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On Integral Invariants and Betti Numbers of Symmetric Riemannian Manifolds, I.

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In his classical paper E. Cartan proved the important theorem that the *p-th Betti-number of a compact symmetric Riemannian manifold M is equal to the number of linearly independent invariant differentials of rank p defined on M.* Now there exist two kinds of compact symmetric Riemannian manifolds. *The first class* consists of the *group-manifolds of simple compact Lie groups.* The fundamental groups of these manifolds are semi-simple. The Poincaré-polynomials of compact simple Lie groups of four classe were first determined by R. Brauer. Pontrjagin investigated the homological properties of these manifolds and determined their homology basis. *The second class* of symmetric Riemannian manifolds are those whose fundamental groups are simple. Among the manifolds of these kinds there exist those which are at the same time *algebraic varieties.* The Betti numbers and the correspondingly homology basic of these manifolds were determined by C. Ehresmann. (The manifolds $S(n)$, $C(n)$, $A(n, k)$)[1]. In this paper I will give the complete table of Betti numbers of compact symmetric Riemannian manifolds by the methode E. Cartan.

Notations.

R_n: n-dimensional real vector space.

P_n: n-dimensional complex vector space.

k: The field of all real numbers, K: That of all complex numbers.

E or E_n: n-dimensional identity matrix, $\delta(ij)$ or δ_{ij}: $E=\delta(ij)$.

I or I_n: The skew matrix $\begin{Vmatrix} & E_{n/2} \\ -E_{n/2} & \end{Vmatrix}$, $\varepsilon(ij)$ or ε_{ij}: $I=\varepsilon(ij)$.

$US(n)$, $U(n)$, $USp(n)$, $O(n)$: unitary unimodular group, unitary group, unitary-symplectic group and real proper orthogonal group respectively.

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(1) These are manifolds in which the Betti numbers of odd dimensions are all zero, see E. Cartan, Selecta, p. 103; Ehresmann, [3].

$R(n, k)$: The set of all k-dimensional linear subspaces of R_n.

$A(n, k)$: The set of all k-dimensional linear subspaces of P_n.

$S(n, k)$: The set of all k-dimensional linear subspaces of P_n invariant with respect to the operation $x \to \hat{x} = I\bar{x}$, $(n = 2m,\ k = 2k')$.

$S(n)$: The set of all zero systems $x \to y = Sx$ such that the skew matrix S is at the same time orthogonal, $S^2 = -1$. $(n = 2 \cdot m)$.

$C(n)$: The set of all m-dimensional linear subspaces of P_n such that $\mathfrak{M} \wedge \hat{\mathfrak{M}} = 0$, where $\hat{\mathfrak{M}}$ is the transform of $\mathfrak{M}$ by $x \to \hat{x}$, $(n = 2 \cdot m)$.

Part I.
Representations; Linear invariants.

Chapter I.
The harmonics of the manifold $A^+(n)$.

1. We denote by $A^+(n)$ the set of all unimodular unitary matrices of n dimensions which are at the same time symmetric: $A^* = A$. $A^+(n)$ is then a symmetric Riemmanian manifold whose group of displacements is $A \to T^*AT$, where $T \epsilon US(n)$. The identity matrix E_n itself is contained in $A(n)$. The group of rotations associated with this point is nothing but the real orthogonal group. The automorphism of the group $US(n)$ associated with E is thus $T \to \bar{T}$, $\bar{T}$ being the complex conjugate of T.

2. The generic matrix A of the manifold $A^+(n)$ satisfy the relation $\bar{A} = A^{-1}$, A is thus a transvection in the sense of E. Cartan. Let us see that any $A \epsilon A^+(n)$ can be written in the form

$$A = R^{-1}\theta R, \quad R \epsilon O(n).$$

where $\theta = \varepsilon_1 + \dots\dots + \varepsilon_n$, $\varepsilon_i = \exp(2\pi\sqrt{-1}\,\theta_i)$. We say that a linear subspace $\mathfrak{M} \epsilon P_n$ is a real linear subspace or simply real if it is invariant by the operation $x \to \bar{x}$ (= the complex conjugate of x) which is invariant under the group $O(n)$. Any real linear subspace can be generated by a number of mutually orthogonal real vectors (the vectors whose components are real). Now let ε_i be an eigenvalue of A, x_i its corresponding eigenvector:

$$Ax_i = \varepsilon_i x_i.$$

By the relation $A^{-1} = \bar{A}$ we obtain $A\bar{x}_i = \varepsilon_i \bar{x}_i$. The eigenspace of ε_i is thus real. We can thus then n real vectors $x_1, \dots\dots, x_n$ such thalt $x_i x_j = \delta_{ij}$ and

$$A\,\|x_1, \ldots\ldots, x_n\| = \|x_1, \ldots\ldots, x_n\| \begin{pmatrix} \varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_n \end{pmatrix}, \quad \varepsilon_1 \ldots\ldots \varepsilon_n = 1$$

We have thus $A = R^{-1}\theta R$, where $R = \|x_1, \ldots\ldots, x_n\|$, $R \in O(n)$.

3. It may be seen that for any $A \in A^+(n)$ there exists a transvection $\theta^{\frac{1}{2}}$ such that A is the transform of E by $\theta^{\frac{1}{2}}$. Indeed, if $A = R^{-1}\theta R$, then by putting $\theta^{\frac{1}{2}} = R^{-1}\theta^{\frac{1}{2}}R(\theta^{\frac{1}{2}} = e^{\pi\sqrt{-1}\theta_1} + \ldots\ldots + e^{\pi\sqrt{-1}\theta_n})$ we obtain $A = \theta^{\frac{1}{2}}E(\theta^{\frac{1}{2}})^*$, and, moreever, $\theta^{\frac{1}{2}}$ is a transvection: $\bar{\theta}^{\frac{1}{2}} = (\theta^{\frac{1}{2}})^{-1}$.

Let $A + dA$ be a point near A. We transform the differential dA back to the point E by the transformation $\theta^{\frac{1}{2}}$: $\delta\theta = \theta^{-\frac{1}{2}}dA\theta^{-\frac{1}{2}}$. A simple calculation shows that $\delta\theta$ can be written as

$$\delta\theta = R^{-1}\left[\theta^{-1}d\theta + (\theta^{\frac{1}{2}}\delta R\theta^{-\frac{1}{2}} - \theta^{-\frac{1}{2}}\delta R\theta^{\frac{1}{2}})\right]R, \quad \delta R = (dR)R^{-1}$$

So that it can easily be seen that $\delta\theta$ is an infinitesimal transvection: $\overline{\delta\theta} = -\delta\theta$. The (i, j)-elements of this matrix is

$$2\pi\sqrt{-1}\,\delta\theta_i. \qquad i = j; \qquad \sum_i \delta\theta_i = 0$$

$$2\sqrt{-1}\sin\frac{\theta_i - \theta_j}{2}\,\delta u_{ij}, \qquad i \neq j$$

The representation of A in the form $R^{-1}\theta R$ is unique in some neighbourhood and we find as the volume-element of the manifold $A^+(n)$ the following expression:

$$\omega = \Delta[\omega]d\theta_1 \ldots\ldots d\theta_{n-1}, \qquad \Delta = \prod_{i<j} \sin\frac{\theta_i - \theta_j}{2}$$

4. Let $a(ij)$ be the elements of the generic matrix $A \in A^+(n)$, $a(ij) = a(ji)$. We construct the quantities

$$a(i_1, \ldots\ldots, i_f; j_1, \ldots\ldots, j_f) = \sum \varepsilon(P)a(i_1j_{1'}) \ldots\ldots a(i_fj_{f'}), \quad (1 \leqq f \leqq n-1)$$

where $\varepsilon(P)$ is the sign of the permutation P: $12\ldots f \rightarrow 1'2'\ldots f'$ and the summation is extended over all P. By means of the generic element of the group $US(n)$, $a(\ldots\ldots;\ldots\ldots)$ are transformed according to the irreducible representation of the group $US(n)$ of signature $(2, \ldots\ldots, 2, 0, \ldots\ldots, 0)$, where 2 appears f times. The quantity

$$a_f = \sum_{i_1 < \ldots\ldots < i_f} a(i_1, \ldots\ldots, i_f; i_1, \ldots\ldots, i_f)$$

stays invariant under the group $O(n)$ and we have

$$a_f = \sum_{i_1 < \cdots\cdots < i_f} \varepsilon_{i_1} \cdots\cdots \varepsilon_{i_f}$$

where $\varepsilon_1, \cdots\cdots, \varepsilon_n$ are the roots of A.

