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COMPACT DOUBLE DIFFERENCES OF COMPOSITION OPERATORS ON THE BERGMAN SPACES OVER THE BALL

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Abstract. Choe et al. have recently characterized compact double differences formed by four composition operators acting on the standard weighted Bergman spaces over the disk of the complex plane. In this paper, we extend such a result to the ball setting. Our characterization is obtained under a suitable restriction on inducing maps, which is automatically satisfied in the case of the disk. We exhibit concrete examples, for the first time even for single composition operators, which shows that such a restriction is essential in the case of the ball.

1. Introduction. The interplay between the operator theoretic and the function theoretic properties of composition operators, acting on classical holomorphic function spaces such as the Hardy space and the standard weighted Bergman spaces over the disk or the ball, has been extensively studied over the past several decades; we refer to standard monographs by Cowen-MacCluer [5] and Shapiro [11] for an overview of various aspects on the theory of composition operators. As is well known, the several-variable theory of composition operators is much more subtle than one-variable theory. For example, one-variable composition operators are always bounded by Littlewood's Subordination Principle on the aforementioned function spaces, which is no longer guaranteed in the several-variable case; see [7]. As a consequence, while quite an extensive study on one-variable theory of composition operators has been established during the past four decades, the several-variable theory has been relatively less known.

Recently, study on differences, or more generally linear combinations, of composition operators has been a topic of growing interest; see [3] and references therein. In the setting of the weighted Bergman spaces over the disk, Moorehouse [9] first characterized compact differences by means of a natural angular derivative cancellation property; note that boundedness is out of question. Moorhouse's characterization is then extended to several-variable settings; see [1] and [2]. In a more recent paper [3] Choe et. al. extended Moorehouse's characterization to the case of double differences. Our goal in this paper is to extend such a characterization for compact double differences to the ball.

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For a fixed positive integer n, let $\mathbf{B} = \mathbf{B}_n$ be the unit ball of the complex n-space $\mathbf{C}^n$. For $\alpha > -1$, let dv be the normalized volume measure on $\mathbf{B}$ and put

$$dv_\alpha(z) := c_\alpha(1 - |z|^2)^\alpha \, dv(z)$$

where the constant c_α is chosen so that $v_\alpha(\mathbf{B}) = 1$. For $0 < p < \infty$, the α-weighted Bergman space $A^p_\alpha(\mathbf{B})$ is the space of all holomorphic function f on $\mathbf{B}$ such that the “norm"

$$\|f\|_{A^p_\alpha} := \left\{ \int_{\mathbf{B}} |f|^p \, dv_\alpha \right\}^{1/p}$$

is finite. As is well known, the space $A^p_\alpha(\mathbf{B})$ equipped with the norm above is a Banach space for $1 \le p < \infty$. On the other hand, it is a complete metric space for $0 < p < 1$ with respect to the translation-invariant metric $(f, g) \mapsto \|f - g\|_{A^p_\alpha}$.

Let $\mathcal{S} = \mathcal{S}(\mathbf{B})$ be the class of all holomorphic self-maps of $\mathbf{B}$. Each function $\varphi \in \mathcal{S}$ induces a composition operator C_φ defined by

$$C_\varphi f := f \circ \varphi$$

for functions f holomorphic on $\mathbf{B}$. Clearly, C_φ takes the space of all holomorphic functions into itself. As is mentioned earlier, C_φ is always bounded on each $A^p_\alpha(\mathbf{B})$ when $n = 1$, but not always when $n > 1$.

We now introduce some notation to be used throughout the paper. We reserve four inducing maps $\varphi_1, \varphi_2, \varphi_3, \varphi_3 \in \mathcal{S}$, not necessarily distinct, to form the double difference

$$T := (C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_4}) .$$

Setting

$$T_{ij} := C_{\varphi_i} - C_{\varphi_j} , \qquad i, j = 1, 2, 3, 4 ,$$

note

$$T = T_{12} - T_{34} = T_{13} - T_{24} . \tag{1.1}$$

Using the pseudohyperbolic distance ρ (see Section 2.2 for the definition) on $\mathbf{B}$, we also put

$$\rho_{ij}(z) := \rho\big(\varphi_i(z), \varphi_j(z)\big) \tag{1.2}$$

and

$$M_{ij}(z) := \left(\frac{1 - |z|}{1 - |\varphi_i(z)|} + \frac{1 - |z|}{1 - |\varphi_j(z)|} \right) \rho_{ij}(z) \tag{1.3}$$

for each $i, j = 1, 2, 3, 4$. Finally, we put

$$M := M_{12} + M_{34} \qquad \text{and} \qquad \widetilde{M} := M_{13} + M_{24}$$

for short. The next theorem is our main result.

THEOREM 1.1. *Let $\alpha > -1$ and $0 < p < \infty$. With the notation as above, consider the following two assertions:*

(a) *T is compact on $A^p_\alpha(\mathbf{B})$;*

(b) $M\widetilde{M} \in C_0(\mathbf{B})$.

Then the implication (a) $\Longrightarrow$ (b) *holds. The implication* (b) $\Longrightarrow$ (a) *also holds, provided that each* C_{φ_j} *is bounded on* $A^q_\beta(\mathbf{B})$ *for some* $\beta \in (-1, \alpha)$ *and* $0 < q < \infty$.

We would like to emphasize that the inducing self-maps in Theorem 1.1 are completely general. So, considering appropriate special cases, one may recover several known results. For example, the special case $\varphi_2 = \varphi_3 = \varphi_4 \equiv 0$ reduces to the compactness characterization for single composition operators as in Corollary 5.1. See Section 5 for more special cases which might be of independent interest. In addition, we exhibit a concrete example showing that the additional boundedness assumption for the implication (b) $\Longrightarrow$ (a) is essential; see Example 5.8. As far as we know, such an example even for single composition operators has not appeared yet in the literature.

In Section 2 we recall some basic facts to be used in later sections. In Section 3 we prove the sufficiency part of Theorem 1.1. In Section 4 we prove the necessity part of Theorem 1.1. In Section 5 we observe some consequences of Theorem 1.1 concerning some special cases. We also exhibit concrete examples showing that the additional boundedness assumption for sufficiency cannot be removed.

Constants. In the rest of the paper we use the same letter C to denote various positive constants which may change at each occurrence. Variables (other than n) indicating the dependency of constants C will be often specified in the parenthesis. We use the notation $X \lesssim Y$ for nonnegative quantities X and Y to mean $X \le CY$ for some inessential constant $C > 0$. Similarly, we use the notation $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

2. Preliminaries. In this section we recall some basic facts which will be used in later sections.

2.1. Compact Operator. It seems better to clarify the notion of compact operators, since when $0 < p < 1$ the spaces under consideration are not Banach spaces. Let X and Y be topological vector spaces whose topologies are induced by complete metrics. A continuous linear operator $L : X \to Y$ is said to be compact if the image of every bounded sequence in X has a convergent subsequence in Y.

We have the following convenient compactness criterion for a linear combination of composition operators acting on the weighted Bergman spaces.

LEMMA 2.1. *Let* $\alpha > -1$ *and* $0 < p < \infty$. *Let* L *be a linear combination of composition operators and assume that* L *is bounded on* $A^p_\alpha(\mathbf{B})$. *Then* L *is compact on* $A^p_\alpha(\mathbf{B})$ *if and only if* $Lf_k \to 0$ *in* $A^p_\alpha(\mathbf{B})$ *for any bounded sequence* $\{f_k\}$ *in* $A^p_\alpha(\mathbf{B})$ *such that* $f_k \to 0$ *uniformly on compact subsets of* $\mathbf{B}$.

A proof can be found in [5, Proposition 3.11] for a single composition operator over the disk and it can be easily modified for a linear combination over the ball.

2.2. Pseudohyperbolic Distance. The pseudohyperbolic distance between $z, w \in \mathbf{B}$ is given by

$$\rho(z, w) := |\sigma_z(w)|$$

where σ_z is the involutive automorphism of $\mathbf{B}$ that exchange 0 and z. More explicitly, we have

$$1 - \rho^2(z,w) = \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2} \tag{2.1}$$

where $\langle z,w\rangle$ denotes the Hermitian inner product of $z, w \in \mathbf{C}^n$. This yields an inequality

$$\frac{1-|z|^2}{|1-\langle z,w\rangle|} \le \sqrt{1-\rho(z,w)^2} \quad \text{for} \quad |w| \le |z|, \tag{2.2}$$

which is useful for our purpose. As is well known, each holomorphic self-map F of $\mathbf{B}$ is a ρ-contraction, i.e.,

$$\rho\big(F(z), F(w)\big) \le \rho(z,w), \qquad z, w \in \mathbf{B}; \tag{2.3}$$

see [10, Theorem 8.1.4].

For $0 < r < 1$ and $z \in \mathbf{B}$, we denote by

$$E_r(z) := \{w \in \mathbf{B} : \rho(z,w) < r\}$$

the pseudo-hyperbolic ball with center z and radius r. Clearly, $E_r(0) = r\mathbf{B}$. Using the well-known automorphism invariance of ρ, we also note $E_r(z) = \sigma_z(r\mathbf{B})$. Given $0 < r < 1$ and $\alpha > -1$, we will use the well-known estimates

$$1-|z| \approx 1-|w| \approx |1-\langle z,w\rangle|, \qquad w \in E_r(z) \tag{2.4}$$

and

$$v_\alpha[E_r(z)] \approx (1-|z|)^{n+1+\alpha}; \tag{2.5}$$

the constants suppressed in these estimates depend only on r and α.

In most of the cases we will work with $r = 1/2$. So, we put

$$E(z) := E_{1/2}(z)$$

for brevity.

2.3. Test Function. Given $\alpha > -1$, we have the submean value type inequality

$$|f(a)|^p \le \frac{C}{(1-|a|^2)^{n+1+\alpha}} \|f\|^p_{A^p_\alpha}, \qquad a \in \mathbf{B}, \tag{2.6}$$

valid for all $f \in A^p_\alpha(\mathbf{B})$ and $0 < p < \infty$. Here the constant C depends only on α. So each point evaluation in $\mathbf{B}$ is a bounded linear functional on the Hilbert space $A^2_\alpha(\mathbf{B})$. Thus, to each $a \in \mathbf{B}$ corresponds a unique reproducing kernel whose explicit formula is actually given by $z \to (1-\langle a,z\rangle)^{-(n+1+\alpha)}$. Suitable powers of those kernel functions will be our test functions in connection with Lemma 2.1. To this end we introduce functions τ_a on $\mathbf{B}$ defined by

$$\tau_a(z) := \frac{1}{1-\langle z,a\rangle} \tag{2.7}$$

for $a \in \mathbf{B}$. When $sp > n+1+\alpha$, we have the optimal norm estimates (see, for example, [12, Theorem 1.12])

$$\|\tau_a^s\|^p_{A^p_\alpha} \approx \frac{1}{(1-|a|^2)^{sp-(n+1+\alpha)}}, \qquad a \in \mathbf{B} \tag{2.8}$$

and thus

$$\frac{\tau_a^s}{\|\tau_a^s\|_{A^p_\alpha}} \to 0 \quad \text{uniformly on compact subsets of } \mathbf{B} \tag{2.9}$$

as $|a| \to 1$.

