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A GROUP-THEORETIC CHARACTERIZATION OF THE FOCK-BARGMANN-HARTOGS DOMAINS

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Abstract. Let M be a connected Stein manifold of dimension N and let D be a Fock-Bargmann-Hartogs domain in $\mathbb{C}^N$. Let $\mathrm{Aut}(M)$ and $\mathrm{Aut}(D)$ denote the groups of all biholomorphic automorphisms of M and D, respectively, equipped with the compact-open topology. Note that $\mathrm{Aut}(M)$ cannot have the structure of a Lie group, in general; while it is known that $\mathrm{Aut}(D)$ has the structure of a connected Lie group. In this paper, we show that if the identity component of $\mathrm{Aut}(M)$ is isomorphic to $\mathrm{Aut}(D)$ as topological groups, then M is biholomorphically equivalent to D. As a consequence of this, we obtain a fundamental result on the topological group structure of $\mathrm{Aut}(D)$.

1. Introduction and results. Let M be a connected complex manifold and $\mathrm{Aut}(M)$ the group of all biholomorphic automorphisms of M. Then, equipped with the compact-open topology, $\mathrm{Aut}(M)$ is a topological group acting continuously on M.

In 1907, Poincaré proved in [25] that there exists no biholomorphic mapping from the unit polydisc Δ^2 onto the unit ball B^2 in $\mathbb{C}^2$ by comparing carefully the topological structures of the isotropy subgroups of $\mathrm{Aut}(\Delta^2)$ and $\mathrm{Aut}(B^2)$ at the origin 0 of $\mathbb{C}^2$. In view of this, for a given complex manifold M it is an interesting problem to bring out some complex analytic nature of M under some topological conditions on M or on $\mathrm{Aut}(M)$. In connection with this, in this paper we would like to study the following characterization problem of a complex manifold M by its holomorphic automorphism group $\mathrm{Aut}(M)$:

QUESTION. *Let M and N be connected complex manifolds and assume that their holomorphic automorphism groups* $\mathrm{Aut}(M)$ *and* $\mathrm{Aut}(N)$ *are isomorphic as topological groups. Then, is M biholomorphically equivalent to N?*

The answer to this question is negative, in general, without any other assumptions on the manifolds M or N. Indeed, consider the following generalized complex ellipsoid

$$E_p = \left\{(z_1, \dots, z_n) \in \mathbb{C}^n\ ;\ \sum_{j=1}^{n} |z_j|^{2p_j} < 1\right\}$$

in $\mathbb{C}^n$, where $n \geq 2$ and $p = (p_1, \dots, p_n) \in \mathbb{R}^n$ with $1 < p_1 < \cdots < p_n$. Then it is known that $\mathrm{Aut}(E_p)$ is a Lie group isomorphic to the n-dimensional torus T^n for any p and further E_p is not biholomorphically equivalent to E_q unless $p = q$ (cf. [23], [13]). However, there exist several articles solving this question affirmatively in the case where manifolds M or N

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are some special domains in $\mathbb{C}^n$. For instance, Isaev-Kruzhilin [7] proved that a connected complex manifold M of dimension n is necessarily biholomorphically equivalent to $\mathbb{C}^n$ if $\mathrm{Aut}(M)$ is isomorphic to $\mathrm{Aut}(\mathbb{C}^n)$ as topological groups. And, Kodama-Shimizu [16] obtained the following: Let k be an integer with $0 \leq k \leq n$ and let M be a connected complex manifold of dimension n that is holomorphically separable and admits a smooth envelope of holomorphy. Assume that $\mathrm{Aut}(M)$ is isomorphic to $\mathrm{Aut}(\mathbb{C}^k \times (\mathbb{C}^*)^{n-k})$ as topological groups. Then M is biholomorphically equivalent to $\mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$. See also [6], [3, 4], [15, 17, 19, 20] for related results. In view of these results, it would be expected that the answer to the question above is affirmative if $\mathrm{Aut}(M)$ is large enough in some sense. However, it should be mentioned the following: Even in the special case where M and N are homogeneous domains in $\mathbb{C}^n$, the answer to our question is not always affirmative. In fact, Miatello [21] proved that, for irreducible homogeneous bounded domains M and N in $\mathbb{C}^n$, $\mathrm{Aut}(M)$ is isomorphic to $\mathrm{Aut}(N)$ as topological groups (and hence as Lie groups) if and only if M is either biholomorphically or anti-biholomorphically equivalent to N. On the other hand, Mukuno-Nagata [22] constructed a concrete example of non-hyperbolic (in the sense of Kobayashi [12]) homogeneous domains M and N in $\mathbb{C}^n$ for every $n \geq 5$ such that $\mathrm{Aut}(M)$ is isomorphic to $\mathrm{Aut}(N)$ as topological groups, while M is not biholomorphically equivalent to N.

In this paper, we study exclusively the Fock-Bargmann-Hartogs domains in $\mathbb{C}^N$ in connection with the question above and establish a group-theoretic characterization of them. In order to state our precise results, let us define the Fock-Bargmann-Hartogs domain $D_{n,m}(\mu)$ according to Yamamori [29] as follows:

$$D_{n,m}(\mu) = \left\{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^N \ ; \ \|w\|^2 < e^{-\mu\|z\|^2}\right\},$$

where $\|\cdot\|$ denotes the Euclidean norm, $0 < \mu \in \mathbb{R}$ and $n, m \in \mathbb{N}$ with $N = n + m$. This is an unbounded strictly pseudoconvex domain in $\mathbb{C}^N$ with real analytic boundary. Since the complex Euclidean space $\mathbb{C}^n$ is now imbedded in $D_{n,m}(\mu)$ in the canonical manner, it is not hyperbolic in the sense of Kobayashi [12]. As we will see in the next section, the holomorphic automorphism group $\mathrm{Aut}(D_{n,m}(\mu))$ of $D_{n,m}(\mu)$ has the structure of a Lie group that acts real analytically on $D_{n,m}(\mu)$. However, it should be noted that $\mathrm{Aut}(D_{n,m}(\mu))$ does not act transitively on $D_{n,m}(\mu)$. After Yamamori [29] gave an explicit formula for the Bergman kernel of $D_{n,m}(\mu)$ in terms of the polylogarithm functions, the Fock-Bargmann-Hartogs domains have been studied from various points of view. For example, Kim-Ninh-Yamamori [10] studied exclusively the structure of $\mathrm{Aut}(D_{n,m}(\mu))$ and succeeded in finding generators of $\mathrm{Aut}(D_{n,m}(\mu))$. Tu-Wang [28] studied proper holomorphic mappings between equidimensional Fock-Bargmann-Hartogs domains and obtained rigidity results on them. In a recent paper [11] by Kim-Yamamori-Zhang, the Fock-Bargmann-Hartogs domains were treated from the complex-geometric point of view: the comparisions among various invariant metrics were discussed, and in [14] Kodama obtained a result on the global extendability of a biholomorphic mapping defined locally near a boundary point of $D_{n,m}(\mu)$. In view of these results, it seems worthwhile to investigate whether the Fock-Bargmann-Hartogs domains can be characterized by their holomorphic automorphism groups. The main purpose of this paper is to clear up this matter. In fact, we can establish the following:

THEOREM. *Let M be a connected Stein manifold of dimension N and let $D_{n,m}(\mu)$ be a Fock-Bargmann-Hartogs domain in $\mathbb{C}^N$ with $N = n + m$. Assume that $m \geq 2$ and the identity component of* $\mathrm{Aut}(M)$ *is isomorphic to* $\mathrm{Aut}(D_{n,m}(\mu))$ *as topological groups. Then M is biholomorphically equivalent to $D_{n,m}(\mu)$.*

Here it should be remarked that $\mathrm{Aut}(D_{n,m}(\mu))$ has the structure of a connected Lie group for every Fock-Bargmann-Hartogs domain $D_{n,m}(\mu)$ (cf. [10], [14]). Moreover, the assumption $m \geq 2$ cannot be dropped. Indeed, consider the following Fock-Bargmann-Hartogs domain D and its subdomain D^*:

$$D = \left\{(z, w) \in \mathbb{C}^n \times \mathbb{C} \,;\ |w|^2 < e^{-\|z\|^2}\right\} \quad \text{and} \quad D^* = D \setminus \varDelta_D\,,$$

where $\varDelta_D = \{(z, w) \in D\,;\ w = 0\} \cong \mathbb{C}^n$. Then D^* as well as D is a pseudoconvex domain in $\mathbb{C}^{n+1}$ and $\mathrm{Aut}(D^*)$ can be naturally identified with $\mathrm{Aut}(D)$. Moreover, D^* is not biholomorphically equivalent to D because D^* is hyperbolic in the sense of Kobayashi [12] and D is not. (For these assertions, see [5], [10] and [14].) Therefore we cannot drop the assumption $m \geq 2$ in the theorem.

As a consequence of our theorem, we can obtain the following fundamental result on the topological group structure of $\mathrm{Aut}(D_{n,m}(\mu))$:

COROLLARY. *Let $D_{n_1,m_1}(\mu_1)$ and $D_{n_2,m_2}(\mu_2)$ be two Fock-Bargmann-Hartogs domains in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ respectively, where $N_j = n_j + m_j$ for $j = 1, 2$. Then* $\mathrm{Aut}(D_{n_1,m_1}(\mu_1))$ *is isomorphic to* $\mathrm{Aut}(D_{n_2,m_2}(\mu_2))$ *as topological groups if and only if $D_{n_1,m_1}(\mu_1)$ is linearly equivalent to $D_{n_2,m_2}(\mu_2)$, that is, there exists a non-singular linear mapping $L : \mathbb{C}^{N_1} \to \mathbb{C}^{N_2}$ such that $L(D_{n_1,m_1}(\mu_1)) = D_{n_2,m_2}(\mu_2)$. Moreover, this can only happen when $(n_1, m_1) = (n_2, m_2)$.*

This paper is organized as follows. In Section 2, we investigate the structure of holomorphic automorphism group $\mathrm{Aut}(D_{n,m}(\mu))$ of a given Fock-Bargmann-Hartogs domain $D_{n,m}(\mu)$. Especially we study the structure of the set of all complete holomorphic vector fields on $D_{n,m}(\mu)$ in detail. For later use, we also recall some standardization of compact group actions on complex manifolds. After these preparations, we prove our theorem and its corollary in Sections 3 and 4, respectively.

2. Preliminaries. In this section, we first study the structure of the holomorphic automorphism group of a given Fock-Bargmann-Hartogs domain $D_{n,m}(\mu)$ in $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^m$. After that, we recall a fact on the standardization of some compact group action on a complex manifold. Also, a well-known fact on Reinhardt domains in $\mathbb{C}^n$ is given. Throughout this section, we write $D = D_{n,m}(\mu)$ for the sake of simplicity.

