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COMBINATORIAL RICCI CURVATURE ON CELL-COMPLEX AND GAUSS-BONNNET THEOREM

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Abstract. In this paper, we introduce a new definition of the Ricci curvature on cell-complexes and prove the Gauss-Bonnnet type theorem for graphs and 2-complexes that decompose closed surfaces. The differential forms on a cell complex are defined as linear maps on the chain complex, and the Laplacian operates this differential forms. Our Ricci curvature is defined by the combinatorial Bochner-Weitzenböck formula. We prove some propositionerties of combinatorial vector fields on a cell complex.

1. Introduction. In this paper, we introduce a new definition of the Ricci curvature on cell-complexes and prove the Gauss-Bonnnet type theorem for graphs and 2-complexes that decompose closed surfaces. In the Riemanian geometry, the curvature plays an important role, and there are many results on the curvature on smooth manifolds. Especially the Gauss-Bonnet theorem is known as a fundamental property of a smooth closed manifold. The curvature on a cell complex was studied in many ways. R. Forman defined the Ricci curvature on a cell complex with the Bochner-Weitzenböck formula on the cochain. With this curvature he also showed Bochner's theorem, Myers' theorem and so on. For the curvature defined by angles, McCormick Paul [7] established the Gauss-Bonnet theorem, but in Forman's way the Gauss-Bonnet theorem does not hold.

R. Forman established the discrete Morse theory in [5]. He studied the function on a cell complex and the relation between critical cells and the homology of the cell complex. He also extended this thoery to the discrete Novikov-Morse theory, and in this theory he defined a differential form on the cell complex. This differential form is not the cochain of the cell complex but a linear map on the chain of the cell complex. In this paper we use these differential forms to define the Ricci curvature. We introduce the L^2 inner product on the space of combinatorial differential forms, and this inner product determines the Laplacian on combinatorial differential forms. Then the Ricci curvature is defined with the combinatorial Bochner-Weitzenböck formula for the combinatorial differential forms.

For the construction of the Bochner-Weitzenböck formula, we need the covariant of a 1-form. For this definition we present the 0- and 2-neighbor vector. These vectors are roughly regarded as "the pararell vectors". We define the covariant of a 1-form as the difference between the components of pararell vectors. Then for the cell complex with constant weights, the Ricci curvature is calculated as combinatorial computation,

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$$\mathrm{Ric}(\omega)(\tau > \sigma) = (2 - \#\{0 - \text{neighbor vector of } (\tau > \sigma)\})(\omega_\sigma^\tau)^2 .$$

This formula means that the Ricci curvature on the cell complex at the cell σ is determined by the structure around σ. For a graph or a 2-dimensional complex that decomposes a closed surface, the Ricci curvature for a unit vector at a vertex (resp. at a face f) is independent of the choice of the unit vector, and we define this value as the Gauss curvature g_v (resp. g_f) at the vertex v (resp. the face f). We have the following Gauss-Bonnet type theorems.

THEOREM 1.1. *Let G be a finite simple graph. Then we have*

$$\sum_v g_v = 2\chi(G), \tag{1.1}$$

where the sum is taken over all vertexes v and $\chi(G)$ is the Euler number of G.

THEOREM 1.2. *Let M be a 2-dimensional quasiconvex cell complex that decomposes a 2-dimensional closed smooth surface. Then we have*

$$\sum_v g_v + \sum_f g_f = 4\chi(M), \tag{1.2}$$

where the sums are taken over all vertexes v and all faces f respectively, and $\chi(M)$ is the Euler number of M.

We also present the vector field on a cell complex. This is defined as a dual of a combinatorial differential 1-form and we prove some properties of this vector field.

2. Combinatorial differential Forms.

2.1. Definition of Combinatorial differential Forms. In this section, we present a differential form on a cell-complex intoroduced in [3]. Let M be a regular cell complex of dimension n, and

$$0 \longrightarrow C_n(M) \xrightarrow{\partial} C_{n-1}(M) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(M) \longrightarrow 0$$

be the real cellular chain complex of M. We set

$$C_*(M) = \bigoplus_p C_p(M) . \tag{2.1}$$

A linear map $\omega : C_*(M) \to C_*(M)$ is said to be of *degree d* if for all $p = 1, \ldots, n$,

$$\omega(C_p(M)) \subset C_{p-d}(M) . \tag{2.2}$$

We say that a linear map ω of degree d is *local* if, for each p and each oriented p-cell α, $\omega(\alpha)$ is a linear combination of oriented $(p-d)$-cells that are faces of α.

DEFINITION 2.1. For $d \geq 0$, we say that a local linear map $\omega : C_*(M) \to C_*(M)$ of degree d is *a combinatorial differential d-form*, and we denote the space of combinatorial differential d-forms by $\Omega^d(M)$.

We define *the differential of combinatorial differential forms*

$$d : \Omega^d(M) \to \Omega^{d+1}(M) \tag{2.3}$$

as follows. For any $\omega \in \Omega^d(M)$ and any p-chain c, we define $(d\omega)(c) \in C_{p-(d+1)}(M)$ by

$$(d\omega)(c) = \partial(\omega(c)) - (-1)^d \omega(\partial c)\,. \tag{2.4}$$

That is,

$$d\omega = \partial \circ \omega - (-1)^d \omega \circ \partial\,. \tag{2.5}$$

LEMMA 2.2 ([3]). *The differential for combinatorial differential forms satisfies the following propositionerties.*

- $d(\Omega^d(M)) \subseteq \Omega^{(d+1)}(M)$.
- $d^2 = 0$.