5. We call "*zonal*" a function defined on $A^+(n)$ which is invariant under the group g. We see that for any A the roots $\varepsilon_1, \cdots\cdots, \varepsilon_n$ are determined uniquely except their order and the $n-1$ quantities $\zeta_1, \cdots\cdots, \zeta_{n-1}$ with

$$\zeta_f = \sum_{i_1 < \cdots\cdots < i_f} \varepsilon_{i_1}, \cdots\cdots, \varepsilon_{i_f} \qquad (1 \leqq f \leqq n-1)$$

constitute a table of basic invariants of the group g. Any zonal can thus be expanded in the form

$$\zeta \sim \sum A_{a_1 \cdots\cdots a_p} \zeta_{a_1 \cdots\cdots a_p}$$

where the right side is a linear combination of the monomials

$$\zeta^*(f_1, \cdots\cdots, f_{n-1}) = (\zeta_{n-1})^{a_1} (\zeta_{n-2})^{a_2} \cdots\cdots (\zeta_1)^{a_{n-1}}$$

We write this monomial as $\zeta^*(f_1, \cdots\cdots, f_{n-1})$, where $(f_1, \cdots\cdots, f_{n-1})$ is the conjugate diagram of $(\overbrace{n-1, \cdots\cdots, n-1}^{2a_1}, \overbrace{1, \cdots\cdots 1}^{2a_{n-1}})$. We say that the monomial $\zeta^*(f_1, \cdots\cdots, f_{n-1})$ is higher than $\zeta^*(f', \cdots\cdots, f_{n-1}')$ if and only if $(f_1, \cdots\cdots, f_{n-1})$ is higher than $(f_1', \cdots\cdots, f_{n-1}')$ in the usual sense.

The angles $(\theta_1, \cdots\cdots, \theta_{n-1})$ and $(\theta_1', \cdots\cdots, \theta_{n-1}')$ are said to be homologous if the $(\varepsilon_1, \cdots\cdots, \varepsilon_n)$ and $(\varepsilon_1', \cdots\cdots, \varepsilon_n')$ constructed from θ and θ' differ only by their order. There exists a polygon P in $(n-1)$-space of $(\theta_1, \cdots\cdots, \theta_{n-1})$ such that no two points of P are homologous and that for any $(\theta_1, \cdots\cdots, \theta_{n-1})$ there exists a point of P which is homologous with (θ); this P may be defined by

$$\theta_1 \leqq \theta_2 \leqq \cdots\cdots \leqq \theta_n ; \quad \theta_n - \theta_1 \leqq 1 ; \quad \theta_1 + \cdots\cdots + \theta_n = 0$$

Two zonals ζ, ζ' are said to be orthogonal if

$$\int_P \zeta \bar{\zeta}' \Delta d\theta_1 \cdots\cdots d\theta_{n-1} = 0$$

Let $\zeta(f_1, \cdots\cdots, f_{n-1})$ be zonals obtained from $\zeta^*(f_1, \cdots\cdots, f_{n-1})$ by means of othogonalization. Then

$$\zeta(f_1, \cdots\cdots, f_{n-1}) = \zeta^*(f_1, \cdots\cdots, f_{n-1}) + (\textit{terms lower than } \zeta^*(f_1, \cdots\cdots, f_{n-1}))$$

6. We have already seen that $n-1$ quantities $a(\ldots\ldots, \ldots\ldots)$ constitute harmonic sets. We $\times$-multiply these quantities and construct the following representation which is also a harmonic set:

$$Z^*(f_1, \ldots\ldots, f_{n-1}) : \quad Z_{n-1}^{a_1}, \ldots\ldots, Z_1^{a_{n-1}}.$$

This representation contains only one irreducible representation of type $(f_1, \ldots\ldots, f_{n-1})$. This is the irreducible representation of the highest weight contained in Z^*. The components of Z may be written in the form

$$A(i_1, \ldots\ldots, i_{f_1}; j_1, \ldots\ldots, j_{f_2}; \ldots\ldots; k_1, \ldots\ldots, k_{f_{n-1}})$$

where A is skew symmetric with respect to the indices $i_1, j_1, \ldots\ldots, k_1$; $i_2, \ldots\ldots; \ldots\ldots$ respectively. A' are polynomials of a $(i_1, \ldots\ldots, i_f; j_2, \ldots\ldots, j_f)$ and the components A of the irreducible representation Z may be obtained from A by means of a definite process containing only symmetrization and alternation corresponding to the operator of Young operating on $(i_1, \ldots\ldots, i_{f_1}; \ldots\ldots)$ in this natural order. It can readily be seen that Z admits a linear invariant with respect to the subgroup g. Indeed, by using the fact that $f_1, \ldots\ldots, f_{n-1}$ are all even we see that

$$\delta(i_1 i_2) \ldots\ldots \delta(j_1 j_2) \ldots\ldots A(i_1 i_2 \ldots\ldots; j_1 j_2 \ldots\ldots; \ldots\ldots)$$

is invariant. This is a linear combination of the monomials $\zeta^*(f_1', \ldots\ldots, f_{n-1}')$. The highest term may be obtained from $A(11\ldots\ldots; 22\ldots\ldots; \ldots\ldots)$ which is equal to $\zeta^*(f_1, \ldots\ldots, f_{n-1})$. Thus this invariant is of the form

$$\zeta'(f_1, \ldots\ldots, f_{n-1}) = \zeta^*(f_1, \ldots\ldots, f_{n-1}) + (\textit{terms lower than } \zeta^*(f_1, \ldots\ldots, f_{n-1}).)$$

Because $\zeta'(f_1, \ldots\ldots, f_{n-1})$ are mutually orthogonal, we see easily that this coincides with $\zeta(f_1, \ldots\ldots, f_{n-1})$ except a constant factor.

7. By the theory of E. Cartan we readily conclude that for any monomial $\zeta^*(f_1, \ldots\ldots, f_{n-1})$ there exists an uniquely determined irreducible harmonic set admitting a linear invariant with respect to the subgroup g which is a polynomial in $\zeta_1, \ldots\ldots, \zeta_{n-1}$ such that its highest terms is $\zeta^*(f_1, \ldots\ldots, f_{n-1})$ and that any harmonic of the manifold $A^+(n)$ can be obtained in this manner. We thus have the following

Theorem 1. The quantities $Z(f_1, \ldots\ldots, f_{n-1})$ constitute a complete harmonic set of the manifold $A^+(n)$. An irreducible representation of the group

$US(n)$ of signature $\zeta(f_1, \ldots\ldots, f_{n-1})$ is contained in a harmonic set of the manifold $A^+(n)$ if and only if $f_1 \equiv f_2 \equiv \ldots\ldots \equiv 0$ mod. 2.

Chapter II.
The harmonics of the manifold $A^-(n)$.