First of all, we have the following fundamental result on $\mathrm{Aut}(D)$:

THEOREM A (Kim-Ninh-Yamamori [10; Theorem 10]). *The automorphism group* Aut(D) *is generated by the following mappings*:

$$\varphi_A : (z, w) \mapsto (Az, w), \quad A \in U(n);$$
$$\varphi_B : (z, w) \mapsto (z, Bw), \quad B \in U(m);$$
$$\varphi_v : (z, w) \mapsto \left(z + v, e^{-\mu\langle z,v\rangle-(\mu/2)\|v\|^2} w\right), \quad v \in \mathbb{C}^n .$$

Hence Aut(D) can be regarded as a closed subgroup of Aut($\mathbb{C}^N$) leaving the boundary ∂D of D invariant and the Aut(D)-action on D (resp. on ∂D) is just the restriction of that on $\mathbb{C}^N$ to D (resp. to ∂D). In particular, via the standard action of the product group $U(n)\times U(m)$ on $\mathbb{C}^n\times\mathbb{C}^m = \mathbb{C}^N$, one may regard $U(n)\times U(m)$ as a compact, connected subgroup of Aut(D). From now on, we shall regard $U(n)$, $U(m)$ and $SU(m)$ as subgroups of $U(n)\times U(m) \subset \mathrm{Aut}(D)$ in the canonical manner, where $SU(m)$ stands for the special unitary group of degree m.

Now, put $\varDelta_D = \{(z, w) \in D\,;\ w = 0\} \cong \mathbb{C}^n$ and $D^* = D \setminus \varDelta_D$. Then it is known [14] that $\varDelta_D$ is just the degeneracy set for the Kobayashi pseudodistance d_D of D, and D^* is hyperbolic in the sense of Kobayashi [12]. Moreover, Aut(D) can be identified with a closed subgroup of the Lie group Aut(D^*). This combined with the proof of Theorem A given in [10] yields that Aut(D) is a connected Lie group of $\dim_{\mathbf{R}} \mathrm{Aut}(D) = n^2 + m^2 + 2n$.

Here we wish to investigate the structure of Aut(D) more closely. For this purpose, let us first introduce the subgroups $\varPi_D$ and G'_D of Aut(D) given by

$$\varPi_D := \text{ the group generated by } \{\varphi_v\,;\ v \in \mathbb{C}^n\} \text{ and}$$
$$G'_D := \text{ the group generated by } \{\varphi_v\,;\ v \in \mathbb{C}^n\} \cup U(n)\,.$$

Let $\mathcal{R} = \{R_\theta\}_{\theta\in\mathbb{R}}$ be the one-parameter subgroup of Aut(D) consisting of all transformations $R_\theta : (z, w) \mapsto (z, e^{i\theta}w)$, $\theta \in \mathbb{R}$. Note that $\mathcal{R}$ is the center of the subgroup $U(m)$ of Aut(D) and $\varphi_v \circ R_\theta = R_\theta \circ \varphi_v$ for all $v \in \mathbb{C}^n$ and all $\theta \in \mathbb{R}$. Moreover, for any two elements v, $v' \in \mathbb{C}^n$, we have

$$\begin{aligned}\varphi_v \circ \varphi_{v'}(z, w) &= \left(z + v + v', e^{-\mu\langle z,v+v'\rangle-(\mu/2)\|v+v'\|^2} e^{(-\mu \mathrm{Im}\langle v',v\rangle)i} w\right)\\ &= \varphi_{v+v'} \circ R_\theta(z, w) \quad \text{with } \theta = -\mu\mathrm{Im}\langle v', v\rangle\,.\end{aligned}$$

Thus, denoting by id_D the identity element of Aut(D), we have $\varphi_0 = \mathrm{id}_D$, $\varphi_v^{-1} = \varphi_{-v}$ and the commutator $[\varphi_v, \varphi_{v'}] := \varphi_v^{-1} \circ \varphi_{v'}^{-1} \circ \varphi_v \circ \varphi_{v'}$ of φ_v and $\varphi_{v'}$ is given by

$$[\varphi_v, \varphi_{v'}] = R_\theta \quad \text{with} \quad \theta = -2\mu\mathrm{Im}\langle v', v\rangle\,.$$

Hence, the set $\varPi := \{\varphi_v\circ R_\theta\,;\ v \in \mathbb{C}^n, \theta \in \mathbb{R}\}$ becomes a connected closed subgroup of Aut(D) of $\dim_{\mathbb{R}} \varPi = 2n + 1$ and, in fact, $\varPi_D$ coincides literally with the group $\varPi$. Consider now the centralizer of $SU(m)$ in Aut(D) and denote it by $C_{\mathrm{Aut}(D)}(SU(m))$. Then it is obvious that $C_{\mathrm{Aut}(D)}(SU(m))$ is generated by the set $\{\varphi_v\,;\ v \in \mathbb{C}^n\}\cup U(n)\cup\mathcal{R}$; so that $G'_D = C_{\mathrm{Aut}(D)}(SU(m))$ and $\mathrm{Aut}(D) = G'_D \cdot SU(m)$. More precisely, since $\varphi_A \circ \varphi_v \circ \varphi_A^{-1} = \varphi_{Av}$ for any $A \in U(n)$ and $v \in \mathbb{C}^n$, $\varPi_D$ is a normal subgroup of G'_D and $G'_D = \varPi_D \cdot U(n)$ with $\varPi_D \cap U(n) = \{\mathrm{id}_D\}$. Obviously $G'_D \cap SU(m) = \mathcal{R} \cap SU(m)$ is a finite subgroup of Aut(D) of order m. Therefore, summarizing our results obtained in the above, we have shown the following:

(2.1) Π_D is a connected closed subgroup of $\mathrm{Aut}(D)$ of $\dim_{\mathbb{R}} \Pi_D = 2n+1$;
(2.2) $G'_D = \Pi_D \cdot U(n)$, $\Pi_D \cap U(n) = \{\mathrm{id}_D\}$ and Π_D is a normal subgroup of G'_D;
(2.3) $G'_D = C_{\mathrm{Aut}(D)}(SU(m))$, the centralizer of $SU(m)$ in $\mathrm{Aut}(D)$;
(2.4) $\mathrm{Aut}(D) = G'_D \cdot SU(m)$ and $G'_D \cap SU(m)$ is a finite group.

Let $\mathcal{L}(\mathrm{Aut}(D))$ be the Lie algebra of $\mathrm{Aut}(D)$. Then we know that $\mathcal{L}(\mathrm{Aut}(D))$ can be identified with the real Lie algebra $\mathfrak{g}(D)$ consisting of all complete holomorphic vector fields on D (cf. [14]). Taking this into account, we would like to fix some basis for $\mathfrak{g}(D)$ for later use. First notice that, for any $v \in \mathbb{C}^n$, the family $\{\phi_t\}_{t\in\mathbb{R}}$ of transformations of $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^m$ given by

$$\phi_t \; : \; (z,w) \mapsto \left(z+tv, e^{-\mu\langle z,tv\rangle - (\mu/2)\|tv\|^2} w\right), \quad t \in \mathbb{R},$$

gives rise to a one-parameter subgroup of $\mathrm{Aut}(D)$ by Theorem A. Thus we have

$$X_v \;:=\; \sum_{j=1}^{n} v_j \frac{\partial}{\partial z_j} - \mu\langle z,v\rangle \sum_{s=1}^{m} w_s \frac{\partial}{\partial w_s} \;\in\; \mathfrak{g}(D) \quad \text{for all } v = (v_1,\dots,v_n) \in \mathbb{C}^n .$$

Let $\mathfrak{u}(L)$ be the linear Lie algebra consisting of all skew Hermitian matrices of degree L and put $\mathfrak{su}(L) = [\mathfrak{u}(L), \mathfrak{u}(L)]$. Then $\mathfrak{u}(L)$ (resp. $\mathfrak{su}(L)$) can be identified with the Lie algebra of $U(L)$ (resp. of $SU(L)$). With this notation, the following vector fields

$$X_A \;:=\; \sum_{j,k=1}^{n} a_{jk} z_k \frac{\partial}{\partial z_j} \quad \text{and} \quad X_B \;:=\; \sum_{s,t=1}^{m} b_{st} w_t \frac{\partial}{\partial w_s}$$

are contained in $\mathfrak{g}(D)$ for all $A = \left(a_{jk}\right) \in \mathfrak{u}(n)$ and all $B = (b_{st}) \in \mathfrak{u}(m)$ by Theorem A. In particular, the vector fields

$$\text{(2.5)} \qquad \begin{aligned} &I_j^z := i z_j \frac{\partial}{\partial z_j} \quad (1 \le j \le n), \quad I_s^w := i w_s \frac{\partial}{\partial w_s} \quad (1 \le s \le m), \\ &I^z := \sum_{j=1}^{n} I_j^z, \quad I^w := \sum_{s=1}^{m} I_s^w \quad \text{and} \quad I := I^z + I^w \end{aligned}$$

are all contained in $\mathfrak{g}(D)$. By the correspondences $X_A \leftrightarrow A, X_B \leftrightarrow B$, we shall often identify $X_A = A$, $X_B = B$, respectively, in this paper.

Among these vector fields, we have the following bracket relations:

$$\text{(2.6)} \qquad \begin{aligned} &[X_v, X_{v'}] = (-2\mu \mathrm{Im}\langle v,v'\rangle) I^w, \quad [X_v, X_A] = X_{Av}, \\ &[I^z, X_v] = -i\left(\sum_{j=1}^{n} v_j \frac{\partial}{\partial z_j} + \mu\langle z,v\rangle \sum_{s=1}^{m} w_s \frac{\partial}{\partial w_s}\right) = -X_{iv}, \\ &[I^w, X_v] = [I^z, X_A] = [I^w, X_A] = [I^z, X_B] = [I^w, X_B] = 0 \end{aligned}$$

for all $v, v' \in \mathbb{C}^n$ and all $A \in \mathfrak{u}(n)$, $B \in \mathfrak{u}(m)$.

Let π_D, $\mathfrak{g}'_D$ be the Lie subalgebras of $\mathfrak{g}(D)$ corresponding to Π_D, G'_D, respectively. Then it is easily verified that

$$\text{(2.7)} \qquad \begin{aligned} &\pi_D = \{X_v \,;\, v \in \mathbb{C}^n\} \oplus \mathbb{R}\{I^w\}, \quad \mathfrak{g}'_D = \pi_D \oplus \mathfrak{u}(n) \;\text{ and }\; [\mathfrak{g}'_D, \pi_D] \subset \pi_D\,; \\ &\mathfrak{g}(D) = \mathfrak{g}'_D \oplus \mathfrak{su}(m) \;\text{ and }\; [\mathfrak{g}'_D, \mathfrak{su}(m)] = \{0\}, \end{aligned}$$

where, for a given subset S of $\mathfrak{g}(D)$, $\mathbb{R}S$ denotes the vector subspace of $\mathfrak{g}(D)$ spanned by S over $\mathbb{R}$ and $\oplus$ means the direct sum of vector spaces. In particular, we have $\dim_{\mathbb{R}} \mathfrak{g}'_D = n^2 + 2n + 1$.

Next we shall recall the following standardization of some compact group actions on complex manifolds. This will be important for the proof of our theorem.