2.4. Carleson Measure. Let $\alpha > -1$ and μ be a positive finite Borel measure on $\mathbf{B}$. For $0 < r < 1$ and $0 < p < \infty$, it is well known (see [12, Section 2.4]) that

$$\text{the embedding } A^p_\alpha(\mathbf{B}) \subset L^p(d\mu) \text{ is bounded} \iff \sup_{z\in\mathbf{B}} \frac{\mu[E_r(z)]}{v_\alpha[E_r(z)]} < \infty .$$

We say that μ is an α-*Carleson measure* if either side of the above holds. Note that the notion of α-Carleson measures is independent of the size of p or r. So, setting

$$\|\mu\|_\alpha = \sup_{z\in\mathbf{B}} \frac{\mu[E(z)]}{v_\alpha[E(z)]},$$

we see that μ is an α-Carleson measure if and only of $\|\mu\|_\alpha < \infty$. Moreover, we have

$$\sup_{\|f\|_{A^p_\alpha}\le 1} \|f\|^p_{L^p(\mu)} \approx \|\mu\|_\alpha \tag{2.10}$$

for α-Carleson measures μ on $\mathbf{B}$; the constants suppressed above depend only on α.

The connection between composition operators and Carleson measures comes from the standard identity (see [6, p. 163])

$$\int_{\mathbf{B}} (h \circ \varphi)\, d\mu = \int_{\mathbf{B}} h\, d(\mu \circ \varphi^{-1}) \tag{2.11}$$

valid for holomorphic self-maps φ of $\mathbf{B}$ and Borel functions $h \ge 0$ on $\mathbf{B}$. Here, $\mu \circ \varphi^{-1}$ denotes the pullback measure defined by $(\mu \circ \varphi^{-1})(E) = \mu[\varphi^{-1}(E)]$ for Borel sets $E \subset \mathbf{B}$. In particular, one can easily see from (2.11) that $C_\varphi : A^p_\alpha(\mathbf{B}) \to L^p(\mu)$ is bounded if and only if $\|\mu \circ \varphi^{-1}\|_\alpha < \infty$.

2.5. Invariant Laplacian. For a function f on $\mathbf{B}$, we define

$$\widetilde{\Delta} f(z) := \Delta(f \circ \sigma_z)(0), \qquad z \in \mathbf{B},$$

where Δ is the ordinary Laplacian. This operator $\widetilde{\Delta}$ is called the *invariant* Laplacian because it is automorphism invariant in the sense that

$$\widetilde{\Delta}(f \circ \sigma) = (\widetilde{\Delta} f) \circ \sigma$$

for all automorphisms σ of $\mathbf{B}$. Given $\alpha > -1$ and $0 < p < \infty$, we will use the norm equivalence

$$\int_{\mathbf{B}} (\widetilde{\Delta}|f|^2)^{p/2}\, dv_\alpha \approx \int_{\mathbf{B}} |f(z) - f(0)|^p\, dv_\alpha(z) \tag{2.12}$$

for function f holomorphic on $\mathbf{B}$; the constants suppressed here depend only on n and α. We refer to [12, Section 2.3] for more details.

The next lemma is taken from [2, Lemma 4.4].

LEMMA 2.2. *Let $\alpha > -1$, $0 < p < \infty$ and $0 < r_1 < r_2 < 1$. Then there is a constant $C = C(\alpha, p, r_1, r_2) > 0$ such that*

$$|f(a) - f(b)|^p \le C \frac{\rho(a,b)^p}{(1-|a|^2)^{n+1+\alpha}} \int_{E_{r_2}(a)} (\widetilde{\Delta}|f|^2)^{p/2} \, dv_\alpha$$

for all $f \in H(\mathbf{B})$ and $a, b \in \mathbf{B}$ with $a \in E_{r_1}(b)$.

2.6. Angular Derivative. The well-known notion of the angular derivatives on the disk has a natural extension to the ball. To state it we first recall some terminology.

Let $\zeta \in \partial\mathbf{B}$. A continuous function $\gamma : [0, 1) \to \mathbf{B}$ with $\lim_{t\to 1} \gamma(t) = \zeta$ is said to be a *restricted ζ-curve* if

$$\lim_{t\to 1} \frac{|\gamma(t) - \langle\gamma(t), \zeta\rangle\zeta|^2}{1 - |\langle\gamma(t), \zeta\rangle|^2} = 0 \quad \text{and} \quad \sup_{0\le t<1} \frac{|\zeta - \langle\gamma(t), \zeta\rangle\zeta|}{1 - |\langle\gamma(t), \zeta\rangle|} < \infty .$$

Note that a ζ-curve contained in the unit disk of the complex line through ζ is restricted if and only if it approaches to ζ nontangentially. We say that $f : \mathbf{B} \to \mathbf{C}$ has *restricted limit* at ζ, denoted by $f(\zeta)$, if

$$\lim_{t\to 1} f(\gamma(t)) = f(\zeta)$$

for every restricted ζ-curve γ. In this case, we write

$$R \lim_{z\to\zeta} f(z) = f(\zeta) .$$

For a holomorphic self-map φ of $\mathbf{B}$, we say that φ has *finite angular derivative* at ζ, denoted by $D\varphi(\zeta)$, if there exists $\eta \in \partial\mathbf{B}$ such that

$$D\varphi(\zeta) = R \lim_{z\to\zeta} \frac{1 - \langle\varphi(z), \eta\rangle}{1 - \langle z, \zeta\rangle} .$$

Note that the above forces φ to have restricted limit η at ζ with $\varphi(\zeta) = \eta$.

By the well-known Julia-Carathéodory Theorem we have

$$\varphi \text{ has finite angular derivative at } \zeta \iff d_\varphi(\zeta) := \liminf_{z\to\zeta} \frac{1 - |\varphi(z)|}{1 - |z|} < \infty$$

where the liminf is taken as $z \to \zeta$ unrestrictedly in $\mathbf{B}$. Moreover, if this is the case, then

$$D\varphi(\zeta) = d_\varphi(\zeta); \tag{2.13}$$

see [5, Theorem 2.81] or [10, Theorem 8.5.6] for details and further results. Thus, the *angular derivative set*

$$A(\varphi) := \left\{\zeta \in \partial\mathbf{B} : d_\varphi(\zeta) < \infty\right\}$$

consists of all boundary points at which φ has finite angular derivatives. Moreover, it is known that

$$\angle \lim_{\lambda\to 1} \frac{1 - |\varphi(\lambda\zeta)|^2}{1 - |\lambda|^2} = d_\varphi(\zeta), \qquad \zeta \in A(\varphi) \tag{2.14}$$

where $\angle \lim$ denotes the nontangential limit; see [2, Lemma 3.1]. Finally, we remark $d_\varphi(\zeta) \ge \frac{1-|\varphi(0)|}{1+|\varphi(0)|} > 0$ by the Schwarz-Pick Lemma (see [10, Theorem 8.1.4]).

3. Necessity for Compactness. In this section we prove the first part of Theorem 1.1. Our proof will be completed by applying the compactness of the operator under consideration to a suitably-chosen collection of test functions. For that purpose we need several auxiliary lemmas.

Before proceeding, we introduce some auxiliary functions. For $a_1, a_2, a_3, a_4 \in \mathbf{B}$ and a positive integer m, put

$$\begin{aligned} I_m(z) &= I_m(z; a_1, a_2, a_3) \\ &:= (1-|z|^2)^m \left|\tau_{a_1}^m(z) - \tau_{a_2}^m(z) - \tau_{a_3}^m(z)\right| \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} J_m(z) &= J_m(z; a_1, a_2, a_3, a_4) \\ &:= (1-|z|^2)^m \left|\tau_{a_1}^m(z) - \tau_{a_2}^m(z) - \tau_{a_3}^m(z) + \tau_{a_4}^m(z)\right| \end{aligned} \tag{3.2}$$

for $z \in \mathbf{B}$; recall that τ_a denotes the function specified in (2.7). We will need certain lower estimates for these functions. We begin with the following inequality.

LEMMA 3.1. *The inequality*

$$\left|\left(\frac{1-|a|^2}{1-\langle a_1, a\rangle}\right)^m - \left(\frac{1-|a|^2}{1-\langle a_2, a\rangle}\right)^m\right| \le m2^{m+2}\rho(a_1,a_2)\left(\frac{1-|a|}{1-|a_1|} + \frac{1-|a|}{1-|a_2|}\right)$$

holds for a, a_1, $a_2 \in \mathbf{B}$ and positive integers m.

PROOF. Fix $a, a_1, a_2 \in \mathbf{B}$ and a positive integer m. Put

$$\lambda := \frac{1-|a|}{1-\langle a_1, a\rangle} \quad \text{and} \quad \eta := \frac{1-|a|}{1-\langle a_2, a\rangle}$$

for short. Since $|\lambda| < 1$ and $|\eta| < 1$, we have

$$\begin{aligned} \left|\left(\frac{1-|a|^2}{1-\langle a_1, a\rangle}\right)^m - \left(\frac{1-|a|^2}{1-\langle a_2, a\rangle}\right)^m\right| &\le 2^m|\lambda^m - \eta^m| \\ &= 2^m|\lambda-\eta|\left|\sum_{j=1}^m \lambda^{m-j}\eta^{j-1}\right| \\ &\le m2^m|\lambda-\eta|\,. \end{aligned}$$

Thus, in order to complete the proof, it is sufficient to show

$$\left|\frac{1}{1-\langle a_1, a\rangle} - \frac{1}{1-\langle a_2, a\rangle}\right| \le 4\rho(a_1,a_2)\left(\frac{1}{1-|a_1|} + \frac{1}{1-|a_2|}\right). \tag{3.3}$$

To prove this inequality, put $a =: r\zeta$ where $r = |a|$ and $\zeta \in \partial\mathbf{B}$. Also, put $\lambda_j = \langle a_j, \zeta\rangle$ for $j = 1, 2$. Using the explicit formula (see [10, Section 2.2] or [12, Section 1.2]) for the involutive automorphisms of $\mathbf{B}$, we note

$$\rho(\lambda_1\zeta, \lambda_2\zeta) = \left|\frac{\lambda_1 - \lambda_2}{1 - \lambda_1\overline{\lambda_2}}\right|.$$

We also note $|1 - \xi| \le 2|1 - r\xi|$ for complex numbers ξ with $|\xi| < 1$. Accordingly, we have

$$\begin{aligned}\left|\frac{1}{1 - r\lambda_1} - \frac{1}{1 - r\lambda_2}\right| &= \frac{r|\lambda_1 - \lambda_2|}{|1 - r\lambda_1||1 - r\lambda_2|} \\ &\le \frac{4|\lambda_1 - \lambda_2|}{|1 - \lambda_1||1 - \lambda_2|} \\ &= 4\rho\big(\lambda_1\zeta, \lambda_2\zeta\big) \cdot \frac{|1 - \lambda_1\overline{\lambda_2}|}{|1 - \lambda_1||1 - \lambda_2|} \\ &\le 4\rho(a_1, a_2) \cdot \frac{|1 - \lambda_1\overline{\lambda_2}|}{|1 - \lambda_1||1 - \lambda_2|}\,;\end{aligned}$$

the last inequality holds by (2.3). Now, using the elementary inequality $|1 - \lambda_1\overline{\lambda_2}| \le |1 - \lambda_1| + |1 - \lambda_2|$, we conclude (3.3), as desired. The proof is complete. □

We now prove the following lower estimate for the functions I_m.