1. The manifold $A^-(n)$ is the set of all skew symmetric unitary unimodular matrices: $A^*+A=0$. The group of displacements is $A \to T^*AT$, $T \in US(n)$. The skew matrix I is contained in $A^-(n)$ and the group of rotation g with center the point I is the group $USp(n)$ consisting of all the elements of the group $US(n)$ which are invariant under the involutive automorphism $\bar{T} \to I^{-1}\bar{T}I$. A transvection is an element of $US(n)$ such that $\theta^{-1}=I^{-1}\bar{\theta}I$. There exists a one to one correspondence between the points of $A^-(n)$ and the set of transvections described by

$$A=\theta I, \qquad \theta=-AI$$

Let θ be a transvection. By the same argument as in the case of the manifold $A^+(n)$ it is possible to show that if ε is an eigenvalue of θ, then the corresponding eigenspace is invariant with respect to the operation $x \to I\bar{x}(=\hat{x})$ which is invariant under the group $USp(n)$ and that θ can be written as

$$\theta=R^{-1}\theta R, \quad R \in USp(n)\,; \quad \theta=\theta_0+\hat{\theta}_0$$

where $\theta_0=\varepsilon_1+\ldots\ldots+\varepsilon_m$, $\varepsilon_i=\exp(2\pi\sqrt{-1}\,\theta_i)$. We call $\varepsilon_1, \ldots\ldots, \varepsilon_m$ the roots of A. For given A, $\varepsilon_1, \ldots\ldots, \varepsilon_m$ are determined uniquely except their order and we see that the quantities $\zeta_1, \ldots\ldots, \zeta_{m-1}$ with $\zeta_f=\sum_{i_1<\cdots<i_f} \varepsilon i_1 \ldots\ldots, \varepsilon i_f$ are fundamental invariants of A with respect to the group g.

2. We now construct the harmonic set of $A^-(n)$ corresponding to the invariant ζ_f. Instead of $a(i_1i_2\ldots\ldots\,; j_1j_2\ldots\ldots)$ of $A^+(n)$ we define

$$Z_f : A(i_1\ldots\ldots i_{2f})=\sum_P \varepsilon(P)Pa(i_1i_2)\ldots\ldots a(i_{s-1}i_s), \quad (s=2f)$$

where $a(ij)$ are the elements of the matrix A, P is the permutation $12\ldots(2f) \to 1'2'\ldots(2f)'$ and the summation is extended over all P. The quantities Z_f constitute a harmonic set and give an irreducible representation of signature $(1, 1, \ldots\ldots, 1, 0, \ldots\ldots 0)$, where 1 appears $2f$ times. We $\times$-multiply these quantities and construct:

$$Z : Z_{m-1}^{a_1} \times \ldots\ldots \times Z_1^{a_{m-1}}$$

Z contains only one irreducible representation of the highest weight. This is of signature $(f_1, \ldots\ldots, f_{n-2})$, where $(f_1, \ldots\ldots, f_{n-2})$ is the conjugate of $(\overbrace{2(m-1), \ldots\ldots, 2(m-1)}^{a_1}, \ldots\ldots, \overbrace{2, \ldots\ldots, 2}^{a_{m-1}})$. We denote this by $Z(f_1, \ldots\ldots, f_{n-2})$. The components of Z are uniquely determined and can be written as $A(i_1 i_2, \ldots\ldots; j_1 j_2, \ldots\ldots; \ldots\ldots)$, A' are skew symmetric with respect to $i_1, i_2, \ldots\ldots; j_1, j_2, \ldots\ldots; \ldots\ldots$ respectively. The zonale of $A^-(n)$ corresponding to Z is

$$\varepsilon(i_1 i_2)\varepsilon(i_3 i_4)\ldots\ldots\ \varepsilon(j_1 j_2)\varepsilon(j_3 j_4)\ldots\ldots\ Z(i_1 i_2\ldots\ldots; j_1 j_2\ldots\ldots; \ldots\ldots)$$

which is a linear combination of the monomials $\zeta_{m-1}^{a_1'}\ldots\ldots\zeta_1^{a_{m-1}'}\ldots\ldots$, the highest term being obtained from $A(12\ldots\ldots; 12\ldots\ldots; \ldots\ldots)$. We denote this by $\zeta(f_1 f_2\ldots\ldots)$.

3. We write Θ in the form $\Theta = R^{-1}\theta R$. If we put $\Theta^{\frac{1}{2}} = R^{-1}\theta^{\frac{1}{2}}R$, where the angles of $\theta^{\frac{1}{2}}$ are just the halves of those of θ, then it can readily be seen that the point $A = \Theta I$ corresponding to Θ is just the transform of I by $\Theta^{\frac{1}{2}}$. Let $\Theta + d\Theta$ be a transvection corresponding to the point $A + dA$ near A. We transform $d\Theta$ back to the point I by the transformation $\Theta^{\frac{1}{2}} : \delta\Theta = \Theta^{-\frac{1}{2}} \cdot d\Theta \cdot \Theta^{-\frac{1}{2}}$. Then

$$\delta\Theta = R^{-1}\left[\theta^{-1}d\theta + (\theta^{\frac{1}{2}}\delta R\theta^{-\frac{1}{2}} - \theta^{-\frac{1}{2}}\delta R\theta^{\frac{1}{2}})\right], \quad \delta R = (dR)R^{-1}.$$

Now the representation of Θ in the form $R^{-1}\theta R$ is not unique. Different R' give the same Θ if they are of the form ρR. For generic θ ρ must be of the form $\rho = A_1 + \ldots\ldots + A_m$, where A_i are matrices of the form

$$\begin{Vmatrix} a & b \\ -\bar{b} & \bar{a} \end{Vmatrix}, \quad a\bar{a} + b\bar{b} = 1$$

(We arrange the indices in the order $1, m+1, 2, m+2, \ldots\ldots, m, m+m = n$). By means of this uncertainty we can normalize $\delta\Theta$ such that $\delta u_{ii} = \delta u_{m+i,\, m+i} = 0$, $(i = 1 \cdot 2, \ldots\ldots, m)$, $\delta u_{i,m+i} = \delta u_{m+i,i} = 0$, $(i = 1, 2, \ldots\ldots, m)$. R has then a definite sense. The (ij)-elements of the matrix $\delta\Theta$ are

$$\delta u_{ii} = \delta u_{m+i,\, m+i} = 2\pi\sqrt{-1}\,\delta\theta_i\,; \qquad \sum_i \delta\theta_i = 0$$

$$\delta u_{im+j} = -\delta u_{m+ij} = \sqrt{-1}\sin(\theta_i - \theta_j/2)\tilde{\omega}_{ii} \qquad (i \neq j)$$

$$\delta u_{ij} = \delta u_{m+i,\, m+j} = \sqrt{-1}\sin(\theta_i - \theta_j)/2)\omega_{ij} \qquad (i \neq j)$$

The volume element of the manifold $A^-(n)$ is thus

$$\omega = \Delta[\omega_j]d\theta_1 \ldots\ldots d\theta_{m-1}\cdot$$

where

$$\Delta = \prod_{i<j} \sin^4 \frac{\theta_i - \theta_j}{2}$$

Theorem 1. 2. *The quantities* $Z(f_1, \ldots\ldots, f_{n-2})$ *constitute a complete harmonic set of the manifold* $A^-(n)$. *The irreducible representation of the group* $US(n)$ *of signature* $(f_1, \ldots\ldots, f_{n-1})$ *is contained in a harmonic set of the manifold* $A^-(n)$ *if and only if the columns of the diagram* $(f_1, \ldots\ldots, f_{n-1})$ *are all even. In particular,* $f_{n-1}=0$.

Chapter III.
The harmonics of the manifold $A(n, k)$; $k \leqq h(=n-k)$.

1. The manifold $A(n, k)$ is the set of all k-dimensional linear subspaces of P_n. Let $\mathfrak{M}$ be an linear subspace of P_n which is contained in $A(n, k)$. A unitary coordinate system $e_1, \ldots\ldots, e_n$ is said to be associated with $\mathfrak{M}$ if $e_1, \ldots\ldots, e_k \in \mathfrak{M}$. With respect to this coordinate system the group of rotations consist of all transformations of the form $T_k + T_h$. The involution of the group $US(n)$ associated with $\mathfrak{M}$ is $A \rightarrow JAJ$, where $J = E_k + -E_h$.