This lemma determines the differential complex

$$\Omega^*(M) : 0 \longrightarrow \Omega^0(M) \xrightarrow{\ d\ } \Omega^1(M) \xrightarrow{\ d\ } \cdots \xrightarrow{\ d\ } \Omega^n(M) \longrightarrow 0.$$

THEOREM 2.3 ([3]). *The cohomology of this complex is isomorphic to the singular cohomology of M. That is,*

$$H^*(\Omega^*(M)) \cong H^*(M)\,. \tag{2.6}$$

2.2. Laplacian for Combinatorial differential Forms. Let us define an inner product on $C_*(M)$. For any two p-cells σ, σ', we set an inner product as

$$\langle \sigma, \sigma' \rangle = \delta_{\sigma,\sigma'} w_\sigma\,, \tag{2.7}$$

where $\delta_{\sigma,\sigma'}$ is the Kronecker's delta, that is, $\delta_{\sigma,\sigma'} = 1$ for $\sigma = \sigma'$ and the others are 0, and $w_\sigma > 0$ is a weight of a cell σ. We define the L^2 inner product for combinatorial differential forms. For two d-forms u, v, we set

$$\langle u, v \rangle = \sum_{\sigma} \frac{1}{w_\sigma} \langle u(\sigma), v(\sigma) \rangle, \tag{2.8}$$

where the sum is taken over all cells σ in M.
Let us consider the adjoint operator of differential with respect to the inner product,

$$d^* : \Omega^d(M) \to \Omega^{d-1}(M)\,. \tag{2.9}$$

That is, for a d-form u and a $(d-1)$-form v we have

$$\langle d^* u, v \rangle = \langle u, dv \rangle\,. \tag{2.10}$$

The space of combinatorial differential d-forms $\Omega^d(M)$ is a sub vector space of the space of linear maps of degree d on the chain $C_*(M)$. Then we set p as the projection on the space of linear maps of degree d on the chain $C_*(M)$ to the space of combinatorial differential d-forms $\Omega^d(M)$.

LEMMA 2.4. *For any d-form ω, we have*

$$d^* = p \circ (\partial^* \circ \omega - (-1)^{(d-1)} \omega \circ \partial^*)\,. \tag{2.11}$$

PROOF. For a p-dimensional cell τ and a $(p-d)$-dimensional cell σ which is a face of τ, we put a d-form e^τ_σ such that

$$e^\tau_\sigma(\tau) = \sigma\,, \tag{2.12}$$

and the value is 0 for other cells. They form a basis of d-forms as a real vector space.

Let $\tau, \sigma, \alpha, \beta$ be cells, and assume that e^τ_σ is a d-form and e^α_β is a $(d-1)$-form. Then we have

$$\begin{aligned}\langle d^* e^\tau_\sigma, e^\alpha_\beta\rangle &= \langle e^\tau_\sigma, de^\alpha_\beta\rangle \\ &= \langle e^\tau_\sigma, \partial \circ e^\alpha_\beta\rangle - (-1)^{(d-1)}\langle e^\tau_\sigma, e^\alpha_\beta \circ \partial\rangle\,.\end{aligned} \tag{2.13}$$

We put A as the right-hand side of equation (2.11), and have

$$\langle A e^\tau_\sigma, e^\alpha_\beta\rangle = \langle e^\tau_\sigma, \partial e^\alpha_\beta\rangle - (-1)^{(d-1)}\langle e^\tau_\sigma \circ \partial^*, e^\alpha_\beta\rangle\,. \tag{2.14}$$

Now we calculate the last term of equation (2.13),

$$\begin{aligned}\langle e^\tau_\sigma, e^\alpha_\beta \circ \partial\rangle &= \sum_{c:\text{cell}} \frac{1}{w_c}\langle e^\tau_\sigma(c), e^\alpha_\beta(\partial c)\rangle \\ &= \frac{1}{w_c}\langle \sigma, e^\alpha_\beta(\partial \tau)\rangle \\ &= \begin{cases} \frac{w_\sigma}{w_\tau}(-1)^{\tau>\alpha} & \text{for } \tau > \alpha,\ \sigma = \beta \\ 0 & \text{otherwise}\,. \end{cases}\end{aligned} \tag{2.15}$$

We calculate the last term of eqation (2.14),

$$\begin{aligned}\langle e^\tau_\sigma \circ \partial^*, e^\alpha_\beta\rangle &= \sum_{c:\text{cell}} \frac{1}{w_c}\langle e^\tau_\sigma(\partial^* c), e^\alpha_\beta(c)\rangle \\ &= \frac{1}{w_\alpha}\langle e^\tau_\sigma(\partial^* \alpha), \beta\rangle \\ &= \begin{cases} \frac{w_\sigma}{w_\tau}(-1)^{\tau>\alpha} & \text{for } \tau > \alpha,\ \sigma = \beta \\ 0 & \text{otherwise}\,. \end{cases}\end{aligned} \tag{2.16}$$

