Tohoku Math. J.
71 (2019), 581–607

L^2 CURVATURE PINCHING THEOREMS AND VANISHING THEOREMS ON COMPLETE RIEMANNIAN MANIFOLDS

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(Received November 17, 2017, revised January 4, 2018)

Abstract. In this paper, by using monotonicity formulas for vector bundle-valued p-forms satisfying the conservation law, we first obtain general L^2 global rigidity theorems for locally conformally flat (LCF) manifolds with constant scalar curvature, under curvature pinching conditions. Secondly, we prove vanishing results for L^2 and some non-L^2 harmonic p-forms on LCF manifolds, by assuming that the underlying manifolds satisfy pointwise or integral curvature conditions. Moreover, by a theorem of Li-Tam for harmonic functions, we show that the underlying manifold must have only one end. Finally, we obtain Liouville theorems for p-harmonic functions on LCF manifolds under pointwise Ricci curvature conditions.

1. Introduction. In the study of Riemannian geometry, locally conformally flat manifolds play an important role. Let us recall that an n-dimensional Riemannian manifold (M^n, g) is said to be locally conformally flat (LCF) if it admits a coordinate covering $\{U_\alpha, \varphi_\alpha\}$ such that the map $\varphi_\alpha : (U_\alpha, g_\alpha) \to (S^n, g_0)$ is a conformal map, where g_0 is the standard metric on S^n. A locally conformally flat manifold may be regarded as a higher dimensional generalization of a Riemann surface. But not every higher dimensional manifold admits a locally conformally flat structure, and it is an interesting problem to give a good classification of locally conformally flat manifolds. By assuming various geometric situations, many partial classification results have been given (see, for examples, [7, 10, 9, 25, 28, 29, 36, 38], etc.).

In the first part, we use the stress-energy tensor to study the rigidity of LCF manifolds. In [13], the authors presented a unified method to establish monotonicity formulas and vanishing theorems for vector-bundled valued p-forms satisfying a conservation law, by means of the stress-energy tensors of various energy functionals in geometry and physics. Later, the authors in [12] established similar monotonicity formulas by using various exhaustion functions. As applications, they proved the Ricci flatness of a Kähler manifold with constant scalar curvature under growth conditions for the Ricci form, and obtained Bernstein type theorems for submanifolds in Euclidean spaces with parallel mean curvature under growth conditions on the second fundamental form. In this paper, we attempt to use monotonicity formulas to study rigidity properties of LCF metric with constant scalar curvature. For these aims, we may interpret the Riemannian (resp. Ricci) curvature tensor as a 2-form (resp. 1-form) with values

2010 *Mathematics Subject Classification.* Primary 53C20; Secondary 53C21, 53C25.
Key words and phrases. Conformally flat, vanishing theorems, L^2 harmonic p-forms, ends, Liouville theorems.
The first author was supported by NSFC grants (No. 11271071, No. 11771087) and LMNS, Fudan. The second author was supported by NSFC grant (No. 11401099), the Natural Science Foundation of Fujian Province, China (Grant No. 2017J01398). The third author was supported by NSF Award No DMS-1447008 and the OU Arts and Sciences TAP Fund.

in the bundle of symmetric endomorphisms of $T(M)$ endowed with its canonical structure of Riemannian vector bundle. For LCF manifolds with constant scalar curvature, the 1-forms corresponding to the Ricci curvature tensor and to the traceless Ricci curvature tensor also satisfy conservation laws. Hence we can establish monotonicity formulas for those one forms, from which L^2 curvature pinching theorems are deduced.

On the other hand, it is an interesting problem in geometry and topology to find sufficient conditions on a LCF manifold M for the vanishing of harmonic forms. When M is compact, the Hodge theory states that the space of harmonic p-forms on M is isomorphic to its p-th de Rham cohomology group. In [4], Bourguignon proved that a compact, $2m$-dimensional, LCF manifold of positive scalar curvature has no non-zero harmonic m-forms, hence its m-th Betti number $\beta_m = 0$. Later, Nayatani [24] generalized Bourguignon's result and proved that a compact LCF manifold M^n with nonnegative scalar curvature satisfies $\beta_p = 0$ for $d + 1 < p < n - p - 1$, where $d = d(M)$ is the Schoen-Yau invariant of M^n. In [16], Guan, Lin and Wang obtained a cohomology vanishing theorem on compact LCF manifolds under a positivity assumption on the Schouten tensor. For the non-compact case, the Hodge theory is no longer true in general. However, it is known that L^2 Hodge theory remains valid for complete non-compact manifolds. Hence it is important to investigate L^2 harmonic forms. In [26], Pigola, Rigoli and Setti showed a vanishing result for bounded harmonic forms of middle degree on complete non-compact LCF manifolds, by adding suitable conditions on scalar curvature and volume growth. In [21], Lin proved some vanishing and finiteness theorems for L^2 harmonic 1-forms on complete non-compact LCF manifolds under integral curvature pinching conditions.

Since the Riemannian curvature of a LCF manifold can be expressed by its Ricci curvature and scalar curvature, we can compute explicitly the Weitzenböck formula for harmonic p-forms. Based on this formula, together with L^2-Sobolev inequality or weighted Poincaré inequality, we shall establish vanishing results for L^2 harmonic p-forms under various $L^{n/2}$-integral curvature or pointwise curvature pinching conditions. In particular, we show that if the Ricci tensor is sufficiently near zero in the integral sense, then $H^p(L^2(M)) = \{0\}$ for all $0 \le p \le n$, where $H^p(L^2(M))$ denotes the space of all L^2 harmonic p-forms on M. Moreover, according to the nonexistence of nontrivial L^2 harmonic 1-forms, we deduce that M has only one end by Li-Tam's harmonic functions theory.

Finally we also consider p-harmonic functions on LCF manifolds. When the scalar curvature of a LCF manifold is negative, it is known that a weighted Poincaré inequality holds. Hence we can use the results of Chang-Chen-Wei [8] to derive some Liouville theorems for p-harmonic functions, by assuming pointwise Ricci curvature bounds.

2. Preliminaries.

Let (M, g) be a complete manifold of dimension $n \ge 3$. Let R_{ijkl} and W_{ijkl} denote respectively the components of the Riemannian curvature tensor and the Weyl curvature tensor of (M, g) in local orthonormal frame fields. A fundamental result in Riemannian geometry is that (see [30])

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(R_{ik}\delta_{jl} - R_{il}\delta_{jk} + R_{jl}\delta_{ik} - R_{jk}\delta_{il}) + \frac{R}{(n-1)(n-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \tag{2.1}$$

where R_{ik} and R denote the Ricci tensor and the scalar curvature respectively. The associated Schouten tensor A with respect to g is defined by

$$A := \frac{1}{n-2}\left(\mathrm{Ric} - \frac{R}{2(n-1)}g\right).$$

It is well known that if $n = 3$, then $W_{ijkl} = 0$, and (M^3, g) is locally conformally flat if and only if the Schouten tensor is Codazzi, i.e., $A_{ik,j} - A_{ij,k} = 0$, where the A_{ij}'s are the components of the Schouten tensor A. If $n \geq 4$, then (M^n, g) is locally conformally flat if and only if the Weyl tensor vanishes, i.e., $W_{ijkl} = 0$. The local conformal flatness and the equation (2.1) yield

$$R_{ijkl} = \frac{1}{n-2}(R_{ik}\delta_{jl} - R_{il}\delta_{jk} + R_{jl}\delta_{ik} - R_{jk}\delta_{il}) - \frac{R}{(n-1)(n-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \tag{2.2}$$

Thus, a locally conformally flat manifold has constant sectional curvature if and only if it is Einstein, that is, $\mathrm{Ric} = \frac{R}{n}g$. As a consequence, by the Hopf classification theorem, space forms are the only locally conformally flat Einstein manifolds.

If the scalar curvature R of a LCF manifold is constant, by (2.2) and the second Bianchi identities, we immediately obtain that the Ricci tensor is Codazzi, that is, $R_{ij,k} = R_{ik,j}$. Therefore, the traceless Ricci tensor $E = \mathrm{Ric} - \frac{R}{n}g$ is Codazzi too.

In order to get vanishing results for L^2 harmonic p-forms on LCF manifols, we need the following L^2-Sobolev inequality. It is known that a simply connected, LCF manifold M^n ($n \geq 3$) has a conformal immersion into $\mathbb{S}^n$, and according to Proposition 2.2 in [29], the Yamabe constant of M^n satisfies $Q(M^n) = Q(\mathbb{S}^n) = \frac{n(n-2)\omega_n^{\frac{2}{n}}}{4}$, where ω_n is the volume of the unit sphere in $\mathbb{R}^n$. Therefore the following inequality

$$Q(\mathbb{S}^n)\left(\int_M f^{\frac{2n}{n-2}}dv\right)^{\frac{n-2}{n}} \leq \int_M |\nabla f|^2 dv + \frac{n-2}{4(n-1)}\int_M Rf^2 dv \tag{2.3}$$

holds for all $f \in C_0^\infty(M)$. If we assume $R \leq 0$, then it follows that

$$Q(\mathbb{S}^n)\left(\int_M f^{\frac{2n}{n-2}}dv\right)^{\frac{n-2}{n}} \leq \int_M |\nabla f|^2 dv, \ \ \forall f \in C_0^\infty(M). \tag{2.4}$$

On the other hand, if $\int_M |R|^{\frac{n}{2}}dv < \infty$, then we can choose a compact set $\Omega \subset M$ large enough such that

$$\left(\int_{M\setminus\Omega} |R|^{\frac{n}{2}}dv\right)^{\frac{2}{n}} \leq \frac{4\varepsilon(n-1)Q(\mathbb{S}^n)}{n-2}$$

for some ε satisfying $0 < \varepsilon < 1$. By the Hölder inequality, the term involving the scalar curvature can be absorbed into the left-hand side of (2.3) to yield

$$(1-\varepsilon)Q(\mathbb{S}^n)\left(\int_{M\setminus\Omega} f^{\frac{2n}{n-2}}dv\right)^{\frac{n-2}{n}} \leq \int_{M\setminus\Omega}|\nabla f|^2 dv,\ \ \forall f\in C_0^\infty(M\setminus\Omega)\,.$$

From the work of G. Carron [6] (one can also consult Theorem 3.2 of [27]), the following L^2-Sobolev inequality

$$C_s\left(\int_M f^{\frac{2n}{n-2}}dv\right)^{\frac{n-2}{n}} \leq \int_M |\nabla f|^2 dv,\ \ \forall f\in C_0^\infty(M) \tag{2.5}$$

holds for some uniform constant $C_s > 0$, which implies a uniform lower bound on the volume of geodesic balls

$$\mathrm{vol}(B_x(\rho)) \geq C\rho^n,\ \ \forall x\in M \tag{2.6}$$

for some constant $C > 0$ (see Proposition 2.1 of [1] for the compact case). Therefore, each end of M has infinite volume.