Let $\mathfrak{M}_i$ be two linear subspaces of P, where dim $\mathfrak{M}_1 = m_1$, dim $\mathfrak{M}_2 = m_2$, $m_1 \leqq m_2$. Then we can take unitary frames $x_1, \ldots\ldots, x_{m_1}$; $y_1, \ldots\ldots, y_{m_2}$ of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ such that $(x_i y_i) = \cos\theta_i$, $(i=1, 2 \ldots\ldots m_1)$, and that all other $(x\bar{y})$ vanish. In fact, let $\mathfrak{M}_1^0$, $\mathfrak{M}_2^0$ be linear subspaces of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ whose vectors are all perpendicular with $\mathfrak{M}_2$ and $\mathfrak{M}_1$ respectively. Then the subspaces $\overline{\mathfrak{M}}_1 = \mathfrak{M}_1 - \mathfrak{M}_1^0$, $\overline{\mathfrak{M}}_2 = \mathfrak{M}_2 - \mathfrak{M}_2^0$ are of the same dimension, say m. Let $P_{\overline{\mathfrak{M}}_1}$, $P_{\overline{\mathfrak{M}}_2}$ be orthogonal projections on $\overline{\mathfrak{M}}_1$ and $\overline{\mathfrak{M}}_2$ respectively. The operator $P_{\overline{\mathfrak{M}}_1}$, $P_{\overline{\mathfrak{M}}_2}$ leaves invariant the subspace $\overline{\mathfrak{M}}_1$, and is hermitian in $\overline{\mathfrak{M}}_1$. Let $x_1, \ldots\ldots, x_m$ be eigenvectors of $P_{\overline{\mathfrak{M}}_1} P_{\overline{\mathfrak{M}}_2}$ in $\overline{\mathfrak{M}}_1$. The projections of x_i into $\mathfrak{M}_2$ are not zero, and if we denote by θ_i the angle between x_i and $P_{\overline{\mathfrak{M}}_2} x_i = \bar{y}_i$, then $\cos^2\theta_i$ are eigenvalues of $P_{\overline{\mathfrak{M}}_1} P_{\overline{\mathfrak{M}}_2}$ in $\mathfrak{M}_1$. The vectors

$y_i=(\cos\theta_i)^{-1}\bar{y}_i$ constitute a certains frame of $\overline{\mathfrak{M}}_2$ and moreover, a system of eigenvectors of $P_{\overline{\mathfrak{M}}_2}P_{\overline{\mathfrak{M}}_1}$ in $\overline{\mathfrak{M}}_2$. The corresponding eigenvalues are again $\cos^2\theta_1$, ..., $\cos^2\theta_m$. If we take in $\mathfrak{M}_1^0$ and $\mathfrak{M}_2^0$ orthogonal frames x_{1+m}, ..., x_{m_1} and y_{1+m}, ..., y_{m_2} arbitrarily, then it can readily be seen that these vectors satisfy the conditions of the assertion. This can be carried out by purely elemental geometric device.

2. A transvection is a transformation θ of $US(n)$ satisfying $J\theta J=\theta^{-1}$. There exists an one to one correspondence between the set of transvections and the linear subspaces of arbitrary dimensions of the space P_n described by

$$S=\theta J,\quad \theta=SJ$$

where S denotes the reflexion with respect to the subspace $\mathfrak{M}$. The correspondense between S and $\mathfrak{M}$ may be described as:

$$\mathfrak{M}=(x;\ 2x=(S+1)x.)\ ;\quad S=2P_{\mathfrak{M}}-1.$$

Let $\mathfrak{M}_0$, $\mathfrak{N}$ be two linear subspaces of P and let dim $\mathfrak{M}_0=k$, dim $\mathfrak{N}=m$. *Then there exisis a coordinate system* e_1, ..., e_n *associated with* $\mathfrak{M}_0(\{e_1,\ \ldots\ldots,\ e_k\}=\mathfrak{M})$ *such that the transvection associated with* $\mathfrak{N}$ *take the form*

$$D(\theta_1)\dot{+}\cdots\cdots\dot{+}D(\theta_k)\dot{+}\begin{Vmatrix} E_p & \\ & -E_p \end{Vmatrix}$$

Suppose first that $k\leqq m\leqq n/2$, then there exists a system of vectors x_1, ..., $x_k\in\mathfrak{M}_0$, $x_ix_j=\delta_{ij}$; y_1, ..., $y_m\in\mathfrak{N}$, $y_iy_j=\delta_{ij}$ such that $x_iy_i=\cos\theta_i$ and that all other x, y vanish. We take cartesian frame associated with $\mathfrak{M}_0$ such that:

$$e_i=x_i\ \ (i=1,\ 2,\ \ldots\ldots,\ k),\ y_i=\cos\theta_ie_i+\sin\theta_i\cdot e_{k+i},$$
$$e_{2k+1},\ \ldots\ldots,\ e_m\in\mathfrak{N},$$

Then the subspace $\mathfrak{N}$ is generated by m vectors which are two rows of the matrix

$$B:\begin{Vmatrix} \cos\theta_1 & \sin\theta_1 & & & & \\ & & \cos\theta_2 & \sin\theta_2 & \cdots\cdots & \\ & & & & & \begin{vmatrix} E_{m-k} & 0 \end{vmatrix} \end{Vmatrix}$$

where the index is arranged in the order 1, $k+1$, 2, $k+2$, ..., k, $2k$, $2k+1$, ..., n. The reflexion with respect to $\mathfrak{N}$ is $S=2B^*B-1$, and the corresponding transvection is $\theta=SJ$. The explicite calculation gives the desired result. The case $k>m$ can be treated in analogous manner.

3. Let $\mathfrak{M}$ be a k-dimensional linear subspace of P, θ be the corresponding transvection. The reflexion with respect to $\mathfrak{M}$ is then θJ. Now we have seen that any θ associated with $\mathfrak{M}$, dim $\mathfrak{M}=k$ can be written as

$$\theta=R^{-1}\theta R,\quad \theta=D(\theta_1)\dot{+}\cdots\cdots\dot{+}D(\theta_k)\dot{+}E_{n-2k}$$

where $R=T_k\dot{+}T_h(\epsilon g)$. Consider the transvection $\theta^{\frac{1}{2}}$ defined by

$$\theta^{\frac{1}{2}}=R^{-1}\theta^{\frac{1}{2}}R,\quad \theta^{\frac{1}{2}}=D(\theta_1/2)\dot{+}\cdots\cdots\dot{+}D(\theta_k/2)\dot{+}E_{n-2k}$$

Then the subspace $\mathfrak{M}$ is the transform of $\mathfrak{M}_0=\{e_1, \cdots\cdots, e_k\}$ by $\theta^{\frac{1}{2}}$. Indeed, the reflection with respect to the transform of $\mathfrak{M}_0$ by $\theta^{\frac{1}{2}}$ is

$$\theta^{\frac{1}{2}}J(\theta^{\frac{1}{2}})^*=\theta^{\frac{1}{2}}J(\theta^{\frac{1}{2}})^*J\cdot J=\theta^{\frac{1}{2}}\cdot\theta^{\frac{1}{2}}J=\theta J=S.$$

Now let $\theta+d\theta$ be a transvection associated with $\mathfrak{M}+d\mathfrak{M}\in A(n, k)$ near $\mathfrak{M}$. We transform $d\theta$ back to $\mathfrak{M}_0$ by the transformation $\theta^{\frac{1}{2}}$:

$$\delta\theta=\theta^{-\frac{1}{2}}d\theta\cdot\theta^{-\frac{1}{2}}.$$

This shows that $\delta\theta$ is an infinitesimal transvection: $J\delta\theta J=-\delta\theta$. We have, in the same manner as in A^+ and A^-, as the volume-element between two zones $(\theta_1, \cdots\cdots, \theta_k)$ and $(\theta_1+d\theta_1, \cdots\cdots, \theta_k+d\theta_k)$ the following expression;

$$\Delta d\theta_1\cdots\cdots d\theta_k\,;\quad \Delta=\Pi(\cos\theta_i-\cos\theta_j)^2\Pi\sin\theta_i\,\Pi\sin^{2(n-2k)}(\theta_i/2)$$