Then we have

$$d^* e^\tau_\sigma = A e^\tau_\sigma\,. \tag{2.17}$$

□

DEFINITION 2.5. We define *the Laplacian for combinatorial differential forms* by

$$\Delta = dd^* + d^* d\,. \tag{2.18}$$

THEOREM 2.6. *Let M be a finite regular cell-complex. Then we have*

$$\operatorname{Ker}(\Delta) \cong H^*(\Omega^*(M)) \cong H^*(M)\,. \tag{2.19}$$

PROOF. The Laplacian is a self-adjoint operator on the finite dimensional vector space $\Omega^*(M)$ and we have $\mathrm{Ker}(\varDelta) = \mathrm{Ker}(d) \cap \mathrm{Ker}(d^*)$. We consider a map

$$\begin{array}{ccc} \mathrm{Ker}(\varDelta) & \to & H^*(\Omega^*(M)) \\ u & \mapsto & [u]\,. \end{array} \tag{2.20}$$

This map is well-defined and injective. Next we prove that this map is surjective. The Laplacian has an eigen decomposition, and we denote the eigenvalues of the Laplacian by $\lambda_0, \ldots, \lambda_k$ are eigenvalues. Let u be a closed form. Then we have an eigen decomposition for the Laplacian

$$\varDelta u = \sum_{i=1,\ldots,k} \lambda_i u_i\,, \tag{2.21}$$

where u_i are eigen vectors for λ_i respectively such that $u = \sum_{i=0,\ldots,k} u_i$. Then putting

$$u' = \sum_{i=1,\ldots,k} \frac{d^* u_i}{\lambda_i}\,, \tag{2.22}$$

we have

$$u = u_0 + du'\,. \tag{2.23}$$

We conclude that the map (2.20) is an isomorphism. □

2.3. Combinatorial function on cell-complex. We realize a combinatorial 0-form as a function. We set $f \in \Omega^0(M)$, that is,

$$f : C^*(M) \to C^*(M)\,. \tag{2.24}$$

For any cell σ, we have

$$f(\sigma) = f_\sigma \sigma\,, \tag{2.25}$$

and we realize $f_\sigma \in \mathbf{R}$ as the value of the function f. For a p-dimensional cell τ, the derivative of f is

$$df(\tau) = \sum_{\sigma:\tau>\sigma} (f(\tau) - f(\sigma))(-1)^{\tau>\sigma}\sigma\,, \tag{2.26}$$

where the sum is taken over all $(p-1)$-dimensional cells σ that are faces of τ, and $(-1)^{\tau>\sigma}$ is the incidence number between τ and σ.

LEMMA 2.7. *Let M be a regular cell-complex and f a function on M, where we identify M with the set of cells of M. f is locally constant if and only if $df = 0$.*

2.4. Combinatorial 1-form on cell-complex. Let $\omega \in \Omega^1(M)$ be a combinatorial 1-form. For a p-dimensional cell τ, we set

$$\omega(\tau) = \sum_{\sigma:\tau>\sigma} \omega_\sigma^\tau (-1)^{\tau>\sigma}\sigma\,, \tag{2.27}$$

where the sum is taken over all $(p-1)$-dimensional cells σ that are faces of τ, and $(-1)^{\tau>\sigma}$ is the incidence number between τ and σ. We call the pair $(\tau > \sigma)$ *a vector* provided that a

p-dimensional cell σ is a face of $(p+1)$-dimensional cell τ. We say that ω has the value ω_σ^τ at the vector $(\tau > \sigma)$. For a p-dimensional cell μ, the derivative of ω is

$$d\omega(\mu) = \sum_{(\mu>\tau,\tau'>\sigma)} (\omega_\tau^\mu + \omega_\sigma^\tau - \omega_{\tau'}^\mu - \omega_\sigma^{\tau'})(-1)^{\mu>\tau}(-1)^{\tau>\sigma}\sigma, \tag{2.28}$$

where the sum is taken over all two $(p-1)$-dimensional cells τ, τ' and $(p-2)$-dimensional cells σ such that

$$\mu > \tau > \sigma,\ \mu > \tau' > \sigma,\ \tau \neq \tau'. \tag{2.29}$$

PROPOSITION 2.8. *For a combinatorial* 1*-form* ω*, we have* $d\omega = 0$ *if and only if*

$$\omega_\tau^\mu + \omega_\sigma^\tau = \omega_{\tau'}^\mu + \omega_\sigma^{\tau'} \tag{2.30}$$

for any p*-dimensional cell* μ*, any two* $(p-1)$ *dimensional cells* τ, τ' *and any* $(p-2)$*-dimensional cell* σ *such that*

$$\mu > \tau > \sigma, \mu > \tau' > \sigma, \tau \neq \tau'. \tag{2.31}$$

For any cell σ, the dual derrivative of ω is

$$d^*\omega(\sigma) = \left(-\sum_{\tau:\tau>\sigma} \frac{w_\sigma}{w_\tau}\omega_\sigma^\tau + \sum_{\rho:\rho<\sigma} \frac{w_\sigma}{w_\rho}\omega_\rho^\sigma\right)\sigma, \tag{2.32}$$

where the first sum is taken over all $(p+1)$-dimensional cells τ that have σ as a face, and the second sum is over all $(p-1)$-dimensional cells ρ that are the faces of σ.

3. Combinatorial Vector field on cell-complex.

DEFINITION 3.1. We call a linear map $X : C_*(M) \to C_{(*+1)}(M)$ *a combinatorial vector field* on a cell-complex M provided that for any cell σ any component of $X(\sigma)$ has σ as a face.

In the same manner as a 1-form, for a p-dimensional cell σ we set

$$X(\sigma) = \sum_{\tau:\tau>\sigma} X_\sigma^\tau (-1)^{\tau>\sigma}\tau, \tag{3.1}$$

where the sum is taken over all $(p-1)$-dimensional cells τ that have σ as a face, and $(-1)^{\tau>\sigma}$ is the incidence number between τ and σ.