3. Monotonicity formulas for curvature tensor and vanishing results. Let (M^n, g) be a Riemannian manifold and $\xi : E \to M$ be a smooth Riemannian vector bundle over (M^n, g) with compatible connection ∇^E, i.e. a vector bundle such that each fiber is equipped with a positive definite inner product $\langle\ ,\ \rangle_E$. Set $A^p(\xi) = \Gamma(\Lambda^p T^*M\otimes E)$ the space of smooth p-forms on M with values in the vector bundle $\xi : E \to M$. The exterior covariant differentiation $d^\nabla : A^p(\xi) \to A^{p+1}(\xi)$ relative to ∇^E is defined by

$$(d^\nabla\omega)(X_1,\ldots,X_{p+1}) = \sum_{i=1}^{p+1}(-1)^{i+1}(\nabla_{X_i}\omega)(X_1,\ldots,\widehat{X_i},\ldots,X_{p+1})\,.$$

The codifferential operator $\delta^\nabla : A^p(\xi) \to A^{p-1}(\xi)$ is characterized as the adjoint of d^∇ if M is compact or ω has a compact support, and is defined by

$$(\delta^\nabla\omega)(X_1,\ldots,X_{p-1}) = -\sum_{i=1}^{n}(\nabla_{e_i}\omega)(e_i,X_1,\ldots,X_{p-1})\,,$$

where $\{e_1,\ldots,e_n\}$ is an orthonormal basis of T_xM.

Given two forms $\omega, \theta\in A^p(\xi)$, the induced inner product is defined as follows:

$$\langle\omega,\theta\rangle = \sum_{i_1,\ldots,i_p=1}^{n}\langle\omega(e_{i_1},\ldots,e_{i_p}),\theta(e_{i_1},\ldots,e_{i_p})\rangle_E\,.$$

Here we are omitting the normalizing factor $\frac{1}{p!}$. For $\omega\in A^p(\xi)$, set $|\omega|^2 = \langle\omega,\omega\rangle$. The energy functional of $\omega\in A^p(\xi)$ is defined by $E(\omega) = \frac{1}{2}\int_M|\omega|^2 dv_g$. Its stress-energy tensor is

$$S_\omega(X,Y) = \frac{|\omega|^2}{2}g(X,Y) - (\omega\odot\omega)(X,Y)\,, \tag{3.1}$$

where $\omega \odot \omega \in \Gamma(A^p(\xi) \otimes A^p(\xi))$ is a symmetric tensor defined by

$$(\omega \odot \omega)(X, Y) = \langle i_X\omega, i_Y\omega\rangle . \tag{3.2}$$

Here $i_X\omega \in A^{p-1}(\xi)$ denotes the interior multiplication by $X \in \Gamma(TM)$. The divergence of S_ω is given by (cf. [35, 2])

$$(\mathrm{div} S_\omega)(X) = \langle \delta^\nabla\omega, i_X\omega\rangle + \langle i_X d^\nabla\omega, \omega\rangle . \tag{3.3}$$

Recall that a 2-tensor field $T \in \Gamma(T^*M \otimes T^*M)$ is a Codazzi tensor if T satisfies

$$(\nabla_Z T)(X, Y) = (\nabla_Y T)(X, Z)$$

for any vector field X, Y and Z. One may regard $T \in \Gamma(T^*M \otimes T^*M)$ as a 1-form $T^\sharp$ with values in T^*M as follows

$$T^\sharp(X) = T(\cdot, X), \tag{3.4}$$

that is, $T^\sharp \in A^1(T^*M)$. Note that the covariant derivative of $T^\sharp$ is given by

$$\begin{aligned}((\nabla_X T^\sharp)(Y))(e) =& \Big(\nabla_X(T^\sharp(Y)) - T^\sharp(\nabla_X Y)\Big)(e) \\ =& \Big(\nabla_X(T^\sharp(Y))\Big)(e) - T(e, \nabla_X Y) \\ =& \nabla_X(T^\sharp(Y)(e)) - T^\sharp(Y)(\nabla_X e) - T(e, \nabla_X Y) \\ =& \nabla_X(T(e, Y)) - T(\nabla_X e, Y) - T(e, \nabla_X Y) \\ =& (\nabla_X T)(e, Y)\end{aligned} \tag{3.5}$$

for any $X, Y \in \Gamma(TM)$ and $e \in T_xM$. Therefore T is a Codazzi tensor if and only if

$$(\nabla_X T^\sharp)(Y) = (\nabla_Y T^\sharp)(X) . \tag{3.6}$$

LEMMA 3.1. *The 2-tensor field T is a Codazzi tensor if and only if $d^\nabla T^\sharp = 0$.*

PROOF. By the definition of d^∇, we have

$$(d^\nabla T^\sharp)(X, Y) = (\nabla_X T^\sharp)(Y) - (\nabla_Y T^\sharp)(X), \ \ \forall X, Y \in \Gamma(TM) .$$

Thus, for any $X, Y \in \Gamma(TM)$, $(\nabla_X T^\sharp)(Y) = (\nabla_Y T^\sharp)(X)$ is equivalent to $d^\nabla T^\sharp = 0$. □

REMARK 3.1. There are many well-known examples of Codazzi tensors. These include any constant scalar multiple of the metric, and more generally any parallel self-adjoint $(1, 1)$ tensor, such as the second fundamental form of submanifolds with parallel mean curvature in a space of constant sectional curvature. Furthermore, the Ricci tensor of a Riemannian manifold M is Codazzi if and only if the curvature tensor of M is harmonic. This is the case, for example, if M is an Einstein manifold.

Now we compute the codifferentiation of $T^\sharp$. Choose an orthonormal frame field $\{e_i\}_{i=1}^n$ around a point $x \in M$ such that $(\nabla e_i)_x = 0$. By (3.5), one gets

$$\delta^\nabla T^\sharp = -\sum_{i=1}^n (\nabla_{e_i} T^\sharp)(e_i) = -\sum_{i=1}^n (\nabla_{e_i} T)(\cdot, e_i) . \tag{3.7}$$

LEMMA 3.2. *Let T be a symmetric Codazzi 2-tensor field. If trT is constant, then* $\delta^\nabla T^\sharp = 0$.

PROOF. Note that $\delta^\nabla T^\sharp \in \Gamma(T^*M)$. For any vector $X \in \Gamma(TM)$, we get from (3.7) that

$$(\delta^\nabla T^\sharp)(X) = -\sum_{i=1}^{n} (\nabla_{e_i} T)(X, e_i)\,.$$

Since T is symmetric and Codazzi, it follows that

$$(\delta^\nabla T^\sharp)(X) = -\sum_{i=1}^{n} (\nabla_X T)(e_i, e_i) = -X(\sum_{i=1}^{n} T(e_i, e_i)) = 0\,.$$

□

Therefore, by (3.4), Lemma 3.1 and Lemma 3.2, we have the following proposition:

PROPOSITION 3.1. *Suppose T is a symmetric Codazzi 2-tensor with constant trace. Then $T^\sharp$ satisfies a conservation law, that is, $divS_{T^\sharp} = 0$ as defined in* (3.3).

For any given vector field X, there corresponds to a dual one form $X^\flat$ such that

$$X^\flat(Y) = g(X, Y), \ \ \forall Y \in \Gamma(TM)\,.$$

The covariant derivative of $X^\flat$ gives a 2-tensor field $\nabla X^\flat$:

$$(\nabla X^\flat)(Y, Z) = (\nabla_Z X^\flat)(Y) = g(\nabla_Z X, Y), \ \ \forall Y, Z \in \Gamma(TM)\,.$$

If $X = \nabla\psi$ is the gradient of some smooth function ψ on M, then $X^\flat = d\psi$ and $\nabla X^\flat = \mathrm{Hess}(\psi)$. A direct computation yields (cf. [35] or Lemma 2.4 of [13])

$$\mathrm{div}(i_X S_\omega) = \langle S_\omega, \nabla X^\flat\rangle + (\mathrm{div} S_\omega)(X), \ \ \forall X \in \Gamma(TM)\,. \tag{3.8}$$

Let D be any bounded domain of M with C^1 boundary. By (3.8) and using the divergence theorem, we immediately have

$$\int_{\partial D} S_\omega(X, \nu) ds_g = \int_D (\langle S_\omega, \nabla X^\flat\rangle + (\mathrm{div} S_\omega)(X)) dv_g\,, \tag{3.9}$$

where ν is the unit outward normal vector field along ∂D. In particular, if ω satisfies the conservation law, i.e. $\mathrm{div} S_\omega = 0$, then

$$\int_{\partial D} S_\omega(X, \nu) ds_g = \int_D \langle S_\omega, \nabla X^\flat\rangle dv_g\,. \tag{3.10}$$

Let $r(x)$ be the geodesic distance function of x relative to some fixed point x_0 and $B_{x_0}(r)$ be the geodesic ball centered at x_0 with radius r. Denote by $\lambda_1(x) \le \lambda_2(x) \le \cdots \le \lambda_n(x)$ the eigenvalues of $\mathrm{Hess}(r^2)$. Let

$$\tau(p) = \frac{1}{2} \inf_{x\in M} \{\lambda_1(x) + \cdots + \lambda_{n-p}(x) - \lambda_{n-p+1}(x) - \cdots - \lambda_n(x)\} \tag{3.11}$$

be a function depending only on the integer p, $1 \le p \le n$.

PROPOSITION 3.2. *Let (M, g) be an n-dimensional complete Riemannian manifold with a pole and let $\xi : E \to M$ be a Riemannian vector bundle on M. If $\tau(p) > 0$ and $\omega \in A^p(\xi)$ satisfies the conservation law, that is, $divS_\omega = 0$, then*

$$\frac{1}{\rho_1^\sigma}\int_{B_{x_0}(\rho_1)} |\omega|^2 dv \le \frac{1}{\rho_2^\sigma}\int_{B_{x_0}(\rho_2)} |\omega|^2 dv \tag{3.12}$$

for any $0 < \rho_1 \le \rho_2$ and $0 < \sigma \le \tau(p)$.