4. The *zonales* are functions depending on k arguments $\cos^2\theta_1/2, \cdots\cdots,$ $\cos^2\theta_k/2$ invariant with respect to any permutation among them and thus are expressible by means of k quantities

$$\zeta_f=\sum_{i_1<\cdots\cdots<i_f}\cos^2(\theta_{i_1}/2)\cdots\cdots\cos^2(\theta_{i_f}/2),\quad (1\leqq f\leqq k)$$

Let $p(i_1, \cdots\cdots, i_k)$ be Plücker coordinates of $A(n, k)$. Consider k quantities $Z_1, \cdots\cdots, Z_k$.

$$Z_f:\ \sum_j p(i_1\cdots\cdots i_f\, j_1\cdots\cdots j_g)\bar{p}(k_1\cdots\cdots k_f\, j_1\cdots\cdots j_g).$$

These quantities constitute harmonic sets of the manifold $A(n, k)$. The zonale of Z_f is just ζ_f. Any harmonic set of the manifold $A(n, k)$ may thus be obtained from $Z_1, \cdots\cdots, Z_k$ by means of $\times$-multiplication. Let $Z(f_1, \cdots\cdots, f_k)$ be the representation of the highest weight contained in

$$Z_k^{a_1} \ldots\ldots Z_1^{a_k}$$

where $(f_1 \ldots\ldots f_k)$ is the conjugate of $(\underbrace{k \ldots\ldots k}_{a_1} \ldots\ldots \underbrace{1 \ldots\ldots 1}_{a_k})$. The zonale of $Z(f_1, \ldots\ldots, f_k)$ is

$$\zeta(f_1 \ldots\ldots f_k) = (\zeta_k)^{a_1} \ldots\ldots (\zeta_1)^{a_k} + (\textit{terms lower than } \zeta_k^{a_1} \ldots\ldots \zeta_1^{a_k}).$$

The representation given by $Z(f_1, \ldots\ldots, f_k)$ is of signature $(h_1 \ldots\ldots h_{n-1})$:

$$h_i = f_i + f_1, \quad (i = 1, 2, \ldots\ldots, k), \quad h_{i+1} = \ldots\ldots = h_{n-2k} = f_1 .$$
$$h_i + h_{n-i+1} = 2f_1$$

and we obtain the following.

Theorem 1. 3. The quantities

$$Z(f_1, \ldots\ldots, f_k) \quad f_1 \geqq f_2 \geqq \ldots\ldots \geqq f_k \geqq 0$$

with any $f_1, f_2, \ldots\ldots, f_k$ constitute a complete harmonic set of the manifold $A(n, k)$. The irreducible representation of the group $US(n)$ of signature $(h_1, \ldots\ldots, h_{n-1})$ is contained in a harmonic set of the manifold $A(n, k)$ if and only if

$$h_1 = h_2 + h_{n-1} = \ldots\ldots = h_k + h_{n-k+1} = 2h_{k+1} = \ldots\ldots = 2h_{n-k}.$$

Chapter IV.
Some lemmas on linear invariants.

1. Let M be a compact homogenious manifold with fundamental group G, and suppose that G is a Lie group. Let $\Gamma(G)$ be an irreducible representation of G, $\Gamma(g)$ be the representation of the group of rotations $g(\subset G)$ derived from $\Gamma(G)$. E. Cartan established the following theorem :

$\Gamma(G)$ is contained in a harmonic set of the manifold M if and only if $\Gamma(g)$ contains a linear invariant. If their exist h linear invariants of $\Gamma(g)$, then there exists h different harmonic sets belonging to the representation $\Gamma(G)$. (see E. Cartan, [2]).

Now we have constructed in the preceding chapters the harmonic sets of the manifolds A^+, A^- by the explicit computation of the zonales. Thus the problems of finding the irreducible representations of the group $US(n)$ admitting a linear invariant with respect to its proper subgroups $0(n)$ or $US(n)$ are completly solved. (This methode may afford a most natural approach to this problem.) We have the following:

Theorem 1. 4. The representation of the group $US(n)$ *of signature* $(f_1, \ldots\ldots, f_{n-1})$ *admits a linear invariant with respect to the proper subgroups* $O(n)$ *and* $USp(n)$ *if and only if* $f_1 \equiv f_2 \equiv \ldots\ldots \equiv 0$ *mod.* 2 *for* $O(n)$, and $g_1 \equiv g_2 \equiv \ldots\ldots \equiv 0$ *mod.* 2 *for* $USp(n)$ *respectively, where* $g_1, g_2, \ldots\ldots$ *is the conjugate of* $f_1, f_2, \ldots\ldots$ *In each cases there exists only one linear invariant.*

In this chapter I will give a direct verification of this fact.

Lemma. Let $P(f_1, \ldots\ldots f_{n-1})$ be an irreducible representation of the real unimodular group $SL(n)$ of signature $(f_1, f_2, \ldots\ldots)$. $P(f_1 \ldots\ldots)$ admits a linear invariant with respect to its proper subgroup $O(n)$ if and only if all f_i are even. Proof. Let $(g_1, g_2, \ldots\ldots)$ be the conjugate of $(f_1, f_2, \ldots, \ldots)$. We take $g = g_1$ vectors $x_1, x_2, \ldots\ldots$ in R_n. The components of x_i may be denoted by $(x_i(1), \ldots\ldots x_i(n))$. The tensors

$$x_\lambda : x(i_1, \ldots\ldots, i_\lambda) = \sum \varepsilon(P) P x(i_1) \ldots\ldots x(i_\lambda)$$

are skew symmetric with respect to $i_1, \ldots\ldots, i_\lambda$ and the tensors defined by

$$X_{g_1} \times X_{g_2} \times \ldots\ldots \times X_{g_f} \quad (f = f_1)$$

or

$$x(i_1 \ldots\ldots i_N) = x(i_1 \ldots\ldots i_{g_1}) \ldots\ldots x(k_1 \ldots\ldots k_{g_f})$$

constitute a basis of the invariant subspace $P(f_1, \ldots\ldots, f_{n-1})$. Now the quantities X_λ does not change if we replace the vectors $x_1, \ldots\ldots, x_g$ by

$$y_1 = x_1,\ y_2 = x_2 + \lambda_{21} x_1,\ \ldots\ldots, y_g = x_g + \lambda_{g1} x_1 + \ldots\ldots + \lambda_{gg} x_g$$

So that we can replace $x_1, \ldots\ldots, x_g$ by mutually orthogonal vectors without changing X. Any tensor of signature $(f_1, \ldots\ldots, f_{n-1})$ is then a linear combination of the form $X = \sum \wedge_\alpha E_\alpha$, where E_α are basic vectors of $P(f_1, \ldots\ldots)$ which are of the form

$$E_\alpha = [x_1 \ldots\ldots x_{g_1}] [x_2 \ldots\ldots x_{g_2}] \ldots\ldots [x_1 \ldots\ldots x_{g_f}]$$

The vectors $x_1, \ldots\ldots, x_g$ are mutualy orthogonal. The only way of constructing orthogonal invariants is the contraction with respect to $\delta(ij)$. We thus obtain as a necessary condition N=even. Any orthogonal invariant is thus a linear combination of the terms

$$\wedge = \delta(i_1 i_2) \ldots\ldots \delta(i_{N-1} i_N) x(i_1, \ldots\ldots i_N)$$

If we use the relation $x_i x_j=0$, $(i \neq j)$, then:

$$\wedge = \nu \sum \wedge_a (x_1x_1)^{S_1}(x_2x_2)^{S_2}\cdots\cdots(x_gx_g)^{S_g},\ 2S_1=f_1, \ldots\ldots, 2S_g=f_g$$

where $\nu(1'2'\ldots\ldots N')$ is a number depending only on the permutation $12, \ldots\ldots N \rightarrow 1'2'\ldots\ldots N'$. From which we conclude at once $f_i=2s_i, \ldots\ldots$ That is, $f_1, \ldots\ldots, f_{n-1}$ are all even. We also see that only one linear invariant can exist. Under this condition the required linear invariant is

$$\delta(i_1i_2)\delta(i_3i_4)\ldots\ldots\delta(j_1j_2)\ldots\ldots X(i_1j_1\ldots\ldots;\ i_2j_2\ldots\ldots;\ \ldots\ldots).$$

This is certainly not identically zero. (Consider the tensor whose components are all zero except $X(12\ldots\ldots g;\ 12\ldots\ldots g;\ \ldots\ldots)$ and its homolog).