For a 1-form ω and a vector field X on M, we define *the pairing*

$$\omega(X)(\sigma) = \sum_{\tau:\tau>\sigma} \omega_\sigma^\tau X_\sigma^\tau. \tag{3.2}$$

Then for a function f on M, we define

$$X(f)(\sigma) = df(X)(\sigma) = \sum_{\tau:\tau>\sigma} X_\sigma^\tau (f(\tau) - f(\sigma)). \tag{3.3}$$

DEFINITION 3.2. Let f be a function on M. We define *the gradient vector field* $\mathrm{grad}(f)$ *of f* by

$$\mathrm{grad}(f)^{\tau}_{\sigma} = \frac{w_\sigma}{w_\tau}(f(\tau) - f(\sigma)) . \tag{3.4}$$

Let X be a vector field on M. We also define *the divergence* $\mathrm{div}(X)$ *of f* by

$$\mathrm{div}(X)(\sigma) = -\sum_{\tau^{(p+1)}:\tau>\sigma} X^{\tau}_{\sigma} + \sum_{\rho^{(p-1)}:\rho>\sigma} X^{\sigma}_{\rho} . \tag{3.5}$$

We define the inner product for vector fields in the same manner as combinatorial differential forms, i.e. for two vector fields X, Y

$$\langle X, Y\rangle(\sigma) = \frac{1}{w_\sigma}\langle X(\sigma), Y(\sigma)\rangle = \sum_{\tau:\tau>\sigma} \frac{w_\tau}{w_\sigma} X^{\tau}_{\sigma} Y^{\tau}_{\sigma} . \tag{3.6}$$

Then we have

$$df(X) = \langle X, \mathrm{grad}(f)\rangle . \tag{3.7}$$

DEFINITION 3.3. For a function f on M, we define *the integral of f over M* by

$$\int_M f = \sum_{\sigma} f(\sigma) , \tag{3.8}$$

where the sum is taken over all cells of M.

THEOREM 3.4. *We assume that M is a finite regular cell-complex. Let f be a function on M and X a vector fieldon M. Then we have*

$$\int_M \langle \mathrm{grad}(f), X\rangle = \int_M f\, \mathrm{div}(X) . \tag{3.9}$$

PROOF. For a cell σ, we have

$$\begin{aligned} \sum_{\sigma}\langle \mathrm{grad}(f), X\rangle(\sigma) &= \sum_{\sigma}\sum_{\tau:\tau>\sigma} \frac{w_\tau}{w_\sigma} X^{\tau}_{\sigma} \cdot \frac{w_\sigma}{w_\tau}(f(\tau) - f(\sigma)) \\ &= \sum_{(\tau>\sigma)} X^{\tau}_{\sigma}(f(\tau) - f(\sigma)) \\ &= \sum_{\sigma} f(\sigma)\left(-\sum_{\tau^{(p+1)}:\tau>\sigma} X^{\tau}_{\sigma} + \sum_{\rho^{(p-1)}:\rho>\sigma} X^{\sigma}_{\rho} \right) \\ &= \int_M f\, \mathrm{div}(X) . \end{aligned} \tag{3.10}$$

□

COROLLARY 3.5. *For any vector field X on M we have*

$$\int_M \operatorname{div}(X) = 0\,. \tag{3.11}$$

PROOF. For a constant function f, the gradient of f vanishes. Then we take a constant function f as

$$f(\sigma) = 1 \tag{3.12}$$

for any cell σ. Then we have the corollary from Theorem 3.4. □

COROLLARY 3.6. *For any function f on M we have*

$$\int_M \Delta f = 0\,. \tag{3.13}$$

PROOF. For any cell σ,

$$\begin{aligned} \Delta f(\sigma) &= \left(- \sum_{\tau^{(p+1)}:\tau>\sigma} (f(\tau) - f(\sigma)) + \sum_{\rho^{(p-1)}:\rho>\sigma} (f(\sigma) - f(\rho)) \right) \\ &= \operatorname{div}(\operatorname{grad}(f))(\sigma)\,. \end{aligned} \tag{3.14}$$

Then we have the corollaryorally from the previous corollary. □

4. Combinatorial Ricci curvature.

4.1. Combinatorial Ricci curvature.

DEFINITION 4.1. Let M be a regular cell complex. We say that M is *quasiconvex* if for every two distinct $(p+1)$-cells τ_1 and τ_2 of M, if $\bar{\tau}_1 \cap \bar{\tau}_2$ contains a p-cell σ, then $\bar{\tau}_1 \cap \bar{\tau}_2 = \bar{\sigma}$. In particular this implies that $\bar{\tau}_1 \cap \bar{\tau}_2$ contains at most one p-cell.

Let M be a regular quasiconvex cell complex.

DEFINITION 4.2. Let τ, σ be two cells of M such that the dimension is $(p+1)$ and p respectively and σ is a face of τ.

We define *a* 0*-neighbor vector of* $(\tau > \sigma)$ as the following.

- The vectors $(\tau' > \sigma)$ for $(p+1)$-cells $\tau' \neq \tau$ such that there are no $(p+2)$-cell μ such that $\mu > \tau, \tau'$.
- The vectors $(\tau > \sigma')$ for p-cells $\sigma' \neq \sigma$ such that there are no $(p-1)$-cell ρ such that $\sigma, \sigma' > \rho$.

We define *a* 2*-neighbor vector of* $(\tau > \sigma)$ as the following.