THEOREM B (Kodama-Shimizu [18; Generalized standardization theorem]). *Let M be a connected complex manifold of dimension n that is holomorphically separable and admits a smooth envelope of holomorphy and let K be a compact connected Lie group of rank n. Assume that an injective continuous group homomorphism ρ of K into* $\mathrm{Aut}(M)$ *is given. Then there exists a biholomorphic mapping F of M onto a Reinhardt domain W in $\mathbb{C}^n$ such that*

$$F\rho(K)F^{-1} = U(n_1) \times \cdots \times U(n_s) \subset \mathrm{Aut}(W), \quad n_1 + \cdots + n_s = n .$$

We finish this section by a well-known fact on Reinhardt domains:

THEOREM C (cf. [26; Chapter II]). *Let f be a holomorphic function on a Reinhardt domain W in $\mathbb{C}^n$. Then f has a Laurent series representation*

$$f(z) = \sum_{\nu \in \mathbb{Z}^n} c_\nu z^\nu , \quad z \in W ,$$

which converges absolutely and uniformly on any compact set in W, where $z = (z_1, \ldots, z_n)$, $\nu = (\nu_1, \ldots, \nu_n)$ and $z^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n}$. Moreover, if $W \cap \{z \in \mathbb{C}^n ; z_i = 0\} \neq \emptyset$ for some $1 \leq i \leq n$, then $c_\nu = 0$ for $\nu_i < 0$. In particular, if W is a pseudoconvex Reinhardt domain in $\mathbb{C}^n$ that is invariant under the standard action of $U(k) \times U(n-k)$ on $\mathbb{C}^n$ and if a point $z_o = (z'_o, z''_o) \in \mathbb{C}^k \times \mathbb{C}^{n-k} = \mathbb{C}^n$ belongs to W, then

$$\{(z', z''_o) \in \mathbb{C}^n ; \|z'\| \leq \|z'_o\|\} \subset W \quad \text{or} \quad \{(z'_o, z'') \in \mathbb{C}^n ; \|z''\| \leq \|z''_o\|\} \subset W$$

according to $k \geq 2$ or $n - k \geq 2$.

3. Proof of Theorem. Let $D_{n,m}(\mu)$ be the Fock-Bargmann-Hartogs domain in $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^m$ as in the theorem and write again $D = D_{n,m}(\mu)$ throughout this section.

Let M be a connected Stein manifold of dimension N and assume that there exists a topological group isomorphism $\Phi : \mathrm{Aut}(D) \to \mathrm{Aut}^o(M)$, the identity component of $\mathrm{Aut}(M)$. Since $K := U(n) \times U(m)$ is a Lie subgroup of $\mathrm{Aut}(D)$, we have the natural injective continuous group homomorphism $\iota : K \to \mathrm{Aut}(D)$. Thus we now obtain an injective continuous group homomorphism $\Phi \circ \iota$ of the compact connected Lie group K of rank $N = n + m$ into $\mathrm{Aut}(M)$. Hence, by Theorem B there exists a biholomorphic mapping F of M into $\mathbb{C}^N$ such that $W := F(M)$ is a Reinhardt domain in $\mathbb{C}^N$ and

$$F(\Phi \circ \iota)(K)F^{-1} = U(N_1) \times \cdots \times U(N_s) \subset \mathrm{Aut}(W), \quad N_1 + \cdots + N_s = N .$$

Let $T^N := (U(1))^N$ be the N-dimensional torus and let $T(D)$ (resp. $T(W)$) be the compact abelian subgroup of $\mathrm{Aut}(D)$ (resp. of $\mathrm{Aut}(W)$) obtained by restricting the usual T^N-action on $\mathbb{C}^N$ to D (resp. to W); so that $T(D)$ as well as $T(W)$ may be naturally identified with T^N. Then, thanks to the conjugacy of the maximal tori $F\Phi(T(D))F^{-1}$ and $T(W)$ in $U(N_1) \times \cdots \times U(N_s)$, we may assume that M is a pseudoconvex Reinhardt domain W in $\mathbb{C}^N$ and we have an isomorphism $\Phi : \mathrm{Aut}(D) \to \mathrm{Aut}^o(W)$ between the topological groups $\mathrm{Aut}(D)$ and $\mathrm{Aut}^o(W)$

such that $\Phi(K) = U(N_1) \times \cdots \times U(N_s)$ and $\Phi(T(D)) = T(W)$. Recall that the commutator group of $U(N_j)$ is $SU(N_j)$ and $SU(N_j)$ is a simple Lie group if $N_j \geq 2$ and that $m \geq 2$ by our assumption. Then, after a suitable permutation of coordinates, if necessary, we may further assume that

$$\Phi(U(n) \times U(m)) = U(n) \times U(m) \quad \text{and} \quad \Phi(SU(m)) = SU(m), \tag{3.1}$$

where we regard $SU(m)$ as a subgroup of $U(n) \times U(m)$ in the canonical manner.

Let C_n, C_m and $C_{n,m}$ be the centers of $U(n), U(m)$ and $U(n) \times U(m)$, respectively, and let $C_{\mathrm{Aut}^o(W)}(SU(m))$ be the centralizer of $SU(m)$ in $\mathrm{Aut}^o(W)$. Obviously, both the groups C_n and C_m are naturally identified with the one-dimensional torus T^1, while $C_{n,m}$ is identified with the two-dimensional torus T^2. And, it follows from (3.1) that

$$\Phi(C_{n,m}) = C_{n,m} \quad \text{and} \quad \Phi(C_{\mathrm{Aut}(D)}(SU(m))) = C_{\mathrm{Aut}^o(W)}(SU(m)). \tag{3.2}$$

Accordingly, putting $G'_W = C_{\mathrm{Aut}^o(W)}(SU(m))$ for simplicity, we obtain by (2.4) that

$$\mathrm{Aut}^o(W) = G'_W \cdot SU(m) \quad \text{and} \quad G'_W \cap SU(m) \text{ is a finite group}. \tag{3.3}$$

Now, the connected topological group $\mathrm{Aut}^o(W)$ can be turned into a Lie group by transfering the Lie group structure from $\mathrm{Aut}(D)$ by means of the topological group isomorphism $\Phi : \mathrm{Aut}(D) \to \mathrm{Aut}^o(W)$. Since the Lie group $\mathrm{Aut}^o(W)$ endowed with the compact-open topology acts continuously on W by biholomorphic transformations, the action is real analytic with respect to the Lie group structure induced from $\mathrm{Aut}(D)$ (cf. [1]). Thus $\mathrm{Aut}^o(W)$ is now a Lie transformation group of W by biholomorphic transformations; accordingly, the Lie algebra of $\mathrm{Aut}^o(W)$ can be identified with the real Lie algebra $\mathfrak{g}(W)$ consisting of all complete holomorphic vector fields on W (cf. [24; p. 103, Theorem VII]). Therefore we obtain the Lie algebra isomorphism

$$\varphi : \mathfrak{g}(D) \to \mathfrak{g}(W) \quad \text{induced by} \quad \Phi : \mathrm{Aut}(D) \to \mathrm{Aut}^o(W). \tag{3.4}$$

Let $\mathfrak{g}'_W$ be the Lie subalgebra of $\mathfrak{g}(W)$ corresponding to the Lie subgroup G'_W of $\mathrm{Aut}^o(W)$. Then, by (3.2) and (3.3), we have

$$\varphi(\mathfrak{g}'_D) = \mathfrak{g}'_W, \quad \mathfrak{g}(W) = \mathfrak{g}'_W \oplus \mathfrak{su}(m) \quad \text{and} \quad [\mathfrak{g}'_W, \mathfrak{su}(m)] = \{0\}. \tag{3.5}$$

Let $\mathfrak{c}_n$, $\mathfrak{c}_m$ and $\mathfrak{c}_{n,m}$ be the Lie algebras of C_n, C_m and $C_{n,m}$, respectively. Then $\mathfrak{c}_n = \mathbb{R}\{I^z\}$, $\mathfrak{c}_m = \mathbb{R}\{I^w\}$ and $\mathfrak{c}_{n,m} = \mathbb{R}\{I^z, I^w\}$. Moreover, $\varphi(\mathfrak{c}_{n,m}) = \mathfrak{c}_{n,m}$ by (3.2) and the restriction $\varphi|_{\mathfrak{c}_{n,m}} : \mathfrak{c}_{n,m} \to \mathfrak{c}_{n,m}$ gives a Lie algebra isomorphism. Hence there exists an element $(a_{\alpha\beta})_{1 \leq \alpha,\beta \leq 2} \in GL(2; \mathbb{Z})$ such that

$$\varphi(I^z) = a_{11} I^z + a_{21} I^w \quad \text{and} \quad \varphi(I^w) = a_{12} I^z + a_{22} I^w.$$

In particular, there exist some elements (A, B) and (a, b) of $\mathbb{Z}^2 \setminus \{0\}$ such that

$$\varphi(I) = A I^z + B I^w \quad \text{and} \quad \varphi(a I^z + b I^w) = I. \tag{3.6}$$

Before proceeding, we here investigate the structure of $\mathfrak{g}(W)$ more closely. Let us denote by $p_1 : \mathbb{C}^N \to \mathbb{C}^n$, $p_2 : \mathbb{C}^N \to \mathbb{C}^m$ the projections given by

$$p_1 : (z, w) \mapsto z, \quad p_2 : (z, w) \mapsto w \quad \text{for } (z, w) \in \mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^N$$

and put $W_1 = p_1(W)$, $W_2 = p_2(W)$, respectively. Then, since W is a pseudoconvex Reinhardt domain in $\mathbb{C}^N$ invariant under the standard $U(n) \times U(m)$-action on $\mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^N$ and since $m \geq 2$, it follows from Theorem C that
(3.7) W_1 is a domain in $\mathbb{C}^n$ invariant under the standard $U(n)$-action on $\mathbb{C}^n$ and W_2 is an open ball $B^m(r_2)$ in $\mathbb{C}^m$ with radius $0 < r_2 \leq +\infty$ and center 0; and
(3.8) $W \subset W_1 \times W_2$.
Notice that $W_1 = \{z \in \mathbb{C}^n; (z, 0) \in W\}$ by Theorem C; accordingly, W_1 can be regarded as a complex submanifold of W. And, if W_1 contains the origin 0 of $\mathbb{C}^n$ (for instance, in the case where $n \geq 2$), W_1 is also an open ball $B^n(r_1)$ in $\mathbb{C}^n$ with radius $0 < r_1 \leq +\infty$ and center 0. With this notation, we first prove the following:

LEMMA 1. *The group G'_W consists of all elements f in* $\mathrm{Aut}^o(W)$ *having the form*

$$f(z, w) = (g(z), \lambda(z)w) \quad \text{for } (z, w) \in W \tag{3.9}$$

with respect to the coordinate system (z, w) in $\mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^N$, where $g \in$ $\mathrm{Aut}(W_1)$ *and λ is a nowhere vanishing holomorphic function on W_1.*

PROOF. Clearly the mapping f written in the form (3.9) belongs to G'_W. Conversely, take an arbitrary element $f \in G'_W$ and express $f = (g, h)$ with respect to the coordinate system (z, w) in $\mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^N$. Then, by the commutativity $f \circ B = B \circ f$ for all $B \in SU(m) \subset \mathrm{Aut}^o(W)$, one has

$$g(z, Bw) = g(z, w), \quad h(z, Bw) = B \cdot h(z, w) \quad \text{for all } (z, w) \in W,\ B \in SU(m). \tag{3.10}$$

