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GENERALIZED KÄHLER EINSTEIN METRICS AND UNIFORM STABILITY FOR TORIC FANO MANIFOLDS

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Abstract. We give a complete criterion for the existence of generalized Kähler Einstein metrics on toric Fano manifolds from view points of a uniform stability in a sense of GIT and the properness of a functional on the space of Kähler metrics.

1. Introduction. In his paper [9], Mabuchi extended the notion of Kähler Einstein metrics for Fano manifolds with non vanishing Futaki character. In this paper, we call them generalized Kähler Einstein metrics. Let X be an n-dimensional Fano manifold and $\omega \in 2\pi c_1(X)$ be a Kähler metric. We denote the Ricci form for ω by $\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\overline{\partial}\log\det\omega^n$. The Ricci potential f_ω for ω is the function satisfying

$$\mathrm{Ric}(\omega) - \omega = \sqrt{-1}\partial\overline{\partial} f_\omega \quad \text{and} \quad \int_X e^{f_\omega}\omega^n = \int_X \omega^n .$$

Then $\omega = \sqrt{-1}g_{i\bar{j}}dz^i \wedge d\bar{z}^j$ is called *generalized Kähler Einstein* if the complex gradient vector field of $1 - e^{f_\omega}$ is holomorphic, that is,

$$\overline{\partial}\left(g^{i\bar{j}}\frac{\partial(1-e^{f_\omega})}{\partial\bar{z}^j}\frac{\partial}{\partial z^i}\right) = 0 .$$

Generalized Kähler Einstein metrics do not necessarily exist. Mabuchi [9] introduced an obstruction for the existence of this metric. Let

$$\tilde{\mathfrak{h}}_\omega := \left\{ f \in C^\infty_{\mathbb{R}}(X) \,\middle|\, \overline{\partial}\left(g^{i\bar{j}}\frac{\partial f}{\partial\bar{z}^j}\frac{\partial}{\partial z^i}\right) = 0 \text{ and } \int_X f\omega^n = 0 \right\} ,$$

and let $\mathrm{pr} : L^2(X,\omega) \to \tilde{\mathfrak{h}}_\omega$ be the L^2-orthogonal projection where $L^2(X,\omega)$ is the Hilbert space of real L^2-functions on (X,ω). Then we can define the holomorphic invariant α_X as follows:

$$\alpha_X := \max_X \mathrm{pr}(1 - e^{f_\omega}) .$$

Indeed α_X is independent of the choice of ω. Furthermore if X admits generalized Kähler Einstein metrics then $\alpha_X < 1$ must hold. See [9] for more details.

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For general Fano manifolds, necessary and sufficient conditions for the existence of generalized Kähler Einstein metics are not known. For toric Fano manifolds, Yao introduced the notion of the *relative Ding stability* in the sense of geometric invariant theory. (After the present paper was submitted on the arXiv, Yao renamed this stability the *uniform relative Ding stability* in his replaced paper [11] independently. His definition of the uniform relative Ding stability is equivalent to the author's one in the present paper.) This stability condition is equivalent to $\alpha_X < 1$, and implies the existence of generalized Kähler Einstein metrics (Yao called them *Mabuchi metrics*). See [11] for more details.

In this paper, we introduce a notion of stability called the *uniform relative Ding stability* for toric Fano manifolds, and establish the complete criterion for the existence of generalized Kähler Einstein metrics from the viewpoint of the properness of a functional on the space of Kähler metrics. These lines to attack the existence problem of generalized Kähler Einstein metrics are expected to extend for general Fano manifolds.

The uniform relative Ding stability can be regarded as a generalization of the uniform Ding stability for Kähler Einstein metrics defined by Berman [2]. For more recent developments around the uniform stability and the Ding stability, see for instance Hisamoto [8] (for toric constant scalar curvature Kähler metrics), Boucksom-Hisamoto-Jonsson [4] (for constant scalar curvature Kähler metrics), Dervan [6] (for twisted constant scalar curvature Kähler metrics) and Fujita [7] (for the volume of Kähler Einstein Fano manifolds).

The following is our main theorem.