PROOF. The proof is similar to that of [13]. We will provide the argument here for completeness of the paper. Take a smooth vector field $X = r\nabla r$ on M. Obviously, $\frac{\partial}{\partial r}$ is an outward unit normal vector field along $\partial B_{x_0}(r)$. Take an orthonormal basis $\{e_i\}_{i=1}^n$ which diagonalizes $\mathrm{Hess}(r^2)$, then

$$\begin{aligned}\langle S_\omega, \nabla X^\flat\rangle =&\frac{1}{2}\sum_{i,j=1}^n S_\omega(e_i, e_j)\mathrm{Hess}(r^2)(e_i, e_j)\\ =&\frac{1}{4}\sum_{i,j=1}^n |\omega|^2\mathrm{Hess}(r^2)(e_i, e_j)\delta_{ij} - \frac{1}{2}\sum_{i,j=1}^n (\omega\odot\omega)(e_i, e_j)\mathrm{Hess}(r^2)(e_i, e_j)\\ =&\frac{|\omega|^2}{4}\sum_{i=1}^n\lambda_i - \frac{1}{2}\sum_{i=1}^n(\omega\odot\omega)(e_i, e_i)\lambda_i\,.\end{aligned}\tag{3.13}$$

For the second term, by (3.2), we have

$$\begin{aligned}\sum_{i=1}^n(\omega\odot\omega)(e_i, e_i)\lambda_i =&\sum_{s=1}^n\langle i_{e_s}\omega, i_{e_s}\omega\rangle\lambda_s\\ =&\sum_{j=1}^p\sum_{i_1,\dots,i_p}\langle\omega(e_{i_1},\dots,e_{i_p}), \omega(e_{i_1},\dots,e_{i_p})\rangle\lambda_{i_j}\\ \le&\sum_{i_1,\dots,i_p}\langle\omega(e_{i_1},\dots,e_{i_p}), \omega(e_{i_1},\dots,e_{i_p})\rangle\sum_{j=n-p+1}^n\lambda_j\\ =&|\omega|^2\sum_{j=n-p+1}^n\lambda_j\,,\end{aligned}$$

where the indices $1 \le i_1, i_2, \dots, i_n \le n$ are distinct with each other in the following discussion. Substituting into (3.13), it follows that

$$\langle S_\omega, \nabla X^\flat\rangle \ge \frac{|\omega|^2}{4}(\lambda_1 + \cdots + \lambda_{n-p} - \lambda_{n-p+1} - \cdots - \lambda_n)\,. \tag{3.14}$$

By the definition of S_ω, we have

$$\begin{aligned}S_\omega(X, \frac{\partial}{\partial r}) =&\frac{|\omega|^2}{2}g(X, \frac{\partial}{\partial r}) - (\omega\odot\omega)(X, \frac{\partial}{\partial r})\\ =&\frac{1}{2}r|\omega|^2 g(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) - r|i_{\frac{\partial}{\partial r}}\omega|^2\end{aligned}\tag{3.15}$$

$$\leq \frac{r|\omega|^2}{2} \quad \text{on } \partial B_{x_0}(r)\,.$$

Since $\mathrm{div} S_\omega = 0$, we get from (3.10), (3.14) and (3.15) that

$$\frac{1}{2}\inf_{x\in M}(\lambda_1+\cdots+\lambda_{n-p}-\lambda_{n-p+1}-\cdots-\lambda_n)\int_{B_{x_0}(r)}|\omega|^2 dv \leq r\int_{\partial B_{x_0}(r)}|\omega|^2 ds\,.$$

Using co-area formula, we have

$$\tau(p)\int_{B_{x_0}(r)}|\omega|^2 dv \leq r\frac{d}{dr}\int_{B_{x_0}(r)}|\omega|^2 dv\,,$$

thus

$$\frac{\frac{d}{dr}\int_{B_{x_0}(r)}|\omega|^2 dv}{\int_{B_{x_0}(r)}|\omega|^2 dv} \geq \frac{\sigma}{r}$$

for any $\sigma \leq \tau(p)$. Integrating the above formula on $[\rho_1, \rho_2]$ yields

$$\frac{1}{\rho_1^\sigma}\int_{B_{x_0}(\rho_1)}|\omega|^2 dv \leq \frac{1}{\rho_2^\sigma}\int_{B_{x_0}(\rho_2)}|\omega|^2 dv\,.$$

□

In the following, we shall use Proposition 3.2 to deduce monotonicity formulas and vanishing results for the curvature tensor of LCF manifolds. For this purpose, we collect the following Lemmas.

LEMMA 3.3 ([15, 13, 17]). *Let (M, g) be a complete Riemannian manifold with a pole x_0 and let r be the distance function relative to x_0. Denote by K_r the radial curvature of M.*

(i) If $-\frac{A}{(1+r^2)^{1+\varepsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\varepsilon}}$ with $\varepsilon > 0$, $A \geq 0$, $0 \leq B < 2\varepsilon$, then

$$\frac{1-\frac{B}{2\varepsilon}}{r}[g - dr\otimes dr] \leq Hess(r) \leq \frac{e^{\frac{A}{2\varepsilon}}}{r}[g - dr\otimes dr]\,.$$

(ii) If $-\frac{a}{1+r^2} \leq K_r \leq \frac{b}{1+r^2}$ with $a \geq 0$, $b \in [0, 1/4]$, then

$$\frac{1+\sqrt{1-4b}}{2r}[g - dr\otimes dr] \leq Hess(r) \leq \frac{1+\sqrt{1+4a}}{2r}[g - dr\otimes dr]\,.$$

(iii) If $-\alpha^2 \leq K_r \leq -\beta^2$ with $\alpha > 0$, $\beta > 0$, then

$$\beta\coth(\beta r)[g - dr\otimes dr] \leq Hess(r) \leq \alpha\coth(\alpha r)[g - dr\otimes dr]\,.$$

Using Lemma 3.3, by a direct calculation we have the following result.

LEMMA 3.4. *Let M^n be a complete manifold of dimension n with a pole x_0. Assume that the radial curvature of M satisfies one of the following conditions:*

(i) $-\frac{A}{(1+r^2)^{1+\varepsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\varepsilon}}$ with $\varepsilon > 0$, $A \geq 0$, $0 \leq B < 2\varepsilon$ and $(n-p-1)(1-\frac{B}{2\varepsilon}) - (p-1)e^{A/2\varepsilon} - \max\{\frac{B}{2\varepsilon}, e^{A/2\varepsilon} - 1\} > 0$;

(ii) $-\frac{a}{1+r^2} \leq K_r \leq \frac{b}{1+r^2}$ with $a \geq 0$, $b \in [0, 1/4]$ and $1 + \frac{n-p-1}{2}(1+\sqrt{1-4b}) - \frac{p}{2}(1 + \sqrt{1+4a}) > 0$;

(iii) $-\alpha^2 \le K_r \le -\beta^2$ *with* $\alpha > 0$, $\beta > 0$ *and* $(n-p-1)\beta - p\alpha \ge 0$. *Then*

$$\tau(p) \ge \sigma(p),$$

where

$$\sigma(p) = \begin{cases} (n-p-1)(1-\frac{B}{2\varepsilon}) - (p-1)e^{\frac{A}{2\varepsilon}} - \max\{\frac{B}{2\varepsilon}, e^{\frac{A}{2\varepsilon}} - 1\} & \text{if } K_r \text{ satisfies } (i), \\ \frac{n-p-1}{2}(1+\sqrt{1-4b}) - \frac{p-1}{2}(1+\sqrt{1+4a}) - \\ \qquad \max\{\frac{1-\sqrt{1-4b}}{2}, \frac{\sqrt{1+4a}-1}{2}\} & \text{if } K_r \text{ satisfies } (ii), \\ n-p-p\frac{\alpha}{\beta} & \text{if } K_r \text{ satisfies } (iii). \end{cases}$$

PROOF. It is known that $\mathrm{Hess}(r^2)$ is given by

$$\mathrm{Hess}(r^2) = 2dr \otimes dr + 2rHess(r).$$

Hence $\lambda_i = 2$ for some $1 \le i \le n$.

When K_r satisfies (i), we divide the discussion into two cases. If $\lambda_i = 2$ for some $1 \le i \le n-p$, by Lemma 3.3 we get

$$\begin{aligned} &\frac{1}{2}\Big(\lambda_1(x) + \cdots + \lambda_{n-p}(x) - \lambda_{n-p+1}(x) - \cdots - \lambda_n(x)\Big) \\ \ge& 1 + (n-p-1)\Big(1 - \frac{B}{2\varepsilon}\Big) - pe^{\frac{A}{2\varepsilon}} \\ =&(n-p-1)\Big(1-\frac{B}{2\varepsilon}\Big) - (p-1)e^{\frac{A}{2\varepsilon}} - (e^{\frac{A}{2\varepsilon}} - 1). \end{aligned} \tag{3.16}$$

If $\lambda_i = 2$ for some $i > n-p$, we have

$$\begin{aligned} &\frac{1}{2}\Big(\lambda_1(x) + \cdots + \lambda_{n-p}(x) - \lambda_{n-p+1}(x) - \cdots - \lambda_n(x)\Big) \\ \ge&(n-p)\Big(1 - \frac{B}{2\varepsilon}\Big) - 1 - (p-1)e^{\frac{A}{2\varepsilon}} \\ =&(n-p-1)\Big(1-\frac{B}{2\varepsilon}\Big) - (p-1)e^{\frac{A}{2\varepsilon}} - \frac{B}{2\varepsilon}. \end{aligned} \tag{3.17}$$

Combining (3.16) and (3.17) gives

$$\tau(p) \ge (n-p-1)\Big(1-\frac{B}{2\varepsilon}\Big) - (p-1)e^{\frac{A}{2\varepsilon}} - \max\Big\{\frac{B}{2\varepsilon}, e^{\frac{A}{2\varepsilon}} - 1\Big\}.$$

When K_r satisfies (ii), the proof is similar to the case (i).