Take an arbitrary point $(z_o, w_o) \in W$ with $w_o \neq 0$ and let $S_{w_o} := SU(m) \cdot w_o$ be the $SU(m)$-orbit passing through w_o. Then S_{w_o} is a real analytic hypersurface in $\mathbb{C}^m$ and $g(z_o, w) = g(z_o, w_o)$ for all $w \in S_{w_o}$ by (3.10). Hence, g does not depend on the variables w (cf. [2; p. 142]); accordingly, g has the form $g(z, w) = g(z)$ on W and g induces a holomorphic automorphism of W_1. Moreover, we assert that h can be written in the form

$$h(z, w) = \lambda(z)w \quad \text{for } (z, w) \in W, \tag{3.11}$$

where λ is a nowhere vanishing holomorphic function on W_1. Indeed, this can be verified as follows. For a given point $z \in W_1$, we set $W(z) = \{w \in \mathbb{C}^m; (z, w) \in W\}$. By Theorem C this is an open ball in $\mathbb{C}^m$ with center 0. Now take a point $z_o \in W_1$ arbitrarily and define a mapping $L : W(z_o) \to \mathbb{C}^m$ by setting $L(w) = h(z_o, w)$ for $w \in W(z_o)$. Then L induces a biholomorphic mapping $L : W(z_o) \to W(g(z_o))$ satisfying the condition

$$L(Bw) = B \cdot L(w) \quad \text{for every } B \in SU(m) \tag{3.12}$$

by (3.10). Thus $L(w) = 0$ if and only if $w = 0$. For an arbitrarily given point $w_o \in W(z_o)$ with $w_o \neq 0$, we put $r_o = \|w_o\|$, $R_o = \|L(w_o)\|$ and consider a biholomorphic mapping $\widehat{L} : W(z_o) \to \mathbb{C}^m$ from the open ball $W(z_o)$ into $\mathbb{C}^m$ defined by

$$\widehat{L}(w) = (r_o/R_o)L(w) \quad \text{for } w \in W(z_o).$$

It then follows from (3.12) that $\widehat{L}$ gives rise to a holomorphic automorphism, say again $\widehat{L}$, of the open ball $B^m(r_o)$ in $\mathbb{C}^m$ with $\widehat{L}(0) = 0$. Consequently, $\widehat{L}$ has to be the restriction of some unitary transformation of $\mathbb{C}^m$ to $B^m(r_o)$, so that there exists an element $U \in U(m)$ such that $L(w) = (R_o/r_o)Uw$ on $W(z_o)$. Then the relation (3.12) tells us that U is a scalar matrix (depending only on z_o). Therefore we obtain the assertion (3.11); completing the proof of Lemma 1. □

Now we consider the mapping $\rho : G'_W \to \mathrm{Aut}(W_1)$ that sends an element $f \in G'_W$ written in the form (3.9) into the element $g \in \mathrm{Aut}(W_1)$. Then it is obvious that

(3.13) $\rho : G'_W \to \mathrm{Aut}(W_1)$ is a continuous group homomorphism.

Moreover, let Y be a complete holomorphic vector field on W contained in $\mathfrak{g}'_W$. It then follows from Lemma 1 that Y can be expressed as

$$Y = \sum_{j=1}^{n} f_j(z)\frac{\partial}{\partial z_j} + \lambda(z)\sum_{s=1}^{m} w_s\frac{\partial}{\partial w_s}, \tag{3.14}$$

where f_j $(1 \leq j \leq n)$ and λ are holomorphic functions on W_1. Let $\mathfrak{X}(W_1)$ be the Lie algebra consisting of all differentiable vector fields on W_1 and consider the mapping $\rho_* : \mathfrak{g}'_W \to \mathfrak{X}(W_1)$ that sends an element $Y \in \mathfrak{g}'_W$ written in the form (3.14) into $\sum_{j=1}^{n} f_j(z)\partial/\partial z_j$. Then Lemma 1 tells us that

(3.15) $\rho_*(Y)$ is a complete holomorphic vector field on W_1 for every $Y \in \mathfrak{g}'_W$.

Let $\mathfrak{g}(W_1)$ be the set of all complete holomorphic vector fields on W_1. Then it should be remarked that ρ (resp. ρ_*) is nothing but the restriction mapping

$$G'_W \ni f \mapsto f|_{W_1} \in \mathrm{Aut}(W_1) \quad (\text{resp. } \mathfrak{g}'_W \ni Y \mapsto Y|_{W_1} \in \mathfrak{g}(W_1))$$

under the natural identification $W_1 = \{(z, w) \in W \,;\, w = 0\}$.

LEMMA 2. *The Reinhardt domain W contains the origin* 0 *of* $\mathbb{C}^N$.

PROOF. If $n \geq 2$, this assertion is an immediate consequence of Theorem C. So let us consider the case where $n = 1$. In this case, it is well-known that $\mathrm{Aut}(W_1)$ has the structure of a Lie group of $\dim_{\mathbb{R}} \mathrm{Aut}(W_1) \leq 4$ and the Lie algebra of $\mathrm{Aut}(W_1)$ is canonically identified with the Lie algebra $\mathfrak{g}(W_1)$ of all complete holomorphic vector fields on W_1. Moreover, being a circular domain in $\mathbb{C}$, W_1 may be one of the following:

$$B^1(r),\ \mathbb{C},\ B^1(r) \setminus \{0\},\ \mathbb{C}^*,\ \{z \in \mathbb{C}\,;\, r < |z| < R\} \text{ or } \{z \in \mathbb{C}\,;\, r < |z| < +\infty\},$$

where r and R are some positive real numbers. Hence, in order to complete the proof of Lemma 2, it suffices to show that $\dim_{\mathbb{R}} \mathrm{Aut}(W_1) \geq 3$ because this inequality can only happen when $W_1 = B^1(r)$ or $W_1 = \mathbb{C}$. For this purpose, we need the following:

SUBLEMMA. *Let D be the Fock-Bargmann-Hartogs domain in* $\mathbb{C} \times \mathbb{C}^m = \mathbb{C}^N$ *and* $\varphi : \mathfrak{g}(D) \to \mathfrak{g}(W)$ *the Lie algebra isomorphism appearing in* (3.4). *Then, for any element*

$$X_v = v\frac{\partial}{\partial z} - \mu\bar{v}z\sum_{s=1}^{m} w_s\frac{\partial}{\partial w_s} \quad \text{with } v \in \mathbb{C}^*,$$

$\varphi(X_v)$ *has the form*

$$\varphi(X_v) = g(z)\frac{\partial}{\partial z} + \lambda(z)\sum_{s=1}^{m} w_s\frac{\partial}{\partial w_s}, \tag{3.16}$$

where g, λ *are holomorphic functions on* W_1 *and* g *is not identically zero on* W_1.

PROOF. Since $X_v \in \mathfrak{g}'_D$, it follows from (3.14) that $\varphi(X_v)$ can be expressed as in (3.16) except for the assertion g is not identically zero on W_1. Assuming that $g(z) \equiv 0$ on W_1 or equivalently $\varphi(X_v)$ has the form $\varphi(X_v) = \lambda(z)\sum_{s=1}^m w_s\partial/\partial w_s$ on W, we wish to derive a contradiction. For this, consider the vector field $I = I^z + I^w \in \mathfrak{g}(D)$ defined in (2.5). Then by (2.6) we have

$$[X_v,[I,X_v]] = -2\mu|v|^2 I^w;\ \text{and so}\quad \varphi([X_v,[I,X_v]]) = -2\mu|v|^2\varphi(I^w) \neq 0\,.$$

On the other hand, writing $\varphi(I) = AI^z + BI^w$ as in (3.6), we have

$$[\varphi(I),\varphi(X_v)] = Az\frac{\partial\lambda(z)}{\partial z}I^w;\ \text{and so}\quad \varphi([X_v,[I,X_v]]) = [\varphi(X_v),[\varphi(I),\varphi(X_v)]] = 0\,.$$

This is a contradiction; thereby $g(z) \neq 0$ at some point $z \in W_1$. □

Let us return to the proof of Lemma 2. Consider the complete holomorphic vector fields

$$X_1 = \frac{\partial}{\partial z} - \mu z\sum_{s=1}^m w_s\frac{\partial}{\partial w_s},\quad X_i = i\left(\frac{\partial}{\partial z} + \mu z\sum_{s=1}^m w_s\frac{\partial}{\partial w_s}\right)$$

on D and write

$$\varphi(X_1) = g_1(z)\frac{\partial}{\partial z} + \lambda_1(z)\sum_{s=1}^m w_s\frac{\partial}{\partial w_s},\quad \varphi(X_i) = g_2(z)\frac{\partial}{\partial z} + \lambda_2(z)\sum_{s=1}^m w_s\frac{\partial}{\partial w_s}$$

as in (3.16). Then $\{\rho_*(\varphi(X_1)), \rho_*(\varphi(X_i)), \rho_*(I^z)\} = \{g_1(z)\partial/\partial z, g_2(z)\partial/\partial z, I^z\}$ is linearly independent in $\mathfrak{g}(W_1)$; and hence, $\dim_{\mathbb{R}}\mathfrak{g}(W_1) \geq 3$. Indeed, assume that

$$a_1\rho_*(\varphi(X_1)) + a_2\rho_*(\varphi(X_i)) + a_3\rho_*(I^z) = 0\quad \text{for some } (a_1,a_2,a_3) \in \mathbb{R}^3 \setminus \{0\}\,.$$

Then, by putting

$$v = a_2 - ia_1\,,\quad \varphi(I) = AI^z + BI^w\,,\quad \varphi(a'I^z + b'I^w) = I^z\,,$$

$$\widehat{X} = a_1X_1 + a_2X_i + a_3(a'I^z + b'I^w)\quad\text{and}\quad \lambda(z) = a_1\lambda_1(z) + a_2\lambda_2(z)\,,$$

where (A,B), (a',b') are some elements of $\mathbb{Z}^2 \setminus \{0\}$, it can be seen that

$$[I,\widehat{X}] = v\frac{\partial}{\partial z} - \mu\bar{v}z\sum_{s=1}^m w_s\frac{\partial}{\partial w_s} = X_v;\ \text{while}\ \ \varphi(\widehat{X}) = \lambda(z)\sum_{s=1}^m w_s\frac{\partial}{\partial w_s}$$

$$\text{and}\quad \varphi(X_v) = [\varphi(I),\varphi(\widehat{X})] = iAz\frac{\partial\lambda(z)}{\partial z}\sum_{s=1}^m w_s\frac{\partial}{\partial w_s}\quad \text{with } v \in \mathbb{C}^*\,.$$

This contradicts the sublemma above. Therefore we conclude that $\dim_{\mathbb{R}}\mathrm{Aut}(W_1) \geq 3$ and W contains the origin 0 of $\mathbb{C}^N$; completing the proof of Lemma 2. □

Writing the coordinate system (z,w) in $\mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^N$ as

$$\zeta = (\zeta_1,\dots,\zeta_N) = (z_1,\dots,z_n,w_1,\dots,w_m) = (z,w)$$

for a while, we put

$$\mathfrak{P}_\nu = \left\{\sum_{k=1}^N p_k(\zeta)\frac{\partial}{\partial\zeta_k}\ ;\ \text{all } p_k\text{'s are homogeneous polynomials in } \zeta \text{ of degree } \nu\right\},$$

the space of all homogeneous polynomial vector fields of degree ν. Clearly one has

$$[\mathfrak{P}_\nu, \mathfrak{P}_\mu] \subset \mathfrak{P}_{\nu+\mu-1} \quad \text{and} \quad [I, Y_\nu] = i(\nu-1)Y_\nu \ \text{ for } \ Y_\nu \in \mathfrak{P}_\nu, \tag{3.17}$$

where $I = I^z + I^w = \sum_{k=1}^N i\zeta_k \partial/\partial\zeta_k \in \mathfrak{P}_1$.