THEOREM 1.1. *Let X be a toric Fano manifold. Then the following conditions are all equivalent.*

1. *X admits a unique toric invariant generalized Kähler Einstein metric.*
2. $\alpha_X < 1$.
3. *X is uniformly relative Ding stable (Definiton* 3.1*).*
4. *The modified Ding functional is proper (Definition* 4.1*).*

The proof will be done by showing $2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$. Recall that $1 \Rightarrow 2$ is the result of Mabuchi [9] for general Fano manifolds. Recall also that $2 \Rightarrow 1$ is the result of Yao [11]. Therefore the author's main contribution is the discussion of the conditions 3 and 4.

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2. Generalized Kähler Einstein metrics on toric Fano manifolds. Let $\triangle$ be an open set in $\mathbb{R}^n$ such that the closure $\overline{\triangle}$ is a reflexive integral Delzant polytope, and $X_\triangle$ be the corresponding toric Fano manifold with the open dense $(\mathbb{C}^*)^n$-action. Note that $0 \in \mathbb{R}^n$ is the only integral point in $\triangle$. In the following, we will just write X as $X_\triangle$ for simplicity. Let $(\mathbb{C}^*)^n = (S^1)^n \times \mathbb{R}^n$ be the standard decomposition, and let $\xi_i := \log |z_i|^2$ be the coordinate of $\mathbb{R}^n$ where $\{z_i\}$ is the standard coordinate of $(\mathbb{C}^*)^n$. The following lemma is well-known.

LEMMA 2.1. *Let ω be an $(S^1)^n$-invariant Kähler metric on X. Then ω is determined by a smooth convex function (called Kähler potential) $\phi = \phi(\xi_1, \ldots, \xi_n)$ on $\mathbb{R}^n$, that is, $\omega = \sqrt{-1}\partial\overline{\partial}\phi$ on $(\mathbb{C}^*)^n$. Moreover its gradient $\nabla\phi$ gives a diffeomorphism from $\mathbb{R}^n$ to $\triangle$.*

Let $\omega \in 2\pi c_1(X)$ be an $(S^1)^n$-invariant reference Kähler metric and ϕ_0 is the Kähler potential of ω. Let u_0 be the Legendre dual of ϕ_0, that is,

$$u_0(x) = \xi \cdot x - \phi_0(\xi) \quad \text{and} \quad x = \nabla\phi_0(\xi)\,.$$

Then u_0 is a smooth convex function (called *symplectic potential*) on $\triangle$. Let

$$C := \left\{ u \in C^0(\overline{\triangle}) \,\middle|\, u \text{ is convex on } \triangle \quad \text{and} \quad u - u_0 \in C^\infty(\overline{\triangle}) \right\}\,.$$

Abreu [1] showed that there is the bijection between C and the space of $(S^1)^n$-invariant Kähler metrics in $[\omega]$ through the Legendre duality.

Let us consider the *modified Ding functional* [11]

$$\mathcal{D}(u) = -\log \int_{\mathbb{R}^n} e^{-(\phi - \inf\phi)} d\xi - u(0) + \int_\triangle u \cdot l dx \quad \text{for } u \in C\,,$$

where ϕ is the Legendre dual of u and l is the unique affine linear function on $\triangle$ such that

$$-u(0) + \int_\triangle u \cdot l dx = 0 \quad \text{for any affine linear } u\,.$$

The critical point of the modified Ding functional is the generalized Kähler Einstein metric. Indeed the derivative of $\mathcal{D}$ at ϕ is (Note that $\delta\phi = -\delta u(\nabla\phi)$ and $u(0) = -\inf\phi$.)

$$\delta\mathcal{D} = \int_{\mathbb{R}^n} \delta\phi \left(\frac{e^{-\phi}}{\int_{\mathbb{R}^n} e^{-\phi}} - l(\nabla\phi)\det(\nabla^2\phi) \right) d\xi, \tag{2.1}$$

and we can see that a smooth solution ϕ of

$$\frac{e^{-\phi}}{\int_{\mathbb{R}^n} e^{-\phi}} = l(\nabla\phi)\det(\nabla^2\phi) \tag{2.2}$$

such that its gradient $\nabla\phi$ gives a diffeomorphism from $\mathbb{R}^n$ to $\triangle$ defines the generalized Kähler Einstein metric (cf. [11, Theorem 14]).

