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VANISHING FOR FROBENIUS TWISTS OF AMPLE VECTOR BUNDLES

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Abstract. We prove several asymptotic vanishing theorems for Frobenius twists of ample vector bundles in positive characteristic. As an application, we improve the Bott-Danilov-Steenbrink vanishing theorem for ample vector bundles on toric varieties.

CONTENTS

1. Introduction.

Let k be a field, and X a finite-type k-scheme. Let $\mathscr{E}$ be a vector bundle on X. In a beautiful series of papers [Ara04, Ara06, Ara11], Arapura studies a measure of the positivity of $\mathscr{E}$ — the Frobenius amplitude of $\mathscr{E}$. Arapura bounds the Frobenius amplitude of ample vector bundles in the case that k has characteristic zero. The purpose of this paper is to prove bounds on the Frobenius amplitude in *positive* characteristic, under favorable hypotheses, and to give some applications to prove strong vanishing theorems on e.g. toric varieties.

DEFINITION 1.0.1 (Frobenius Amplitude). Let k be a field of characteristic $p > 0$ and X a finite-type k-scheme. If $\mathscr{E}$ is a vector bundle on X, we define the *Frobenius amplitude* of $\mathscr{E}$, denoted $\phi(\mathscr{E})$, to be the least integer l such that for any coherent sheaf $\mathscr{F}$ on X, there exists $N \gg 0$ such that

$$H^i(X, \mathscr{E}^{(p^n)} \otimes \mathscr{F}) = 0$$

for all $i > l$ and $n > N$.

Now let k be a field of characteristic 0 and X a finite-type k-scheme. For $\mathscr{E}$ a vector bundle on X, we say that $\phi(\mathscr{E}) < l$ if there exists a finite-type $\mathbb{Z}$-algebra R, a finite-type R-scheme $\mathcal{X}$, and a vector bundle $\widetilde{\mathscr{E}}$ on $\mathcal{X}$ such that

(1) there is a ring map $R \to k$ and isomorphisms $X \simeq \mathcal{X}_k, \mathscr{E} \simeq \widetilde{\mathscr{E}}_k$, and

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(2) $\phi(\widetilde{\mathscr{E}_{\mathfrak{q}}}) < l$ for all closed points $\mathfrak{q} \subset \mathrm{Spec}(R)$.

See [Ara04, Section 1] for more details. Arapura proves

THEOREM 3.0.2 (Frobenius Amplitude of Ample Bundles, [Ara04, Theorem 6.1]). *Let k be a field of characteristic zero and X a projective variety over k. Let $\mathscr{E}$ be an ample vector bundle on X. Then*

$$\phi(\mathscr{E}) < \mathrm{rk}(\mathscr{E})\,.$$

We give a simple new proof of this theorem in Section 3.

This theorem implies several strong vanishing theorems [Ara04]. It motivates the following question:

QUESTION 1.0.2. Let X be a proper variety over a field k of characteristic $p > 0$. Let $\mathscr{E}$ be an ample vector bundle on X. Is

$$\phi(\mathscr{E}) < \mathrm{rk}(\mathscr{E})?$$

The goal of this paper is to answer Question 1.0.2 under favorable hypotheses — in particular, we give a new short proof of Theorem 3.0.2, and answer Question 1.0.2, if X lifts to $W_2(k)$, and admits a lift of Frobenius.

Explicitly, our main results is:

THEOREM 2.2.1 (Main Theorem). *Let k be a perfect field of characteristic $p > 0$ and X a projective k-scheme admitting a flat lift X_2 over $W_2(k)$. Let $\mathscr{E}$ be an ample vector bundle on X with $\mathrm{rk}(\mathscr{E}) < p$, and suppose that for some $N > 0$, $\mathscr{E}^{(p^N)}$ lifts to X_2. Then for $j > 0, n \geq j$ or $j = 0, n > 0$ we have*

$$\phi\left(\mathrm{Sym}^{n-j}(\mathscr{E}) \otimes \bigwedge^{j}(\mathscr{E})\right) \leq \mathrm{rk}(\mathscr{E}) - j\,.$$

Taking $n = 1, j = 1$ we obtain

$$\phi(\mathscr{E}) < \mathrm{rk}(\mathscr{E}),$$

answering Question 1.0.2 in this case.