Let $Y = \sum_{k=1}^N f_k(\zeta)\partial/\partial\zeta_k$ be an arbitrary element of $\mathfrak{g}(W)$. Then, being a holomorphic function on the pseudoconvex Reinhardt domain W in $\mathbb{C}^N$ containing the origin 0, every component function f_k can now be expanded uniquely as

$$f_k(\zeta) = \sum_{\nu=0}^{\infty} p_\nu^k(\zeta), \quad \zeta \in W,$$

which converges absolutely and uniformly on compact subsets of W, where p_ν^k is a homogeneous polynomial in ζ of degree ν. Thus Y can be expressed as a convergent series $Y = \sum_{\nu=0}^{\infty} Y_\nu$ with $Y_\nu = \sum_{k=1}^N p_\nu^k(\zeta)\partial/\partial\zeta_k \in \mathfrak{P}_\nu$. Notice that the complex Lie algebra $\mathfrak{g}$ spanned by $\mathfrak{g}(W)$ is finite-dimensional and contains the vector field $d := \sum_{k=1}^N \zeta_k\partial/\partial\zeta_k = -iI$. Then, with exactly the same argument as in the proof of [9; Theorem 1], one can show the following:

LEMMA 3. *Every element Y in $\mathfrak{g}(W)$ can be written in the form*

$$Y = \sum_{\nu=0}^{\nu_o} Y_\nu \quad \text{with } \ Y_\nu \in \mathfrak{P}_\nu, \ \ 0 \le \nu \le \nu_o,$$

where ν_o is a positive integer depending only on $\mathfrak{g}(W)$.

More precisely, we would like to show the following:

LEMMA 4. *Every element Y in $\mathfrak{g}(W)$ can be written in the form*

$$Y = Y_0 + Y_1 + Y_2 \quad \text{with } \ Y_\nu \in \mathfrak{P}_\nu, \ \ 0 \le \nu \le 2.$$

PROOF. Notice that

$$\mathfrak{g}(W) = \mathbb{R}\{\varphi(X_v), \varphi(X_A), \varphi(X_B)\,;\, v \in \mathbb{C}^n, A \in \mathfrak{u}(n), B \in \mathfrak{u}(m)\}$$

and $\varphi(X_A)$, $\varphi(X_B)$ are polynomial vector fields of degree one by (3.1), where X_v, X_A and X_B are complete holomorphic vector fields on D defined in the preceding section. Thus it suffices to show the lemma for every element $\varphi(X_v)$. To this end, we first verify the following assertion:
(3.18) Let $v \in \mathbb{C}^n$ and assume that $\varphi(X_v)$ has the form

$$\varphi(X_v) = Y_3 + \cdots + Y_{\nu_o} \quad \text{with } \ Y_\nu \in \mathfrak{P}_\nu, \ \ 3 \le \nu \le \nu_o,$$

where ν_o is the integer appearing in Lemma 3. Then we have $v = 0$ and $Y_\nu = 0$ for all $3 \le \nu \le \nu_o$.

Indeed, assume to the contrary that $v \ne 0$. Let ν' be the least integer ≥ 3 such that $Y_{\nu'} \ne 0$ in the expression of Y in (3.18). We now verify the assertion (3.18) only in the case where

$\nu' = 3$, since the verification in the general case is almost identical. Let

(3.19) $$\varphi(aI^z + bI^w) = I \quad \text{as in (3.6) and put} \quad \widehat{I} = aI^z + bI^w .$$

Then, by direct computations, we obtain that

$$[\widehat{I}, X_v] = -aX_{iv}, \quad [\widehat{I}, [\widehat{I}, X_v]] = -a^2 X_v \quad \text{and} \quad [I, Y_\nu] = i(\nu - 1)Y_\nu$$

for every ν; and hence

$$-a^2(Y_3 + \cdots + Y_{\nu_o}) = \varphi([\widehat{I}, [\widehat{I}, X_v]]) = -\left\{2^2 Y_3 + \cdots + (\nu_o - 1)^2 Y_{\nu_o}\right\} .$$

Since $Y_3 \neq 0$, this implies that $a^2 = 2^2 \neq 0$ and $\varphi(X_\nu) = Y_3$; accordingly

$$\varphi([X_v, [\widehat{I}, X_v]]) = [\varphi(X_v), [I, \varphi(X_v)]] = [Y_3, 2iY_3] = 0 .$$

On the other hand, since

$$[X_v, [\widehat{I}, X_v]] = [X_v, -aX_{iv}] = -2a\mu\|v\|^2 I^w$$

by (2.6), we have

(3.20) $$\varphi([X_v, [\widehat{I}, X_v]]) = -2a\mu\|v\|^2 \varphi(I^w) \neq 0 ,$$

a contradiction. Therefore we conclude that $v = 0$; proving the assertion (3.18).

Now take an arbitrary element $\varphi(X_v)$ and write

$$\varphi(X_v) = \sum_{\nu=0}^{\nu_o} Y_\nu \quad \text{with} \ Y_\nu \in \mathfrak{P}_\nu, \ 0 \leq \nu \leq \nu_o ,$$

according to Lemma 3. By routine computations, we then have

$$\begin{aligned} \varphi\big(-a(1 - a^2)X_{iv}\big) &= \varphi\big([\widehat{I}, X_v + [\widehat{I}, [\widehat{I}, X_v]]]\big) \\ &= 2i(1 - 2^2)Y_3 + \cdots + (\nu_o - 1)i\left\{1 - (\nu_o - 1)^2\right\} Y_{\nu_o} . \end{aligned}$$

Consequently, $-a(1 - a^2)iv = 0$ and $Y_3 = \cdots = Y_{\nu_o} = 0$ by (3.18). Therefore we have shown that $\varphi(X_v) = Y_0 + Y_1 + Y_2$; completing the proof of Lemma 4. □

Let $T_0 W$ (resp. $T_0 W_1$) be the holomorphic tangent space to W (resp. to W_1) at 0, where $0 \in W_1 \subset W$ is the origin of $\mathbb{C}^N$. As usual, by making use of the standard basis $\{(\partial/\partial\zeta_1)_0, \ldots, (\partial/\partial\zeta_N)_0\}$ for $T_0 W$, one may identify $T_0 W = \mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^m$ and $T_0 W_1 = \mathbb{C}^n$. With this notation, we can prove the following:

LEMMA 5. *The vector field $\varphi(X_v)$ can be written in the form*

$$\varphi(X_v) = Y_0^v + Y_2^v \quad \textit{for every} \ v \in \mathbb{C}^n ,$$

where

(1) $Y_\nu^v \in \mathfrak{P}_\nu$, $\nu = 0, 2$; *and*

(2) $Y_0^v = 0$ *if and only if* $Y_2^v = 0$.

In particular, we have $\{\varphi(X_v)_0 \,;\, v \in \mathbb{C}^n\} = T_0 W_1$, where $\varphi(X_v)_0$ denotes the value of the vector field $\varphi(X_v)$ at 0.

PROOF. By Lemma 4 we know that every $\varphi(X_v)$ can be written in the form

$$\varphi(X_v) = Y_0^v + Y_1^v + Y_2^v \quad \text{with} \quad Y_\nu^v \in \mathfrak{P}_\nu, \quad \nu = 0, 1, 2\,.$$

Assume here that $Y_1^v \neq 0$ for some $v \in \mathbb{C}^n$. Then

$$\varphi\big((1-a^2)X_v\big) = \varphi\big(X_v + [\widehat{I}, [\widehat{I}, X_v]]\big) = Y_1^v \neq 0\,; \quad \text{and}$$

$$-a(1-a^2)\varphi(X_{iv}) = \varphi\big([\widehat{I}, (1-a^2)X_v]\big) = [I, Y_1^v] = 0\,,$$

where $\widehat{I}$ is the vector field on D appearing in (3.19). Thus $a = 0$ and $\varphi(bI^w) = I$. Since $[I^w, \mathfrak{g}(D)] = \{0\}$, this implies that $[I, \mathfrak{g}(W)] = \{0\}$. Note that I is the complete holomorphic vector field on W induced by the one-parameter subgroup $\{R_\theta\}_{\theta\in\mathbb{R}}$ of $\mathrm{Aut}(W)$ given by $R_\theta : (z, w) \mapsto (e^{i\theta}z, e^{i\theta}w)$, $\theta \in \mathbb{R}$. It then follows that $f \circ R_\theta = R_\theta \circ f$ for all $f \in \mathrm{Aut}^o(W)$ and all $\theta \in \mathbb{R}$. In particular, for any $f \in G'_W$ written in the form (3.9), we have

$$g(e^{i\theta}z) = e^{i\theta}g(z), \quad \lambda(e^{i\theta}z)e^{i\theta}w = e^{i\theta}\lambda(z)w \quad \text{for all } (z, w) \in W,\ \theta \in \mathbb{R}. \tag{3.21}$$

Therefore, by the standard argument using the power series expansions of holomorphic functions on the Reinhardt domain W_1, it can be shown that g is a linear automorphism of W_1 and λ is a constant function on W_1. Together with (3.3), this tells us that every $f \in \mathrm{Aut}^o(W)$ is a linear automorphism of W, that is, f can be expressed in the form

$$f(z, w) = (Az, Bw) \quad \text{for } (z, w) \in W\,,$$

where A and B are suitable non-singular matrices. Thus $[\mathfrak{c}_{n,m}, \mathfrak{g}(W)] = \{0\}$ or equivalently $[\mathfrak{c}_{n,m}, \mathfrak{g}(D)] = \{0\}$ by (3.2). However, since $I^z \in \mathfrak{c}_{n,m}$ and $[I^z, X_v] = -X_{iv} \neq 0$, this is impossible. Therefore, $Y_1^v = 0$ and $\varphi(X_v) = Y_0^v + Y_2^v$ for every $v \in \mathbb{C}^n$, as desired.

To prove (2), assume that $Y_0^v = 0$ and $Y_2^v \neq 0$. Then $v \neq 0$ and

$$-2a\mu\|v\|^2\varphi(I^w) = \varphi([X_v, [\widehat{I}, X_v]]) = [Y_2^v, iY_2^v] = 0\,.$$

However, this is absurd because $a \neq 0$ as we proved in the preceding paragraph. Therefore $Y_2^v = 0$ if $Y_0^v = 0$. A similar argument shows that $Y_0^v = 0$ if $Y_2^v = 0$, as desired.