3. Uniform relative Ding stability. Notation of this section is the same as in the previous sections. Let us consider the *relative Ding-Futaki invariant* [11]

$$\mathcal{I}(u) := -u(0) + \int_\triangle u \cdot l dx \quad \text{for } u \in C\,.$$

This is nothing but the linear term of the modified Ding functional. We introduce the uniform relative Ding stability by the following uniform lower bound estimate of $\mathcal{I}$.

DEFINITION 3.1. A toric Fano manifold X is *uniformly relative Ding stable* if there is a constant $\lambda > 0$ such that

$$\mathcal{I}(u) \geq \lambda \int_\triangle u dx \tag{3.1}$$

holds for every *normalized* convex function $u \in \tilde{C} := \{ u \in C \mid u \geq u(0) = 0 \}$.

Although we can also define another stability by the lower bound estimate by $\lambda \int_{\partial\triangle} u d\sigma$ as the analog of the uniform K-stability defined by Donaldson [6], this condition is stronger than that of Definition 3.1. Indeed there is the uniform estimate as follows [6]: there is a constant $C > 0$ such that $\int_{\triangle} u dx \leq C \int_{\partial\triangle} u d\sigma$ holds for all $u \in \tilde{C}$. Our uniform stability follows from the condition $\alpha_X < 1$ immediately.

LEMMA 3.2. *Let X be a toric Fano manifold. If the condition $\alpha_X < 1$ holds then X is uniformly relative Ding stable.*

PROOF. As Yao pointed out in [11], the invariant α_X is given explicitly by

$$\alpha_X = \max_{\overline{\triangle}}\{1 - |\triangle| l\},$$

where $|\triangle|$ is the volume of $\triangle$. Therefore for any $u \in \tilde{C}$, we have

$$\mathcal{I}(u) = \int_{\triangle} u \cdot l dx \geq \frac{1 - \alpha_X}{|\triangle|} \int_{\triangle} u dx .$$

□

4. Properness of the modified Ding functional. First we recall a functional on the space of Kähler metrics. Let (X, ω) be an n-dimensional compact Kähler manifold. Let G be any maximal compact subgroup of $\mathrm{Aut}(X)$. If ω is G-invariant then we define

$$\mathcal{M}_G(\omega) = \left\{\phi \in C^\infty(X) \,\middle|\, \omega_\phi := \omega + \sqrt{-1}\partial\overline{\partial}\phi > 0 \text{ and } \phi \text{ is } G\text{-invariant.}\right\}.$$

For any $\phi \in \mathcal{M}_G(\omega)$, we let

$$J(\phi) = \int_0^1 \int_X \dot{\phi}_t(\omega^n - \omega_{\phi_t}^n)\frac{1}{n!} \wedge dt ,$$

where $\{\phi_t\}_{t\in[0,1]}$ is a path in $\mathcal{M}_G(\omega)$ connecting 0 to ω.

In general, the properness of a functional on $\mathcal{M}_G(\omega)$ is defined as a uniform lower bound estimate by the functional J [10, Definition 6.8]. In the case of previous sections where X is toric Fano manifold and $\omega \in 2\pi c_1(X)$ is $(S^1)^n$-invariant reference Kähler metirc, there exists a uniform constant $C > 0$ such that

$$\left|J(\phi) - \int_{\triangle} u dx\right| \leq C \tag{4.1}$$

holds for all $u \in \tilde{C}$ [12, Lemma 2.2], where ϕ is the $(S^1)^n$-invariant function associated from the Kähler metric defined by u . Thus we can define the properness of the modified Ding functional for toric Fano manifolds as follows:

DEFINITION 4.1 (cf. [10, Definition 6.8]). In the same notation as in Section 2, the modified Ding functional $\mathcal{D}$ is *proper* if there exists an increasing function $\mu(r)$ on $\mathbb{R}$ with the property

$$\lim_{r\to\infty} \mu(r) = \infty ,$$

such that

$$\mathcal{D}(u) \geq \mu\Big(\int_{\triangle} u dx\Big) \tag{4.2}$$

holds for all $u \in \tilde{C}$.