Lemma. The irreducible representation of the group $US(n)$ of signature $(f_1, \ldots\ldots, f_{n-1})$ admits a linear invariant with respect to the subgroup $USp(n)$ if and only if all g_i are even.

We take a generic tensor X of $P(f_1, \ldots\ldots, f_{n-1})$ in the form

$$X=\sum [x_1\ldots\ldots x_{g_1}][x_1\ldots\ldots x_{g_2}]\ldots\ldots$$

Any invariant may be obtained by contraction with respect to $\varepsilon(ij)$. If the term $[x, \ldots\ldots x][x, \ldots\ldots x]\ldots\ldots$ does not vanish by this contraction, then it is possible to show that $x, \ldots\ldots x$ can be taken such that:

$$x_{1+i}=y_{1+i},\ x_{2+i}=y_{2+i}+z_{2+i},\ z_{2+i}\perp\{y_2, y_1, \ldots\ldots, y_{2+i}, y_{1+i}\},\ y_{2+i}=1\bar{y}_{1+i}$$
$$(i=0, 2, \ldots\ldots).$$

From this follows at once the required result.

From these lemmas we conclude not only the theorem 1. 4, but also the following.

Theorem 1. 5. The irreducible representation of the group $GL(n, K)$ of signature $(f_1, \ldots\ldots, f_n)$ admits a linear invariant with respect to its proper subgroups $0^+(n)$ or $Sp(n)$ if aud only if $f_i-f_n\equiv 0$ mod 2 for $0(n)$ and $g_1\equiv g_2\equiv\ldots\ldots\equiv 0$ mod 2 for $Sp(n)$.

Part II.
The Betti-numbers.

Chapter I.
Formal preperation.

1. We consider a set (Ω) of all (h, k) matrices in k or K.:

$$\Omega : (\omega_{i\lambda}), \quad (1 \leqq i \leqq h,\ 1 \leqq \lambda \leqq k).$$

In Ω we consider a group Γ isomorphic with $GL(k) \times GL(h)$:

$$\Omega \rightarrow A\Omega B^*, \quad A \epsilon GL(k), \quad B \epsilon GL(h)$$

The outer products of p elements of Ω :

$$\Gamma_p \cdot \omega_{i_1\lambda_1} \cdots\cdots \omega_{i_p\lambda_p}$$

give a representation of the group Γ. Concerning the decomposition of $\Gamma(p)$ the following fact is known (Ehresmann, [3]) :

$\Gamma(p)$ decomposes into the irreducible representations according to the following sheme :

$$\Gamma(p) \sim \sum P(f_1, \cdots\cdots, f_k) \times P(g_1, \cdots\cdots, g_h).$$

where $P(f_1, \cdots\cdots, f_k)$ is the irreducible representation of the group $GL(k)$ of signature $f_1, \cdots\cdots, f_k$ $(h \geqq f_1 \geqq f_2 \geqq \cdots\cdots \geqq f_k \geqq 0)$, $P(g_1, \cdots\cdots, g_h)$ is the irreducible represenation of the group $GL(h)$ of signature $(g_1, \cdots\cdots, g_h)$, $(k \geqq g_1 \geqq g_2 \geqq \cdots \geqq g_h \geqq 0$. Moreover, $(f_1, \cdots\cdots, f_k)$ and $(g_1, \cdots\cdots, g_h)$ are mutually conjugate. The term of the highest wheight in $P(f_1, \cdots\cdots, f_k) \times P(g_1, \cdots\cdots, g_h)$ is

$$F_0 \quad \omega_{11} \times \omega_{21} \times \cdots\cdots \times \omega_{f_11} \times \omega_{12} \cdots\cdots \times \omega_{f_22} \times \cdots\cdots \times \omega_{1g_1} \times \omega_{2g_1} \cdots\cdots \times \omega_{f_1g_1}.$$

The components of $P(\cdots\cdots) \times P(\cdots\cdots)$ may be obtained from $\omega_{i_1\lambda_1} \cdots\cdots \omega_{i_p\lambda_p}$ by operating the operator of Young according to the definite manner indicated by the leading term F_0. By the aid of theorem 1. 5. we obtain the following.

Lemma 1. By descending from $GL(k) \times GL(h)$ to its subgroups $O(k) \times O(h)$ or $Sp(k) \times Sp(h)$ the representation $P(f_1, \cdots\cdots, f_k) \times P(g_1, \cdots\cdots, g_h)$ admits a linear invariant if and only if $f_1 - f_k \equiv f_2 - f_k \equiv \cdots\cdots \equiv 0$ mod. 2, $g_1 - g_h \equiv g_2 - g_h \equiv \cdots\cdots \equiv 0$ mod. 2 for $O(k) \times O(h)$ and $f_1 \equiv f_2 \equiv \cdots\cdots \equiv 0$ mod. 2, $g_1 \equiv g_2 \equiv \cdots\cdots \equiv 0$ mod. 2 for $Sp(k) \times Sp(h)$ respectively.

2. We denote by (Ω) the set of all skew symmetric matrices Ω in K:

$$\Omega = (\omega_{ij}), \quad \omega_{ij} + \omega_{ji} = 0, \quad (i, j \cdots\cdots = 1, 2, \cdots\cdots n).$$

Consider in (Ω) the group Γ isomorphic with $GL(n)$: $\Omega \rightarrow A^*\Omega A$, $A \epsilon GL(n)$. Concerning the representation given by the outer products $\Gamma(p)$: $\omega_{i_1j_1} \times \cdots$

$\ldots\omega_{i_p j_p}$ the following fact is known ([3]) :

$\Gamma(p)$ decomposes into the irreducible representations according to

$$\Gamma(p) \sim \sum \Gamma(f_1, \ldots\ldots, f_l)$$

where $n > f_1 > f_2 > \ldots\ldots > f_l > 0$. $p = f_1 + \ldots\ldots + f_l$. The representation $\Gamma(f_1, \ldots \ldots, f_l)$ is of signature $(f_1, f_2+1, \ldots\ldots, f_l+l-1, h_1, h_2, \ldots\ldots)$, $(h_1, \ldots\ldots)$ is the conjugate of $(f_1-l+1, f_2-l+2, \ldots\ldots, f_l)$. The highest term in $(f_1, \ldots\ldots f_l)$ is

$$F_0 : \omega_{12} \times \omega_{13} \times \ldots\ldots \times \omega_{1\, f_1+1} \times \omega_{23} \ldots\ldots \times \omega_{2 f_2+2} \times \ldots\ldots \times \omega_{l\, l+1} \times \ldots\ldots \times \omega_{l\, l+f_l}$$

The components of $\Gamma(f_1, \ldots\ldots, f_l)$ may be obtained from $\Gamma(p)$ by means of the definite process indicated by the leading term F_0. By descending from $GL(n)$ to its subgroup $Sp(n)$ this representation admits a linear invariant if and only if the columns of the diagram $(f_1, f_2+1, \ldots\ldots, h_1, h_2, \ldots\ldots)$ are all even. Under this condition the numbers

$$k_1 = f_1 + f_2,\ k_2 = f_3 + f_4,\ \ldots\ldots,\ k_p = f_{2p-1} + f_{2p},\quad (l = 2p)$$
$$k_1 = f_1 + f_2,\ k_2 = f_3 + f_4,\ \ldots\ldots,\ k_p = f_{2p-1} + f_{2p},\ k_{p+1} = f_{2p+1}\quad (l = 2p+1)$$

are all odd numbers satisfying the condition

$$k_1 \equiv k_2 \equiv \ldots\ldots \equiv 1 \text{ mod. } 4, \text{ and } k_{p+1} = f_{2p+1} = 1,\ 1 \leqq k_i \leqq 2n.$$

By this fact we readily see the following.