LEMMA 3.2. *Given* $0 < \varepsilon < 1$, *there is a collection of positive integers* $\{N_k\}_{k=1}^4$ *such that*

$$\max_{\substack{1 \le k \le 4 \\ 1 \le j \le 3}} I_{N_k}(a_j) \ge \frac{1}{2} \tag{3.4}$$

whenever $a_1, a_2, a_3 \in \mathbf{B}_n$ *satisfy*

$$\varepsilon \le \frac{1 - |a_i|}{1 - |a_j|} \le \frac{1}{\varepsilon}, \qquad i, j = 1, 2, 3\,. \tag{3.5}$$

In addition, the integers N_k*'s can be chosen arbitrarily large.*

PROOF. Fix $0 < \varepsilon < 1$ and suppose that $a_1, a_2, a_3 \in \mathbf{B}$ satisfy (3.5). For simplicity we put

$$r_{ij} := \rho(a_i, a_j), \qquad i, j = 1, 2, 3$$

and

$$d := \min\{r_{12}, r_{13}\}\,.$$

Now we fix any positive integer m. We consider the following three cases separately:

(i) $m2^{m+4}d \le \varepsilon$;
(ii) $m2^{m+4}d > \varepsilon$ and $|a_1| \ge \max\{|a_2|, |a_3|\}$;
(iii) $m2^{m+4}d > \varepsilon$ and $|a_1| < \max\{|a_2|, |a_3|\}$.

Case (i) : First, assume $d = r_{12}$. By Lemma 3.1, we have

$$\begin{aligned} I_m(a_3) &\geq 1 - \left| \left(\frac{1-|a_3|^2}{1-\langle a_1, a_3\rangle} \right)^m - \left(\frac{1-|a_3|^2}{1-\langle a_2, a_3\rangle} \right)^m \right| \\ &\geq 1 - m2^{m+2} r_{12} \left(\frac{1-|a_3|}{1-|a_1|} + \frac{1-|a_3|}{1-|a_2|} \right) \\ &\geq 1 - \frac{\varepsilon}{4} \cdot \frac{2}{\varepsilon} \\ &= \frac{1}{2}\,. \end{aligned}$$

Similarly, we obtain $I_m(a_2) \geq 1/2$ if $d = r_{13}$. So, choosing $N_1 = m$, we obtain

$$\max_{2\leq j\leq 3} I_{N_1}(a_j) \geq \frac{1}{2}\,. \tag{3.6}$$

Case (ii) : Since $m2^{m+4}d > \varepsilon$ and $|a_1| \geq \max\{|a_2|, |a_3|\}$, we have by (2.2)

$$\frac{1-|a_1|^2}{|1-\langle a_j, a_1\rangle|} \leq \sqrt{1-d^2} \leq \sqrt{1 - \frac{\varepsilon^2}{m^2 4^{m+4}}}$$

for $j = 2, 3$. So, we have

$$I_\nu(a_1) \geq 1 - \sum_{j=2}^{3} \left(\frac{1-|a_1|^2}{|1-\langle a_j, a_1\rangle|} \right)^\nu \geq 1 - 2\left(1 - \frac{\varepsilon^2}{m^2 4^{m+4}}\right)^{\nu/2}$$

for any ν. Thus, choosing $N_2 = N_2(\varepsilon, m) \geq m$ so that

$$\left(1 - \frac{\varepsilon^2}{m^2 4^{m+4}}\right)^{N_2} \leq \frac{1}{16}\,, \tag{3.7}$$

we obtain

$$I_{N_2}(a_1) \geq \frac{1}{2}\,. \tag{3.8}$$

Case (iii) : Without loss of generality, we assume $|a_2| \geq |a_3|$. Since $r_{12} \geq d > \frac{\varepsilon}{m2^{m+4}}$ and $|a_2| > |a_1|$, we have by (2.2)

$$\frac{1-|a_2|^2}{|1-\langle a_1, a_2\rangle|} \leq \sqrt{1-r_{12}^2} \leq \sqrt{1 - \frac{\varepsilon^2}{m^2 4^{m+4}}}\,.$$

For N_2 chosen in (3.7) and integers $\nu \geq N_2$, we obtain

$$\begin{aligned} I_\nu(a_2) &\geq \left| 1 + \left(\frac{1-|a_2|^2}{1-\langle a_3, a_2\rangle} \right)^\nu \right| - \left(\frac{1-|a_2|^2}{|1-\langle a_1, a_2\rangle|} \right)^\nu \\ &\geq \left| 1 + \left(\frac{1-|a_2|^2}{1-\langle a_3, a_2\rangle} \right)^\nu \right| - \left(1 - \frac{\varepsilon^2}{m^2 4^{m+4}}\right)^{\nu/2} \end{aligned}$$

$$\geq \frac{3}{4} + \mathrm{Re}\left(\frac{1-|a_2|^2}{1-\langle a_3, a_2\rangle}\right)^\nu .$$

In conjunction with the estimate above, we note that for any complex number λ at least one of λ, λ^2 and λ^3 has the nonnegative real part. Consequently, we have

$$\max_{2\leq k\leq 4} I_{N_k}(a_2) \geq \frac{3}{4}, \tag{3.9}$$

where $N_3 = 2N_2$ and $N_4 = 3N_2$.

Now, combining inequalities (3.6), (3.8) and (3.9), we conclude (3.4). In addition, recalling that m is an arbitrary positive integer and $N_k \geq m$ for each k, we see that the integers N_k's can be chosen arbitrary large. The proof is complete. □

For the functions J_m, we prove the following lower estimate.

LEMMA 3.3. *Given $0 < \varepsilon < 1$, there is a collection of positive integers $\{N_k\}_{k=1}^6$ such that*

$$\max_{\substack{1\leq k\leq 6\\ 1\leq j\leq 4}} J_{N_k}(a_j) \geq \frac{1}{2} \tag{3.10}$$

whenever $a_1, a_2, a_3, a_4 \in \mathbf{B}_n$ satisfy

$$\min_{1\leq j\leq 3} |a_j| \geq \frac{1}{2}, \quad \varepsilon \leq \min\{\rho(a_1, a_2), \rho(a_1, a_3)\} \tag{3.11}$$

and

$$\varepsilon \leq \frac{1-|a_i|}{1-|a_j|} \leq \frac{1}{\varepsilon}, \qquad i, j = 1, 2, 3\,. \tag{3.12}$$

In addition, the integers N_k's can be chosen arbitrarily large.

PROOF. Fix $0 < \varepsilon < 1$ and choose a positive integer $m = m(\varepsilon)$ such that

$$(1-\varepsilon^2)^m \leq \frac{1}{16} \quad \text{and} \quad \left(1 + \frac{1-\varepsilon}{3\varepsilon}\right)^{-m} \leq \frac{1}{6}\,. \tag{3.13}$$

Note that m can be chosen arbitrary large. For $a_1, a_2, a_3, a_4 \in \mathbf{B}$ satisfying (3.11) and (3.12), put

$$r_{ij} := \rho(a_i, a_j), \qquad i, j = 1, 2, 3, 4$$

for simplicity.

In case $|a_1| \geq \max\{|a_2|, |a_3|\}$, we obtain by (2.2)

$$\frac{1-|a_1|^2}{|1-\langle a_j, a_1\rangle|} \leq \sqrt{1-r_{1j}^2} \leq \sqrt{1-\varepsilon^2},$$

for $j = 2, 3$. It follows that

$$\begin{aligned} J_\nu(a_1) &\geq \left|1 + \left(\frac{1-|a_1|^2}{1-\langle a_4, a_1\rangle}\right)^\nu\right| - \sum_{j=2}^{3}\left(\frac{1-|a_1|^2}{|1-\langle a_j, a_1\rangle|}\right)^\nu \\ &\geq \left|1 + \left(\frac{1-|a_1|^2}{1-\langle a_4, a_1\rangle}\right)^\nu\right| - 2(1-\varepsilon^2)^{\nu/2} \end{aligned}$$

$$\geq \frac{1}{2} + \operatorname{Re}\left(\frac{1-|a_1|^2}{1-\langle a_4, a_1\rangle}\right)^{\nu}$$

for all integers $\nu \geq m$. Thus, by the same argument as in the proof of Case (iii) of Lemma 3.2, we obtain

$$\max_{1\leq k\leq 3} J_{N_k}(a_1) \geq \frac{1}{2} \tag{3.14}$$

where $N_1 = m, N_2 = 2m$ and $N_3 = 3m$.

We now consider the case when $\max\{|a_2|, |a_3|\} > |a_1|$ for the rest of the proof. By symmetry we may further assume $|a_2| \geq |a_3|$ so that $|a_2| \geq \max\{|a_1|, |a_3|\}$. Note from (2.2) and (3.11)

$$\frac{1-|a_2|^2}{|1-\langle a_1, a_2\rangle|} \leq \sqrt{1-r_{12}^2} \leq \sqrt{1-\varepsilon^2}\,. \tag{3.15}$$

Fixing an integer $N_4 = N_4(\varepsilon, m) \geq m$ such that

$$\left(1-\frac{\varepsilon^4}{m^2 4^{m+5}}\right)^{N_4} \leq \frac{1}{16}, \tag{3.16}$$

we claim

$$\max_{\substack{1\leq k\leq 6\\ 1\leq j\leq 4}} J_{N_k}(a_j) \geq \frac{1}{2} \tag{3.17}$$

where $N_5 := 2N_4$ and $N_6 := 3N_4$. This estimate, together with (3.14), completes the proof.

In order to prove (3.17), we consider the following three cases separately:

(i) $\dfrac{1-|a_2|}{1-|a_4|} > \dfrac{1}{\varepsilon}$ or $\dfrac{1-|a_2|}{1-|a_4|} < \varepsilon$;

(ii) $\min\{r_{24}, r_{34}\} > \varepsilon^2(m2^{m+5})^{-1}$;

(iii) $\varepsilon \leq \dfrac{1-|a_2|}{1-|a_4|} \leq \dfrac{1}{\varepsilon}$ and $\min\{r_{24}, r_{34}\} \leq \varepsilon^2(m2^{m+5})^{-1}$.