We view this as a Manivel-type vanishing theorem. Unfortunately, the hypotheses are rather strong — in particular, we must assume that $\mathscr{E}^{(p^N)}$ lifts for some $N > 0$ (note that $N = 0$ does not suffice). This is a difficult condition to verify — the main situation in which it may be checked is when $\mathscr{E}$ admits a lift to X_2, and X_2 admits a lift of Frobenius:

COROLLARY 2.2.3 (Main Corollary). *Let k be a perfect field of characteristic $p > 0$ and X a projective k-scheme admitting a flat lift X_2 over $\mathbb{Z}/p^2\mathbb{Z}$; suppose X_2 admits a lift of absolute Frobenius. Let $\mathscr{E}$ be an ample vector bundle on X with $\mathrm{rk}(\mathscr{E}) < p$, and suppose that $\mathscr{E}$ lifts to X_2. Then for $j > 0, n \geq j$ or $j = 0, n > 0$ we have*

$$\phi\left(\mathrm{Sym}^{n-j}(\mathscr{E}) \otimes \bigwedge^{j}(\mathscr{E})\right) \leq \mathrm{rk}(\mathscr{E}) - j\,.$$

The condition that X_2 admits a lift of Frobenius is quite restrictive (see e.g. [Mig93]), but it holds for e.g. toric varieties [BTLM97] and canonical lifts of ordinary Abelian varieties.

As an application of these methods, we prove a generalization of the Bott-Steenbrink-Danilov vanishing theorem:

THEOREM 4.0.2 (Generalization of Bott-Steenbrink-Danilov). *Let X be a normal projective toric variety over a perfect field k, and $\mathscr{E}_1, \dots, \mathscr{E}_m$ ample vector bundles on X. Let $j : U \hookrightarrow X$ be the inclusion of the smooth locus. Then if a_i, b_i are non-negative integers not all equal to zero, $q \geq 0$, and*

(1) $\mathrm{char}(k) = 0$, *or*

(2) $\mathrm{char}(k) = p > \sum_i \mathrm{rk}(\mathscr{E}_i)$, *and each $\mathscr{E}_i$ lifts to the canonical (toric) lift of X to $W_2(k)$, then*

$$H^s\left(X, j_*\Omega^q_U \otimes \mathrm{Sym}^{a_1}\mathscr{E}_1 \otimes \cdots \otimes \mathrm{Sym}^{a_m}(\mathscr{E}_m) \otimes \bigwedge^{b_1}\mathscr{E}_1 \otimes \cdots \otimes \bigwedge^{b_m}\mathscr{E}_m\right) = 0$$

for $s > \sum_{i=1}^m (\mathrm{rk}(\mathscr{E}_i) - b_i)$.

Here as usual $\Omega^i_U = \bigwedge^i \Omega^1_U$.

This result generalizes several results in the literature. Danilov states (without proof) the special case where $m = 1$, $\mathscr{E}_1 = \mathscr{L}$ is an ample line bundle and $a_1 = 0, b_1 = 1$ [Dan78, Theorem 7.5.2]. This is proven in the case X is simplicial by Batyrev and Cox [BC94] and in general in [BTLM97]. Manivel [Man96] proves a version of this theorem in characteristic zero, when X is smooth. Theorem 4.0.2 is a strengthening (for toric varieties) of his famous vanishing theorem [Man97]. See also [Bri09, Fuj07, Mat02, Mav08, Mus02, HMP10] for related work.

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2. Proof of Theorem 2.2.1 and Corollary 2.2.3.

2.1. The Cartier Isomorphism.

We first recall the main theorem of [DI87]:

THEOREM 2.1.1 (Deligne-Illusie). *Let k be a perfect field of characteristic $p > 0$ and S a k-scheme. Suppose that S admits a flat lift $\widetilde{S}$ over $W_2(k)$. Let X be a smooth S-scheme with $\dim(X/S) < p$. Then there is an isomorphism*

$$F_{X/S*}\Omega^\bullet_{X/S} \simeq \bigoplus \Omega^i_{X'/S}[-i]$$

in $D(X')$ if and only if X' admits a flat lift over $\widetilde{S}$.

Here X' is the Frobenius twist of X, i.e. it fits into the Cartesian diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S & \xrightarrow{\mathrm{Frob}} & S \end{array}$$

and $F_{X/S} : X \to X'$ is the relative Frobenius.