When K_r satisfies (iii), using $\beta r \coth(\beta r) > 1$ and $\frac{\coth(\alpha r)}{\coth(\beta r)} < 1$ for $0 < \beta < \alpha$, we get

$$\begin{aligned} \tau(p) =&\frac{1}{2}\inf_{x\in M}\{\lambda_1(x) + \cdots + \lambda_{n-p}(x) - \lambda_{n-p+1}(x) - \cdots - \lambda_n(x)\} \\ \ge& 1 + (n-p-1)\beta r\coth(\beta r) - p\alpha r\coth(\alpha r) \\ =& 1 + \beta r \coth(\beta r)\left[(n-p-1) - p\frac{\alpha r\coth(\alpha r)}{\beta r\coth(\beta r)}\right] \\ \ge& 1 + (n-p-1) - p\frac{\alpha}{\beta} \end{aligned}$$

$$=n-p-p\frac{\alpha}{\beta},$$

provided that $(n-p-1)\beta - p\alpha \geq 0$. □

Let (M^n, g) be a Riemannian manifold of dimension n, and let $V \to M$ be the vector bundle of skew-symmetric endomorphisms of TM endowed with its canonical Riemannian structure. Then the curvature tensor Rm can be seen as a V-valued 2-form and thus the second Bianchi identity can be equivalently expressed as $d^\nabla Rm = 0$. Actually, using moving frame method, we may compute

$$\begin{aligned}(d^\nabla R_{ij})_{klm} =& R_{ijlm,k} - R_{ijkm,l} + R_{ijkl,m}\\ =& R_{ijlm,k} + R_{ijmk,l} + R_{ijkl,m} = 0\,.\end{aligned}$$

LEMMA 3.5. *Let (M^n, g), $n \geq 3$, be a LCF Riemannian manifold with constant scalar curvature. Then the curvature tensor Rm is a harmonic V-valued 2-form and thus Rm satisfies a conservation law, that is, $divS_{Rm} = 0$ as defined in* (3.3).

PROOF. We only need to prove that $d^\nabla Rm = 0$ and $\delta^\nabla Rm = 0$. We have already pointed out that the first property is just the second Bianchi identity. In terms of the condition that M is a LCF manifold with constant scalar curvature, we find that

$$\begin{aligned}(\delta^\nabla Rm)_{jkl} = R_{ijkl,i} =& \nabla_i R_{ijkl}\\ =& -\nabla_k R_{ijli} - \nabla_l R_{ijik}\\ =& \nabla_k R_{jl} - \nabla_l R_{jk} = 0\,.\end{aligned}$$

□

REMARK 3.2. It is well known that (see [11])

$$\delta^\nabla W = \frac{n-3}{n-2} d^\nabla \left(\mathrm{Ric} - \frac{R}{2(n-1)} g\right)$$

for any Riemannian manifold (M, g), $n \geq 3$. By the relation (2.1), the Weyl curvature tensor W of an Einstein manifold is also a harmonic V-valued 2-form. Thus, W also satisfies a conservation law.

For the Ricci tensor Ric, we can consider Ric to be a 1-form $\mathrm{Ric}^\sharp$ with values in the tangent vector bundle at every point $x \in M$, that is, for every $X \in T_x M$, $\mathrm{Ric}^\sharp(X)$ satisfies

$$\langle \mathrm{Ric}^\sharp(X), Y\rangle = \mathrm{Ric}(X, Y),\ \forall Y \in T_x M\,.$$

$E^\sharp$ satisfies $\langle E^\sharp(X), Y\rangle = E(X, Y)$, $\forall Y \in T_x M$, where E is the traceless Ricci tensor given by $E = \mathrm{Ric} - \frac{R}{n} g$. Thus if M is a conformally flat Riemannian manifold with constant scalar curvature, then by Proposition 3.1, $\mathrm{Ric}^\sharp$ and $E^\sharp$ satisfy conservation laws, that is, $\mathrm{div}S_{\mathrm{Ric}^\sharp} = 0$ and $\mathrm{div}S_{E^\sharp} = 0$ as defined in (3.3). Let $|\,\mathrm{Ric}\,|$ be the norm of Ricci tensor Ric and $|E|$ be the norm of the traceless Ricci tensor E given by $|\operatorname{Ric}| = (\sum_{i,j=1}^{n} R_{ij}^2)^{\frac{1}{2}}$ and $|E| = \Big(\sum_{i,j=1}^{n} (R_{ij} - \frac{R}{n}\delta_{ij})^2\Big)^{\frac{1}{2}}$ respectively. Summarizing the previous discussions, we have the following results.

THEOREM 3.1. *Let (M^n, g) be a complete, locally conformally flat Riemannian manifold with a pole. Assume that M has constant scalar curvature and the radial curvature of M satisfies the conditions of Lemma* 3.4. *Then for any* $0 < \rho_1 \le \rho_2$,

$$\frac{1}{\rho_1^{\sigma(2)}}\int_{B_{x_0}(\rho_1)} |Rm|^2 dv \le \frac{1}{\rho_2^{\sigma(2)}}\int_{B_{x_0}(\rho_2)} |Rm|^2 dv\,,$$

and

$$\frac{1}{\rho_1^{\sigma(1)}}\int_{B_{x_0}(\rho_1)} |Ric|^2 dv \le \frac{1}{\rho_2^{\sigma(1)}}\int_{B_{x_0}(\rho_2)} |Ric|^2 dv\,,$$

and

$$\frac{1}{\rho_1^{\sigma(1)}}\int_{B_{x_0}(\rho_1)} |E|^2 dv \le \frac{1}{\rho_2^{\sigma(1)}}\int_{B_{x_0}(\rho_2)} |E|^2 dv\,,$$

where $\sigma(p)$, $p = 1, 2$, satisfies

$$\sigma(p) = \begin{cases} (n-p-1)(1-\frac{B}{2\varepsilon}) - (p-1)e^{\frac{A}{2\varepsilon}} - \max\{\frac{B}{2\varepsilon}, e^{\frac{A}{2\varepsilon}} - 1\} & \text{if } K_r \text{ satisfies } (i), \\ \frac{n-p-1}{2}(1+\sqrt{1-4b}) - \frac{p-1}{2}(1+\sqrt{1+4a}) - \\ \qquad \max\{\frac{1-\sqrt{1-4b}}{2}, \frac{\sqrt{1+4a}-1}{2}\} & \text{if } K_r \text{ satisfies } (ii), \\ n-p-p\frac{\alpha}{\beta} & \text{if } K_r \text{ satisfies } (iii). \end{cases}$$

Letting $p = 1$ in Theorem 3.1, we have the following corollaries.

COROLLARY 3.1. *Let M^n, $n \ge 3$, be a complete, locally conformally flat Riemannian manifold with a pole x_0 and zero scalar curvature. Assume the Ricci curvature of M satisfies one of the following conditions:*

(i) $-\frac{n-2}{2}\frac{A}{(1+r^2)^{1+\varepsilon}} \le Ric \le \frac{n-2}{2}\frac{B}{(1+r^2)^{1+\varepsilon}}$ *with* $\varepsilon > 0$, $A \ge 0$, $0 \le B < 2\varepsilon$ *and* $\sigma(1) = (n-2)(1-\frac{B}{2\varepsilon}) - \max\{\frac{B}{2\varepsilon}, e^{A/2\varepsilon} - 1\} > 0$;

(ii) $-\frac{n-2}{2}\frac{a}{1+r^2} \le Ric \le \frac{n-2}{2}\frac{b}{1+r^2}$ *with* $a \ge 0$, $b \in [0, 1/4]$ *and* $\sigma(1) = \frac{n-2}{2}(1+\sqrt{1-4b}) - \max\{\frac{1-\sqrt{1-4b}}{2}, \frac{\sqrt{1+4a}-1}{2}\} > 0$.

Assume further that

$$\int_{B_{x_0}(\rho)} |Ric|^2 dv_g = o(\rho^{\sigma(1)}) \ \ as \ \ \rho \to +\infty\,.$$

Then M is flat.

PROOF. By the relation (2.2), the radial curvature of M satisfies (i), (ii) of Lemma 3.4. Hence Theorem 3.1 and the growth condition of $|\mathrm{Ric}|^2$ implies that (M, g) is Ricci-flat. Therefore, it follows immediately from (2.2) that $Rm = 0$. □

COROLLARY 3.2. *Let M^n, $n \geq 3$, be a complete, locally conformally flat Riemannian manifold with constant scalar curvature R. Assume M has a pole x_0 and its radial curvature satisfies $-\alpha^2 \leq K \leq -\beta^2$ with $\alpha > 0$, $\beta > 0$ and $(n-2)\beta - \alpha \geq 0$. If*

$$\int_{B_{x_0}(\rho)} |E|^2 dv_g = o(\rho^{n-1-\frac{\alpha}{\beta}}) \ \ as \ \ \rho \to +\infty,$$

then M is of constant curvature $\frac{R}{n(n-1)}$.

4. Vanishing theorems for L^2 harmonic p-forms on LCF manifolds. Let (M^n, g) be a complete, locally conformally flat Riemannian manifold, and let $\triangle$ be the Hodge Laplace-Beltrami operator of M^n acting on the space of differential p-forms. The Weitzenböck formula ([34]) gives

$$\triangle = \nabla^\star \nabla - \mathcal{R}_p\,, \tag{4.1}$$

where $\nabla^\star\nabla$ is the Bochner Laplacian and $\mathcal{R}_p$ is an endomorphism depending upon the curvature tensor of M^n. Using an orthonormal basis $\{\theta^1, \ldots, \theta^n\}$ dual to $\{e_1, \ldots, e_n\}$, the curvature term $\mathcal{R}_p$ can be expressed as

$$\langle \mathcal{R}_p(\theta), \theta \rangle = \langle \sum_{j,k=1}^{n} \theta^k \wedge i_{e_j} R(e_k, e_j)\theta, \theta \rangle$$

for any p-form θ. Let ω be any harmonic p-form, which may be expressed in a local coordinate system as

$$\omega = \alpha_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}\,.$$

By (4.1), we deduce that

$$\frac{1}{2}\triangle|\omega|^2 = |\nabla\omega|^2 + \langle \sum_{j,k=1}^{n} \theta^k \wedge i_{e_j} R(e_k, e_j)\omega, \omega \rangle \tag{4.2}$$

$$= |\nabla\omega|^2 + pF(\omega)\,, \tag{4.3}$$

where

$$F(\omega) = R_{ij}\alpha^{ii_2\cdots i_p}\alpha^{j}_{i_2\cdots i_p} - \frac{p-1}{2} R_{ijkl}\alpha^{iji_3\cdots i_p}\alpha^{kl}_{i_3\cdots i_p}\,.$$

Here, repeated indices are contracted and summed. Substituting (2.2) into the above equality, we obtain

$$\frac{1}{2}\triangle|\omega|^2 = |\nabla\omega|^2 + p\Big[\frac{n-2p}{n-2} R_{ij}\alpha^{ii_2\cdots i_p}\alpha^{j}_{i_2\cdots i_p} + \frac{(p-1)}{(n-1)(n-2)} R|\omega|^2\Big] \tag{4.4}$$

$$= |\nabla\omega|^2 + p\Big[\frac{n-2p}{n-2}\Big(R_{ij} - \frac{R}{n}\delta_{ij}\Big)\alpha^{ii_2\cdots i_p}\alpha^{j}_{i_2\cdots i_p} + \frac{n-p}{n(n-1)} R|\omega|^2\Big]\,. \tag{4.5}$$

Using the method of Lagrange multipliers, one has the following lemma.