Case (i): First, assume $\frac{1-|a_2|}{1-|a_4|} > \frac{1}{\varepsilon}$. Since $|a_2| \geq \max\{|a_1|, |a_3|\}$, we have

$$\frac{|1-\langle a_j, a_4\rangle|}{1-|a_4|} \geq \frac{1-|a_j||a_4|}{1-|a_4|} \geq \frac{1-|a_2||a_4|}{1-|a_4|}$$

for $j = 1, 2, 3$. Meanwhile, since $|a_2| \geq \frac{1}{2}$ by (3.11), we have $1-|a_4| < \varepsilon(1-|a_2|) < \frac{1}{2}$ and thus $|a_4| > \frac{1}{2}$. So, we obtain

$$\frac{1-|a_2||a_4|}{1-|a_4|^2} = 1 + \frac{|a_4|}{1+|a_4|}\left(\frac{1-|a_2|}{1-|a_4|} - 1\right) > 1 + \frac{1-\varepsilon}{3\varepsilon} \tag{3.18}$$

and thus

$$\begin{aligned} J_{N_1}(a_4) &= J_m(a_4) \\ &\geq 1 - \sum_{j=1}^{3}\left(\frac{1-|a_4|^2}{|1-\langle a_j, a_4\rangle|}\right)^m \\ &\geq 1 - 3\left(1+\frac{1-\varepsilon}{3\varepsilon}\right)^{-m} \end{aligned}$$

$$\geq \frac{1}{2};$$

the last inequality holds by (3.13).

Now, we consider the case $\frac{1-|a_2|}{1-|a_4|} < \varepsilon$. In this case, exchanging the roles of a_2 and a_4 in (3.18), we have

$$\frac{1-|a_4||a_2|}{1-|a_2|^2} > 1 + \frac{1-\varepsilon}{3\varepsilon}.$$

This, together with (3.13) and (3.15), yields

$$\begin{aligned} J_\nu(a_2) &\geq \left|1 + \left(\frac{1-|a_2|^2}{1-\langle a_3, a_2\rangle}\right)^\nu\right| - \left(\frac{1-|a_2|^2}{|1-\langle a_1, a_2\rangle|}\right)^\nu - \left(\frac{1-|a_2|^2}{|1-\langle a_4, a_2\rangle|}\right)^\nu \\ &\geq \left|1 + \left(\frac{1-|a_2|^2}{1-\langle a_3, a_2\rangle}\right)^\nu\right| - (1-\varepsilon^2)^{\nu/2} - \left(1 + \frac{1-\varepsilon}{3\varepsilon}\right)^{-\nu} \\ &\geq \frac{7}{12} + \operatorname{Re}\left(\frac{1-|a_2|^2}{1-\langle a_3, a_2\rangle}\right)^\nu \end{aligned}$$

for all integers $\nu \geq m$. We thus conclude

$$\max_{1\leq k\leq 3} J_{N_k}(a_2) \geq \frac{1}{2}$$

as in the proof of (3.14). This completes the proof of (3.17) for Case (i).

Case (ii): We first consider the case of $|a_2| > |a_4|$. Since $m2^{m+5}r_{24} > \varepsilon^2$, we have by (2.2)

$$\frac{1-|a_2|^2}{|1-\langle a_4, a_2\rangle|} \leq \sqrt{1-r_{24}^2} \leq \sqrt{1-\frac{\varepsilon^4}{m^2 4^{m+5}}}.$$

Thus we have by (3.15) and (3.16)

$$\begin{aligned} J_\nu(a_2) &\geq \left|1 + \left(\frac{1-|a_2|^2}{1-\langle a_3, a_2\rangle}\right)^\nu\right| - \left(\frac{1-|a_2|^2}{|1-\langle a_1, a_2\rangle|}\right)^\nu - \left(\frac{1-|a_2|^2}{|1-\langle a_4, a_2\rangle|}\right)^\nu \\ &\geq \left|1 + \left(\frac{1-|a_2|^2}{1-\langle a_3, a_2\rangle}\right)^\nu\right| - (1-\varepsilon^2)^{\nu/2} - \left(1 - \frac{\varepsilon^4}{m^2 4^{m+5}}\right)^{\nu/2} \\ &\geq \frac{1}{2} + \operatorname{Re}\left(\frac{1-|a_2|^2}{1-\langle a_3, a_2\rangle}\right)^\nu, \end{aligned}$$

for all $\nu \geq N_4$. We thus conclude

$$\max_{4\leq k\leq 6} J_{N_k}(a_2) \geq \frac{1}{2}$$

as in the proof of (3.14).

We now consider the case $|a_2| \le |a_4|$. This time we have by (2.2)

$$\frac{1-|a_4|^2}{|1-\langle a_j, a_4\rangle|} \le \sqrt{1-r_{j4}^2} \le \sqrt{1-\frac{\varepsilon^4}{m^2 4^{m+5}}}$$

for $j = 2, 3$. Thus, repeating a similar argument, we have

$$J_\nu(a_4) \ge \frac{1}{2} + \mathrm{Re}\left(\frac{1-|a_4|^2}{1-\langle a_1, a_4\rangle}\right)^\nu, \quad \nu \ge N_4$$

and thus conclude

$$\max_{4\le k\le 6} J_{N_k}(a_4) \ge \frac{1}{2}.$$

This completes the proof of (3.17) for Case (ii).

Case (iii): First, we consider the case $r_{24} \le r_{34}$. Since $\frac{1-|a_2|}{1-|a_4|} < \frac{1}{\varepsilon}$ and $m2^{m+2}r_{24} \le \frac{\varepsilon^2}{8}$, we have by Lemma 3.1 and (3.12)

$$\begin{aligned}\left|\left(\frac{1-|a_1|^2}{1-\langle a_2, a_1\rangle}\right)^m - \left(\frac{1-|a_1|^2}{1-\langle a_4, a_1\rangle}\right)^m\right| &\le m2^{m+2}r_{24}\left(\frac{1-|a_1|}{1-|a_2|} + \frac{1-|a_1|}{1-|a_4|}\right)\\ &\le \frac{\varepsilon^2}{8}\left(\frac{1-|a_1|}{1-|a_2|} + \frac{1-|a_1|}{1-|a_2|}\cdot\frac{1-|a_2|}{1-|a_4|}\right)\\ &< \frac{\varepsilon^2}{8}\left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon^2}\right)\\ &\le \frac{1}{4}.\end{aligned}$$

Note that the same estimate holds with a_3 in place of a_1. Also, recall $|a_2| \ge \max\{|a_1|, |a_3|\}$. Accordingly, we may assume $|a_1| \ge |a_3|$ so that

$$\frac{1-|a_1|^2}{|1-\langle a_3, a_1\rangle|} \le \sqrt{1-r_{13}^2} \le \sqrt{1-\varepsilon^2} \tag{3.19}$$

by (2.2) and (3.11). It follows that

$$\begin{aligned}J_{N_1}(a_1) &= J_m(a_1)\\ &\ge 1 - \left(\frac{1-|a_1|^2}{|1-\langle a_3, a_1\rangle|}\right)^m - \left|\left(\frac{1-|a_1|^2}{1-\langle a_2, a_1\rangle}\right)^m - \left(\frac{1-|a_1|^2}{1-\langle a_4, a_1\rangle}\right)^m\right|\\ &\ge \frac{3}{4} - (1-\varepsilon^2)^{m/2}\\ &\ge \frac{1}{2};\end{aligned}$$

the last inequality holds by (3.13).

Next, we consider the case $r_{34} \le r_{24}$. In this case a similar argument using Lemma 3.1 yields

$$\left| \left(\frac{1 - |a_2|^2}{1 - \langle a_3, a_2 \rangle} \right)^m - \left(\frac{1 - |a_2|^2}{1 - \langle a_4, a_2 \rangle} \right)^m \right| \le \frac{1}{4}\,.$$

Thus, using (3.15) in place of (3.19), we obtain

$$J_{N_1}(a_2) = J_m(a_2) \ge \frac{1}{2}\,.$$

This completes the proof of (3.17) for Case (iii). The proof is complete. □

Having Lemmas 3.2 and 3.3, we now proceed to the proof of the first part of in Theorem 1.1, which can be restated as follows.

PROPOSITION 3.4. *With the notation as in Theorem 1.1, assume that T is compact on $A^p_\alpha(\mathbf{B})$. Then $M\widetilde{M} \in C_0(\mathbf{B})$.*

PROOF. We assume $M\widetilde{M} \notin C_0(\mathbf{B})$ and complete the proof by deriving a contradiction to the compactness of T.

Since $M\widetilde{M} \notin C_0(\mathbf{B})$, there is a sequence $\{z_k\} \in \mathbf{B}$ with $|z_k| \to 1$ and

$$\inf_k M(z_k)\widetilde{M}(z_k) > 0 \tag{3.20}$$

as $k \to \infty$. For simplicity we introduce some temporary notation associated with this sequence. Put

$$a_{jk} := \varphi_j(z_k) \quad \text{and} \quad Q_{jk} := \frac{1 - |z_k|}{1 - |a_{jk}|}$$

for each j and k. By the Schwarz-Pick Lemma, we have

$$\sup_k Q_{jk} \le \frac{1 + |\varphi_j(0)|}{1 - |\varphi_j(0)|} \tag{3.21}$$

for each j. Thus, after passing to a subsequence of $\{z_k\}$ if necessary, we may assume that the sequence $\{Q_{jk}\}_k$ converges for each j. So, we note

$$V := \{j : \lim_{k\to\infty} Q_{jk} > 0\} \ne \emptyset$$

by (3.20). Thus, for each $j \in V$, we have $1 - |z_k| \approx 1 - |a_{jk}|$ for all k and thus $|a_{jk}| \to 1$ as $k \to \infty$. Finally, put

$$r_{ijk} := \rho_{ij}(z_k) = \rho(a_{ik}, a_{jk})$$

for each i, j and k. With these notations, we have

$$M_{ij}(z_k) = (Q_{ik} + Q_{jk})r_{ijk}$$

for each i, j and k. Here, ρ_{ij} and M_{ij} are the functions specified in (1.2) and (1.3).

Now, we introduce our test function. Let Λ be the set of all positive integers greater than $\frac{n+1+\alpha}{p}$. Given $j \in V$ and $m \in \Lambda$, let

$$f_k^{j,m}(z) := \frac{(1-|a_{jk}|^2)^{m-\frac{n+1+\alpha}{p}}}{(1-\langle z, a_{jk}\rangle)^m}, \qquad z \in \mathbf{B}$$

for positive integers k. For fixed j and m, note from (2.8) and (2.9) that $\{f_k^{j,m}\}_k$ is bounded in $A^p_\alpha(\mathbf{B})$ and that $f_k^{j,m} \to 0$ uniformly on compact subsets of $\mathbf{B}$ as $k \to \infty$. So, we have

$$\lim_{k\to\infty} \|Tf_k^{j,m}\|_{A^p_\alpha} = 0$$

by Lemma 2.1. This, together with (2.6), yields

$$\begin{aligned}\|Tf_k^{j,m}\|_{A^p_\alpha} &\gtrsim |Tf_k^{j,m}(z_k)|(1-|z_k|^2)^{\frac{n+1+\alpha}{p}}\\ &\approx |Tf_k^{j,m}(z_k)|(1-|a_{jk}|^2)^{\frac{n+1+\alpha}{p}}\\ &= J_m(a_{jk})\end{aligned}$$

where $J_m = J_m(\,\cdot\,; a_{1k}, a_{2k}, a_{3k}, a_{4k})$ is the function introduced in (3.2). Accordingly, we have

$$\lim_{k\to\infty} J_m(a_{jk}) = 0\,.$$

Note that this holds for each $j \in V$ and $m \in \Lambda$. As a consequence we obtain

$$\lim_{k\to\infty}\left[\sum_{m\in\Lambda_F}\sum_{j\in V} J_m(a_{jk})\right] = 0 \tag{3.22}$$

for any finite set $\Lambda_F \subset \Lambda$. On the other hand, we will prove below that this is not possible by (3.20), which is a contradiction.