Let k be a perfect field of characteristic $p > 0$, and S a k-scheme admitting a flat lift $\widetilde{S}$ to $W_2(k)$. Now suppose $\mathscr{E}$ is a vector bundle on S and set

$$\mathbb{E}^\vee = \underline{\mathrm{Spec}}_S(\mathrm{Sym}^*(\mathscr{E}))$$

to be the total space of $\mathscr{E}^\vee$; let

$$\pi : \mathbb{E}^\vee \to S$$

be the structure morphism. Then the Frobenius twist $\mathbb{E}^{\vee\prime}$ of $\mathbb{E}^\vee$ is

$$\mathbb{E}^{\vee\prime} \simeq \underline{\mathrm{Spec}}_S(\mathrm{Sym}^*(\mathscr{E}^{(p)}));$$

let π' be the structure map for $\mathbb{E}^{\vee\prime}$. Consider the $\mathscr{O}_S$-linear complex

$$\pi_*\Omega^\bullet_{\mathbb{E}^\vee/S} : 0 \to \mathrm{Sym}^*(\mathscr{E}) \to \mathrm{Sym}^*(\mathscr{E}) \otimes \mathscr{E} \to \mathrm{Sym}^*(\mathscr{E}) \otimes \bigwedge^2 \mathscr{E} \to \cdots$$

The natural action of the multiplicative group $\mathbb{G}_m$ on $\mathbb{E}^\vee$ makes this into a graded complex; let

$$\Omega^\bullet_{\mathbb{E}^\vee/S}(n) : 0 \to \mathrm{Sym}^n(\mathscr{E}) \to \mathrm{Sym}^{n-1}(\mathscr{E}) \otimes \mathscr{E} \to \cdots \to \mathrm{Sym}^{n-i}(\mathscr{E}) \otimes \bigwedge^i \mathscr{E} \to \cdots$$

be the n-th graded piece. The analogous complex for $\mathbb{E}^{\vee\prime}$ is

$$\pi_*\Omega^\bullet_{\mathbb{E}^{\vee\prime}/S} : 0 \to \mathrm{Sym}^*(\mathscr{E}^{(p)}) \to \mathrm{Sym}^*(\mathscr{E}^{(p)}) \otimes \mathscr{E} \to \mathrm{Sym}^*(\mathscr{E}^{(p)}) \otimes \bigwedge^2 \mathscr{E}^{(p)} \to \cdots$$

which is also naturally a graded complex.

By a result of Cartier (see e.g. [Kat70, Theorem 7.2], taking $X = \mathbb{E}^\vee$), there is a natural isomorphism

$$\mathscr{H}^i(\pi_*\Omega^\bullet_{\mathbb{E}^\vee/S}) \simeq \pi'_*\Omega^i_{\mathbb{E}^{\vee\prime}/S} = \mathrm{Sym}^*\, \mathscr{E}^{(p)} \otimes \bigwedge^i \mathscr{E}^{(p)}$$

and keeping track of the grading we find that

$$\mathscr{H}^i(\Omega^\bullet_{\mathbb{E}^\vee/S}(pn)) \simeq \mathrm{Sym}^{n-i}(\mathscr{E}^{(p)}) \otimes \bigwedge^i \mathscr{E}^{(p)}.$$

By Theorem 2.1.1, this isomorphism may be promoted to an isomorphism in $D^b(S)$ if $\mathscr{E}^{(p)}$ admits a lift to $\widetilde{S}$, and $\mathrm{rk}(\mathscr{E}) < p$. In this case, we have an isomorphism

$$\Omega^\bullet_{\mathbb{E}^\vee/S}(pn) \simeq \bigoplus_i \mathrm{Sym}^{n-i}(\mathscr{E}^{(p)}) \otimes \bigwedge^i \mathscr{E}^{(p)}[-i]$$

in $D^b(S)$.

