independent from t.

Proof. By the definition

$$\Psi_k(t) = \int_t^{\pi} \Psi_{k-1}(u) \frac{du}{u},$$

$$\begin{aligned} \Psi^{\alpha+1}_{k-1}(t) &= \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-u)^{\alpha} \Psi_{k-1}(u)\, du \\ &= \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-u)^{\alpha} u \Psi_k'(u)\, du \end{aligned}$$

and then

$$\Psi_{k-1}^{\alpha+1}(t) = -\frac{1}{\Gamma(\alpha+1)}\int_0^t \{t(t-u)^\alpha - (t-u)^{\alpha+1}\}\Psi_k'(u)du$$
$$= t\Psi_k^\alpha(t) - (\alpha+1)\Psi_k^{\alpha+1}(t).$$

Lemma 5. *For any non-negative k and $\alpha \geqq 0$,*

$$\phi(t) = o\left(\log\frac{1}{t}\right)^k \quad (C,\ \alpha),$$

implies

$$S_n(t) = o(\log n)^k \quad (C,\ \alpha+\delta)$$

for any $\delta > 0$.

This lemma is due to Kawata and Wang[2].

Lemma 6. *If $\mathfrak{S}[\phi]$ is (C, k)-summable, then $\phi(t)$ is $(2, k)$-continuous.*

This is due to Hardy[1].

4. Proof of Theorem. Necessity. We have already seen that

$$R_k(\phi,\ \omega) = S_\omega(\Psi_k) + o(\log \omega)^k$$

and

$$R_k^\alpha(\phi,\ \omega) = S_\omega^\alpha(\Psi_k) + o(\omega^\alpha(\log \omega)^k).$$

By the hypothesis $\mathfrak{S}[\phi]$ is (C, k)-summable, and then

$$S_\omega(\Psi_k) = o(\log \omega)^k$$

as $\omega \to \infty$. Consequently by Lemma 6 $\phi(t)$ is $(2, k)$-continuous.

Sufficiency. Since for some β_0 $\phi(t)$ is (β_0, k)-continuous

$$\Psi_k^{\beta_0}(t) = o\left(t^{\beta_0}\left(\log\frac{1}{t}\right)^k\right)$$

as $t \to 0$. By Lemma 5

$$(3) \qquad S_\omega(\Psi_k) = o(\log \omega)^k \quad (C,\ \beta_0+\delta).$$

That is $\mathfrak{S}[\phi]$ is $(\beta_0+\delta, k)$-summable for any $\delta > 0$.

On the other hand, by the condition that $\phi(t)$ is (β_0, k)-continuous, $\phi(t)$ is (β_0+1, k)-continuous, that is,

$$(4) \qquad \Psi_k^{\beta_0+1}(t) = o\left(t^{\beta_0+1}\left(\log\frac{1}{t}\right)^k\right).$$

From Lemma 4 and 3 we get

(5) $$\Psi_{k-1}^{\beta_0+1}(t)=o\left(t^{\beta_0+1}\left(\log\frac{1}{t}\right)^k\right).$$

By Lemma 5 and (5) we have

(6) $$S_\omega(\Psi_{k-1})=o(\log\omega)^k \quad (C, \beta_0+1+\delta)$$

for any $\delta>0$. We can now easily prove that

$$\Psi_{k-1}(t)=O\left(\log\frac{1}{t}\right)^k \quad (C, 1)$$

implies

$$S_\omega(\Psi_{k-1})=O(\log\omega)^k \quad (C, \varepsilon)$$

for any $\varepsilon>0$. (6), (7) and the Andersen's theorem give us

(8) $$S_\omega(\Psi_{k-1})=o(\log\omega)^k \quad (C, \varepsilon).$$

Since (8) and (3) satisfy the condition of Lemma 3, $\mathfrak{S}[\phi]$ is $(\varepsilon-1, k)$-summable for any $\varepsilon>0$. Thus the theorem is completely proved.

References.

1) Hardy. Quart. Journ., (2) 1 (1936).
2) Kawata-Wang, Tohoku Math. Journ., 45 (1935).
3) Bosanquet-Offord, Compositio Math., 1 (1934).

Mathematical Institute
Tôhoku University.