Note from (3.21) that both M and $\widetilde{M}$ are bounded above on $\mathbf{B}$. We also note from (3.20) that M and $\widetilde{M}$ both bounded below along the sequence $\{z_k\}$ by some positive number, say $2c$. So, we have

$$\max\{M_{12}(z_k), M_{34}(z_k)\} \ge c \quad \text{and} \quad \max\{M_{13}(z_k), M_{24}(z_k)\} \ge c$$

for each k. Thus, for each k, at least one of the following four cases holds:

(a) $\min\{M_{12}(z_k), M_{13}(z_k)\} \ge c$;
(b) $\min\{M_{12}(z_k), M_{24}(z_k)\} \ge c$;
(c) $\min\{M_{34}(z_k), M_{13}(z_k)\} \ge c$;
(d) $\min\{M_{34}(z_k), M_{24}(z_k)\} \ge c$.

First, consider Case (a). Restating Case (a) more explicitly, we have

$$(Q_{1k} + Q_{2k})r_{12k} \geq c \quad \text{and} \quad (Q_{1k} + Q_{3k})r_{13k} \geq c \tag{3.23}$$

for each k. This, together with (3.21), yields $\delta > 0$ such that

$$\min\{r_{12k}, r_{13k}\} \geq \delta \tag{3.24}$$

for each k. Note that the sequence $\{Q_{jk}\}_k$ is bounded below by some positive number for each $j \in V$. Thus we may further assume

$$\min_{j \in V} Q_{jk} > \delta$$

for each k. This, together with (3.21), yields

$$\delta < \frac{1 - |z_k|}{1 - |a_{jk}|} \leq s := \max_{1 \leq i \leq 4} \frac{1 + |\varphi_i(0)|}{1 - |\varphi_i(0)|}$$

so that

$$\varepsilon \leq \frac{1 - |a_{ik}|}{1 - |a_{jk}|} \leq \frac{1}{\varepsilon} \qquad \text{where} \quad \varepsilon := \frac{\delta}{s} < 1 \tag{3.25}$$

for $i, j \in V$ and for each k. We also have by (3.24)

$$\varepsilon \leq \min\{r_{12k}, r_{13k}\} \tag{3.26}$$

for each k.

We now split the proof according to the number $\sharp V$ of the elements of the set V:

Subcase (a1): Assume $\sharp V = 4$ so that $V = \{1, 2, 3, 4\}$. For $j \in V$, note $|a_{jk}| \geq \frac{1}{2}$ for k sufficiently large, because $|a_{jk}| \to 1$ as $k \to \infty$. Having (3.25) and (3.26), we may apply Lemma 3.3 to find a finite set $\Lambda_1 \subset \Lambda$, independent of a_{jk}'s, such that

$$\max_{\substack{m \in \Lambda_1 \\ j \in V}} J_m(a_{jk}) \geq \frac{1}{2}$$

for k sufficiently large.

Subcase (a2): Assume $\sharp V = 3$. We provide details for $V = \{1,2,3\}$; other cases can be treated in a similar way. Note

$$\frac{1-|a_{jk}|^2}{|1-\langle a_{4k},a_{jk}\rangle|} \le \frac{2}{\delta}\frac{1-|z_k|}{|1-\langle a_{4k},a_{jk}\rangle|} \le \frac{2}{\delta}Q_{4k} \to 0\,, \qquad j\in V$$

as $k\to\infty$. This yields

$$J_m(a_{jk}) \ge I_m(a_{jk}) - \left(\frac{2Q_{4k}}{\delta}\right)^m \ge I_m(a_{jk}) - \frac{1}{4}\,, \qquad j\in V,\ m\in\Lambda$$

for k sufficiently large. Accordingly, having (3.25), we may apply Lemma 3.2 to find a finite set $\Lambda_2\subset\Lambda$, independent of a_{jk}'s, such that

$$\max_{\substack{m\in\Lambda_2\\ j\in V}} I_m(a_{jk}) \ge \frac{1}{2}$$

where $I_m = I_m(\,\cdot\,; a_{1k}, a_{2k}, a_{3k})$ is the function introduced in (3.2). This in turn yields

$$\max_{\substack{m\in\Lambda_2\\ j\in V}} J_m(a_{jk}) \ge \frac{1}{4}$$

for k sufficiently large.

Subcase (a3): Assume $\sharp V = 2$. In this case we claim that there is some $m_1\in\Lambda$ such that

$$\max_{\substack{m\in\Lambda_3\\ j\in V}} J_m(a_{jk}) \ge \frac{1}{4} \qquad \text{where} \quad \Lambda_3 := \{m_1, 2m_1, 3m_1\} \tag{3.27}$$

for k sufficiently large. We provide below details for the cases $V = \{1,2\}$ and $V = \{1,4\}$; other cases can be treated in a similar way. In conjunction with this we note from (3.23) that $V\neq\{2,4\}$ and $V\neq\{3,4\}$.

First, consider the case $V=\{1,2\}$. In this case we may assume (after passing to a subsequence if necessary) $|a_{1k}|\ge|a_{2k}|$ by symmetry so that

$$\frac{1-|a_{1k}|^2}{|1-\langle a_{1k},a_{2k}\rangle|} \le \sqrt{1-r_{12k}^2} \le \sqrt{1-\delta^2}$$

by (2.2) and (3.24). Now, since $Q_{3k}+Q_{4k}\to 0$ as $k\to\infty$, we have as in the proof of Subcase (a2)

$$\begin{aligned} J_m(a_{1k}) &\ge 1 - \left(\frac{1-|a_{1k}|^2}{|1-\langle a_{1k},a_{2k}\rangle|}\right)^m - \frac{1}{4}\,, \qquad m\in\Lambda \\ &\ge \frac{3}{4} - (1-\delta^2)^{m/2} \end{aligned}$$

for k sufficiently large. Accordingly, choosing $m_1\in\Lambda$ sufficiently large so that $(1-\delta^2)^{m_1}\le\frac{1}{4}$, we conclude (3.27).

We now consider the case $V=\{1,4\}$. Since $Q_{2k}+Q_{3k}\to 0$ as $k\to\infty$, we obtain as in the proof of Subcase (a2)

$$J_m(a_{1k}) \ge \left|1 + \left(\frac{1-|a_{1k}|^2}{1-\langle a_{4k},a_{1k}\rangle}\right)^m\right| - \frac{1}{4}$$

$$\geq \frac{3}{4} + \mathrm{Re}\left(\frac{1-|a_{1k}|^2}{1-\langle a_{4k}, a_{1k}\rangle}\right)^m, \qquad m \in \Lambda$$

for k sufficiently large. We thus conclude (3.27) as in the proof of (3.14).

Subcase (a4): Assume $\sharp V = 1$. We provide details for $V = \{1\}$; other cases can be treated in a similar way. Since $Q_{2k} + Q_{3k} + Q_{4k} \to 0$ as $k \to \infty$, we have as in the proof of Subcase (a2)

$$J_m(a_{1k}) \geq \frac{1}{4}, \qquad m \in \Lambda$$

for k sufficiently large. Thus, fixing any $m_3 \in \Lambda$, we conclude

$$\max_{\substack{m\in\Lambda_4\\ j\in V}} J_m(a_{jk}) \geq \frac{1}{4} \qquad \text{where} \quad \Lambda_4 := \{m_3\}$$

for k sufficiently large.

Now, putting

$$\Lambda_a := \bigcup_{\ell=1}^{4} \Lambda_\ell$$

where Λ_ℓ's are the finite sets obtained in Subcases (a1)–(a4), we conclude for Case (a)

$$\max_{\substack{m\in\Lambda_a\\ j\in V}} J_m(a_{jk}) \geq \frac{1}{4}$$

for k sufficiently large.

So far we have constructed a finite set $\Lambda_a \subset \Lambda$ that corresponds to Case (a). One may repeat similar arguments to find finite sets Λ_b, Λ_c and Λ_d that correspond to Case (b), Case (c) and Case (d), respectively. Finally, setting

$$\Lambda_{F_0} := \Lambda_a \cup \Lambda_b \cup \Lambda_c \cup \Lambda_c,$$

we obtain

$$\max_{\substack{m\in\Lambda_{F_0}\\ j\in V}} J_m(a_{jk}) \geq \frac{1}{4}$$

for k sufficiently large, which is a contradiction to (3.22). The proof is complete. □

4. Sufficiency for Compactness. In this section we prove the second part of Theorem 1.1. We first recall a sufficient condition for boundedness of a composition operator on $A^p_\alpha(\mathbf{B})$. The next lemma is taken from [4, Theorem 3.3].

LEMMA 4.1. *Let $\beta > -1$ and $0 < q < \infty$. Let $\varphi \in \mathcal{S}$. If C_φ is bounded on $A^q_\beta(\mathbf{B})$, then C_φ is bounded on $A^p_\alpha(\mathbf{B})$ for any $\alpha \geq \beta$ and $0 < p < \infty$.*

Given $\alpha > -1$ and a bounded nonnegative Borel function W on $\mathbf{B}$, put

$$dW_\alpha := W dv_\alpha .$$

Associated with this measure is the weighted pullback measure $d(W_\alpha \circ \varphi^{-1})$ defined by

$$(W_\alpha \circ \varphi^{-1})(E) := W_\alpha[\varphi^{-1}(E)]$$

for Borel subsets E of $\mathbf{B}$.

In the setting of the disk the next lemma is implicit in the proof of [9, Lemma 1]. The proof below is included for completeness.

LEMMA 4.2. *Let $\alpha > \beta > -1$ and $0 < p, q < \infty$. Put $\gamma := \min\{\alpha - \beta, 1\}$. Let $\varphi \in \mathcal{S}$ and assume that C_φ is bounded on $A^q_\beta(\mathbf{B})$. Let $\varepsilon > 0$ and $W : \mathbf{B} \to [0, 1]$ be a Borel function. If*

$$\sup_{z\in\mathbf{B}} \left[W(z) \frac{1 - |z|}{1 - |\varphi(z)|} \right] \leq \varepsilon, \tag{4.1}$$

then there is a constant $C = C(\alpha, \beta) > 0$ such that

$$\int_{\mathbf{B}} |f \circ \varphi|^p W \, dv_\alpha \leq C\varepsilon^\gamma \|f\|^p_{A^p_\alpha}$$

for functions $f \in A^p_\alpha(\mathbf{B})$.