2.2. The Main Theorem. We are now ready to prove:

THEOREM 2.2.1 (Main Theorem). *Let k be a perfect field of characteristic $p > 0$ and X a projective k-scheme admitting a flat lift X_2 over $W_2(k)$. Let $\mathscr{E}$ be an ample vector bundle on X with* $\mathrm{rk}(\mathscr{E}) < p$, *and suppose that for some $N > 0$, $\mathscr{E}^{(p^N)}$ lifts to X_2. Then for $j > 0, n \geq j$ or $j = 0, n > 0$ we have*

$$\phi\left(\mathrm{Sym}^{n-j}(\mathscr{E}) \otimes \bigwedge^j(\mathscr{E})\right) \leq \mathrm{rk}(\mathscr{E}) - j\,.$$

PROOF. Let $\mathscr{F}$ be any coherent sheaf on X and $j \geq 0$ an integer; we wish to show that for $i > \mathrm{rk}(\mathscr{E}) - j$, and $m \gg 0$,

$$H^i\left(X, \mathrm{Sym}^{n-j}(\mathscr{E}^{(p^m)}) \otimes \bigwedge^j \mathscr{E}^{(p^m)} \otimes \mathscr{F}\right) = 0\,.$$

Choose $r \gg 0$ so that

$$H^i\left(X, \mathrm{Sym}^{p^r n-j}(\mathscr{E}^{(p^N)}) \otimes \bigwedge^j \mathscr{E}^{(p^N)} \otimes \mathscr{F}\right) = 0$$

for all j and all $i > 0$. Such an r exists by the ampleness of $\mathscr{E}^{(p^N)}$.

We claim that

$$H^i\left(X, \mathrm{Sym}^{p^{r-s} n-j}(\mathscr{E}^{(p^{N+s})}) \otimes \bigwedge^j \mathscr{E}^{(p^{N+s})} \otimes \mathscr{F}\right) = 0 \tag{2.2.2}$$

for all $0 \leq s \leq r$ and all $i > \mathrm{rk}(\mathscr{E}) - j$. We will prove this by induction on s; the case $s = 0$ is immediate from our choice of r.

Assume that Equation 2.2.2 holds for some s; we will prove it for $s + 1$. From the first hypercohomology spectral sequence (associated to the stupid filtration of $\Omega^\bullet_{\mathbb{E}^\vee/X}(p^{r-s}n)^{(p^{N+s})}$, i.e. $F^i\Omega^\bullet_{\mathbb{E}^\vee/X}(p^{r-s}n)^{(p^{N+s})} = \Omega^{\bullet\geq i}_{\mathbb{E}^\vee/X}(p^{r-s}n)^{(p^{N+s})}$),

$$E_1^{u,v} = H^u\left(X, \mathrm{Sym}^{p^{r-s} n-v}(\mathscr{E}^{(p^{N+s})}) \otimes \bigwedge^v \mathscr{E}^{(p^{N+s})} \otimes \mathscr{F}\right) \Longrightarrow \mathbb{H}^{u+v}(X, \Omega^\bullet_{\mathbb{E}^\vee/X}(p^{r-s}n)^{(p^{N+s})} \otimes \mathscr{F})$$

we have that

$$\mathbb{H}^i(X, \Omega^\bullet_{\mathbb{E}^\vee/X}(p^{r-s}n)^{(p^{N+s})} \otimes \mathscr{F}) = 0$$

for $i > \mathrm{rk}(\mathscr{E})$, by the induction hypothesis, as $E_1^{u,v} = 0$ for $u + v > \mathrm{rk}(\mathscr{E})$.

Now consider the second hypercohomology spectral sequence

$$E_2^{u,v} = H^u(X, \mathscr{H}^v(\Omega^\bullet_{\mathbb{E}^\vee/X}(p^{r-s}n)^{(p^{N+s})} \otimes \mathscr{F})) \implies \mathbb{H}^{u+v}(X, \Omega^\bullet_{\mathbb{E}^\vee/X}(p^{r-s}n)^{(p^{N+s})} \otimes \mathscr{F})\,.$$

As $\mathscr{E}^{(p^N)}$ is assumed to admit a lift to W_2, this spectral sequence is degenerate by Theorem 2.1.1. Thus $E_2^{u,v} = 0$ for $u + v > \mathrm{rk}(\mathscr{E})$. But by the Cartier isomorphism,

$$\begin{aligned} E_2^{u,v} &= H^u(X, \mathscr{H}^v(\Omega^\bullet_{\mathbb{E}^\vee/X}(p^{r-s}n)^{(p^{N+s})} \otimes \mathscr{F})) \\ &= H^u\left(X, \mathrm{Sym}^{p^{r-s-1} n-v}(\mathscr{E}^{(p^{N+s+1})}) \otimes \bigwedge^v \mathscr{E}^{(p^{N+s+1})} \otimes \mathscr{F}\right) \end{aligned}$$

completing the induction step and proving the claim in Equation 2.2.2. Taking $r = s$ in 2.2.2 completes the proof. □

