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On the characterisation of the normal population by the independence of the sample mean and the sample variance.

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1. Let $X_1, X_2, \ldots\ldots, X_n$ $(n \geqq 2)$ be the sample variables from a certain population, that is, let X_i $(i=1, 2, \ldots\ldots, n)$ be independent random variables having same distribution $F(x)$. In the mathematical statistics, the following fact is well known and is of fundamental importance in the theory of exact sampling.

If $F(x)$ is the normal distribution function, then the two statistics

$$(1\cdot1) \qquad \bar{X}=\frac{1}{n}\sum_{i=1}^{n} X_i$$

and

$$(1\cdot2) \qquad S=\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2$$

are statistically independent.

R. C. Geary(1) has proved the converse of this theorem and given the characterisation of the normal population by using the formulae(2) due to Fisher for relations between semi-invariants of various algebraic forms of sample variables. The object of the present paper is to give another proof, under the more general conditions assuming nothing about the moments of X_i, while Geary has supposed the existence of moments of every order.

2. We restate the theorem.

Theorem. Let $X_1, X_2, \ldots\ldots, X_n$ $(n \geqq 2)$ be the independent random variables whose distributions are equal to the same $F(x)$. If two random variables $Y=\sum_{i=1}^{n} X_i$, $Z=\sum_{i=1}^{n}(X_i-\bar{X})^2$ $(\bar{X}=\frac{1}{n}\sum^{n} X_i)$ are independently distributed,

(1) R. C. Geary, The distribution of "Student's" ratio for non-normal samples. Journ. Royal Statist. Soc., Supplement 3 (1936).

(2) R. A. Fisher, Moments and product moments of sampling distributions. Proc. London Math. Soc. 30 (1929).

then $F(x)$ must be the normal distribution function, excluding the unit distribution.

We consider the characteristic function[3] of the simultaneous variable (X_i, X_i^2)

$$(2\cdot1) \qquad f(t, s)=\int_{-\infty}^{\infty} e^{itx+isx^2}dF(x),$$

where t is a real number but we consider s as a complex number, $s=\sigma+i\tau$, $\tau>0$. $f(t, s)$ is obviously an analytic function of s regular in the upper half -plane $\tau>0$. Since $X_i(i=1, 2, \ldots\ldots, n)$ are independent variables, the characteristic function of variable (Y, Σ), Σ being $\sum_{i=1}^{n} X_i^2$, is $\{f(t, s)\}^n$ which noticing that $\Sigma\geqq0$, can also be written as

$$(2\cdot2) \qquad \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{it\eta+is\theta}dF(\eta, \theta),$$

where $F(\eta, \theta)$ is the distribution function of (Y, Σ).

Since $Z+\frac{1}{n}Y^2=\Sigma$, denoting the distribution of (Y, Z) as $G(\eta, \zeta)$, we have further

$$(2\cdot3) \qquad \{f(t, s)\}^n=\int_{-\infty}^{\infty}\int_{0}^{\infty} e^{it\eta+is(\frac{\eta^2}{n}+\zeta)}dG(\eta, \zeta).$$

The statistical independence of Y and Z shows that

$$(2\cdot4) \qquad dG(\eta, \zeta)=dG_1(\eta)dG_2(\zeta),$$

$G_1(\eta)$ and $G_2(\zeta)$ being the distribution function of Y and Z respectively. Hence we can write $(2\cdot3)$ as

$$(2\cdot5) \qquad \{f(t, s)\}^n=\int_{-\infty}^{\infty} e^{it\eta+i\frac{s}{n}\eta^2}dG_1(\eta)\cdot\int_{0}^{\infty} e^{is\zeta}dG_2(\zeta).$$

Now we observe that, putting $\phi\left(t, \frac{s}{n}\right)=\int_{-\infty}^{\infty} e^{it\eta+i\frac{s}{n}\eta^2}dG_1(\eta)$ and $u(s)=u_n(s)=\int_{0}^{\infty} e^{is\zeta}dG_2(\zeta)$,

(3) In the ordinary sense of the characteristic function, t, s are real, but in this paper, we use the same terminology in the case where s is complex.

$$(2\cdot6)\qquad -i\frac{\partial^2}{\partial t^2}f(t,s)=\frac{\partial}{\partial s}f(t,s);\quad \tau>0,$$

$$(2\cdot7)\qquad -\frac{i}{n}\frac{\partial^2}{\partial t^2}\psi\left(t,\frac{s}{n}\right)=\frac{\partial}{\partial s}\psi\left(t,\frac{s}{n}\right),\quad \tau>0,$$

and

$$(2\cdot8)\qquad -ia'(s)\geqq 0\quad \text{for}\quad \sigma=0,\ \tau>0.$$

We differentiate both sides of (2·5) with respect to s, we have

$$n\{f(t,s)\}^{n-1}\frac{\partial}{\partial s}f(t,s)=a(s)\frac{\partial}{\partial s}\psi\left(t,\frac{s}{n}\right)+\psi\left(t,\frac{s}{n}\right)a'(s).^{(4)}$$

which becomes, by (2·6) and (2·7),

$$(2\cdot9)\qquad n\{f(t,s)\}^{n-1}\frac{\partial^2}{\partial t^2}f(t,s)=\frac{1}{n}a(s)\frac{\partial^2}{\partial t^2}\psi\left(t,\frac{s}{n}\right)+i\psi\left(t,\frac{s}{n}\right)a'(s).$$