- The vectors $(\mu > \tau')$ for $(p+1), (p+2)$-cells τ' and μ such that $\mu > \tau > \sigma$, $\mu > \tau' > \sigma$ and $\tau \neq \tau'$.

- The vectors $(\sigma' > \rho)$ for $(p-1), p$-cells ρ and σ' such that $\tau > \sigma > \rho$, $\tau > \sigma' > \rho$ and $\sigma \neq \sigma'$.

DEFINITION 4.3. For a combinatorial 1-form ω on M, we define *the combinatorial covariant derivative* as

$$|\nabla\omega|^2(\tau > \sigma) = \sum_{(\mu>\tau');\text{2-neighbor}} \frac{w_\sigma}{w_\mu}(\omega^\tau_\sigma - \omega^\mu_{\tau'})^2 + \sum_{(\sigma'>\rho);\text{2-neighbor}} \frac{w_\rho}{w_\tau}(\omega^\tau_\sigma - \omega^{\sigma'}_\rho)^2$$
$$+ \sum_{(\tau'>\sigma);\text{0-neighbor}} \frac{(w_\sigma)^2}{w_\tau w_{\tau'}}(\omega^\tau_\sigma + \omega^{\tau'}_\sigma)^2 + \sum_{(\tau>\sigma');\text{0-neighbor}} \frac{w_\sigma w_{\sigma'}}{(w_\tau)^2}(\omega^\tau_\sigma + \omega^\tau_{\sigma'})^2 ,$$

where the sums are taken over all 2-neighbor vectors and 0-neighbor vectors for $(\tau > \sigma)$ respectively.

DEFINITION 4.4. For a combinatorial 1-form ω on M, we define *the Laplacian of* $|\omega|^2$ as

$$\Delta^\flat|\omega|^2(\tau > \sigma)$$
$$= \sum_{(\mu>\tau');\text{2-neighbor}} \frac{w_\sigma}{w_\mu}((\omega^\tau_\sigma)^2 - (\omega^\mu_{\tau'})^2) + \sum_{(\sigma'>\rho);\text{2-neighbor}} \frac{w_\rho}{w_\tau}((\omega^\tau_\sigma)^2 - (\omega^{\sigma'}_\rho)^2)$$
$$+ \sum_{(\tau'>\sigma);\text{0-neighbor}} \frac{(w_\sigma)^2}{w_\tau w_{\tau'}}((\omega^\tau_\sigma)^2 + (\omega^{\tau'}_\sigma)^2) + \sum_{(\tau>\sigma');\text{0-neighbor}} \frac{w_\sigma w_{\sigma'}}{(w_\tau)^2}((\omega^\tau_\sigma)^2 + (\omega^\tau_{\sigma'})^2) ,$$

where the sums are taken over all 2-neighbor vectors and 0-neighbor vectors for $(\tau > \sigma)$ respectively.

This Laplacian is symmetric for vectors, hence we have

$$\sum_{(\tau>\sigma)} \Delta^\flat|\omega|^2(\tau > \sigma) = 0 , \tag{4.1}$$

where the sum is taken over all vectors.

DEFINITION 4.5. For a combinatorial 1-form ω, we define *the Ricci curvature on a vector* $(\tau > \sigma)$ as

$$\mathrm{Ric}(\omega)(\tau > \sigma) = \langle \Delta\omega, \omega\rangle(\tau > \sigma) - \frac{1}{2}|\nabla\omega|^2(\tau > \sigma) + \frac{1}{2}\Delta^\flat|\omega|^2(\tau > \sigma) .$$

LEMMA 4.6. *For any combinatorial* 1*-form* ω *on* M*, we have*

$$\langle \Delta\omega, \omega\rangle(\tau > \sigma) = - \sum_{(\mu>\tau');\text{2-neighbor}} \frac{w_\sigma}{w_\mu}\omega^\tau_\sigma\omega^\mu_{\tau'} - \sum_{(\sigma'>\rho);\text{2-neighbor}} \frac{w_\rho}{w_\tau}\omega^\tau_\sigma\omega^{\sigma'}_\rho$$
$$+ \sum_{(\tau'>\sigma);\text{0-neighbor}} \frac{(w_\sigma)^2}{w_\tau w_{\tau'}}\omega^\tau_\sigma\omega^{\tau'}_\sigma + \sum_{(\tau>\sigma');\text{0-neighbor}} \frac{w_\sigma w_{\sigma'}}{(w_\tau)^2}\omega^\tau_\sigma\omega^\tau_{\sigma'}$$
$$+ \sum_{(\mu>\tau');\text{2-neighbor}} \left(\frac{(w_\sigma)^2}{w_\tau w_{\tau'}} - \frac{w_\sigma}{w_\mu}\right)\omega^\tau_\sigma\omega^{\tau'}_\sigma + \sum_{(\sigma'>\rho);\text{2-neighbor}} \left(\frac{w_\sigma w_{\sigma'}}{(w_\tau)^2} - \frac{w_\rho}{w_\tau}\right)\omega^\tau_\sigma\omega^\tau_{\sigma'}$$

$$+(\#\{\text{2-neighbor vector}\}+2)\left(\frac{w_\sigma}{w_\tau}\right)^2(\omega_\sigma^\tau)^2 .$$