The corollary follows easily:

COROLLARY 2.2.3 (Main Corollary). *Let k be a perfect field of characteristic $p > 0$ and X a projective k-scheme admitting a flat lift X_2 over $\mathbb{Z}/p^2\mathbb{Z}$; suppose X_2 admits a lift of* *absolute Frobenius. Let* $\mathscr{E}$ *be an ample vector bundle on* X *with* $\mathrm{rk}(\mathscr{E}) < p$*, and suppose that* $\mathscr{E}$ *lifts to* X_2*. Then for* $j > 0, n \geq j$ *or* $j = 0, n > 0$ *we have*

$$\phi\left(\mathrm{Sym}^{n-j}(\mathscr{E}) \otimes \bigwedge^{j}(\mathscr{E})\right) \leq \mathrm{rk}(\mathscr{E}) - j\,.$$

PROOF. By Theorem 2.2.1, it suffices to show that $\mathscr{E}^{(p)}$ lifts to X_2. But $\mathscr{E}$ lifts to a vector bundle $\mathscr{E}_2$ on X_2, and Frobenius lifts to a $\mathbb{Z}/p^2\mathbb{Z}$-endomorphism of X_2, say F_2. Then $F_2^*\mathscr{E}_2$ is a lift of $\mathscr{E}^{(p)}$, as desired. □

COROLLARY 2.2.4. *Let* k, X, X_2 *be as in Corollary 2.2.3. Let* $\mathscr{E}_1, \ldots, \mathscr{E}_m$ *be ample vector vector bundles on* X *which lift to* X_2*, with* $\sum_s \mathrm{rk}(\mathscr{E}_s) < \mathrm{char}(k)$*. Then for* a_i, b_i *non-negative integers not all equal to zero,*

$$\phi\left(\mathrm{Sym}^{a_1}(\mathscr{E}_1) \otimes \cdots \otimes \mathrm{Sym}^{a_m}(\mathscr{E}_m) \otimes \bigwedge^{b_1} \mathscr{E}_1 \otimes \cdots \otimes \bigwedge^{b_m} \mathscr{E}_m\right) \leq \sum_{i=1}^{m}(\mathrm{rk}(\mathscr{E}_i) - b_i)\,.$$

PROOF. Let $\mathscr{E} = \bigoplus_i \mathscr{E}_i$; let $a = \sum_i a_i, b = \sum_i b_i$. Then by Corollary 2.2.3, we have that

$$\phi\left(\mathrm{Sym}^{a}(\mathscr{E}) \otimes \bigwedge^{b} \mathscr{E}\right) \leq \mathrm{rk}(\mathscr{E}) - b = \sum_{i=1}^{m}(\mathrm{rk}(\mathscr{E}_i) - b_i)\,.$$

But

$$\mathrm{Sym}^{a_1}(\mathscr{E}_1) \otimes \cdots \otimes \mathrm{Sym}^{a_m}(\mathscr{E}_m) \otimes \bigwedge^{b_1} \mathscr{E}_1 \otimes \cdots \otimes \bigwedge^{b_m} \mathscr{E}_m$$

is a direct summand of $\mathrm{Sym}^a(\mathscr{E}) \otimes \bigwedge^b \mathscr{E}$ so the result follows. □

3. Proof of Theorem 3.0.2.

We now give a short proof of Theorem 3.0.2, originally due to Arapura [Ara04, Theorem 6.1]. We require a lemma, essentially from [Ara04]:

LEMMA 3.0.1. *Let* $\mathscr{E}$ *be an ample vector bundle, and* $\mathscr{F}$ *any vector bundle. Then for* $N \gg 0$*,*

$$\phi(\mathrm{Sym}^N(\mathscr{E}) \otimes \mathscr{F}) = 0\,.$$

PROOF. By [Ara04, Lemma 3.3], it suffices to observe that the Castelnuovo-Mumford regularity of $\mathrm{Sym}^N(\mathscr{E}) \otimes \mathscr{F}$ tends to $-\infty$ as N tends to ∞. But this is immediate from the ampleness of $\mathscr{E}$. □