Finally, take an arbitrary element $\varphi(X_v) = Y_0^v + Y_2^v$. Then, owing to (3.14), Y_0^v can be expressed as $Y_0^v = \sum_{k=1}^n \alpha_k^v \partial/\partial z_k$ with $\alpha_k^v \in \mathbb{C}$ $(1 \leq k \leq n)$. Hence

$$\varphi(X_v)_0 = \big(Y_0^v\big)_0 = \big(\alpha_1^v, \ldots, \alpha_n^v\big) \in \mathbb{C}^n = T_0W_1\,. \tag{3.22}$$

Let $\{v_1, \ldots, v_{2n}\}$ be an $\mathbb{R}$-basis for $\mathbb{C}^n$. It then follows from the assertion (2) that $\big\{Y_0^{v_1}, \ldots, Y_0^{v_{2n}}\big\}$ is linearly independent in $\mathfrak{P}_0$. Together with (3.22), this yields at once that

$$T_0W_1 = \mathbb{R}\big\{\varphi(X_{v_1})_0, \ldots, \varphi(X_{v_{2n}})_0\big\}\,.$$

Therefore we obtain the last assertion; completing the proof of Lemma 5. □

As an immediate consequence of Lemma 5, we have the following:

(3.23) Every element Y in $\mathfrak{g}'_W \cap (\mathfrak{P}_0 \oplus \mathfrak{P}_2)$ can be written in the form $Y = \varphi(X_v)$ with some $v \in \mathbb{C}^n$.

Indeed, since $\varphi(\mathfrak{g}'_D) = \mathfrak{g}'_W$, there exist some elements $v \in \mathbb{C}^n$, $A \in \mathfrak{u}(n)$ and $a \in \mathbb{R}$ such that $\varphi(X_v + X_A + aI^w) = Y$. Here we know that $\varphi(X_A + aI^w) \in \mathfrak{P}_1$ and $\varphi(X_v) \in \mathfrak{P}_0 \oplus \mathfrak{P}_2$ by Lemma 5. So, if $Y \in \mathfrak{P}_0 \oplus \mathfrak{P}_2$, then we conclude that $\varphi(X_A + aI^w) = 0$ and $\varphi(X_v) = Y$, as required.

LEMMA 6. *Let $\rho : G'_W \to \mathrm{Aut}(W_1)$ be the group homomorphism appearing in* (3.13). *Then $\rho(G'_W)$ acts transitively on W_1.*

PROOF. By (3.7) and Lemma 2, we see that $W_1 = B^n(r_1)$ with $0 < r_1 \leq +\infty$. Consider first the case where $W_1 = B^n(r_1)$ with $0 < r_1 < +\infty$. Then it is well-known that $\mathrm{Aut}(W_1)$ is a Lie group of $\dim_{\mathbb{R}} \mathrm{Aut}(W_1) = n^2 + 2n$. In this case, $\rho : G'_W \to \mathrm{Aut}(W_1)$ is a Lie group homomorphism and $\rho_* : \mathfrak{g}'_W \to \mathfrak{g}(W_1)$ is the Lie algebra homomorphism induced by ρ. Take any element $\varphi(X_v)$ and write $\varphi(X_v) = Y_0^v + Y_2^v$ as in Lemma 5. Then, by (3.14) $\rho_*(\varphi(X_v))$ can be expressed as

$$\rho_*(\varphi(X_v)) = Y_0^v + \widehat{Y}_2^v, \quad \widehat{Y}_2^v = \sum_{j=1}^n p_j^v(z)\frac{\partial}{\partial z_j},$$

where each p_j^v is a homogeneous polynomial in z of degree two; and hence

$$\dim_{\mathbb{R}}\{\rho_*(\varphi(X_v))\,;\, v \in \mathbb{C}^n\} = 2n$$

by the last assertion in Lemma 5. Since ρ_* is injective on $\mathfrak{u}(n)$ and $\mathfrak{u}(n) \cap \varphi(\pi_D) = \{0\}$, this implies that $\dim_{\mathbb{R}} \rho_*(\mathfrak{g}'_W) \geq n^2 + 2n$. Thus, $\dim_{\mathbb{R}} \rho_*(\mathfrak{g}'_W) = n^2 + 2n$ and $\rho(G'_W) = \mathrm{Aut}(W_1)$ because both the Lie groups are connected and have the same dimension. Since $\mathrm{Aut}(W_1)$ acts transitively on the ball $W_1 = B^n(r_1)$, so does $\rho(G'_W)$, as desired.

Consider next the case where $r_1 = +\infty$ or $W_1 = \mathbb{C}^n$ and let $V_1 := \rho(G'_W) \cdot 0$ be the $\rho(G'_W)$-orbit passing through the origin 0 of $\mathbb{C}^n$. Notice that V_1 is open in $\mathbb{C}^n$ by Lemma 5 and that $U(n) \subset \rho(G'_W)$. Then $V_1 = B^n(r)$ with some $0 < r \leq +\infty$. Here we assert that $V_1 = \mathbb{C}^n$ or $r = +\infty$. Indeed, assume not. Then V_1 is a bounded ball $B^n(r)$ and $\rho(G'_W)$ can be regarded as a subgroup of $\mathrm{Aut}(B^n(r))$. By the same reasoning as above, we then have $\rho(G'_W) = \mathrm{Aut}(B^n(r))$. However, this is impossible because every element of the subgroup $\rho(G'_W)$ of $\mathrm{Aut}(\mathbb{C}^n)$ must be holomorphic on the whole of $\mathbb{C}^n$, while $\mathrm{Aut}(B^n(r))$ contains an element that is not holomorphic on $\mathbb{C}^n$. Therefore we have shown that $V_1 = \mathbb{C}^n = W_1$ and $\rho(G'_W)$ acts transitively on W_1; completing the proof of Lemma 6. □

LEMMA 7. *The domain W_2 is an open ball $B^m(r_2)$ in $\mathbb{C}^m$ with $0 < r_2 < +\infty$.*

PROOF. By (3.7) we know that $W_2 = B^m(r_2)$ with $0 < r_2 \leq +\infty$. Assume here that $r_2 = +\infty$ or $W_2 = \mathbb{C}^m$. Then

$$W \supset \{(z,w) \in W\,;\, z = 0\} = \{0\} \times W_2 = \{0\} \times \mathbb{C}^m$$

by Lemma 2 and Theorem C. So, if we take an arbitrary element $f \in G'_W$ and represent $f(z,w) = (g(z), \lambda(z)w)$ as in (3.9), then

$$W \supset f(\{0\} \times \mathbb{C}^m) = \{g(0)\} \times \mathbb{C}^m$$

because $\lambda(0) \neq 0$. Thus, since $\rho(G'_W)$ acts transitively on W_1 by Lemma 6, it follows that

$$W_1 \times \mathbb{C}^m \supset W \supset \bigcup_{g \in \rho(G'_W)} (\{g(0)\} \times \mathbb{C}^m) = W_1 \times \mathbb{C}^m\,;$$

consequently, $W = W_1 \times \mathbb{C}^m$ and $\mathrm{Aut}^o(W)$ does not have the structure of a Lie group. However, this contradicts the fact that our $\mathrm{Aut}^o(W)$ is now a Lie group isomorphic to $\mathrm{Aut}(D)$. Therefore we conclude that $r_2 \neq +\infty$; completing the proof of Lemma 7. □

For the pseudoconvex Reinhardt domain W contained in $W_1 \times W_2 \subset \mathbb{C}^n \times \mathbb{C}^m$, we set

$$\partial^* W = (W_1 \times \mathbf{C}^m) \cap \partial W,$$

which is an open subset of the boundary ∂W of W. Then, by using the facts in (3.3) and Lemmas 1, 6 and 7, the following four assertions are verified:

(3.24) $\mathrm{Aut}^o(W)$ can be regarded as a subgroup of $\mathrm{Aut}(W_1 \times \mathbb{C}^m)$ leaving $\partial^* W$ invariant;

(3.25) $\mathrm{Aut}^o(W)$ acts transitively on $\partial^* W$;

(3.26) $\mathrm{Aut}^o(W) \cdot 0 = W_1$ (think of W_1 as a complex submanifold of W);

(3.27) For any point $z_o \in W_1$, $p_1^{-1}(z_o) \cap \partial^* W$ can be written in the form

$$p_1^{-1}(z_o) \cap \partial^* W = \{(z_o, w)\,;\ \|w\| = r(z_o)\} \quad \text{with some } \ 0 < r(z_o) \le r_2\,.$$

Therefore $\partial^* W$ is a connected real analytic hypersurface in $p_1^{-1}(W_1) \subset \mathbb{C}^N$ given by the $\mathrm{Aut}^o(W)$-orbit passing through a point of $\partial^* W$. Note that $\mathrm{Aut}^o(W)$ contains $U(n) \times U(m)$ as its subgroup and that every point (z, w) of $\partial^* W$ is mapped by a suitable element $(A, B) \in U(n) \times U(m)$ to some point of the set

$$\partial^* W_{(z_1, w_1)} := \left\{(z, w) \in \partial^* W\,;\ z_j = 0\ (2 \le j \le n),\ w_s = 0\ (2 \le s \le m)\right\},$$

the cross-section of $\partial^* W$ by the $z_1 w_1$-coordinate space in $\mathbb{C}^N$. Thus the shape of $\partial^* W$ is completely determined by that of $\partial^* W_{(z_1, w_1)}$. Consequently, since $\partial^* W_{(z_1, w_1)}$ can be naturally regarded as a Reinhardt hypersurface in $\mathbb{C}^2$, one can choose a small $0 < \delta \le r_1$ and a real analytic function H on the open interval $I_\delta := (-\delta, \delta)$ with $H(t) > 0$ on I_δ and $H(0) = (r_2)^2$ in such a way that $p_1^{-1}(B^n(\delta)) \cap \partial^* W$ can be described as

$$p_1^{-1}(B^n(\delta)) \cap \partial^* W = \left\{(z, w) \in B^n(\delta) \times \mathbb{C}^m\,;\ \|w\|^2 = H(\|z\|^2)\right\}. \tag{3.28}$$

(For the Reinhardt hypersurfaces in $\mathbb{C}^L$, see e.g., [8; Chapter 1].) Thus, by (3.24) every element V in $\mathfrak{g}(W)$ satisfies the following tangency condition:

$$\mathrm{Re}\big((V\rho)(z, w)\big) = 0 \quad \text{whenever } \ \rho(z, w) = 0,$$

where we have put $\rho(z, w) = \|w\|^2 - H(\|z\|^2)$.

By Lemma 7, we now have two possibilities as follows:

$$\text{CASE I}: \quad W_1 = \mathbb{C}^n,\ W_2 = B^m(r_2) \quad \text{with } \ 0 < r_2 < +\infty; \text{ and}$$

$$\text{CASE II}: \quad W_1 = B^n(r_1),\ W_2 = B^m(r_2) \quad \text{with } \ 0 < r_1, r_2 < +\infty\,.$$

LEMMA 8. CASE II *does not occur.*

PROOF. Assuming contrarily that this case occurs, we shall derive a contradiction. After a suitable change of coordinates of the form $(\widetilde{z}, \widetilde{w}) = (sz, tw)$ with $0 < s, t \in \mathbb{R}$, if necessary, we may assume that $W_1 = B^n$ and $W_2 = B^m$, the unit balls in $\mathbb{C}^n$ and in $\mathbb{C}^m$, respectively. The proof will be divided into two cases where $n = 1$ and $n \ge 2$.