Then we have the following:

PROPOSITION 4.2. *Let X be a toric Fano manifold. Suppose X is uniform relative Ding stable, that is, there exists a constant $\lambda > 0$ satisfying $\mathcal{I}(u) \geq \lambda \int_{\triangle} u dx$ for all $u \in \tilde{C}$. Then there exists $\delta > 0$ depending only on λ such that*

$$\mathcal{D}(u) \geq \delta \int_{\triangle} u dx - C_\delta$$

holds for all $u \in \tilde{C}$. In particular the modified Ding functional is proper.

PROOF. For a fixed $v_0 \in C$ and its Legendre dual ψ_0, we define a smooth function A by

$$A(\nabla\psi_0) = \frac{e^{-\psi_0}}{\int_{\mathbb{R}^n} e^{-\psi_0}} \det(\nabla^2\psi_0)^{-1} .$$

Then v_0 minimizes the following functional on C:

$$\mathcal{D}_A(u) := -\log \int_{\mathbb{R}^n} e^{-(\phi - \inf\phi)} d\xi - u(0) + \int_{\triangle} u \cdot A dx$$

where ϕ is the Legendre dual of u. Indeed v_0 is the critical point of $\mathcal{D}_A$ (cf. (2.1)), and the nonlinear term of $\mathcal{D}_A$ is convex on any affine line in C (cf. [3, Proposition 2.15]). Let $-C_0 := \mathcal{D}_A(v_0)$ and let

$$\mathcal{I}_A(u) := -u(0) + \int_{\triangle} u \cdot A dx .$$

Now we compute the difference between $\mathcal{I}$ and $\mathcal{I}_A$. Taking a $\delta > 0$ and for any $u \in \tilde{C}$, we have

$$\begin{aligned} |\mathcal{I}(u) - \mathcal{I}_A(u)| &\leq C \int_{\triangle} u dx \\ &= C\left((1+\delta)\int_{\triangle} u dx - \delta \int_{\triangle} u dx\right) \\ &\leq C_{\delta,\lambda}\mathcal{I}(u) - C\delta \int_{\triangle} u dx. \end{aligned}$$

The last inequality follows from the assumption of the uniform relative Ding stability. It follows that

$$(1 + C_{\delta,\lambda})\mathcal{I}(u) \geq \mathcal{I}_A(u) + C\delta \int_{\triangle} u dx .$$

Then we have

$$\mathcal{D}(u) = -\log \int_{\mathbb{R}^n} e^{-(\phi - \inf\phi)} d\xi + \mathcal{I}(u) \tag{4.3}$$

$$\geq -\log \int_{\mathbb{R}^n} e^{-(\phi - \inf \phi)} d\xi + \mathcal{I}_A\left(\frac{u}{1 + C_{\delta,\lambda}}\right) + \frac{C\delta}{1 + C_{\delta,\lambda}} \int_\triangle u dx$$

$$\geq -\log \int_{\mathbb{R}^n} \exp\left(-\frac{\phi - \inf \phi}{1 + C_{\delta,\lambda}}\right) d\xi + \mathcal{I}_A\left(\frac{u}{1 + C_{\delta,\lambda}}\right) + \frac{C\delta}{1 + C_{\delta,\lambda}} \int_\triangle u dx$$

$$= \mathcal{D}_A\left(\frac{u}{1 + C_{\delta,\lambda}}\right) - n \log(1 + C_{\delta,\lambda}) + \frac{C\delta}{1 + C_{\delta,\lambda}} \int_\triangle u dx \,.$$

Note that the convex function $\frac{u}{1+C_{\delta,\lambda}}$ is not in C, since $\frac{u}{1+C_{\delta,\lambda}} - u_0$ is not in $C^\infty(\overline{\triangle})$. Thus we need to show that $\mathcal{D}_A(\frac{u}{1+C_{\delta,\lambda}})$ is uniformly bounded from below for any $u \in \tilde{C}$. Let

$$u' := \frac{1}{(1 + C_{\delta,\lambda})^2} u + \left(1 - \frac{1}{(1 + C_{\delta,\lambda})^2}\right) u_0 \,.$$