We are now ready to prove:

THEOREM 3.0.2 (Frobenius Amplitude of Ample Bundles, [Ara04, Theorem 6.1]). *Let* k *be a field of characteristic zero and* X *a projective variety over* k*. Let* $\mathscr{E}$ *be an ample vector bundle on* X*. Then*

$$\phi(\mathscr{E}) < \mathrm{rk}(\mathscr{E})\,.$$

PROOF. By Lemma 3.0.1, there exists N_0 such that for all $n > N_0$,

$$\phi\left(\mathrm{Sym}^{n-i}(\mathscr{E}) \otimes \bigwedge^{i} \mathscr{E}\right) = 0$$

for all i (using the ampleness of $\mathscr{E}$). Now let R be a finite-type $\mathbb{Z}$-algebra, with a map $R \to k$ and $(\mathcal{X}, \widetilde{\mathscr{E}})$ a finite-type R-scheme with a vector bundle so that $\mathcal{X}_k \simeq X, \widetilde{\mathscr{E}}_k \simeq \mathscr{E}$, and such that

$$\phi\left(\mathrm{Sym}^{n-i}(\widetilde{\mathscr{E}}_{\mathfrak{q}}) \otimes \bigwedge^{i} \widetilde{\mathscr{E}}_{\mathfrak{q}}\right) = 0$$

for all closed points $\mathfrak{q} \in \mathrm{Spec}(R)$ (such a model $(R, \mathcal{X}, \widetilde{\mathscr{E}})$ exists by the definition of ϕ).

Let $\mathscr{F}$ be any coherent sheaf on X; we may assume $\mathscr{F}$ extends to a coherent sheaf $\widetilde{\mathscr{F}}$ on $\mathcal{X}$. We claim that for closed points $\mathfrak{q} \in \mathrm{Spec}(R)$ with $p = \mathrm{char}(\kappa(\mathfrak{q})) > N_0$, $j \geq \mathrm{rk}(\mathscr{E})$, and for $m \gg 0$,

$$H^j(\mathcal{X}_{\mathfrak{q}}, \widetilde{\mathscr{E}}_{\mathfrak{q}}^{(p^m)} \otimes \widetilde{\mathscr{F}}_{\mathfrak{q}}) = 0\,,$$

which clearly suffices.

Consider the hypercohomology spectral sequence

$$E_2^{i,j} = H^i(X_{\mathfrak{q}}, \mathscr{H}^j(\Omega^\bullet_{\widetilde{\mathbb{E}_{\mathfrak{q}}}/X_{\mathfrak{q}}}(p)^{(p^T)} \otimes \widetilde{\mathscr{F}}_{\mathfrak{q}})) \implies \mathbb{H}^{i+j}(X_{\mathfrak{q}}, \Omega^\bullet_{\widetilde{\mathbb{E}_{\mathfrak{q}}}/X_{\mathfrak{q}}}(p)^{(p^T)} \otimes \widetilde{\mathscr{F}}_{\mathfrak{q}})\,.$$

By the first paragraph of this proof, the right hand side vanishes for $T \gg 0$, $i + j > \mathrm{rk}(\mathscr{E})$. On the other hand,

$$\mathscr{H}^j(\Omega^\bullet_{\widetilde{\mathbb{E}_{\mathfrak{q}}}/X_{\mathfrak{q}}}(p)^{(p^T)} \otimes \widetilde{\mathscr{F}}_{\mathfrak{q}}) = \begin{cases} \widetilde{\mathscr{E}}_{\mathfrak{q}}^{(p^{T+1})} \otimes \widetilde{\mathscr{F}}_{\mathfrak{q}} & \text{if } j = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the only non-zero differentials are on the E_2 page for degree reasons, i.e.

$$d_2^i : H^i(X_{\mathfrak{q}}, \widetilde{\mathscr{E}}_{\mathfrak{q}}^{(p^{T+1})} \otimes \widetilde{\mathscr{F}}_{\mathfrak{q}}) \to H^{i+2}(X_{\mathfrak{q}}, \widetilde{\mathscr{E}}_{\mathfrak{q}}^{(p^{T+1})} \otimes \widetilde{\mathscr{F}}_{\mathfrak{q}})\,.$$