Lemma II. By decending from $GL(n)$ to i's subgroup $Sp(n)$ the representation $\Gamma(f_1 \ldots\ldots f_l)$ admits B_p linear invariants. B_p is the coefficients of the Polynomial

$$P(z) \equiv (1+z)(1+z^5)\ldots\ldots(1+z^{4\nu+3}) \quad n = 2\nu$$

Let $(\Omega)_0$ be the set of all skew symmetric matrices in K satisfying the relation $\sum \varepsilon_{ij}\omega_{ij} = 0$. This relation is invariant with respect to the group $Sp(n)$ and we have the

Lemma II′. By descending from $GL(n)$ to its subgroup $Sp(n)$ the number of linear invariants contained in the product $\Gamma(p) : \omega_{i_1 j_1} \times \ldots\ldots \times \omega_{i_p j_p}$ is the coefficient of z^p in the polynomial

$$P_0(z) \equiv (1+z^5)(1+z^9)\ldots\ldots(1+z^{4\nu-3}). \quad n = 2\nu$$

Proof. Consider the invariant decomposition of ω_{ij}, where ω_{ij} are the elements of the generic matrix of (Ω) :

$$\omega_{ij}=\overset{0}{\omega}_{ij}+\omega'_{ij}, \quad \omega'_{ij}=\varepsilon_{ij}\omega, \quad \omega=\frac{1}{n}\sum \varepsilon_{ab}\omega_{ab}.$$

The product $\Gamma(p)$ is

$$\omega_{i_1j_1}\cdots\cdots\omega_{i_pj_p}=\overset{0}{\omega}_{i_1j_1}\cdots\cdots\overset{0}{\omega}_{i_pj_p}+p[\overset{1}{\omega}_{i_1j_1};\ \overset{0}{\omega}_{i_2j_2}\cdots\cdots\overset{0}{\omega}_{i_pj_p}]$$

The two terms in the right side of this equation are unitary orthogonal, so that if we denote by P, P^0, P^1 the vector spaces spanned by $\omega_{i_1j_1}\cdots\cdots\omega_{i_pj_p}$, $\overset{0}{\omega}_{i_1j_1}\cdots\cdots\overset{0}{\omega}_{i_pj_p}\cdots\cdots$ and $[\overset{1}{\omega}_{i_1j_1}\overset{0}{\omega}_{i_2j_2}\cdots\cdots]$ respectively, then $P=P^0+P^1$. Let χ, χ^0, χ^1 be the characters of the group $Sp(n)$ induced in P, P^0, P^1. Then $\chi=\chi^0+\chi^1$. By integrating over the unitary restricted group $Sp(n)$ we have

$$\int\chi=\int\chi^0+\int\chi^1.$$

The left side of this equation indicates the number of linear invariants contained in $\Gamma(p)$, say B_p. The first term of the right side indicates the required number, say B_p^0. It is easy to see that the second term of the right side is just B_{p-1}^0. Thus we see $B_p=B_p^0+B_{p-1}^0$. B_p is thus the coefficient of the Polynomial $P(z)/(1+z)$.

The adjoint group of the group $OL(n)$ is just $\Omega\rightarrow A^*\Omega A$, where $A\in OL(n)$. The representation $\Gamma(f_1, \cdots\cdots, f_l)$ admits a linear invariant with respect to this group if and anly if the numbers $f_1, f_2+1, \cdots\cdots, f_l+l-1$; $h_1, h_2, \cdots\cdots$ are all congruent with each other. We have thus as in the case of lemma II the

Theorem II. The Poincaré Polynomial of the group $OL(n)$ is

$$P(z)\equiv(1+z^3)(1+z^7)\cdots\cdots(1+z^{4\nu-1}) \qquad n=2\nu+1.$$

$$P(z)\equiv(1+z^3)(1+z^7)\cdots\cdots(1+z^{4\nu-5})(1+z^{2\nu-1}) \quad n=2\nu.$$

3. In this paragraph (Ω) is the set of all symmetric matrices in K.

$$\Omega:\ (\omega_{ij})\ ;\ i,j\cdots\cdots=1, 2, \cdots\cdots, n, \quad \omega_{ij}=\omega_{ji}$$

The outer product

$$\Gamma(p):\ \omega_{i_1j_1}\cdots\cdots\omega_{i_pj_p}$$

decomposes according to the following sheme :

$$\Gamma(p)\sim\sum\Gamma(f_1, \cdots\cdots, f_l) \quad n+1>f_1>f_2>\cdots\cdots>f_l>0.$$

$\Gamma(f_1, \ldots\ldots, f_l)$ is the irreducible representation of the group $GL(n)$ of signature $(f_1+1, \ldots\ldots, f_l+l, h_1, \ldots\ldots, h_{n-l})$, where $h_1, \ldots\ldots, h_{n-l}$ is the conjugate of $(f_1-l, f_2-l-1, \ldots\ldots, f_l-1)$. The highest term in $(f_1, \ldots\ldots, f_l)$ is

$$\omega_{11} \times \ldots\ldots \times \omega_{1f_1} \times \omega_{22} \times \ldots\ldots \omega_{2, f_2+1} \times \ldots\ldots \times \omega_{ll} \times \ldots\ldots \times \omega l f_l+l-1 \quad .$$

By theorem 1.5 we have the following

Lemma III. By descending from $GL(n)$ to its subgroup $O(n)$ the representation $\Gamma(p)$ admits B_p linear invariants, where B_p is the coefficient of

$$P(z) \equiv (1+z)(1+z^5)\ldots\ldots(1+z^{4\nu-3})(1+z^{2\nu}) \quad \text{for } n=2\nu.$$

$$P(z) \equiv (1+z)(1+z^5)\ldots\ldots(1+z^{4\nu+1}) \qquad \text{for } n=2\nu+1.$$

Let $(\Omega)_0$ be the set of all symmetric matrices in K satisfying the relation $\sum \omega_{0j}=0$ which is invariant under the group $OL(n)$. By using the invariant decomposition $\omega_{ij}=\omega^0_{ij}+\omega'_{ij}$, where $n\omega'_{ij}=\delta_{ij}\sum \omega_{aa}$, we have in the same way as in Lemma II$'$ the followin

Lemma III$'$. The Polynomial $P_0(Z)$ corresponding to ω^0_{ij} is given by $P_0(z) \equiv P(z)/(1+z)$, where $P(z)$ is the Polynomial of Lemma III.

The adjoint group of the group $USp(n)$ can be reduced to the form $\Omega \rightarrow A^*\Omega A$, $A \in USp(n)$. The representation $\Gamma(f_1, \ldots\ldots, f_l)$ admits a linear invariant with respect to this subgroup if and only if the columns of the diagram $(f_1+1, f_2+2. \ldots\ldots, f_l+l, h_1, \ldots\ldots, h_{n-l})$ are even. We thus see that the numbers $k_1=f_1+f_2$, $k_2=f_2+f_3$, ... are all odd numbers such that $k_1 \equiv k_2 \equiv \ldots\ldots \equiv -1 \bmod 4$; $2n > k_i \geqq 3$, and we have the following

Theorem III. The Poincaré Polynomial of the group $Sp(n)$ is

$$P(z) \equiv (1+z^3)(1+z^7)\ldots\ldots(1+z^{4\nu-1}).$$

Chapter II.
The determination of Betti-numbers.