In differentiating two times (2·5) with respect to t we get

$$(2\cdot10)\qquad n\{f(t,s)\}^{n-1}\frac{\partial^2}{\partial t^2}f(t,s)+n(n-1)\{f(t,s)\}^{n-1}\left\{\frac{\partial}{\partial t}f(t,s)\right\}^2$$
$$=a(s)\frac{\partial^2}{\partial t^2}\psi\left(t,\frac{s}{n}\right).$$

The elimination of $a(s)\frac{\partial^2}{\partial t^2}\psi\left(t,\frac{s}{n}\right)$ from (2·9) and (2·10) gives

$$\{f(t,s)\}^{n-1}\frac{\partial^2}{\partial t^2}f(t,s)-\{f(t,s)\}^{n-1}\left\{\frac{\partial}{\partial t}f(t,s)^2\right\}=i\frac{a'}{n-1}\psi\left(t,\frac{s}{n}\right).$$

By (2·5), this becomes further

$$(2\cdot11)\qquad \{f(t,s)\}^{n-1}\frac{\partial^2}{\partial t^2}f(t,s)-\{f(t,s)\}^{n-2}\left\{\frac{\partial}{\partial t}f(t,s)\right\}^2$$
$$=i\{f(t,s)\}^{n}\frac{1}{n-1}\frac{a'(s)}{a(s)}.$$

From this equation we can easily prove that in the t-interval, for fixed s, such that $f(t,s)\neq 0$,

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(4) The dash in $a'(s)$ means the differentiation with respect to s.

$$(2\cdot12)\qquad f(t, s)=\exp\left[\frac{i}{n-1}\frac{\alpha'(s)}{\alpha(s)}\left\{\frac{t^2}{2}+C(s)t+D(s)\right\}\right].$$

But since $f(t, s)$ is a continuous function of t and the right side of (2·12) has no zeros as a function of t, we see that (2·12) holds for all values of t.

Now we take $\sigma=0$, and thus $s=i\tau$. Then it holds that

$$(2\cdot13)\qquad \lim_{\tau\to+0} f(t, i\tau)=\int_{-\infty}^{\infty} e^{itx}\,dF(x),$$

for every t, since

$$\left|\int_{-\infty}^{\infty} e^{itx}dF(x)-\int_{-\infty}^{\infty} e^{itx-\tau x^2}dF(x)\right|\leqq\left|\int_{-A}^{A} e^{itx}(1-e^{-\tau x^2})dF(x)\right|$$

$$+\left|\int_{|x|>A} e^{itx}dF(x)\right|+\left|\int_{|x|>A} e^{itx-\tau x^2}dF(x)\right|\leqq \tau A^2\int_{-A}^{A}dF(x)+2\int_{|x|>A}dF(x)<\varepsilon,$$

if we take A such that $2\int_{|x|>A} dF(x)<\frac{\varepsilon}{2}$ and then take τ so small that

$$\tau A^2\int_{-A}^{A}dF(x)<\frac{\varepsilon}{2}.$$

Now if we take $t=0$ in (1·12) and let τ tend to zero, then by (2·13) $f(0, i\tau)\to1$, and hence

$$\lim_{\tau\to0}\frac{\alpha'(i\tau)}{\alpha(i\tau)}D(i\tau)=0.$$

Next noticing the existence of $\lim_{\tau\to0} f(t, i\tau)f(-t, i\tau)$, $(t\neq0)$, we can show the existence of $\lim_{\tau\to0}\alpha'(i\tau)$. And hence we also get the existence of $\lim_{\tau\to0} C(i\tau)$.

Let

$$\lim_{\tau\to0}\frac{i}{n-1}\frac{\alpha'(i\tau)}{\alpha(i\tau)}=-\alpha_n,\quad \lim_{\tau\to0}\frac{i}{n-1}\frac{\alpha'(i\tau)}{\alpha(i\tau)}C(i\tau)=\beta_n.$$

If $\alpha_n\neq0$, then letting $s=i\tau\to0$ in (2·12), we have

$$(2\cdot14)\qquad f(t)=f(t, 0)=e^{-\frac{\alpha_n}{2}t^2+\beta_n t}.$$

But since the left side is independent of n, α_n and β_n are constants independent of n(5) and thus we can put $\alpha_n=\alpha$, $\beta_n=i\beta$, where β is real, for $f(t)=\overline{f(-t)}$.

If $\alpha \neq 0$, then by (2·8), $\alpha>0$ and (2·14) can be written as

$$f(t)=e^{-\frac{\alpha}{2}t^2+i\beta t}$$

which shows that $F(x)$ is a normal distribution function.

If $\alpha=0$, then $f(t)=e^{i\beta t}$. This shows that $F(x)$ is an unit distribution function having only one point spectrum at $x=\beta$.

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(5) This can also be proved explicitly. From the existence of lim α_n we can show that the variance of Z is finite, and $\frac{1}{i}\lim_{\tau\to 0}\alpha'(i\tau)=E(Z)$, the mean value of Z, which is $(n-1)\sigma^2$, σ^2 being the variance of X_i. This is a well known fact in the sampling theory. Hence α_n is independent of n. For β_n, we can also prove its independence of n directly.