Note that this convex function u' is in C. Since the convexity of the nonlinear term of $\mathcal{D}_A$ holds on any affine line in the set of bounded convex functions on $\overline{\triangle}$ as well as on any one in C (cf. [3, Proposition 2.15]), then we have

$$\mathcal{D}_A(u') \leq \frac{1}{1 + C_{\delta,\lambda}} \mathcal{D}_A\left(\frac{u}{1 + C_{\delta,\lambda}}\right) + \left(1 - \frac{1}{1 + C_{\delta,\lambda}}\right) \mathcal{D}_A\left(\left(1 + \frac{1}{1 + C_{\delta,\lambda}}\right) u_0\right).$$

It follows that

$$\begin{aligned} \mathcal{D}_A\left(\frac{u}{1 + C_{\delta,\lambda}}\right) &\geq (1 + C_{\delta,\lambda}) \mathcal{D}_A(u') + C_{\delta,\lambda} \mathcal{D}_A\left(\left(1 + \frac{1}{1 + C_{\delta,\lambda}}\right) u_0\right) \\ &\geq -(1 + C_{\delta,\lambda}) C_0 + C_{\delta,\lambda} \mathcal{D}_A\left(\left(1 + \frac{1}{1 + C_{\delta,\lambda}}\right) u_0\right) \\ &= \text{Const}\,. \end{aligned} \tag{4.4}$$

Combining (4.3) with (4.4), and replacing $\frac{C\delta}{1+C_{\delta,\lambda}}$ in (4.3) by δ, this completes the proof. □

5. Remaining proofs of Theorem 1.1. By Lemma 3.2 and Proposition 4.2, it remains to show that the properness of the modified Ding functional implies the solvability of the equation (2.2). Then we have the following.

PROPOSITION 5.1. *Under the assumption of the properness of the modified Ding functional, we have*

$$l > 0 \quad on \ \overline{\triangle}\,.$$

PROOF. Since l is affine linear, it suffice to show that $l(p) > 0$ for any vertex p of $\overline{\triangle}$. Let $\{v_i\}_{i\gg 1}$ be a sequence of smooth convex functions on $\overline{\triangle}$ satisfying the followings:

- $v_i \geq v_i(0) = 0$.
- v_i tends to the K (> 0) times the Dirac function for p as $i \to \infty$.

A construction of v_i is written in the end of this proof.

Then the convex function $u_i := \tilde{u}_0 + v_i$ is in $\tilde{C}$, where $\tilde{u}_0 \in \tilde{C}$ is the normalization of the symplectic potencial $u_0 \in C$ of the $(S^1)^n$-invariant reference Kähler metric on X. Let ϕ_i be the Legendre dual of u_i, that is, $\phi_i(\xi) = \sup_{x\in\triangle}(x\cdot\xi - u_i(x))$. Note that

$$\inf_{\mathbb{R}^n}\phi_i = 0 \quad \text{and} \quad \phi_i(\xi) \leq \sup_{x\in\triangle}(x\cdot\xi)$$

since $\inf_{\mathbb{R}^n}\phi_i = -u_i(0)$ by a property of the Legendre duality, and $u_i \geq u_i(0) = 0$ by the definition of u_i. It follows that

$$\log\int_{\mathbb{R}^n} e^{-(\phi_i - \inf\phi_i)}d\xi \geq \log\int_{\mathbb{R}^n} e^{-\sup_{x\in\triangle}(x\cdot\xi)}d\xi\,.$$

By the properness of the modified Ding functional (4.2), we thus have

$$\int_\triangle u_i\cdot l dx \geq \mu\left(\int_\triangle u_i dx\right) + \log\int_{\mathbb{R}^n} e^{-\sup_{x\in\triangle}(x\cdot\xi)}d\xi\,.$$

By taking $i\longrightarrow\infty$, we have

$$K\cdot l(p) + \int_\triangle l\cdot\tilde{u}_0 dx \geq \mu\left(K + \int_\triangle \tilde{u}_0 dx\right) + \log\int_{\mathbb{R}^n} e^{-\sup_{x\in\triangle}(x\cdot\xi)}d\xi\,.$$

Namely, $K\cdot l(p) \geq \mu(K + C_1) - C_2$ holds for some constants C_i independent of K. Hence, by taking K sufficiently large, we get $l(p) > 0$.