LEMMA 4.1 ([31]). *Let $(a_{ij})_{n\times n}$ be a real symmetric matrix with $\sum_{i=1}^{n} a_{ii} = 0$, then*

$$\sum_{i,j=1}^{n} a_{ij}x_i x_j \geq -\sqrt{\frac{n-1}{n}}\Big(\sum_{i,j=1}^{n} a_{ij}^2\Big)^{\frac{1}{2}} \sum_{i=1}^{n} x_i^2$$

where $x_i \in \mathbb{R}$.

By Lemma 4.1, it follows from (4.5) that

$$|\omega|\triangle|\omega| \geq |\nabla\omega|^2 - |\nabla|\omega||^2 - \frac{p|n-2p|}{n-2}\sqrt{\frac{n-1}{n}}|E||\omega|^2 + \frac{p(n-p)}{n(n-1)}R|\omega|^2\,, \tag{4.6}$$

where $|E|$ is the norm of the traceless Ricci tensor E. When the underlying manifold M is compact or the harmonic form ω is squared integrable, then ω is closed and coclosed (cf. [37]). According to [5], we have the refined Kato's inequality

$$|\nabla\omega|^2 - |\nabla|\omega||^2 \geq K_p|\nabla|\omega||^2\,, \tag{4.7}$$

where

$$K_p = \begin{cases} \frac{1}{n-p} & \text{if } 1 \leq p \leq n/2\,, \\ \frac{1}{p} & \text{if } n/2 \leq p \leq n-1\,. \end{cases}$$

Therefore, the relation (4.6) reduces to

$$|\omega|\triangle|\omega| \geq K_p|\nabla|\omega||^2 - \frac{p|n-2p|}{n-2}\sqrt{\frac{n-1}{n}}|E||\omega|^2 + \frac{p(n-p)}{n(n-1)}R|\omega|^2\,. \tag{4.8}$$

Now, using the inequality (4.6) to compact locally conformally flat Riemannian manifold, we have the following theorem, generalizing Corollary 8.8 of [4], for the case $R(x) > 0$ for every $x \in M$ and $p = m = \frac{n}{2}$.

THEOREM 4.1. *Let (M^n, g), $n \geq 3$, be a compact locally conformally flat Riemannian manifold satisfying*

$$R(x) \geq \sqrt{\frac{n-1}{n}}\frac{n(n-1)|n-2p|}{(n-p)(n-2)}|E|(x) \tag{4.9}$$

for every $x \in M$, $1 \leq p \leq n$. Assume that (4.9) *is strict at some point. Then the Betti number $\beta_p(M) = 0$. In particular, if M is a $2m$-dimensional compact LCF Riemannian manifold with nonnegative scalar curvature $R \geq 0$, and $R > 0$ holds at some point, then $\beta_m(M) = 0$.*

PROOF. For any given harmonic p-form ω, we have via (4.6) and the hypothesis (4.9) on the scalar curvature R,

$$\frac{1}{2}\triangle|\omega|^2 \geq |\nabla\omega|^2 + \Big[\frac{p(n-p)}{n(n-1)}R - \frac{p|n-2p|}{n-2}\sqrt{\frac{n-1}{n}}|E|\Big]|\omega|^2 \geq 0\,. \tag{4.10}$$

By the compactness of M and the maximum principle, $|\omega| = \text{const}$. Substituting this into (4.10) and using the hypothesis on R again, we have $\omega = 0$. Therefore, by Hodge's Theorem, $\beta_p(M) = 0$. □

REMARK 4.1. It is well known that a compact orientable conformally flat Riemannian manifold with positive Ricci curvature must satisfy $\beta_p(M) = 0$ for all $1 \leq p \leq n-1$ (see [14]).

THEOREM 4.2. *Let (M^n, g), $n \geq 3$, be a complete non-compact, simply connected, locally conformally flat Riemannian manifold. For any $0 \leq p \leq n$, there exists a positive constant C_p such that if*

$$\left(\int_M |Ric|^{\frac{n}{2}} dv\right)^{\frac{2}{n}} < C_p \,, \tag{4.11}$$

then every closed and coclosed p-form ω on M with $\liminf_{r\to\infty} \frac{1}{r^2} \int_{B_{x_0}(r)} |\omega|^2 dv = 0$ vanishes identically. In particular, $H^p(L^2(M)) = \{0\}$.

PROOF. Let ω be a closed and coclosed p-form on M with $\liminf_{r\to\infty} \frac{1}{r^2} \int_{B_{x_0}(r)} |\omega|^2 dv = 0$. When $1 \leq p \leq n-1$, by (4.4) and using the fact that $R^2 \leq n|\mathrm{Ric}|^2$, we have

$$\frac{1}{2}\triangle|\omega|^2 \geq |\nabla\omega|^2 - \frac{p|n-2p|}{n-2}|\mathrm{Ric}||\omega|^2 - \frac{p(p-1)\sqrt{n}}{(n-1)(n-2)}|\mathrm{Ric}||\omega|^2 \,.$$

Combining this with (4.7), we deduce that

$$|\omega|\triangle|\omega| + \frac{p}{n-2}\left(|n-2p| + \frac{(p-1)\sqrt{n}}{n-1}\right)|\mathrm{Ric}||\omega|^2 \geq K_p|\nabla|\omega||^2 \,. \tag{4.12}$$

Fix a point $x_0 \in M$ and let $\rho(x)$ be the geodesic distance on M from x_0 to x. Let us choose $\eta \in C_0^\infty(M)$ satisfying

$$\eta(x) = \begin{cases} 1 & \text{if } \rho(x) \leq r\,, \\ 0 & \text{if } 2r < \rho(x) \end{cases}$$

and

$$|\nabla\eta|(x) \leq \frac{2}{r} \ \text{ if } \ r < \rho(x) \leq 2r \tag{4.13}$$

for $r > 0$. Multiplying (4.12) by η^2 and integrating by parts over M, we obtain

$$\begin{aligned} 0 \leq & \int_M (\eta^2|\omega|\triangle|\omega| - K_p\eta^2|\nabla|\omega||^2)dv \\ & + \frac{p}{n-2}\left(|n-2p| + \frac{(p-1)\sqrt{n}}{n-1}\right)\int_M |\mathrm{Ric}|\eta^2|\omega|^2 dv \\ = & -2\int_M \eta|\omega|\langle\nabla\eta, \nabla|\omega|\rangle dv - (1+K_p)\int_M \eta^2|\nabla|\omega||^2 dv \\ & + \frac{p}{n-2}\left(|n-2p| + \frac{(p-1)\sqrt{n}}{n-1}\right)\int_M |\mathrm{Ric}|\eta^2|\omega|^2 dv \,. \end{aligned} \tag{4.14}$$

By the hypothesis (4.11), we have

$$\int_M |R|^{\frac{n}{2}} dv \leq n^{n/4}\int_M |\mathrm{Ric}|^{\frac{n}{2}} dv < \infty \,,$$

which implies that the L^2-Sobolev inequality (2.5) holds for some constant $C_s > 0$. Hence it follows from (2.5) and the Hölder inequality that

$$\begin{aligned}
(4.15)\qquad \int_M |\mathrm{Ric}|\eta^2|\omega|^2 dv \leq & \left(\int_{\mathrm{supp}(\eta)} |\mathrm{Ric}|^{\frac{n}{2}} dv\right)^{\frac{2}{n}} \left(\int_M (\eta|\omega|)^{\frac{2n}{n-2}} dv\right)^{\frac{n-2}{n}} \\
\leq & R(\eta) \int_M |\nabla(\eta|\omega|)|^2 dv \\
= & R(\eta) \int_M (\eta^2|\nabla|\omega||^2 + |\omega|^2|\nabla\eta|^2) dv \\
& + 2R(\eta) \int_M \eta|\omega|\langle\nabla\eta, \nabla|\omega|\rangle dv,
\end{aligned}$$

where $R(\eta) = \frac{1}{C_s}\left(\int_{\mathrm{supp}(\eta)} |\mathrm{Ric}|^{\frac{n}{2}} dv\right)^{\frac{2}{n}}$. Substituting (4.15) into (4.14) yields

$$\begin{aligned}
0 \leq & (2A-2)\int_M \eta|\omega|\langle\nabla\eta, \nabla|\omega|\rangle dv - (1+K_p-A)\int_M \eta^2|\nabla|\omega||^2 dv + A\int_M |\omega|^2|\nabla\eta|^2 dv \\
\leq & (-1-K_p+A+|A-1|\varepsilon)\int_M \eta^2|\nabla|\omega||^2 dv + \left(A + \frac{|A-1|}{\varepsilon}\right)\int_M |\omega|^2|\nabla\eta|^2 dv
\end{aligned}$$

for all $\varepsilon > 0$, where

$$A = \frac{p}{n-2}\left(|n-2p| + \frac{(p-1)\sqrt{n}}{n-1}\right)R(\eta).$$

Now let us choose the integral bound C_p in (4.11) satisfying

$$C_p = \frac{n-2}{p}\left(|n-2p| + \frac{(p-1)\sqrt{n}}{n-1}\right)^{-1}(1+K_p)C_s.$$

Then we can take sufficiently small $\varepsilon > 0$ such that $1 + K_p - A - |A-1|\varepsilon > 0$. Therefore,

$$\begin{aligned}
(1+K_p-A-|A-1|\varepsilon)\int_{B_{x_0}(r)} |\nabla|\omega||^2 dv \leq & (1+K_p-A-|A-1|\varepsilon)\int_M \eta^2|\nabla|\omega||^2 dv \\
\leq & \left(A + \frac{|A-1|}{\varepsilon}\right)\int_M |\omega|^2|\nabla\eta|^2 dv \\
\leq & \left(A + \frac{|A-1|}{\varepsilon}\right)\frac{4}{r^2}\int_{B_{x_0}(2r)} |\omega|^2 dv.
\end{aligned}$$

Letting $r \to \infty$, we have $\nabla|\omega| = 0$ on M, i.e., $|\omega|$ is constant. Since $\liminf_{r\to\infty} \frac{1}{r^2}\int_{B_{x_0}(r)} |\omega|^2 dv = 0$ and the volume growth (2.6) implies $\frac{\mathrm{vol}(B_{x_0}(r))}{r^2} \geq Cr^{n-2} \to \infty$ as $r \to \infty$, we conclude that $\omega = 0$.