PROOF. Assume (4.1). Since $\beta_1 := \alpha - \gamma \geq \beta$, we note $\|v_{\beta_1} \circ \varphi^{-1}\|_{\beta_1} < \infty$ by Lemma 4.1 and (2.10). Also, note $W^{1-\gamma} \leq 1$, because $\gamma \leq 1$. We thus have for $z \in \mathbf{B}$

$$\begin{aligned}
(W_\alpha \circ \varphi^{-1})[E(z)] &= \int_{\varphi^{-1}[E(z)]} W(w) \, dv_\alpha(w) \\
&\approx \int_{\varphi^{-1}[E(z)]} W(w)^{1-\gamma} \left[W(w)(1 - |w|^2) \right]^\gamma \, dv_{\beta_1}(w) \\
&\leq \varepsilon^\gamma \int_{\varphi^{-1}[E(z)]} (1 - |\varphi(w)|)^\gamma \, dv_{\beta_1}(w) \\
&\approx \varepsilon^\gamma (1 - |z|)^\gamma v_{\beta_1}[\varphi^{-1}(E(z))] \, ;
\end{aligned}$$

the last estimate holds by (2.4). This, together with (2.5), yields

$$\frac{W_\alpha \circ \varphi^{-1}[E(z)]}{v_\alpha[E(z)]} \lesssim \varepsilon^\gamma \frac{(v_{\beta_1} \circ \varphi^{-1})[E(z)]}{v_{\beta_1}[E(z)]} \, ;$$

the constant suppressed here depends only on n, α and β. It follows that

$$\|W_\alpha \circ \varphi^{-1}\|_\alpha \leq C\varepsilon^\gamma \|v_{\beta_1} \circ \varphi^{-1}\|_{\beta_1}$$

for some constant $C = C(\alpha, \beta) > 0$. We therefore conclude the lemma by (2.10) and (2.11). The proof is complete. □

The following lemma is a key step in the proof of sufficiency part.

LEMMA 4.3. *Let $\alpha > \beta > -1$ and $0 < p, q < \infty$. Let $\varphi, \psi \in \mathcal{S}$ and assume that C_φ, C_ψ are bounded on $A^q_\beta(\mathbf{B})$. Let $K \subset \mathbf{B}$ be a Borel set. If*

$$\sup_{z\in K} \left[\left(\frac{1 - |z|}{1 - |\varphi(z)|} + \frac{1 - |z|}{1 - |\psi(z)|} \right) \rho\big(\varphi(z), \psi(z)\big) \right] \leq \varepsilon, \tag{4.2}$$

then there exists a constant $h(\varepsilon) = h(\varepsilon, p, \alpha, \beta, \varphi, \psi) > 0$ *such that*

$$\lim_{\varepsilon\to 0} h(\varepsilon) = 0$$

and

$$\int_K |f\circ\varphi - f\circ\psi|^p \, dv_\alpha \le h(\varepsilon)\|f\|^p_{A^p_\alpha}$$

for functions $f \in A^p_\alpha(\mathbf{B})$.

PROOF. Assume (4.2) and let $f \in A^p_\alpha(\mathbf{B})$. For a number $\delta = \delta(\varepsilon) \in (0, \frac{1}{3})$ to be fixed later, we decompose the integral under consideration into two parts as

$$\int_K |f\circ\varphi - f\circ\psi|^p \, dv_\alpha = \int_{K_\delta} + \int_{K'_\delta} \tag{4.3}$$

where

$$K_\delta := \{z \in K : \rho(\varphi(z), \psi(z)) < \delta\} \quad \text{and} \quad K'_\delta := K \setminus K_\delta.$$

First, we estimate the first term in the right-hand side of (4.3). Applying Lemma 2.2 (with $r_1 = \frac{1}{3}$ and $r_2 = \frac{1}{2}$) and then Fubini's Theorem, we obtain

$$\begin{aligned}\int_{K_\delta} &\lesssim \int_{\mathbf{B}} \frac{\delta^p}{(1-|\varphi(z)|^2)^{n+1+\alpha}} \left[\int_{E(\varphi(z))} \big(\widetilde{\Delta}|f|^2(w)\big)^{p/2} \, dv_\alpha(w)\right] dv_\alpha(z) \\ &= \delta^p \int_{\mathbf{B}} \big(\widetilde{\Delta}|f|^2(w)\big)^{p/2} \left[\int_{\varphi^{-1}[E(w)]} \frac{dv_\alpha(z)}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right] dv_\alpha(w);\end{aligned}$$

recall $E(\cdot) = E_{1/2}(\cdot)$. Meanwhile, noting that $\rho\big(\varphi(z), w\big) < \frac{1}{2}$ for $z \in \varphi^{-1}[E(w)]$, we obtain by (2.4) and (2.5)

$$\int_{\varphi^{-1}[E(w)]} \frac{dv_\alpha(z)}{(1-|\varphi(z)|^2)^{n+1+\alpha}} \approx \frac{(v_\alpha\circ\varphi^{-1})[E(w)]}{v_\alpha[E(w)]} \le \|v_\alpha\circ\varphi^{-1}\|_\alpha$$

for all $w \in \mathbf{B}$. In addition, since C_φ is bounded on $A^p_\alpha(\mathbf{B})$ by Lemma 4.1, we note $\|v_\alpha\circ\varphi^{-1}\|_\alpha < \infty$ by (2.10) and (2.11). Combining these observations and then using (2.12), we obtain

$$\int_{K_\delta} \lesssim \delta^p \int_{\mathbf{B}} |f(w) - f(0)|^p \, dv_\alpha(w) \lesssim \delta^p \|f\|^p_{A^p_\alpha}. \tag{4.4}$$

Next, we estimate the second term in the right-hand side of (4.3). Note $\delta\chi_{K'_\delta} \le \rho(\varphi, \psi)$, where $\chi_{K'_\delta}$ denotes the characteristic function of K'_δ. By (4.2), we have

$$\chi_{K'_\delta}(z)\left(\frac{1-|z|}{1-|\varphi(z)|} + \frac{1-|z|}{1-|\psi(z)|}\right) \le \frac{\varepsilon}{\delta}$$

for all $z \in \mathbf{B}$. It follows from Lemma 4.2 that

$$\int_{K'_\delta} \lesssim \int_{\mathbf{B}} (|f\circ\varphi|^p + |f\circ\psi|^p)\chi_{K'_\delta} \, dv_\alpha \lesssim \left(\frac{\varepsilon}{\delta}\right)^\gamma \|f\|^p_{A^p_\alpha} \tag{4.5}$$

where $\gamma := \min\{\alpha - \beta, 1\}$.

Now, we deduce from (4.5) and (4.4)

$$\int_K |f\circ\varphi - f\circ\psi|^p \, dv_\alpha \lesssim \left[\left(\frac{\varepsilon}{\delta}\right)^\gamma + \delta^p\right] \|f\|^p_{A^p_\alpha}, \qquad f\in A^p_\alpha(\mathbf{B}).$$

One may keep track of the constant suppressed above to see that it is independent of f, ε and δ. Consequently, a suitable choice of δ, say $\delta = \frac{\sqrt{\varepsilon}}{1+3\sqrt{\varepsilon}}$, completes the proof. □

Now, we are ready to prove the second part of Theorem 1.1.

PROPOSITION 4.4. *With the notation as in Theorem 1.1, assume $M\widetilde{M}\in C_0(\mathbf{B})$. Then T is compact on $A^p_\alpha(\mathbf{B})$, provided that each C_{φ_j} is bounded on $A^q_\beta(\mathbf{B})$ for some $\beta\in(-1,\alpha)$ and $0<q<\infty$.*

PROOF. Assume that each C_{φ_j} is bounded on $A^q_\beta(\mathbf{B})$ for some $\beta\in(-1,\alpha)$ and $0<q<\infty$. Consider an arbitrary sequence $\{f_k\}$ in $A^p_\alpha(\mathbf{B})$ such that $\|f_k\|_{A^p_\alpha}\le 1$ and $f_k\to 0$ uniformly on compact subsets of $\mathbf{B}$. We claim

$$Tf_k \to 0 \quad \text{in} \quad A^p_\alpha(\mathbf{B}). \tag{4.6}$$

Note that, with this claim granted, the asserted compactness of T follows from Lemma 2.1.

We now proceed to the proof of (4.6). Let $\varepsilon>0$. Since $M\widetilde{M}\in C_0(\mathbf{B})$ by assumption, there is some $r\in(0,1)$ such that

$$M(z)\widetilde{M}(z)\le\varepsilon^2$$

for z with $|z|\ge r$. Thus, setting

$$U_\varepsilon := \{z\in\mathbf{B} : M(z)\le\varepsilon\} \quad \text{and} \quad \widetilde{U}_\varepsilon := \{z\in\mathbf{B} : \widetilde{M}(z)\le\varepsilon\},$$

we note

$$\mathbf{B}\setminus r\mathbf{B}\subset U_\varepsilon\cup\widetilde{U}_\varepsilon.$$

According to this observation, we obtain

$$\int_{\mathbf{B}} |Tf_k|^p \, dv_\alpha \le \int_{r\mathbf{B}} + \int_{U_\varepsilon} + \int_{\widetilde{U}_\varepsilon} =: I_{1k} + I_{2k} + \widetilde{I}_{2k} \tag{4.7}$$

for each k.

Note $f_k \to 0$ uniformly on the set $\cup_{j=1}^4 \varphi_j(r\mathbf{B})$ which is relatively compact in $\mathbf{B}$. So, for the integral over $r\mathbf{B}$ in (4.7), we conclude

$$I_{1k} \to 0 \tag{4.8}$$

as $k \to \infty$. Meanwhile, note

$$M(z) = M_{12}(z) + M_{34}(z) \le \varepsilon$$

for $z \in U_\varepsilon$. So, for the integral over U_ε in (4.7), we see by Lemma 4.3 that there is a constant $h(\varepsilon) > 0$ satisfying $h(\varepsilon) \to 0$ as $\varepsilon \to 0$ and

$$I_{2k} \lesssim \int_{U_\varepsilon} |T_{12} f_k|^p \, dv_\alpha + \int_{U_\varepsilon} |T_{34} f_k|^p \, dv_\alpha \le h(\varepsilon)$$

for all k. It follows that

$$\limsup_{k\to\infty} I_{2k} \lesssim h(\varepsilon). \tag{4.9}$$

Recalling $T = T_{13} - T_{24}$ from (1.1), one may repeat the same argument to conclude

$$\limsup_{k\to\infty} \widetilde{I}_{2k} \lesssim \widetilde{h}(\varepsilon) \tag{4.10}$$

for some constant $\widetilde{h}(\varepsilon) > 0$ such that $\widetilde{h}(\varepsilon) \to 0$ as $\varepsilon \to 0$. One may check that the constants suppressed in (4.9) and (4.10) are independent of k and ε. Thus, combining the observations in (4.8), (4.9) and (4.10), we obtain

$$\limsup_{k\to\infty} \int_{\mathbf{B}} |T f_k|^p \le C[h(\varepsilon) + \widetilde{h}(\varepsilon)]$$

for some constant $C > 0$ independent of ε. Finally, taking the limit $\varepsilon \to 0$, we conclude (4.6), as required. The proof is complete. □

5. Remarks. Note that Theorem 1.1 can be applied even when some of the operators C_{φ_j}'s coincide or already compact. So, in this section we consider three special cases to recover or derive some consequences which might be of independent interest.