PROOF. For a 1-form ω, we have

$$(d^*d\omega)_\sigma^\tau = \sum_{(\mu>\tau');\text{2-neighbor}} \frac{w_\mu}{w_\tau}(\omega_\tau^\mu+\omega_\sigma^\tau-\omega_{\tau'}^\mu-\omega_\sigma^{\tau'}) + \sum_{(\sigma'>\rho);\text{2-neighbor}} \frac{w_\sigma}{w_\rho}(\omega_\sigma^\tau+\omega_\rho^\sigma-\omega_{\sigma'}^\tau-\omega_\rho^{\sigma'}) \tag{4.2}$$

$$(dd^*\omega)_\sigma^\tau = -\sum_{\mu>\tau}\frac{w_\mu}{w_\tau}\omega_\tau^\mu + \sum_{\tau>\sigma'}\frac{w_\tau}{w_{\sigma'}}\omega_{\sigma'}^\tau + \sum_{\tau'>\sigma}\frac{w_{\tau'}}{w_\sigma}\omega_\sigma^{\tau'} - \sum_{\sigma>\rho}\frac{\sigma}{\rho}\omega_\rho^\sigma .$$

Then the Laplacian of ω is

$$\begin{aligned}(\Delta\omega)_\sigma^\tau = &-\sum_{(\mu>\tau');\text{2-neighbor}}\frac{w_\mu}{w_\tau}\omega_{\tau'}^\mu - \sum_{(\sigma'>\rho);\text{2-neighbor}}\frac{w_\sigma}{w_\rho}\omega_\rho^{\sigma'}\\ &+\sum_{(\tau'>\sigma);\text{0-neighbor}}\frac{w_{\tau'}}{w_\sigma}\omega_\sigma^{\tau'} + \sum_{(\tau>\sigma');\text{0-neighbor}}\frac{w_\tau}{w_{\sigma'}}\omega_{\sigma'}^\tau\\ &+\sum_{(\mu>\tau');\text{2-neighbor}}\left(\frac{w_\sigma}{w_{\tau'}}-\frac{w_\tau}{w_\mu}\right)\omega_\sigma^{\tau'} + \sum_{(\sigma'>\rho);\text{2-neighbor}}\left(\frac{w_{\sigma'}}{w_\tau}-\frac{w_\rho}{w_\sigma}\right)\omega_{\sigma'}^\tau\\ &+(\#\{\text{2-neighbor vector}\}+2)\frac{w_\sigma}{w_\tau}(\omega_\sigma^\tau) .\end{aligned}$$

Takeing the inner product of ω and $\Delta\omega$, we have the lemma. □

THEOREM 4.7. *Let M be a regular quasiconvex cell-complex, and $(\tau>\sigma)$ a vector on M. For a combinatorial 1-form ω on M, the Ricci curvature* $\mathrm{Ric}(\omega)$ *is reprenseted by*

$$\mathrm{Ric}(\omega)(\tau>\sigma) = (2-\#\{\text{0-neighbor vector of }(\tau>\sigma)\})\left(\frac{w_\sigma}{w_\tau}\right)^2(\omega_\sigma^\tau)^2 \tag{4.3}$$
$$+\sum_{(\mu>\tau');\text{2-neighbor}}\left(\frac{(w_\sigma)^2}{w_\tau w_{\tau'}}-\frac{w_\sigma}{w_\mu}\right)\omega_\sigma^\tau\omega_\sigma^{\tau'} + \sum_{(\sigma'>\rho);\text{2-neighbor}}\left(\frac{w_\sigma w_{\sigma'}}{(w_\tau)^2}-\frac{w_\rho}{w_\tau}\right)\omega_\sigma^\tau\omega_{\sigma'}^\tau .$$

In particular, with the assumption that the weight of each cell is constant, we have

$$\mathrm{Ric}(\omega)(\tau>\sigma) = (2-\#\{\text{0-neighbor vector of }(\tau>\sigma)\})\left(\frac{w_\sigma}{w_\tau}\right)^2(\omega_\sigma^\tau)^2 .$$

PROOF. With the above lemma, we have

$$\begin{aligned}&2\langle\Delta\omega,\omega\rangle(\tau>\sigma)\\ &= -\sum_{(\mu>\tau');\text{2-neighbor}}\frac{w_\sigma}{w_\mu}((\omega_\sigma^\tau-\omega_{\tau'}^\mu)^2-((\omega_\sigma^\tau)^2-(\omega_{\tau'}^\mu)^2)-2(\omega_\sigma^\tau)^2)\\ &\quad-\sum_{(\sigma'>\rho);\text{2-neighbor}}\frac{w_\rho}{w_\tau}((\omega_\sigma^\tau-\omega_\rho^{\sigma'})^2-((\omega_\sigma^\tau)^2-(\omega_\rho^{\sigma'})^2)-2(\omega_\sigma^\tau)^2)\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{(\tau'>\sigma);0\text{-neighbor}} \frac{(w_\sigma)^2}{w_\tau w_{\tau'}}((\omega_\sigma^\tau + \omega_\sigma^{\tau'})^2 - ((\omega_\sigma^\tau)^2 + (\omega_\sigma^{\tau'})^2) - 2(\omega_\sigma^\tau)^2) \\
&+ \sum_{(\tau>\sigma');0\text{-neighbor}} \frac{w_\sigma w_{\sigma'}}{(w_\tau)^2}((\omega_\sigma^\tau + \omega_{\sigma'}^\tau)^2 - ((\omega_\sigma^\tau)^2 + (\omega_{\sigma'}^\tau)^2) - (\omega_\sigma^\tau)^2) \\
&+ 2(\#\{\text{2-neighbor vector}\} + 2)(\omega_\sigma^\tau)^2 \\
&+ 2 \sum_{(\mu>\tau');2\text{-neighbor}} \left(\frac{(w_\sigma)^2}{w_\tau w_{\tau'}} - \frac{w_\sigma}{w_\mu}\right)\omega_\sigma^\tau \omega_\sigma^{\tau'} + 2 \sum_{(\sigma'>\rho);2\text{-neighbor}} \left(\frac{w_\sigma w_{\sigma'}}{(w_\tau)^2} - \frac{w_\rho}{w_\tau}\right)\omega_\sigma^\tau \omega_{\sigma'}^\tau \\
=&|\nabla\omega|^2(\tau > \sigma) - \Delta^\flat|\omega|^2(\tau > \sigma) + 2(2 - \#\{\text{0-neighbor vector}\})(\omega_\sigma^\tau)^2 . \\
&+2 \sum_{(\mu>\tau');2\text{-neighbor}} \left(\frac{(w_\sigma)^2}{w_\tau w_{\tau'}} - \frac{w_\sigma}{w_\mu}\right)\omega_\sigma^\tau \omega_\sigma^{\tau'} + 2 \sum_{(\sigma'>\rho);2\text{-neighbor}} \left(\frac{w_\sigma w_{\sigma'}}{(w_\tau)^2} - \frac{w_\rho}{w_\tau}\right)\omega_\sigma^\tau \omega_{\sigma'}^\tau .
\end{aligned}
$$