When $p = 0$, let f be a harmonic function with $\liminf_{r\to\infty} \frac{1}{r^2}\int_{B_{x_0}(r)} |\omega|^2 dv = 0$. According to [33], f is constant. Since $\frac{\mathrm{vol}(B_{x_0}(r))}{r^2} \geq Cr^{n-2}$, we have $f = 0$. When $p = n$, we consider $*\omega$, where $*$ is the Hodge Star. Then $*\omega$ is a harmonic function with $|\omega| = |*\omega|$. By the previous result, $*\omega = 0$ and so is $\omega = 0$. It follows that $H^p(L^2(M)) = \{0\}$ for all $0 \le p \le n$. This completes the proof. □

REMARK 4.2. Since the constant C_s in the Sobolev inequality (2.5) can not be explicitly computed, we can't also give the explicit value of C_p in (4.11).

THEOREM 4.3. *Let (M^n, g), $n \ge 3$, be a complete non-compact, simply connected, locally conformally flat Riemannian manifold with $R \ge 0$. Assume that*

$$\Big(\int_M |E|^{\frac{n}{2}} dv\Big)^{\frac{2}{n}} < C(p), \tag{4.16}$$

where $C(p) = \frac{(n-2)\sqrt{n}}{p|n-2p|\sqrt{n-1}} \min\left\{1 + K_p, \frac{4p(n-p)}{n(n-2)}\right\} Q(\mathbb{S}^n)$ for every $1 \le p \le n-1$ but $p \ne \frac{n}{2}$. Then every closed and coclosed p-form ω on M with $\liminf_{r\to\infty} \frac{1}{r^2}\int_{B_{x_0}(r)} |\omega|^2 dv = 0$ vanishes identically. In particular, $H^p(L^2(M)) = \{0\}$ for $1 \le p \le n-1$ but $p \ne \frac{n}{2}$.

PROOF. Let ω be a closed and coclosed p-form on M with $\liminf_{r\to\infty} \frac{1}{r^2}\int_{B_{x_0}(r)} |\omega|^2 dv = 0$. Let $\eta \in C_0^\infty(M)$ be a smooth function on M with compact support. Multiplying (4.8) by η^2 and integrating over M, we obtain

$$\begin{aligned}\int_M \eta^2|\omega|\triangle|\omega| dv \ge& K_p \int_M \eta^2|\nabla|\omega||^2 dv - \frac{p|n-2p|}{n-2}\sqrt{\frac{n-1}{n}}\int_M |E|\eta^2|\omega|^2 dv \\ &+ \frac{p(n-p)}{n(n-1)}\int_M R\eta^2|\omega|^2 dv\,.\end{aligned} \tag{4.17}$$

Integrating by parts and using the Cauchy-Schwarz inequality gives

$$\begin{aligned}\int_M \eta^2|\omega|\triangle|\omega| dv =& -2\int_M \eta|\omega|\langle\nabla\eta, \nabla|\omega|\rangle dv - \int_M \eta^2|\nabla|\omega||^2 dv \\ \le& (b-1)\int_M \eta^2|\nabla|\omega||^2 dv + \frac{1}{b}\int_M |\omega|^2|\nabla\eta|^2 dv\end{aligned}$$

for all $b > 0$. Substituting the above inequality into (4.17) yields

$$\begin{aligned}(1 + K_p - b)\int_M \eta^2|\nabla|\omega||^2 dv \le& \frac{1}{b}\int_M |\omega|^2|\nabla\eta|^2 dv + \frac{p|n-2p|}{n-2}\sqrt{\frac{n-1}{n}}\int_M |E|\eta^2|\omega|^2 dv \\ &- \frac{p(n-p)}{n(n-1)}\int_M R\eta^2|\omega|^2 dv\,.\end{aligned} \tag{4.18}$$

On the other hand, using (2.3) together with the Hölder and Cauchy-Schwarz inequalities, we have

$$\begin{aligned}\int_M |E|\eta^2|\omega|^2 dv \le& \left(\int_{\mathrm{supp}(\eta)} |E|^{\frac{n}{2}} dv\right)^{\frac{2}{n}} \left(\int_M (\eta|\omega|)^{\frac{2n}{n-2}} dv\right)^{\frac{n-2}{n}} \\ \le& \frac{1}{Q(\mathbb{S}^n)}\left(\int_{\mathrm{supp}(\eta)} |E|^{\frac{n}{2}} dv\right)^{\frac{2}{n}} \int_M \Big[|\nabla(\eta|\omega|)|^2 + \frac{n-2}{4(n-1)}R\eta^2|\omega|^2\Big] dv\end{aligned}$$

$$
\begin{aligned}
=&T(\eta)\int_M\left[\eta^2|\nabla|\omega||^2+|\omega|^2|\nabla\eta|^2+\frac{n-2}{4(n-1)}R\eta^2|\omega|^2\right]dv\\
&+2T(\eta)\int_M\eta|\omega|\langle\nabla\eta,\nabla|\omega|\rangle dv\\
\le&T(\eta)\int_M\left[(1+\gamma)\eta^2|\nabla|\omega||^2+(1+\frac{1}{\gamma})|\omega|^2|\nabla\eta|^2+\frac{n-2}{4(n-1)}R\eta^2|\omega|^2\right]dv
\end{aligned}
$$

for all $\gamma>0$, where $\mathrm{supp}(\eta)$ is the support of η on M, and $T(\eta)=\frac{1}{Q(\mathbb{S}^n)}(\int_{\mathrm{supp}(\eta)}|E|^{\frac{n}{2}}dv)^{\frac{2}{n}}$. Substituting the above inequality into (4.18), we conclude that

$$
B\int_M\eta^2|\nabla|\omega||^2dv\le C\int_M|\omega|^2|\nabla\eta|^2dv+D\int_M R\eta^2|\omega|^2dv, \tag{4.19}
$$

where

$$
\begin{aligned}
B=&1+K_p-b-\frac{p|n-2p|}{n-2}\sqrt{\frac{n-1}{n}}T(\eta)(1+\gamma),\\
C=&\frac{1}{b}+\frac{p|n-2p|}{n-2}\sqrt{\frac{n-1}{n}}T(\eta)(1+\frac{1}{\gamma}),\\
D=&\frac{p|n-2p|}{4(n-1)}\sqrt{\frac{n-1}{n}}T(\eta)-\frac{p(n-p)}{n(n-1)}.
\end{aligned}
$$

It follows from the hypothesis (4.16) that for $1\le p\le n-1$ but $p\ne\frac{n}{2}$,

$$
T(\eta)=\frac{1}{Q(\mathbb{S}^n)}\Big(\int_{\mathrm{supp}(\eta)}|E|^{\frac{n}{2}}dv\Big)^{\frac{2}{n}}<\frac{n-2}{p|n-2p|}\sqrt{\frac{n}{n-1}}\min\left\{1+K_p,\frac{4p(n-p)}{n(n-2)}\right\},
$$

which implies that $D<0$ and $1+K_p-\frac{p|n-2p|}{n-2}\sqrt{\frac{n-1}{n}}T(\eta)>0$. Hence we can choose γ and b small enough such that

$$
B=1+K_p-b-\frac{p|n-2p|}{n-2}\sqrt{\frac{n-1}{n}}T(\eta)(1+\gamma)>0.
$$

Let η be the cut-off function defined by (4.13). Substituting η into (4.19) and noting the hypothesis $R\ge 0$, we have

$$
\begin{aligned}
B\int_{B_{x_0}(r)}|\nabla|\omega||^2dv\le&B\int_M\eta^2|\nabla|\omega||^2dv\\
\le&\frac{4C}{r^2}\int_{B_{x_0}(2r)}|\omega|^2dv+D\int_{B_{x_0}(r)}R|\omega|^2dv.
\end{aligned}
$$

Letting $r\to\infty$, and noting $\liminf_{r\to\infty}\frac{1}{r^2}\int_{B_{x_0}(r)}|\omega|^2dv=0$, we conclude that

$$
\nabla|\omega|=0\quad\text{and}\quad R|\omega|=0
$$

on M. Hence, $|\omega|=\text{const}$. If $|\omega|$ is not identically zero, then $R=0$, which implies that the L^2-Sobolev inequality (2.4) holds, and $\frac{\mathrm{vol}(B_{x_0}(r))}{r^2}\ge Cr^{n-2}\to\infty$ as $r\to\infty$. This would contradict $\liminf_{r\to\infty} \frac{1}{r^2}\int_{B_{x_0}(r)} |\omega|^2 dv = 0$. Therefore, $\omega = 0$. It follows that $H^p(L^2(M)) = \{0\}$ for $1 \le p \le n-1$ but $p \ne \frac{n}{2}$. This completes the proof. □

For the middle degree case, we deduce the following vanishing theorem without assumptions on E.

THEOREM 4.4. *Let (M^n, g), $n = 2m > 3$, be a complete non-compact, simply connected, locally conformally flat Riemannian manifold with $R \ge 0$. Then every closed and coclosed m-form ω on M with $\liminf_{r\to\infty} \frac{1}{r^2}\int_{B_{x_0}(r)} |\omega|^2 dv = 0$ vanishes identically. In particular, $H^m(L^2(M)) = \{0\}$.*

PROOF. Taking $p = m = \frac{n}{2}$ in (4.8), we have

$$|\omega|\triangle|\omega| \ge \frac{1}{m}|\nabla|\omega||^2 + \frac{m}{2(2m-1)}R|\omega|^2 . \tag{4.20}$$

Let η be the cut-off function defined by (4.13). Multiplying (4.20) by η^2 and integrating by parts over M, we obtain

$$\begin{aligned}
&\frac{m+1}{m}\int_M |\nabla|\omega||^2\eta^2 dv + \frac{m}{2(2m-1)}\int_M R|\omega|^2\eta^2 dv \\
&\le \frac{1}{2}\int_M \triangle|\omega|^2\eta^2 dv \\
&= -2\int_M \langle|\omega|\nabla\eta, \eta\nabla|\omega|\rangle dv \\
&\le m\int_M |\omega|^2|\nabla\eta|^2 dv + \frac{1}{m}\int_M |\nabla|\omega||^2\eta^2 dv ,
\end{aligned}$$

which implies that

$$\begin{aligned}
\int_{B_{x_0}(r)} |\nabla|\omega||^2 dv + \frac{m}{2(2m-1)}\int_{B_{x_0}(r)} R|\omega|^2 dv &\le m\int_M |\omega|^2|\nabla\eta|^2 dv \\
&\le \frac{4m}{r^2}\int_{B_{x_0}(2r)} |\omega|^2 dv .
\end{aligned}$$

Having established this fact, the rest of the proof is completely analogous to that of Theorem 4.3. □

REMARK 4.3. Pigola, Rigoli and Setti [26] proved a vanishing theorem for bounded harmonic m-forms on a $2m$-dimensional complete LCF manifold by putting some assumptions on the scalar curvature and volume growth.