1. $R(n, k)$ Infinitesimal transvection is

$$\delta\theta : \begin{pmatrix} & S \\ -S^* & \end{pmatrix}, \quad S=(\omega_{i\lambda}), \quad 1 \leqq i \leqq h, \quad 1 \leqq \lambda \leqq k.$$

The adjoint transformation is $S \rightarrow T_k^* S T_h$, where $T_k \in O(k)$, $T_h \in O(h)$.

(*The case of lemma I.*)

$S(n,k)$. Infinitesimal transvection is of the same form as $R(n,k)$ but

$$S=\begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \quad A=(\omega_{i\lambda}), \quad B=(\tilde{\omega}_{i\lambda}), \quad 1 \leqq i \leqq k/2, \quad 1 \leqq \lambda \leqq k/2.$$

This is also reduced to *the case of lemma I*, because the adjoint transformations are defined by $S \to T_k{}^* S T_h$, where T_k, T_h are unitary symplectic.

$A(n,k)$. The adjoint transformations are $S \to T_k{}^* S T_h$, where T_k, T_h are unitary. *This case was treated by Ehresmann.*

By lemma I we obtain the following

Theorem IV. The p-th Betti-number of the manifolds $R(n,k)$, $S(n,k)$, $A(n,k)$ are

$R(n,k)$: *The number of partitions* $p=f_1+\cdots\cdots+f_k$ such that f_i-f_k, g_i-g_h are all even, where $(g_1, g_2 \cdots\cdots g_h)$ is the conjugate of $(f_1, f_2 \cdots\cdots f_k)$.

$S(n,k)$: $B_p=0$ *if* $p \neq 4s$. *For* $p \neq 4s$ B_p *is equal to the number of partitions of* s *in* $k/2$ *non negative intergers less than or equal to* $h/2$: $S=f_1+\cdots\cdots+f_{k/2}$, $h/2 \geqq f_1 \geqq f_2 \geqq \cdots\cdots \geqq f_{k/2} \geqq 0$.

$A(n,k)$: $B_p=0$ *if* $p \neq 2s$. *For* $p=2s$ B_p *is equal to the number of partitions of* s *in* k *non negative integers less than* $h+1$.

2. The adjoint transformations of the manifolds A^+, A^-, S, C are

$A^+(n)$: $\Omega \to T^*\Omega T$, $T \epsilon O(n)$, where $\Omega=(\omega_{ij})$, $\omega_{ij}=\omega_{ji}$, $\sum \omega_{ii}=0$.

$A^-(n)$: $\Omega \to T^*\Omega T$, $T \epsilon USp(n)$, where $\Omega=(\omega_{ij})$, $\omega_{ij}+\omega_{ji}=0$, and $\sum \varepsilon_{ij}\omega_{ij}=0$ (*The cases of lemma III' and lemma II' respectively.*)

$S(n)$: $\Omega \to U\Omega U^*$, $\Omega+\Omega^*=0$; $U \epsilon U(n/2)$,

$C(n)$: $\Omega \to U\Omega U^*$, $\Omega=\Omega^*$; $U \epsilon U(n/2)$,

(*The cases treated by Ehresmann*).

By lemma II and III we obtain the following

Theorem V. The Poincaré Polynomials of the manifolds A^+, A^-, S, C *are*

$A^+(n)$: $(1+z^5)(1+z^9)\cdots\cdots(1+z^{4\nu+1})$, $\quad n=2\nu+1$

$(1+z^5)(1+z^9)\cdots\cdots(1+z^{4\nu-3})(1+z^{2\nu})$, $\quad n=2\nu$.

$A^-(n)$: $(1+z^5)(1+z^9)\cdots\cdots(1+z^{4\nu-3})$

$S(n)$: $(1+z^2)(1+z^4)\cdots\cdots(1+z^{2(\nu-1)})$.

$C(n)$: $(1+z^2)(1+z^4)\cdots\cdots(1+z^{2n})$.

3. We denote by $\tilde{A}^+(n)$, $\tilde{A}^-(n)$ the set of all symmettic or skew symmetric unitary matrices. These are also symmetric Riemannian manifolds and the Betti numbers of these manifolds can be determined in the same manner as above by the aid of lemmas II and III :.

Theorem VI. The Poincaré polynomials of the manifolds $\tilde{A}^+(n)$, $\tilde{A}^-(n)$ can be represented as $P(z)=P_0(z)/(1+z)$, where $P_0(z)$ are the Poincaré polynomials of $A^+(n)$ and $A^-(n)$ respectively.

4. The adjoint transformations of the group $GL(n)$ are

$$\Omega \rightarrow A^{-1}\Omega A, \quad \Omega=(\omega_i^j)$$

consider the outer product $\Gamma(p)$: $\omega_{i_1}^{j_1}\cdots\cdots\omega_{i_p}{}^{j_p}$. We operate the operator of Young corresponding to the partition $(f_1, \cdots\cdots, f_n)$ to the indices $i_1, i_2, \cdots\cdots, i_p$. The result is not zerro if and only if $n \geqq f_1$. With respect to the indices $j_1, \cdots\cdots j_p$ this quantity gives the irreducible representation of signature $(-g_n, \cdots\cdots, -g_1)$. Thus $\Gamma(p)$ decomposes as $\Gamma(p) \sim \sum_{n\geqq f_1 \geqq \cdots\cdots \geqq f_n \geqq 0} P(f_1\cdots$ $\cdots f_n) \times P(-g_n, \cdots\cdots, -g_1)$. By the lemma stated in Ehresmann, 3 we see that $P(f_1, \cdots\cdots, f_n) \times P(-g_n, \cdots\cdots, -g_1)$ admits a linear invariant if and only if the diagram $(f_1, f_2, \cdots\cdots, f_n)$ is self-conjugate. By examining the property of the self-conjugate diagram we readily see the following

Theorem I. The Poincaré polynomicals of the groups $GL(n)$ and $SL(n)$ are

$$(1+z)(1+z^3)\cdots\cdots(1+z^{2n-1})$$

and

$$(1+z^3)(1+z^5)\cdots\cdots(1+z^{2n-1}).$$

Remark. After all the discussions given above it remains finally to show that the invariants given above does not vanish identically. But this can easily bc carried out by means of the leading term F. Consider, for example the case of the group $GL(n)$. The leading term of the representation $P(f_1, \cdots\cdots, f_n) \times P(-f_n, \cdots\cdots, -f_1)$ is

$$F_0: \begin{array}{l} \omega_1^1\,\omega_1^2 \cdots\cdots \omega_1^{f_1} \\ \times\, \omega_2^1\,\omega_2^2 \cdots\cdots \omega_2^{f_2} \\ \times\, \omega_{g_1}^1\,\omega_{g_2}^2 \cdots\cdots \end{array} \quad (f_1=g_1, \cdots\cdots)$$

Consider the infinitesimal matrix Ω in which all ω_i^j are zero except those appearing in F_0. The only non vanishing term is F_0 and this gives at the same time the value of the corresponding invariant.

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Reference.

E. Cartan. [1]. Sur les invariants intégraux de certains espaces homogènes clos et les propriétés topologiques de ces espaces (Ann. Soc. pol Math., t. 8.)

[2]. Sur la détermination d'un systeme orthogonal complet daes un espace de Riemann symétrique clos (Rend. Circ. Math. Palermo, t. 53).

C. Ehresmann. [3]. Sur la topologie de certains espaces homogènes. (Ann. of Math, vol. 35.)

H. Weyl. [4]. The classical groups, Princeton, 1939.

[5]. Harmonics on homogenious manifolds. (Ann. of Math, vol. 35.)