Then we have the equation (4.3). □

5. Combinatorial Gauss-Bonnet Theorem.

5.1. Gauss-Bonnet Theorem for graph. Let $G = (V, E)$ be a finite simple graph, where V is the set of vertexes and E the set of edges. We realize G as 1-dimensional cell complex, i.e. vertexes are 0-cells and edges are 1-cells.

LEMMA 5.1. *Let v and e be a vertex and an edge of G respectively such that $e > v$. We take a combinatorial 1-form ω on G. Then we have*

$$
\mathrm{Ric}(\omega)(e > v) = (2 - \deg(v)) \left(\frac{w_v}{w_e}\right)^2 (\omega_v^e)^2, \tag{5.1}
$$

where $\deg(v)$ *is the degree of* v.

PROOF. Let v and e be a vertex and an edge of G respectively such that $e > v$. For the definition of 0-neighbor vector, we find two vectors $(e' > v)$ and $(e > v')$ such that there are no 2-cell f such that $f > e, e'$, and there are no (-1)-cell ρ such that $v, v' > \rho$ respectively.

Since there are no 2-cells in G, the vector $(e' > v)$ is a 0-neighbor vector for any e' which has the vertex v except for the edge e. Hence there are exactly $\deg(v) - 1$ edges that satisfy the above condition. Since there are no (-1)-cells, there is only one vertex v' such that $(e > v')$ is a 0-neighbor vector for $(e > v)$. Then the number of 0-neighbor vectors for $(e > v)$ is $\deg(v)$.

With the definition of a 2-neighbor vecotor, there are not 2-neighbor vectors for the vector $(e > v)$. We have the lemma from the equation (4.3). □

With this lemma, we immediately have the following lemma.

LEMMA 5.2. *We take a combinatorial 1-form ω on G such that for any vertex v*

$$
\sum_{e;e>v} \left(\frac{w_v}{w_e}\right)^2 (\omega_v^e)^2 = 1, \tag{5.2}
$$

where the sum is taken over all edges e such that $e > v$. Then for any vertex v we have

$$\sum_{e;e>v} \mathrm{Ric}(\omega)(e > v) = 2 - \deg(v)\,. \tag{5.3}$$

For a smooth surface the Gauss curvature at a point p equal to the Ricci curvature for a unit vector at p. The following definition is an analogue to this fact.

DEFINITION 5.3. We define *the Gauss curvature for a vertex* v by

$$g_v = 2 - \deg(v)\,. \tag{5.4}$$

PROOF OF THEOREM 1.1. From the definition of the Gauss curvature, we have

$$\begin{aligned}\sum_v g_v &= \sum_v (2 - \deg(v)) \\ &= 2\#V - \sum_v \deg(v) \\ &= 2\#V - 2\#E \\ &= 2\chi(G)\,.\end{aligned} \tag{5.5}$$

□

5.2. Gauss-Bonnet Theorem for 2-complex. Let M be a 2-dimensional quasiconvex cell complex that decomposes a 2-dimensional closed smooth surface.

LEMMA 5.4. *Let v and e be a vertex and an edge on M respectively such that $e > v$. We take a combinatorial 1-form ω on M. Then we have*

$$\mathrm{Ric}(\omega)(e > v) = (4 - \deg(v)) \left(\frac{w_v}{w_e}\right)^2 (\omega_v^e)^2 + \sum_{(f>e');\text{2-neighbor}} \left(\frac{(w_v)^2}{w_e w_{e'}} - \frac{w_v}{w_f}\right) \omega_v^e \omega_v^{e'}\,,$$

where $\deg(v)$ *is the degree of v and the sum is taken over all 2-neighbor vectors of the vector* $(e > v)$.

PROOF. Let v and e be a vertex and an edge on M respectively such that $e > v$. For the definition of a 0-neghbor vecotor, we find the vectors $(e' > v)$ and $(e > v')$ such that there are no 2-cell f such that $f > e, e'$ and there are no (-1)-cell ρ such that $v, v' > \rho$ respectively.