Combining Theorems 4.3 and 4.4, we immediately have

COROLLARY 4.1. *Let (M^n, g), $n \ge 3$, be a complete non-compact, simply connected, locally conformally flat Riemannian manifold with $R \ge 0$. Then there exists a positive constant C such that if*

$$\int_M |E|^{\frac{n}{2}} dv < C ,$$

then every closed and coclosed p-form ω *on* M *with* $\liminf_{r\to\infty} \frac{1}{r^2}\int_{B_{x_0}(r)} |\omega|^2 dv = 0$ *vanishes identically for every* $1 \leq p \leq n-1$*. In particular,* $H^p(L^2(M)) = \{0\}$.

THEOREM 4.5. *Let* (M^n, g) *be a complete non-compact, simply connected, locally conformally flat Riemannian manifold of dimension* $n = 2m > 3$*. Then there exists* $C > 0$ *such that if*

$$\int_M |R|^m dv < C,$$

then every closed and coclosed m-form ω *on* M *with* $\liminf_{r\to\infty} \frac{1}{r^2}\int_{B_{x_0}(r)} |\omega|^2 dv = 0$ *vanishes identically. In particular,* $H^m(L^2(M)) = \{0\}$.

PROOF. It follows from (4.20) that

$$|\omega|\triangle|\omega| + \frac{m}{2(2m-1)}|R||\omega|^2 \geq \frac{1}{m}|\nabla|\omega||^2 .$$

By an analogue argument of Theorem 4.3, we prove that $|\omega| = \text{const}$. Using also (2.5) we immediately complete the proof. □

Let us recall that a Riemannian manifold M is said to have nonnegative isotropic curvature if

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \geq 0$$

for every orthonormal 4-frame $\{e_1, e_2, e_3, e_4\}$. From [22], we know that if M is conformally flat and has nonnegative isotropic curvature, then $F(\omega) \geq 0$ for any $2 \leq p \leq [\frac{n}{2}]$. Thus, it follows from the relations (4.3) and (4.7) that

$$|\omega|\triangle|\omega| \geq \frac{1}{n-p}|\nabla|\omega||^2 .$$

Therefore, using the previous argument and the duality generated by the star operator $*$, we have the following result.

THEOREM 4.6. *Let* (M^n, g)*,* $n \geq 4$*, be a complete locally conformally flat manifold with nonnegative isotropic curvature. Then for* $2 \leq p \leq n-2$*,* (i) *if* $\liminf_{r\to\infty} \frac{\mathrm{vol}(B_{x_0}(r))}{r^2} > 0$*, then every closed and coclosed p-form* ω *on* M *with* $\liminf_{r\to\infty} \frac{1}{r^2}\int_{B_{x_0}(r)} |\omega|^2 dv = 0$ *vanishes identically;* (ii) *if* M *has infinite volume then* $H^p(L^2(M)) = \{0\}$.

For a LCF Riemannian manifold, a direct computation from (2.2) gives

$$R_{ijkl} = 0 \tag{4.21}$$

if i, j, k, l are different indexes, and

$$R_{ijij} = \frac{1}{n-2}\left(\mathrm{Ric}_{ii} + \mathrm{Ric}_{jj} - \frac{R}{n-1}\right)$$

for all distinct i, j. Thus if

$$\mathrm{Ric} \geq \frac{R}{2(n-1),}$$

then $R_{ijij} \geq 0$, which combining with (4.21) implies that M has nonnegative isotropic curvature. Applying Theorem 4.6, we have the following corollary.

COROLLARY 4.2. *Let (M^n, g), $n \geq 4$, be a complete non-compact locally conformally flat Riemannian manifold. Assume that*

$$Ric(x) \geq \frac{1}{2(n-1)} R(x)$$

for all $x \in M$. Then $H^p(L^2(M)) = \{0\}$ for all $2 \leq p \leq n-2$.

PROOF. According to the previous discussion, M is of nonnegative Ricci curvature. Since M is complete non-compact, we conclude from [37] that M has infinite volume. Hence the conclusion follows immediately from Theorem 4.6 (ii). □

For an oriented four-manifold M^4 the bundle of two-forms splits $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ into the $+1$-eigenspace of the Hodge $*$-operator (self-dual two-forms) and -1-eigenspace (anti-self-dual two-forms). This allows us to conclude that the Weyl tensor W is an endomorphism of $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ such that $W = W^+ \oplus W^-$. An oriented four-manifold M^4 is said to be half-conformally flat if either $W^+ = 0$ or $W^- = 0$ (see [3, Chapter 13, Section C] for a nice overview on half conformally flat manifolds). Without loss of generality, we assume that $W^+ = 0$.

By the property of W^-, for any $k, l = 1, 2, 3, 4$, we have

$$W^-_{12kl} = -W^-_{34kl}, \quad W^-_{13kl} = -W^-_{42kl}, \quad W^-_{14kl} = -W^-_{23kl}\,.$$

Combining with the first Bianchi identity, we compute

$$\begin{aligned} & W^-_{1313} + W^-_{1414} + W^-_{2323} + W^-_{2424} - 2W^-_{1234} \\ & = -W^-_{4213} - W^-_{2314} - W^-_{1423} - W^-_{3124} - 2W^-_{1234} \\ & = -2W^-_{1342} - 2W^-_{1423} - 2W^-_{1234} \\ & = 0\,. \end{aligned}$$

Hence, the assumption $W^+ = 0$ and the relation (2.1) imply that

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} = \frac{1}{3}R\,. \tag{4.22}$$

THEOREM 4.7. *Let (M^4, g) be a complete, half-conformally flat Riemannian manifold with $R \geq 0$ and with infinite volume. Then $H^2(L^2(M)) = \{0\}$.*

PROOF. By (4.22), the condition $R \geq 0$ implies that M has nonnegative isotropic curvature. From the proof of Theorem 2.1 in [23], we have $\langle \mathcal{R}_p(\theta), \theta \rangle \geq 0$ for any p-form θ. Thus, for any harmonic p-form ω, the relations (4.2) and (4.7) give

$$|\omega| \triangle |\omega| \geq \frac{1}{n-p} |\nabla|\omega||^2 .$$

Therefore, an analogous argument as Theorem 4.6 completes the proof. □

We say that M supports a weighted Poincaré inequality $\left(P_\rho\right)$, if there exists a positive function $\rho(x)$ a.e. on M such that

$$\left(P_\rho\right) \qquad \int_M \rho(x) f^2(x)\, dv \leq \int_M |\nabla f(x)|^2\, dv, \ \ \forall f \in W_0^{1,2}(M) \ .$$

If M^n is a simply connected locally conformal flat manifold with $R \leq 0$, then (2.3) implies that M supports a weighted Poincaré inequality

$$\int_M \left(|\nabla \phi|^2 - \frac{n-2}{4(n-1)} |R| \phi^2 \right) dv \geq 0, \ \ \forall \phi \in C_0^\infty(M), \tag{4.23}$$

which is equivalent to the nonnegative eigenvalue of the Schrödinger operator $\triangle - \frac{n-2}{4(n-1)}|R|$. Thus under a lower bound condition of Ricci curvature, we can deduce the following vanishing theorem.

THEOREM 4.8. *Let (M^n, g), $n \geq 4$, be a complete, simply connected, locally conformally flat manifold with $R \leq 0$. Suppose the Ricci curvature of M satisfies the lower bound*

$$Ric(x) \geq \frac{(n-2)^2 - 4p(p-1)}{4p(n-1)(n-2p)} R(x) \tag{4.24}$$

for $1 \leq p < [\frac{n}{2}]$ at every $x \in M$. Then every closed and coclosed p-form ω on M with $\liminf_{r\to\infty} \frac{1}{r^2} \int_{B_{x_0}(r)} |\omega|^2 dv = 0$ vanishes identically. In particular, $H^p(L^2(M)) = \{0\}$.

PROOF. Let ω be a closed and coclosed p-form on M with $\liminf_{r\to\infty} \frac{1}{r^2} \int_{B_{x_0}(r)} |\omega|^2 dv = 0$. Substituting (4.24) into (4.4), and using (4.7), we have

$$\begin{aligned} |\omega| \triangle |\omega| \geq & \frac{1}{n-p} |\nabla|\omega||^2 + \frac{p(n-2p)}{n-2} \frac{(n-2)^2 - 4p(p-1)}{4p(n-1)(n-2p)} R|\omega|^2 \\ & + \frac{p(p-1)}{(n-1)(n-2)} R|\omega|^2 \\ \geq & \frac{1}{n-p} |\nabla|\omega||^2 + \frac{n-2}{4(n-1)} R|\omega|^2 . \end{aligned} \tag{4.25}$$

Let η be the cut-off function defined by (4.13). Choosing $\phi = \eta|\omega|$ in (4.23), using (4.25) and integrating by parts, we compute

$$0 \leq \int_M \left(|\nabla(\eta|\omega|)|^2 - \frac{n-2}{4(n-1)} |R| \eta^2 |\omega|^2 \right) dv$$

$$
\begin{aligned}
&= \int_M \Big(-\eta|\omega|\triangle(\eta|\omega|) - \frac{n-2}{4(n-1)}|R|\eta^2|\omega|^2\Big)dv \\
&= -\int_M \eta|\omega|(|\omega|\triangle\eta + \eta\triangle|\omega| + 2\langle\nabla\eta, \nabla|\omega|\rangle)dv - \frac{n-2}{4(n-1)}\int_M |R|\eta^2|\omega|^2 dv \\
&= -\int_M \Big[\eta^2|\omega|\triangle|\omega| + \frac{n-2}{4(n-1)}|R||\omega|^2\Big]dv - 2\int_M \eta|\omega|\langle\nabla\eta, \nabla|\omega|\rangle dv \\
&\quad - \int_M |\omega|^2\eta\triangle\eta dv \\
&\leq -\frac{1}{n-p}\int_M \eta^2|\nabla|\omega||^2 dv + \int_M |\omega|^2|\nabla\eta|^2 dv \\
&\leq -\frac{1}{n-p}\int_{B_{x_0}(r)} |\nabla|\omega||^2 dv + \frac{4}{r^2}\int_{B_{x_0}(2r)} |\omega|^2 dv\,.
\end{aligned}
$$

Letting $r \to \infty$ and using $\liminf\limits_{r\to\infty} \frac{1}{r^2}\int_{B_{x_0}(r)} |\omega|^2 dv = 0$, we infer

$$
\nabla|\omega| = 0\,.
$$

Hence $|\omega|$ is constant. Since $\frac{\mathrm{vol}(B_{x_0}(r))}{r^2} \geq Cr^{n-2} \to \infty$ as $r \to \infty$ by the assumption $R \leq 0$, we conclude that $\omega = 0$. □

5. Topology of LCF Riemannian manifolds. According to the vanishing theorem in Section 4, we can study the topology at infinity of LCF manifolds.