Case (II-1). $n = 1$: We set $W_1 = \Delta$ (the unit disc) for a while. Recall that the one-parameter subgroup $\{\psi_t\}_{t \in \mathbb{R}}$ of $\mathrm{Aut}(\Delta)$ given by

$$\psi_t\ :\ z \longmapsto \frac{(\cosh t)z + \sinh t}{(\sinh t)z + \cosh t}, \qquad t \in \mathbb{R},$$

induces the complete holomorphic vector field $V := (1 - z^2)\partial/\partial z$ on Δ.

First we assert that there exists an element $v \in \mathbb{C}$ such that $\varphi(X_v)$ has the form

$$\varphi(X_v) = (1 - z^2)\frac{\partial}{\partial z} + cz\sum_{s=1}^{m} w_s \frac{\partial}{\partial w_s}\,. \tag{3.29}$$

Indeed, since $\mathfrak{g}(\Delta) = \rho_*(\mathfrak{g}'_W) = \rho_*(\varphi(\mathfrak{g}'_D))$ by the proof of Lemma 6 and since $V \in \mathfrak{g}(\Delta)$, one can choose some constants $\alpha, \beta, \gamma \in \mathbb{R}$ and $v, c \in \mathbb{C}$ in such a way that

$$\varphi(X_v + \alpha I^z + \beta I^w) = (1 - z^2)\frac{\partial}{\partial z} + \gamma I^w + cz\sum_{s=1}^{m} w_s \frac{\partial}{\partial w_s}\,.$$

Put $X = X_v + \alpha I^z + \beta I^w$ for a while and let $\widehat{I} = aI^z + bI^w$ be the vector field on D appearing in (3.19). Then we have

$$\varphi(a^2 X_v) = -\varphi([\widehat{I}, [\widehat{I}, X]]) = (1 - z^2)\frac{\partial}{\partial z} + cz\sum_{s=1}^{m} w_s \frac{\partial}{\partial w_s}\,.$$

Therefore, after taking $a^2 v$ instead of v, if necessary, we obtain (3.29).

Next we wish to show that the constant c in (3.29) is real. To this end, recall that $\varphi(I^w) = a_{12}I^z + a_{22}I^w$ with some $(a_{12}, a_{22}) \in \mathbb{Z}^2 \setminus \{0\}$. It then follows from (3.20) that

$$\varphi([X_v, [\widehat{I}, X_v]]) = -2a\mu|v|^2(a_{12}I^z + a_{22}I^w)\,. \tag{3.30}$$

On the other hand, if we put $\varphi(X_v) = Y$ in (3.29), then routine computations show that

$$[Y, [I, Y]] = -4I^z + 2cI^w\,. \tag{3.31}$$

Thus, comparing the coefficients of I^w on the right-hand sides in (3.30) and (3.31), we obtain that $c = -a\mu|v|^2 a_{22} \in \mathbb{R}$, as desired.

Now, recall that

$$p_1^{-1}(\Delta(\delta)) \cap \partial^* W = \{(z, w) \in \Delta(\delta) \times \mathbb{C}^m\,;\, \rho(z, w) = 0\}\,, \tag{3.32}$$

where $\Delta(\delta) = \{z \in \mathbb{C}\,;\, |z| < \delta \leq 1\}$ and $\rho(z, w) = \|w\|^2 - H(|z|^2)$. Then, from the tangency condition $\mathrm{Re}(\varphi(X_v)\rho) = 0$ on $p_1^{-1}(\Delta(\delta)) \cap \partial^* W$ for the vector field $\varphi(X_v)$ in (3.29), we obtain the differential equation

$$cH(|z|^2) = H'(|z|^2)(1 - |z|^2) \quad \text{on } \Delta(\delta) \text{ with } H(0) = 1\,;$$

consequently, $H(r^2) = (1 - r^2)^{-c}$ for all $0 \leq r < \delta$. Therefore, by analytic continuation we have

$$\partial^* W = \left\{(z, w) \in \Delta \times \mathbb{C}^m\,;\, \|w\|^2 = (1 - |z|^2)^{-c}\right\} \quad \text{and so}$$
$$W = \left\{(z, w) \in \Delta \times \mathbb{C}^m\,;\, \|w\|^2 < (1 - |z|^2)^{-c}\right\}\,.$$

Here, if $c = 0$, then $W = \Delta \times B^m$ and $\dim_{\mathbb{R}} \mathrm{Aut}^o(W) = m^2 + 2m + 3 > \dim_{\mathbb{R}} \mathrm{Aut}(D)$. This is absurd because $\mathrm{Aut}^o(W)$ is now isomorphic to $\mathrm{Aut}(D)$ as Lie groups. If $c > 0$, then $\lim_{|z|\uparrow 1}(1 - |z|^2)^{-c} = \infty$; so that $W \subset W_1 \times W_2 = \Delta \times B^m \subsetneqq W$, a contradiction. Hence, c

must be a negative constant. So, putting $p := -1/c > 0$, we conclude that

$$W = \left\{(z, w) \in \mathbb{C} \times \mathbb{C}^m \,;\, |z|^2 + \|w\|^{2p} < 1\right\},$$

a generalized complex ellipsoid, say $\mathcal{E}_p$, in $\mathbb{C} \times \mathbb{C}^m = \mathbb{C}^N$. In this case, we know [13] that $\mathrm{Aut}(\mathcal{E}_p)$ contains the one-parameter subgroup $\{\tau_t\}_{t\in\mathbb{R}}$ given by

$$\tau_t \,:\, (z, w) \longmapsto \left(\frac{(\cosh t)z + \sinh t}{(\sinh t)z + \cosh t}, \frac{w}{\big((\sinh t)z + \cosh t\big)^{1/p}}\right), \quad t \in \mathbb{R},$$

which induces the complete holomorphic vector field

$$Y := (1 - z^2)\frac{\partial}{\partial z} - \frac{1}{p} z \sum_{s=1}^{m} w_s \frac{\partial}{\partial w_s} \,\in\, \mathfrak{g}'_W \cap (\mathfrak{P}_0 \oplus \mathfrak{P}_2)\,.$$

Thus, by (3.23) there exists an element $v \in \mathbb{C}^*$ such that $\varphi(X_v) = Y$. Recall that

$$[X_v, [\widehat{I}, X_v]] = -2a\mu|v|^2 I^w\,; \quad \text{and hence,} \quad [X_v, [X_v, [\widehat{I}, X_v]]] = 0\,.$$

Then we arrive at a contradiction:

$$\varphi([X_v, [X_v, [\widehat{I}, X_v]]]) = [Y, [Y, [I, Y]]] = -4i\left\{(1 + z^2)\frac{\partial}{\partial z} + \frac{1}{p} z \sum_{s=1}^{m} w_s \frac{\partial}{\partial w_s}\right\} \neq 0\,.$$

Therefore we have shown that Case II does not occur in the case where $n = 1$.

Case (II-2). $n \geq 2$: Note that the generalized complex ellipsoid $\mathcal{E}_1$ in $\mathbb{C} \times \mathbb{C}^{n-1} = \mathbb{C}^n$ is nothing but the unit ball B^n in $\mathbb{C}^n$. Then it is obvious that $\rho_*(\mathfrak{g}'_W) = \mathfrak{g}(B^n)$ contains the complete holomorphic vector field $V := (1 - z_1^2)\partial/\partial z_1 - z_1 \sum_{k=2}^{n} z_k \partial/\partial z_k$. Hence, by the same method used in the proof of (3.29), one can choose an element $v \in \mathbb{C}^n$ in such a way that $\varphi(X_v)$ has the form

$$\varphi(X_v) = (1 - z_1^2)\frac{\partial}{\partial z_1} - z_1 \sum_{k=2}^{n} z_k \frac{\partial}{\partial z_k} + \sum_{1\leq l\leq n, 1\leq s\leq m} c_l z_l w_s \frac{\partial}{\partial w_s}\,.$$

Put again $\varphi(X_v) = Y$. Then

$$-[I_1^z, [I_1^z, Y]] = (1 - z_1^2)\frac{\partial}{\partial z_1} - z_1 \sum_{k=2}^{n} z_k \frac{\partial}{\partial z_k} + c_1 z_1 \sum_{s=1}^{m} w_s \frac{\partial}{\partial w_s}$$

and this is obviously an element of $\mathfrak{g}'_W \cap (\mathfrak{P}_0 \oplus \mathfrak{P}_2)$. Hence, by the fact (3.23) we may assume from the beginning that

$$Y = (1 - z_1^2)\frac{\partial}{\partial z_1} - z_1 \sum_{k=2}^{n} z_k \frac{\partial}{\partial z_k} + c_1 z_1 \sum_{s=1}^{m} w_s \frac{\partial}{\partial w_s}\,.$$

As in (3.30) and (3.31), it then follows that

$$\varphi([X_v, [\widehat{I}, X_v]]) = -2a\mu\|v\|^2(a_{12}I^z + a_{22}I^w) \;\text{ and}$$

$$[Y, [I, Y]] = -4I_1^z - 2\sum_{k=2}^{n} I_k^z + 2c_1 I^w\,.$$

Thus, comparing the coefficients of I_1^z and I_2^z on the right-hand sides in the above two equations, one obtains that $a\mu\|v\|^2 a_{12} = 2$ and $a\mu\|v\|^2 a_{12} = 1$, a contradiction. Therefore we have shown that Case II does not occur also in the case where $n \geq 2$, completing the proof of Lemma 8. □

LEMMA 9. *In* CASE I, *W is biholomorphically equivalent to D.*

PROOF. Without loss of generality, we may assume that $W_2 = B^m$ in Case I. We divide again the proof into two cases.

Case (I-1). $n = 1$: Since $\{\varphi(X_v)_0\,;\ v \in \mathbb{C}\} = T_0\mathbb{C}$ by Lemma 5, there exists an element $v \in \mathbb{C}$ such that $\varphi(X_v)$ has the form

$$\varphi(X_v) = \frac{\partial}{\partial z} + p(z)\frac{\partial}{\partial z} + cz\sum_{s=1}^{m} w_s\frac{\partial}{\partial w_s}\,, \tag{3.33}$$

where p is a homogeneous polynomial in z of degree two and $c \in \mathbb{C}$. Here we claim that $p(z) = 0$ on $W_1 = \mathbb{C}$ and $c \in \mathbb{R}^*$. Indeed, since

$$\mathfrak{g}(\mathbb{C}) = \left\{(\alpha z + \beta)\frac{\partial}{\partial z}\,;\ \alpha, \beta \in \mathbb{C}\right\} \quad \text{and} \quad (1 + p(z))\frac{\partial}{\partial z} = \rho_*(\varphi(X_v)) \in \mathfrak{g}(\mathbb{C})\,,$$

we have $p(z) = 0$ on $\mathbb{C}$. Moreover, by combining the fact (2) of Lemma 5 with the same computations as in (3.30) and (3.31), one can verify that $c \in \mathbb{R}^*$, as claimed.