For the edge e there are exactly two faces that have the edge e. For exactly two edges e' the vectors $(e' > v)$ are not 0-neghbor vecotors of $(e > v)$. The number of 0-neghbor vecotors of $(e > v)$ is $\deg(v) - 3$. Since there are no (-1)-cells in M, there is only one vertex v' such that $(e > v')$ is a 0-neighbor vector for $(e > v)$. Then the number of 0-neighbor vecotors for $(e > v)$ is $\deg(v) - 2$. We have the lemma from the equation (4.3). □

For a vertex v we consider the sum

$$\mathrm{Ric}(\omega)(v) := \sum_{e;e>v} \mathrm{Ric}(\omega)(e > v)\,, \tag{5.6}$$

where the sum is taken over all edges e that have the vertex v. This is a quadratic form for real bases $\{\frac{w_v}{w_e}\omega_v^e\}_{e>v}$. The trace with this bases is $\deg(v)(4 - \deg(v))$. For a smooth manifold the scalar curvature is a trace of the Ricci curvature. The next definition is an analogue to this fact.

DEFINITION 5.5. We define *the scalar curvature $S(v)$ at a vertex v* as

$$S(v) = \operatorname{trace}\operatorname{Ric}(\omega)(v) = \deg(v)(4 - \deg(v))\,. \tag{5.7}$$

For a smooth surface the scalar curvature is the twice of the Gauss curvature. The next definition is analogue to this fact.

DEFINITION 5.6. We define *the Gauss curvature at a vetex v* as

$$g_v = \frac{S(v)}{\deg(v)} = 4 - \deg(v)\,. \tag{5.8}$$

LEMMA 5.7. *Let e and f be an edge and a face of M respectively such that $f > e$. We take a combinatorial 1-form ω on M. Then we have*

$$\operatorname{Ric}(\omega)(f > e) = (4 - \deg(f))\left(\frac{w_e}{w_f}\right)^2 (\omega_e^f)^2 + \sum_{(e'>v);2\text{-neighbor}} \left(\frac{w_e w_{e'}}{(w_f)^2} - \frac{w_v}{w_f}\right)\omega_e^f \omega_{e'}^f\,,$$

where $\deg(f)$ *is the degree of f, that is, the number of edges of f and the sum is taken over the all 2-neighbor vector of the vector $(f > e)$.*

PROOF. Let e and f be an edge and a face of M respectively such that $f > e$. For the definition of a 0-neghbor vecotor, we find two vectors $(f' > e)$ and $(f > e')$ such that there are no 3-cell σ such that $\sigma > f, f'$ and there are no 0-cell v such that $e, e' > v$ respectively.

For the edge e there are exactly two faces that have e as an edge. Then for only one face f' the vector $(f' > e)$ is a 0-neighbor vector of $(f > e)$. For edges of the face f, exactly two edges e_1, e_2 intersect with the edge e, then the two vectors $(f > e_1)$ and $(f > e_2)$ are not 0-neighbor vectors of $(f > e)$. For the other edges e' of the face f, the vector $(f > e')$ is a 0-neighbor vector of $(f > e)$. Then the number of 0-neighbor vecotors for $(e > v)$ is $\deg(f) - 2$. We have the lemma from the eqation (4.4). □

For a face f we consider the next sum,

$$\operatorname{Ric}(\omega)(f) := \sum_{e;f>e} \operatorname{Ric}(\omega)(f > e)\,, \tag{5.9}$$

where the sum is taken over the all edges e contained in the boundary of the face f.

DEFINITION 5.8. We define *the scalar curvature $S(f)$ at a face f* as

$$S(f) = \operatorname{trace}\operatorname{Ric}(\omega)(f) = \deg(f)(4 - \deg(f))\,. \tag{5.10}$$

We define *the Gauss curvature at a face f* as

$$g_f = \frac{S(f)}{\deg(f)} = 4 - \deg(f)\,. \tag{5.11}$$

If all weights of cells of M are constants, we conclude the following lemma that is an analogue to the smooth surface.

LEMMA 5.9. *Let M be a 2-dimensional quasiconvex cell complex that decomposes a 2-dimensional closed smooth surface. We assume that all weights of cells of M are constants.*

(1) *Let v be a vertex of M. We take a combinatorial 1-form ω on M such that*

$$\sum_{e;e>v} (\omega_v^e)^2 = 1\,, \tag{5.12}$$

where the sum is taken over all edges e such that $e > v$. Then we have

$$\sum_{e;e>v} \mathrm{Ric}(\omega)(e > v) = 4 - \deg(v) = g_v\,. \tag{5.13}$$

(2) *Let f be a face of M. We take a combinatorial 1-form ω on M such that*

$$\sum_{e;f>e} (\omega_e^f)^2 = 1\,, \tag{5.14}$$

where the sum is taken over all edges e such that $f > e$. Then we have

$$\sum_{e;f>e} \mathrm{Ric}(\omega)(f > e) = 4 - \deg(f) = g_f\,. \tag{5.15}$$

PROOF OF THEOREM 1.2. We denote V, E and F by the numbers of vertexes, edges and faces in M respectively.

From the definition of the Gauss curvature, we have

$$\begin{aligned} \sum_v g_v + \sum_f g_f =& \sum_v (4 - \deg(v)) + \sum_f (4 - \deg(f)) \\ =& 4V - \sum_v \deg(v) + 4F - \sum_f \deg(f) \\ =& 4V - 2E + 4F - 2E \\ =& 4(V - E + F) \\ =& 4\chi(M)\,. \end{aligned} \tag{5.16}$$

□

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