THEOREM 5.1. *Let (M^n, g), $n \geq 3$, be a complete, simply connected, locally conformally flat Riemannian manifold. Then there exists a constant $C > 0$ such that if*

$$
\int_M |Ric|^{\frac{n}{2}} dv < C\,, \tag{5.1}
$$

then M has only one end.

PROOF. By the hypothesis, it follows from Theorem 4.2 that $H^1(L^2(M)) = \{0\}$. The assumption (5.1) implies that the following Sobolev inequality

$$
C_s\Big(\int_M |f|^{\frac{2n}{n-2}} dv\Big)^{\frac{n-2}{n}} \leq \int_M |\nabla f|^2 dv, \ \ \forall f \in C_0^\infty(M)
$$

holds for some $C_s > 0$. Hence M has infinite volume. According to Corollary 4 of [19], each end of M is non-parabolic. By the important result in [18], the number of non-parabolic ends of M is at most the dimension of the space of harmonic functions with finite Dirichlet integral. Observe that if f is a harmonic function with finite Dirichlet integral then its exterior df is an L^2 harmonic 1-form. Therefore, M has only one end. □

Considering the case of $p = 1$ in Theorem 4.3, using an analogous method as above, we have

THEOREM 5.2. *Let (M^n, g), $n \geq 3$, be a complete non-compact, simply connected, locally conformally flat Riemannian manifold with $R \geq 0$. Assume that*

$$\left(\int_M |E|^{\frac{n}{2}} dv\right)^{\frac{2}{n}} < C(n), \tag{5.2}$$

where $C(n) = \sqrt{\frac{n}{n-1}} \min\left\{\frac{n}{n-1}, \frac{4(n-1)}{n(n-2)}\right\} Q(\mathbb{S}^n)$, then M has only one non-parabolic end.

REMARK 5.1. In [21], H.Z. Lin proved a one-end theorem for LCF manifolds by assuming that $R \leq 0$ and $(\int_M |E|^n dv)^{\frac{2}{n}} < C(n)$ for some explicit constant $C(n) > 0$.

From Theorem 4.8 and the Sobolev inequality (2.4), we have the following one end theorem under pointwise condition.

THEOREM 5.3. *Let (M^n, g), $n \geq 4$, be a complete, simply connected, locally conformally flat Riemannian manifold with $R \leq 0$. Suppose that*

$$Ric(x) \geq \frac{n-2}{4(n-1)} R(x) \tag{5.3}$$

for all $x \in M$. Then M has only one end.

PROOF. Suppose contrary, there were at least two ends, then by the method in [27, p.681-683], there would exist a nonconstant bounded harmonic function f with finite energy on M. Hence df would be a nonconstant L^2 harmonic 1-form on M. That is, $H^1(L^2(M)) \neq \{0\}$, contradicting Theorem 4.8 in which $p = 1$. □

REMARK 5.2. In [20], Li-Wang proved that for a complete, simply connected, LCF manifold M^n $(n \geq 4)$ with $R \leq 0$, if the Ricci curvature Ric $\geq \frac{1}{4}R$ and the scalar curvature satisfies some decay condition, then either M has only one end, or $M = \mathbb{R} \times N$ with a warped product metric for some compact manifold N.

6. Liouville theorems of p-harmonic functions on LCF manifolds with negative scalar curvature. We recall a real-valued C^3 function u on a Riemannian M is said to be *strongly p-harmonic* if u is a (strong) solution of the p-Laplace equation

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0 \tag{6.1}$$

for $p > 1$. A function $u \in W^{1,p}_{loc}(M)$ is said to be *weakly p-harmonic* if

$$\int_M |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle\, dv = 0, \quad \forall \phi \in C_0^\infty(M) .$$

It is well known that the p-Laplace equation (6.1) arises as the Euler-Lagrange equation of the p-energy functional $E_p(u) = \int_M |\nabla u|^p\, dv$. To study the topology of the Riemannian manifold with respect to the p-harmonic theory, let us recall the following definition (see also [8, 27]):

DEFINITION 6.1. An end E of the Riemannian manifold M is said to be p-hyperbolic if for every compact set $K \subset \bar{E}$,

$$\operatorname{cap}_p(K, E) := \inf \int_E |\nabla f|^p > 0,$$

where the infimum is taken with respect to all $f \in C_0^\infty(\bar{E})$ such that $f \geq 1$ on K.

In [8], Chang, Chen and Wei introduce and study an approximate solution of the p-Laplace equation, and a linearlization $\mathcal{L}_\varepsilon$ of a perturbed p-Laplace operator. They prove a Liouville type theorem for weakly p-harmonic functions with finite p-energy on a complete noncompact manifold M which supports a weighted Poincaré inequality (P_ρ) and satisfies a curvature assumption. This nonexistence result, when combined with an existence theorem, implies that such an M has at most one p-hyperbolic end. More precisely, the following is proved:

THEOREM A ([8]). *Let M be a complete non-compact Riemannian n-manifold, $n \geq 2$, supporting a weighted Poincaré inequality (P_ρ) with Ricci curvature*

$$Ric(x) \geq -\tau\rho(x) \tag{6.2}$$

for all $x \in M$, where τ is a constant such that

$$\tau < \frac{4(p-1+\kappa)}{p^2}, \tag{6.3}$$

in which $p > 1$, and

$$\kappa = \begin{cases} \max\left\{\frac{1}{n-1}, \min\left\{\frac{(p-1)^2}{n}, 1\right\}\right\} & \text{if} \quad p > 2, \\ \frac{(p-1)^2}{n-1} & \text{if} \quad 1 < p \leq 2. \end{cases} \tag{6.4}$$

Then every weakly p-harmonic function u with finite p-energy $E_p(u)$ is constant. Moreover, M has at most one p-hyperbolic end.

Moreover, a Liouville type theorem for strongly p-harmonic functions with finite q-energy on Riemannian manifolds is obtained:

THEOREM B ([8]). *Let M be a complete non-compact Riemannian n-manifold, $n \geq 2$, satisfying $\left(P_\rho\right)$, with Ricci curvature*

$$Ric(x) \geq -\tau\rho(x) \tag{6.5}$$

for all $x \in M$, where τ is a constant such that

$$\tau < \frac{4(q-1+\kappa+b)}{q^2}, \tag{6.6}$$

in which

$$\kappa = \min\left\{\frac{(p-1)^2}{n-1}, 1\right\} \text{ and } b = \min\{0, (p-2)(q-p)\}, \text{ where } p > 1. \tag{6.7}$$

Let $u \in C^3(M)$ be a strongly p-harmonic function with finite q-energy $E_q(u)$.
(I) Then u is constant under each one of the following conditions:
(1) $p = 2$ and $q > \frac{n-2}{n-1}$,

(2) $p = 4$, $q > \max\{1, 1 - \kappa - b\}$,
(3) $p > 2$, $p \neq 4$, *and either*

$$\max\left\{1, p - 1 - \frac{\kappa}{p-1}\right\} < q \leq \min\left\{2, p - \frac{(p-4)^2 n}{4(p-2)}\right\}$$

or

$$\max\{2, 1 - \kappa - b\} < q.$$

(II) *u does not exist for* $1 < p < 2$ *and* $q > 2$.

We recall from Section 4, if M is a locally conformal flat manifold with scalar curvature $R < 0$, a.e., then M supports a weighted Poincaré inequality (4.23) or (P_ρ) in which $\rho = -\frac{n-2}{4(n-1)}R$. Applying Theorems A and B, we have

THEOREM 6.1. *Let (M^n, g), $n \geq 3$, be a complete non-compact, simply connected, locally conformal flat Riemannian manifold with scalar curvature $R < 0$, a.e. and Ricci curvature satisfying*

$$Ric(x) \geq aR(x) \tag{6.8}$$

for all $x \in M$, where a is a constant such that

$$a < \frac{(n-2)(p-1+\kappa)}{(n-1)p^2}, \tag{6.9}$$

in which $p > 1$, and κ is as in (6.4). Then every weakly p-harmonic function u with finite p-energy $E_p(u)$ is constant. Moreover, M has at most one p-hyperbolic end.

PROOF. Since M supports a weighted Poincaré inequality (4.23) or $\left(P_\rho\right)$ in which $\rho = -\frac{n-2}{4(n-1)}R$, the inequalities (6.8) and (6.9) are equivalent to the inequalities (6.2) and (6.3) respectively. Indeed, $\mathrm{Ric} \geq -\tau\rho = \frac{n-2}{4(n-1)}\tau R = aR$, (6.2) $\Longleftrightarrow$ (6.8), in which $a = \frac{n-2}{4(n-1)}\tau$, and

$$(6.3) \quad \tau < \frac{4(p-1+\kappa)}{p^2} \quad \Longleftrightarrow \quad (6.9) \quad a < \frac{n-2}{4(n-1)} \cdot \frac{4(p-1+\kappa)}{p^2}.$$

Now the assertion follows immediately from Theorem A. □

THEOREM 6.2. *Let (M^n, g), $n \geq 3$, be a complete non-compact, simply connected, locally conformal flat Riemannian manifold with scalar curvature $R < 0$, a.e. and Ricci curvature satisfying*

$$Ric(x) \geq aR(x) \tag{6.10}$$

for all $x \in M$, where a is a constant such that

$$a < \frac{n-2}{n-1} \cdot \frac{q-1+\kappa+b}{q^2}, \tag{6.11}$$

in which $p > 1$, and κ is as in (6.7).

Let $u \in C^3(M)$ be a strongly p-harmonic function with finite q-energy $E_q(u)$. Then the conclusions (I) and (II) as in Theorem B hold.

PROOF. Arguing as before, the inequalities (6.10) and (6.11) are equivalent to the inequalities (6.5) and (6.6) respectively, and the assertion follows immediately from Theorem B. □

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