When $\varphi_2 = \varphi_3 = \varphi_4 \equiv 0$, note

$$M\widetilde{M} = M_{12}^2 .$$

In conjunction with this, we also note $\rho_{12} = \rho\big(\varphi_1(z), 0\big) = |\varphi_1(z)|$ so that

$$\begin{aligned} M_{12}(z) &= \left[\frac{1-|z|}{1-|\varphi_1(z)|} + (1-|z|) \right] |\varphi_1(z)| \\ &= \frac{1-|z|}{1-|\varphi_1(z)|} - (1-|z|)(1-|\varphi_1(z)|). \end{aligned} \tag{5.1}$$

We thus recover the following characterization due to Zhu [13, Theorem 11].

COROLLARY 5.1. *Let $\alpha > -1$ and $0 < p < \infty$. Let $\varphi \in \mathcal{S}$ and assume that C_φ is bounded on $A^q_\beta(\mathbf{B})$ for some $\beta \in (-1, \alpha)$ and $0 < q < \infty$. Then C_φ is compact on $A^p_\alpha(\mathbf{B})$ if and* *only if*

$$\lim_{|z|\to 1} \frac{1-|z|}{1-|\varphi(z)|} = 0\,.$$

Meanwhile, when $\varphi_1 = \varphi_4$ and $\varphi_2 = \varphi_3$, note

$$M\widetilde{M} = 4M_{12}^2\,.$$

Thus we recover the following result (see [2, Theorem 4.7]), which is a ball version of Moorhouse's characterization for compact differences over the disk.

COROLLARY 5.2. *Let $\alpha > -1$ and $0 < p < \infty$. Let $\varphi, \psi \in \mathcal{S}$ and assume that C_φ and C_ψ are both bounded on $A^q_\beta(\mathbf{B})$ for some $\beta \in (-1,\alpha)$ and $0 < q < \infty$. Then $C_\varphi - C_\psi$ is compact on $A^p_\alpha(\mathbf{B})$ if and only if*

$$\lim_{|z|\to 1} \left(\frac{1-|z|}{1-|\varphi(z)|} + \frac{1-|z|}{1-|\psi(z)|}\right) \rho\big(\varphi(z),\psi(z)\big) = 0\,.$$

Also, consider the case $\varphi_1 = \varphi_4$. In this case we have

$$M\widetilde{M} = (M_{12} + M_{13})^2\,.$$

Thus, as a consequence of Corollary 5.2, we obtain following result.

COROLLARY 5.3. *Let $\alpha > -1$ and $0 < p < \infty$. For $j = 1,2,3$ let $\varphi_j \in \mathcal{S}$ and assume that C_{φ_j} is bounded on $A^q_\beta(\mathbf{B})$ for some $\beta \in (-1,\alpha)$ and $0 < q < \infty$. Then the following two assertions are equivalent*:

(a) $2C_{\varphi_1} - C_{\varphi_2} - C_{\varphi_3}$ *is compact on* $A^p_\alpha(\mathbf{B})$*;*

(b) $C_{\varphi_1} - C_{\varphi_2}$ *and* $C_{\varphi_1} - C_{\varphi_3}$ *are both compact on* $A^p_\alpha(\mathbf{B})$.

Finally, we consider the case when one of the operators is already compact. Recall that $A(\varphi)$ denotes the angular derivative set of $\varphi \in \mathcal{S}$.

COROLLARY 5.4. *Let $\alpha > -1$ and $0 < p < \infty$. For $j = 1,2,3$ let $\varphi_j \in \mathcal{S}$ and assume that C_{φ_j} is bounded on $A^q_\beta(\mathbf{B})$ for some $\beta \in (-1,\alpha)$ and $0 < q < \infty$. Then the following three assertions are equivalent:*

(a) $C_{\varphi_1} - C_{\varphi_2} - C_{\varphi_3}$ *is compact on* $A^p_\alpha(\mathbf{B})$;

(b) $\displaystyle\lim_{|z|\to 1} \left[M_{12}(z) + \frac{1-|z|}{1-|\varphi_3(z)|}\right]\left[M_{13}(z) + \frac{1-|z|}{1-|\varphi_2(z)|}\right] = 0;$

(c) $A(\varphi_1) = A(\varphi_2) \cup A(\varphi_3)$ *and* $A(\varphi_2) \cap A(\varphi_3) = \emptyset$*. Moreover,*

$$\lim_{z\to\zeta} M_{1j}(z) = 0\,, \qquad \zeta \in A(\varphi_j) \tag{5.2}$$

for $j = 2,3$.

PROOF. As in (5.1), we have

$$M_{j4}(z) = \frac{1-|z|}{1-|\varphi_j(z)|} - (1-|z|)(1-|\varphi_j(z)|)$$

for $j = 2,3$. Thus the condition $M\widetilde{M} \in C_0(\mathbf{B})$ reduces to Assertion (b). So, (a) and (b) are equivalent by Theorem 1.1. While the equivalence of (a) and (c) is contained in [2, Theorem 5.4], we include below another proof by establishing the equivalence of (b) and (c).

Assume that (c) holds. Since $A(\varphi_1)$ is the disjoint union of $A(\varphi_2)$ and $A(\varphi_3)$, we have

$$\text{either} \quad \frac{1-|z|}{1-|\varphi_2(z)|} \to 0 \quad \text{or} \quad \frac{1-|z|}{1-|\varphi_3(z)|} \to 0 \quad \text{as} \quad z \to \zeta$$

for each $\zeta \in A(\varphi_1)$. Thus we have by (5.2)

$$\lim_{z\to\zeta}\left[M_{12}(z) + \frac{1-|z|}{1-|\varphi_3(z)|}\right]\left[M_{13}(z) + \frac{1-|z|}{1-|\varphi_2(z)|}\right] = 0 \tag{5.3}$$

for any $\zeta \in A(\varphi_1)$. Note

$$\frac{1-|z|}{1-|\varphi_2(z)|} \to 0 \quad \text{and} \quad \frac{1-|z|}{1-|\varphi_3(z)|} \to 0 \quad \text{as} \quad z \to \zeta$$

for $\zeta \in \partial\mathbf{B} \setminus A(\varphi_1)$. Thus, (5.3) remains valid for $\zeta \in \partial\mathbf{B} \setminus A(\varphi_1)$ by (5.2). So, (b) holds, as asserted.

Conversely, assume that (b) holds. Since (b) implies

$$\lim_{z\to\zeta} \frac{1-|z|}{1-|\varphi_2(z)|} \cdot \frac{1-|z|}{1-|\varphi_3(z)|} = 0, \qquad \zeta \in \partial\mathbf{B},$$

we note $A(\varphi_2) \cap A(\varphi_3) = \emptyset$ by the Julia-Carathéodory Theorem. Next, assume $\zeta \in A(\varphi_2)$ but $\zeta \notin A(\varphi_1)$. We then have by (2.1)

$$\frac{1-\rho_{12}^2(\lambda\zeta)}{4} \le \frac{1-|\varphi_2(\lambda\zeta)|}{1-|\varphi_1(\lambda\zeta)|} = \frac{1-|\varphi_2(\lambda\zeta)|}{1-|\lambda\zeta|} \cdot \frac{1-|\lambda\zeta|}{1-|\varphi_1(\lambda\zeta)|}$$

for $\lambda \in \mathbf{B}_1$. So we have by (2.13) and (2.14)

$$\angle \lim_{\lambda\to 1} \rho_{12}(\lambda\zeta) = 1$$

and thus

$$\angle \lim_{\lambda\to 1} \left(M_{12}(\lambda\zeta) \cdot \frac{1-|\lambda|}{1-|\varphi_2(\lambda\zeta)|} \right) = \frac{1}{d_{\varphi_2}^2(\zeta)} > 0,$$

which contradicts to (b). We thus conclude $A(\varphi_2) \setminus A(\varphi_1) = \emptyset$, i.e., $A(\varphi_2) \subset A(\varphi_1)$. Similarly, we have $A(\varphi_3) \subset A(\varphi_1)$. On the other hand, if $\zeta \in A(\varphi_1)$ but $\zeta \notin A(\varphi_2) \cup A(\varphi_3)$, then a similar argument yields

$$\angle \lim_{\lambda\to 1} M_{12}(\lambda\zeta) M_{13}(\lambda\zeta) = \frac{1}{d_{\varphi_1}^2(\zeta)} > 0,$$

which again contradicts to (b). Accordingly, we conclude $A(\varphi_1) = A(\varphi_2) \cup A(\varphi_3)$.

We now show (5.2). By symmetry it is enough to consider the case $j = 2$ only. In order to derive a contradiction, suppose that (5.2) fails for some $\zeta \in A(\varphi_2)$. We then have

$$\inf_k M_{12}(z_k) > 0 \tag{5.4}$$

for some sequence $\{z_k\} \subset \mathbf{B}$ converging to ζ. This implies

$$\inf_k \rho_{12}(z_k) > 0\,. \tag{5.5}$$

Since

$$\lim_{k\to\infty} M_{12}(z_k)\left(M_{13}(z_k) + \frac{1-|z_k|}{1-|\varphi_2(z_k)|}\right) = 0$$

by (b), we also have by (5.4)

$$\lim_{k\to\infty} M_{13}(z_k) = 0 \tag{5.6}$$

and

$$\lim_{k\to\infty} \frac{1-|z_k|}{1-|\varphi_2(z_k)|} = 0\,. \tag{5.7}$$

It follows from (5.4), (5.5) and (5.7) that

$$\inf_k \frac{1-|z_k|}{1-|\varphi_1(z_k)|} > 0\,, \tag{5.8}$$

which, together with (5.6), in turn yields

$$\lim_{k\to\infty} \rho_{13}(z_k) = 0\,.$$

In particular, by (2.4), we have $1 - |\varphi_1(z_k)| \approx 1 - |\varphi_3(z_k)|$ for all k and thus obtain by (5.8)

$$\limsup_{k\to\infty} \frac{1-|z_k|}{1-|\varphi_3(z_k)|} > 0\,.$$

Consequently, we conclude $\zeta \in A(\varphi_3)$, which contradicts to the fact that $A(\varphi_2)$ and $A(\varphi_3)$ are disjoint. The proof is complete. □

We now turn to the construction of explicit examples showing that the additional boundedness assumption in Theorem 1.1(b) cannot be removed. For simplicity we take $n = 2$ for the rest of the paper. We introduce some notation. For the rest of the paper we use the notation

$$h(z) := z_1 + \frac{z_2^2}{2}$$

for $z = (z_1, z_2) \in \mathbf{B}_2$. Given $0 < \varepsilon < 1$, we put

$$\psi_\varepsilon(z) := \left(1 - \left(\frac{1-h(z)}{2}\right)^{\varepsilon}, 0\right)\,.$$

Since $h(\mathbf{B}_2) \subset \mathbf{B}_1$, we have $\psi_\varepsilon \in \mathcal{S}(\mathbf{B}_2)$.