Describe now $p_1^{-1}(\Delta(\delta)) \cap \partial^* W$ in the same form as in (3.32). Then, for the complete holomorphic vector field $\varphi(X_v)$ on W in (3.33) with $p(z) \equiv 0$ and $c \in \mathbb{R}^*$, the tangency condition $\mathrm{Re}(\varphi(X_v)\rho) = 0$ on $p_1^{-1}(\Delta(\delta)) \cap \partial^* W$ with the initial condition $H(0) = 1$ yields that $H(r^2) = e^{cr^2}$ for all $0 \le r < \delta$. Therefore, by anlytic continuation we obtain that

$$\partial W = \partial^* W = \left\{(z, w) \in \mathbb{C} \times \mathbb{C}^m\,;\ \|w\|^2 = e^{c|z|^2}\right\}\,.$$

Moreover, by the same reasoning as in the proof of Lemma 8, Case (II-1), c has to be a negative constant. Thus, putting $\nu = -c > 0$, we conclude that W can be described as

$$W = \left\{(z, w) \in \mathbb{C} \times \mathbb{C}^m\,;\ \|w\|^2 < e^{-\nu|z|^2}\right\}\,;$$

accordingly, the non-singular linear mapping $L : \mathbb{C} \times \mathbb{C}^m \to \mathbb{C} \times \mathbb{C}^m$ defined by

$$L : (z, w) \mapsto (\xi, \eta) = \left(\sqrt{\nu/\mu}\,z, w\right) \quad \text{for } (z, w) \in \mathbb{C} \times \mathbb{C}^m$$

gives rise to a linear equivalence between W and $D = D_{1,m}(\mu)$; completing the proof of Lemma 9 in the case where $n = 1$.

Case (I-2). $n \ge 2$: Since $\{\varphi(X_v)_0\,;\ v \in \mathbb{C}^n\} = T_0\mathbb{C}^n$ by Lemma 5, one can choose an element $v \in \mathbb{C}^n$ in such a way that $\varphi(X_v)$ has the form

$$\varphi(X_v) = \frac{\partial}{\partial z_1} + \sum_{k=1}^{n} p_k(z)\frac{\partial}{\partial z_k} + \mu(z)\sum_{s=1}^{m} w_s\frac{\partial}{\partial w_s}\,,$$

where p_k's (resp. μ) are homogeneous polynomials in z of degree two (resp. of degree one). Putting $\varphi(X_v) = Y$, we here assert that

$$[I_k^z, Y] = 0 \quad \text{for all } k = 2, \dots, n\,. \tag{3.34}$$

Indeed, a straightforward computation shows that

$$[I_k^z, Y] = i\left\{\sum_{j=1}^{n}\left(z_k\frac{\partial p_j(z)}{\partial z_k} - \delta_{jk}p_k(z)\right)\frac{\partial}{\partial z_j} + z_k\frac{\partial \mu(z)}{\partial z_k}\sum_{s=1}^{m} w_s\frac{\partial}{\partial w_s}\right\}$$

for each $2 \le k \le n$, where δ_{jk} denotes the Kronecker symbol. Thus $[I_k^z, Y] \in \mathfrak{g}'_W \cap \mathfrak{P}_2$; hence, by (3.23) there exists an element $u \in \mathbb{C}^n$ such that $[I_k^z, Y] = \varphi(X_u)$ and $\varphi(X_u) \in \mathfrak{P}_2$.

Consequently, $[I_k^z, Y] = \varphi(X_u) = 0$ by Lemma 5, as asserted. On the other hand, one can check that Y satisfies the equations (3.34) only when Y has the form

$$Y = \frac{\partial}{\partial z_1} + \sum_{k=1}^{n} a_k z_1 z_k \frac{\partial}{\partial z_k} + c z_1 \sum_{s=1}^{m} w_s \frac{\partial}{\partial w_s} \tag{3.35}$$

with some constants $a_k, c \in \mathbb{C}$. Moreover, in exactly the same way as in the proof of Lemma 8, Case (II-1), it can be shown that $c \in \mathbb{R}$ (and also $a_k \in \mathbb{R}$ for all k).

Now we put

$$\mathbb{C}_{(z_1,w)} = \left\{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m \,;\, z_j = 0 \ (2 \le j \le n)\right\},$$
$$W_{(z_1,w)} = W \cap \mathbb{C}_{(z_1,w)} \quad \text{and} \quad \partial^* W_{(z_1,w)} = \partial^* W \cap \mathbb{C}_{(z_1,w)}\,.$$

Then $\mathbb{C}_{(z_1,w)}$ can be naturally identified with $\mathbb{C}^{m+1}$ with the coordinate system (z_1, w) and $W_{(z_1,w)}$ can be regarded as a domain in $\mathbb{C}^{m+1}$ with real analytic boundary $\partial^* W_{(z_1,w)}$. Moreover, we have

$$p_1^{-1}(\Delta(\delta)) \cap \partial^* W_{(z_1,w)} = \left\{(z_1, w) \in \Delta(\delta) \times \mathbb{C}^m \,;\, \|w\|^2 = H(|z_1|^2)\right\} \tag{3.36}$$

under the natural identification $\{z \in B^n(\delta)\,;\, z_j = 0 \ (2 \le j \le n)\} = \Delta(\delta)$, the open disc with radius δ and center 0 in the z_1-coordinate space $\mathbb{C}$; and

(3.37) the holomorphic vector field Y on $\mathbb{C}^n \times \mathbb{C}^m$ in (3.35) is tangent to $W_{(z_1,w)}$; hence, its restriction $Y^{(1)} := (1 + a_1 z_1^2)\partial/\partial z_1 + c z_1 \sum_{s=1}^{m} w_s \partial/\partial w_s$ to $W_{(z_1,w)}$ induces a complete holomorphic vector field on $W_{(z_1,w)}$.

Clearly the vector field $Y^{(1)}$ is tangent to $\{(z_1, w) \in W_{(z_1,w)}\,;\, w = 0\} = \mathbb{C}$, a complex submanifold of $W_{(z_1,w)}$; accordingly, $(1 + a_1 z_1^2)\partial/\partial z_1$ gives now a complete holomorphic vector field on $\mathbb{C}$. Thus $a_1 = 0$ and

$$Y^{(1)} = \frac{\partial}{\partial z_1} + c z_1 \sum_{s=1}^{m} w_s \frac{\partial}{\partial w_s} \ \in \mathfrak{g}(W_{(z_1,w)})\,.$$

So, by repeating exactly the same argument as in Case (I-1), one can verify that $W_{(z_1,w)}$ can be written in the form

$$W_{(z_1,w)} = \left\{(z_1, w) \in \mathbb{C} \times \mathbb{C}^m \,;\, \|w\|^2 < e^{c|z_1|^2}\right\} \quad \text{with} \quad c < 0\,.$$

Therefore, by the invariance of W under the standard $U(n) \times U(m)$-action on $\mathbb{C}^n \times \mathbb{C}^m$, we now conclude that W has the representation

$$W = \left\{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m \,;\, \|w\|^2 < e^{-\nu\|z\|^2}\right\} \quad \text{with} \ \ \nu := -c > 0\,,$$

which is linearly equivalent to $D = D_{n,m}(\mu)$ as in Case (I-1); thereby completing the proof of Lemma 9 in the case where $n \ge 2$. □

Summarizing our results obtained in the above, we have shown that only Case I occurs and the domain W is, in fact, biholomorphically equivalent to the model domain $D = D_{n,m}(\mu)$. We have thus completed the proof of our theorem. □

4. Proof of the Corollary. For the sake of simplicity, we write $D_1 = D_{n_1,m_1}(\mu_1)$ and $D_2 = D_{n_2,m_2}(\mu_2)$ in this section.

It is trivial that $\mathrm{Aut}(D_1)$ and $\mathrm{Aut}(D_2)$ are isomorphic as topological groups if D_1 and D_2 are linearly equivalent. So, assuming that there exists a topological group isomorphism $\Phi : \mathrm{Aut}(D_1) \to \mathrm{Aut}(D_2)$, we would like to prove that D_1 and D_2 are linearly equivalent. We have now two cases to consider.

CASE 1. $N_1 = N_2$: In this case, if $m_1 \geq 2$ or $m_2 \geq 2$, there exists a biholomorphic mapping $f : D_1 \to D_2$ by our theorem. It then follows that $f(\Delta_{D_1}) = \Delta_{D_2}$ and f induces a biholomorphic mapping from $\Delta_{D_1} \cong \mathbb{C}^{n_1}$ onto $\Delta_{D_2} \cong \mathbb{C}^{n_2}$ because the degeneracy sets for Kobayashi pseudodistances are invariant under biholomorphic mappings, in general. Therefore, $n_1 = n_2$ and so $m_1 = m_2$. If $m_1 = m_2 = 1$, it is trivial that $n_1 = n_2$. Anyway we have $(n_1, m_1) = (n_2, m_2)$ in Case 1; and hence, a non-singular linear mapping $L : \mathbb{C}^{n_1} \times \mathbb{C}^{m_1} \to \mathbb{C}^{n_2} \times \mathbb{C}^{m_2}$ can be defined by

$$L(z, w) = \left(\sqrt{\mu_1/\mu_2}\, z, w\right) \quad \text{for} \quad (z, w) \in \mathbb{C}^{n_1} \times \mathbb{C}^{m_1} .$$

Clearly this L gives now a linear equivalence between D_1 and D_2, as desired.

CASE 2. $N_1 \neq N_2$: We assert that this case does not occur. Indeed, assuming that this case occurs, we wish to derive a contradiction. For this, let us recall the following:

FACT ([15; Lemma 2.1]). *Let M be a connected Stein manifold of dimension n. If $N > n$, then there is no injective continuous group homomorphism of the N-dimensional torus T^N into the topological group* $\mathrm{Aut}(M)$.

Without loss of generality, we may assume that $N_1 > N_2$. Then, under the identification $T^{N_1} = T(D_1)$, our isomorphism Φ gives now an injective continuous group homomorphism of T^{N_1} into $\mathrm{Aut}(D_2)$. Since D_2 is a connected Stein manifold of dimension $N_2 < N_1$, this contradicts the Fact above; thereby, Case 2 does not occur, as asserted.

Finally, by the argument in Case 1 above, it is obvious that D_1 is linearly equivalent to D_2 if and only if $(n_1, m_1) = (n_2, m_2)$.

Therefore we have completed the proof of our corollary. □

REMARK. Let D_1 and D_2 be two Reinhardt domains in $\mathbb{C}^N$ and assume that $\mathrm{Aut}(D_1)$ has the structure of a Lie group with respect to the compact-open topology. Then we know the following result due to Shimizu [27; Section 4]: If D_1 and D_2 are holomorphically equivalent, then they are algebraically equivalent. In addition to this, if D_1 contains the origin 0 of $\mathbb{C}^N$, then D_1 and D_2 are linearly equivalent.

Now let us consider the special case where D_1 and D_2 are our Fock-Bargmann-Hartogs domains in the above proof of the Corollary. Then we know that $\mathrm{Aut}(D_j)$ has the structure of a Lie group with respect to the compact-open topology and D_j contains the origin 0 of $\mathbb{C}^{N_j}$ for $j = 1, 2$. Accordingly, Shimizu's result also assures us that D_1 and D_2 are linearly equivalent if they are holomorphically equivalent.

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