For $T \gg 0$, and $i > \mathrm{rk}(\mathscr{E}) - 1$, these differentials are thus isomorphisms. Thus for $T \gg 0$, $i > \mathrm{rk}(\mathscr{E}) - 1$

$$H^i(X_{\mathfrak{q}}, \widetilde{\mathscr{E}}_{\mathfrak{q}}^{(p^{T+1})} \otimes \widetilde{\mathscr{F}}_{\mathfrak{q}}) = 0$$

by backwards induction on i, as desired (using that H^i vanishes for $i > \dim(X)$). □

REMARK 3.0.3. In [Ara04, Proof of Theorem 6.1], Arapura gives a proof of Theorem 3.0.2 using a resolution of $\mathscr{E}^{(p)}$ by Schur functors, from [CL74]. This complex is a subcomplex of $\Omega^\bullet_{\widetilde{\mathbb{E}}/X}(p)$. Arapura's proof requires the Kempf vanishing theorem as input; we are able to avoid this input because of our use of symmetric powers as opposed to other Schur functors.

4. Applications. Let k be a perfect field and X a variety over k. We say that X admits a Frobenius lift if X admits a flat lift X_2 to $W_2(k)$, and if absolute Frobenius lifts to X_2 as a $\mathbb{Z}/p^2\mathbb{Z}$-morphism. We first prove the following easy theorem:

THEOREM 4.0.1. *Let k be a perfect field of characteristic $p > 0$, and X a normal projective k-variety which admits a Frobenius lift; let $j : U \to X$ be the inclusion of the* *non-singular locus. Let $\mathscr{E}$ be a vector bundle on X. Then for all i and all $q > \phi(\mathscr{E})$,*

$$H^q(X, \mathscr{E} \otimes j_* \Omega^i_{U/k}) = 0\,.$$

PROOF. We follow the proof of [BTLM97, Theorem 3]. By [BTLM97, Theorem 2], there is a split monomorphism of abelian sheaves

$$0 \to \Omega^i_U \to F_* \Omega^i_U\,,$$

and hence pushing forward to X, a split monomorphism

$$0 \to j_* \Omega^i_U \to F_* j_* \Omega^i_U$$

(using that j and F commute). Tensoring with $\mathscr{E}^{(p^r)}$, we obtain injections

$$H^s(X, j_* \Omega^i_U \otimes \mathscr{E}^{(p^r)}) \hookrightarrow H^s(X, F_* j_* \Omega^i_U \otimes \mathscr{E}^{(p^r)}) \simeq H^s(X, j_* \Omega^i_U \otimes \mathscr{E}^{(p^{r+1})})$$

for all s, where the latter isomorphism uses the projection formula and the fact that F is affine. Now we are done by backwards induction on r, by the definition of $\phi(\mathscr{E})$. □

We may immediately conclude:

THEOREM 4.0.2 (Generalization of Bott-Steenbrink-Danilov). *Let X be a normal projective toric variety over a perfect field k, and $\mathscr{E}_1, \ldots, \mathscr{E}_m$ ample vector bundles on X. Let $j : U \hookrightarrow X$ be the inclusion of the smooth locus. Then if a_i, b_i are non-negative integers not all equal to zero, $q \geq 0$, and*

(1) $\mathrm{char}(k) = 0$*, or*

(2) $\mathrm{char}(k) = p > \sum_i \mathrm{rk}(\mathscr{E}_i)$*, and each $\mathscr{E}_i$ lifts to the canonical (toric) lift of X to $W_2(k)$,*

then

$$H^s\left(X, j_* \Omega^q_U \otimes \mathrm{Sym}^{a_1} \mathscr{E}_1 \otimes \cdots \otimes \mathrm{Sym}^{a_m}(\mathscr{E}_m) \otimes \bigwedge^{b_1} \mathscr{E}_1 \otimes \cdots \otimes \bigwedge^{b_m} \mathscr{E}_m\right) = 0$$

for $s > \sum_{i=1}^m (\mathrm{rk}(\mathscr{E}_i) - b_i)$.

PROOF. By a standard spreading-out argument, it suffices to prove the characteristic p statement. But this is immediate from Theorem 4.0.1 and Corollary 2.2.4. □

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