Put

$$\mathbf{e} := (1, 0)\,.$$

Since $|\mathrm{Arg}(1 - \langle\psi_\varepsilon(z), \mathbf{e}\rangle)| \le \frac{\varepsilon\pi}{2}$, we also note that $\langle\psi_\varepsilon(z), \mathbf{e}\rangle$ is contained in a nontangential region with vertex at 1. So, there is a constant $c = c(\varepsilon) > 1$ such that

$$|1 - \langle\psi_\varepsilon(z), \mathbf{e}\rangle| < c(1 - |\langle\psi_\varepsilon(z), \mathbf{e}\rangle|) \tag{5.9}$$

for all $z \in \mathbf{B}_2$. Finally, we put

$$S_\delta(\zeta) := \{z \in \mathbf{B}_2 : |1 - \langle z, \zeta\rangle| < \delta\}$$

for $0 < \delta < 1$ and $\zeta \in \partial\mathbf{B}_2$.

We need some preliminary lemmas. First, we observe that each function ψ_ε does not have any finite angular derivatives.

LEMMA 5.5. *Let* $0 < \varepsilon < 1$. *Then*

$$\lim_{|z|\to 1} \frac{1 - |z|}{1 - |\psi_\varepsilon(z)|} = 0\,.$$

PROOF. By the Julia-Carathéodory Theorem it suffice to prove that ψ_ε does not have any finite angular derivatives. Noting $\psi_\varepsilon^{-1}(\partial\mathbf{B}_2) = \{\mathbf{e}\}$, we consider a restricted $\mathbf{e}$-curve γ in $\mathbf{B}_2$ given by

$$\gamma(t) = \left(t, \frac{1-t}{2}\right), \qquad 0 \le t < 1\,.$$

Note that $\psi_\varepsilon(\mathbf{e}) = \mathbf{e}$ and

$$\begin{aligned} \frac{1 - \langle\psi_\varepsilon(\gamma(t)), \mathbf{e}\rangle}{1 - \langle\gamma(t), \mathbf{e}\rangle} &= \frac{1}{1-t}\left[\frac{1 - h(\gamma(t))}{2}\right]^\varepsilon \\ &= \frac{1}{1-t}\left(\frac{1-t}{2}\right)^\varepsilon \left(\frac{3+t}{4}\right)^\varepsilon \\ &\longrightarrow \infty \end{aligned}$$

as $t \longrightarrow 1$. This shows that ψ_ε does not have angular derivative at $\mathbf{e}$. So, we see that ψ_ε does not have any finite angular derivatives, as asserted. The proof is complete. □

Next, we investigate the relation of weight parameters α and β for which an operator of the form

$$L := \sum_{j=1}^{N} a_j C_{\varphi_j} \quad \text{where} \quad \varphi_j := \lambda_j \psi_\varepsilon \tag{5.10}$$

is bounded from $A^p_\alpha(\mathbf{B}_2)$ into $A^p_\beta(\mathbf{B}_2)$. Here, N is a positive integer, $\lambda_1, \ldots, \lambda_N$ are distinct unimodular complex numbers and $a_1, \ldots, a_N$ are nonzero complex numbers. To this end we recall the following optimal estimate which is implicit in the proof of [8, Proposition 4.4].

LEMMA 5.6. *Let* $\alpha > -1$. *Then there is a constant* $C = C(\alpha) > 0$ *such that*

$$C^{-1} \le \frac{(v_\alpha \circ h^{-1})\big[\widetilde{S}_\delta(1)\big]}{\delta^{3+\alpha-1/4}} \le C$$

for $0 < \delta < 1$ *where* $\widetilde{S}_\delta(1) = \{\lambda \in \mathbf{B}_1 : |1 - \lambda| < \delta\}$.

LEMMA 5.7. *Let* $\alpha, \beta > -1$ *and* $0 < p < \infty$. *Let* $0 < \varepsilon < 1$ *and L be an operator as in* (5.10). *Then* $L : A^p_\alpha(\mathbf{B}_2) \to A^p_\beta(\mathbf{B}_2)$ *is bounded if and only if* $\beta + 3 \geq \varepsilon(\alpha + 3) + \frac{1}{4}$.

PROOF. We first establish the optimal estimate

$$\sup_{\zeta \in \partial \mathbf{B}_2} \frac{(v_\beta \circ \psi_\varepsilon^{-1})[S_\delta(\zeta)]}{v_\alpha[S_\delta(\zeta)]} \approx \delta^{(3+\beta-1/4)/\varepsilon-3-\alpha} \tag{5.11}$$

for $0 < \delta < 1$. In conjunction with this estimate, we recall the well-known estimate

$$v_\alpha[S_\delta(\zeta)] \approx \delta^{3+\alpha}, \qquad 0 < \delta < 1 \tag{5.12}$$

uniformly in $\zeta \in \partial \mathbf{B}_2$; see, for example, [5, Exercise 2.2.8].

Let $\zeta \in \partial \mathbf{B}_2$. To avoid triviality assume $\psi_\varepsilon^{-1}[S_\delta(\zeta)] \neq \emptyset$ and let $z \in \psi_\varepsilon^{-1}[S_\delta(\zeta)]$. Since

$$|\langle \psi_\varepsilon(z), \zeta \rangle| \leq |\psi_\varepsilon(z)| = |\langle \psi_\varepsilon(z), \mathbf{e} \rangle|,$$

we note from (5.9)

$$|1 - \langle \psi_\varepsilon(z), \mathbf{e} \rangle| \leq c|1 - \langle \psi_\varepsilon(z), \zeta \rangle| < c\delta.$$

In other words, we have

$$\psi_\varepsilon^{-1}[S_\delta(\zeta)] \subset \psi_\varepsilon^{-1}[S_{c\delta}(\mathbf{e})],$$

which allows us to focus on the case $\zeta = \mathbf{e}$. Note

$$S_\delta(\mathbf{e}) = \{z \in \mathbf{B}_2 : |1 - z_1| < \delta\}$$

and thus

$$\psi_\varepsilon^{-1}[S_\delta(\mathbf{e})] = \{z \in \mathbf{B}_2 : |1 - h(z)| < 2\delta^{1/\varepsilon}\} = h^{-1}\left[\widetilde{S}_{2\delta^{1/\varepsilon}}(1)\right].$$

We thus have by Lemma 5.6

$$(v_\beta \circ \psi_\varepsilon^{-1})[S_\delta(\mathbf{e})] \approx \delta^{(3+\beta-1/4)/\varepsilon}$$

for $0 < \delta < 1$. By this and (5.12) we conclude (5.11), as required.

We now proceed to the proof of the lemma. Note by (5.11) and the well-known Carleson Measure Criteria (see, for example, [7, Proposition 3.1]) that

$$C_{\varphi_j} : A^p_\alpha(\mathbf{B}_2) \to A^p_\beta(\mathbf{B}_2) \text{ is bounded } \iff \beta + 3 \geq \varepsilon(\alpha + 3) + \frac{1}{4} \tag{5.13}$$

for each j. So, the lemma holds for $N = 1$. Also, this implies the sufficiency part of the lemma.

We now prove the necessity part of the lemma for the case $N \geq 2$. So, suppose that $L : A^p_\alpha(\mathbf{B}_2) \to A^p_\beta(\mathbf{B}_2)$ is bounded. We may assume $\lambda_1 = 1$ so that $\varphi_1 = \psi_\varepsilon$. We employ test functions f_δ given by

$$f_\delta(z) := \frac{\delta^{1/p}}{[1 - (1 - \delta)z_1]^{(\alpha+4)/p}}, \qquad z = (z_1, z_2) \in \mathbf{B}_2$$

for $0 < \delta < 1$. Note $\|f_\delta\|_{A^p_\alpha} \approx 1$ for all δ by (2.8). It follows that

$$1 \gtrsim \|Lf_\delta\|^p_{A^p_\beta}$$

$$\gtrsim |a_1|^p \int_{\mathbf{B}_2} |f_\delta \circ \varphi_1|^p \, dv_\beta - \sum_{j=2}^{N} |a_j|^p \int_{\mathbf{B}_2} |f_\delta \circ \varphi_j|^p \, dv_\beta$$
$$=: I - II\,.$$

To estimate the first integral of the above, we note

$$|1-(1-\delta)z_1| \le 2\delta, \qquad z \in S_\delta(\mathbf{e})$$

and thus $|f_\delta|^p \gtrsim \delta^{-(\alpha+3)}$ on $S_\delta(\mathbf{e})$. Accordingly, we obtain

$$\begin{aligned} I &= |a_1|^p \int_{\mathbf{B}_2} |f_\delta|^p \, d(v_\beta \circ \psi_\varepsilon^{-1}) \\ &\ge |a_1|^p \int_{S_\delta(\mathbf{e})} |f_\delta|^p \, d(v_\beta \circ \psi_\varepsilon^{-1}) \\ &\gtrsim \frac{(v_\beta \circ \psi_\varepsilon^{-1})[S_\delta(\mathbf{e})]}{\delta^{\alpha+3}} \\ &\approx \delta^{(3+\beta-1/4)/\varepsilon-3-\alpha}\,; \end{aligned}$$

the last estimate holds by (5.11). Meanwhile, since

$$(f_\delta \circ \varphi_j)(\mathbf{e}) = f_\delta(\lambda_j \mathbf{e}) = O(\delta^{1/p})$$

for each $j \neq 1$, we have

$$II = O(\delta)$$

for $0 < \delta < 1$. Combining these observations, we obtain

$$\delta^{(3+\beta-1/4)/\varepsilon-3-\alpha} = O(1),$$

which yields $\beta + 3 \ge \varepsilon(\alpha+3) + \frac{1}{4}$, as required. This completes the proof. □

We now close the paper with the following example in connection with Theorem 1.1 and its corollaries in this section.

EXAMPLE 5.8. *Let $\alpha > -1$ and $1 - \frac{1}{4(\alpha+3)} < \varepsilon < 1$. Let L be an operator as in* (5.10)*. Then*

$$\lim_{|z|\to 1} \frac{1-|z|}{1-|\varphi_j(z)|} = 0 \tag{5.14}$$

for each j, but L is not bounded on $A^p_\alpha(\mathbf{B})$ for any $0 < p < \infty$.

PROOF. Clearly, (5.14) holds by Lemma 5.5. Meanwhile, since $\alpha + 3 < \varepsilon(\alpha+3) + \frac{1}{4}$, we see from Lemma 5.7 that L is not bounded on $A^p_\alpha(\mathbf{B})$ for any $0 < p < \infty$. □

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