Finally, for the construction of v_i, we take an affine linear function w_i on $\mathbb{R}^n$ satisfying the followings:

1. $w_i(p) = Ki$.
2. $\int_{\overline{\triangle}\cap\{w_i\geq 0\}} w_i dx = K$.
3. There exists $r_i > 0$ such that $\lim_i r_i = 0$ and $\left\{\, x\in\overline{\triangle}\;\middle|\; w_i(x)\geq 0\,\right\} \subset \left\{\, x\in\overline{\triangle}\;\middle|\; \|x-p\|_{\mathrm{Euc}} < r_i\,\right\}$.

Let $\hat{w}_i := \max_{\mathbb{R}^n}\{0, w_i\}$. Note that $\hat{w}_i|_{\overline{\triangle}}$ tends to the K times the Dirac function for p as $i\longrightarrow\infty$. To take a smoothing of $\hat{w}_i|_{\overline{\triangle}}$, let us consider the convolution $\hat{w}_i\star\rho_\varepsilon$, where $\rho_\varepsilon\geq 0$ is the smooth mollifier on $\mathbb{R}^n$ whose support is in $B_\varepsilon(0)$. For sufficiently large i and small $\varepsilon > 0$, it is easy to see that $\hat{w}_i\star\rho_\varepsilon$ is smooth and convex on $\overline{\triangle}$, and satisfies $\hat{w}_i\star\rho_\varepsilon \geq \hat{w}_i\star\rho_\varepsilon(0) = 0$ on $\overline{\triangle}$. Note that $\hat{w}_i\star\rho_\varepsilon|_{\overline{\triangle}}$ uniformly converges to $\hat{w}_i|_{\overline{\triangle}}$ as $\varepsilon\longrightarrow 0$. Thus we can take $\varepsilon_i > 0$ such that

$$\|\hat{w}_i\star\rho_{\varepsilon_i} - \hat{w}_i\|_{L^\infty(\overline{\triangle})} \leq \frac{1}{i}\,.$$

Then we define v_i as $\hat{w}_i\star\rho_{\varepsilon_i}|_{\overline{\triangle}}$. □

The condition $l > 0$ guarantees the non-degeneracy of the equation (2.2). Hence the solvability of the equation of (2.2) follows immediately from the result for real Monge-Ampère equations by Berman and Berndtsson [3, Theorem 1.1]. This completes the proof of Theorem 1.1.

REMARK 5.2. By the same argument as in Proposition 5.1, we can prove directly that the uniform relative Ding stability implies the condition $l > 0$ on $\overline{\triangle}$ (see also [11, Proposition 12]). Indeed, we apply the inequality (3.1) of the uniform relative Ding stability to the function $u_i = \tilde{u}_0 + v_i \in \tilde{C}$ in Proposition 5.1. By taking $i \to \infty$, we then have

$$K \cdot l(p) + \int_{\triangle} \tilde{u}_0 \cdot l dx \geq \lambda \cdot K + \lambda \int_{\triangle} \tilde{u}_0 dx\,.$$

Hence, by taking K sufficiently large, we get $l(p) > 0$.

REFERENCES

[1] M. ABREU, Kähler geometry of toric varieties and extremal metrics, Internat. J. Math. 9 (1998), 641–651.
[2] R. BERMAN, K-polystability of $\mathbb{Q}$-Fano varieties admitting Kähler-Einstein metrics, Invent. Math. 203 (2016), no. 3, 973–1025.
[3] R. BERMAN AND B. BERNDTSSON, Real Monge-Ampère equations and Kähler-Ricci solitons on toric log Fano varieties, Ann. Fac. Sci. Toulouse Math. (6) 22 (2013), no. 4, 649–711.
[4] S. BOUCKSOM, T. HISAMOTO AND M. JONSSON, Uniform K-stability, Duistermaat-Heckman Measures and singularities of pairs, Ann. Inst. Fourier (Grenoble) 67 (2017), no. 2, 743–841.
[5] R. DERVAN, Uniform stability of twisted constant scalar curvature Kähler metric, Int. Math. Res. Not. IMRN 2016, no. 15, 4728–4783.
[6] S. DONALDSON, Scalar curvature and stability of toric varieties, J. Differential Geom (2) 62 (2002), 289–349.
[7] K. FUJITA, Optimal Bounds for the volumes of Kähler-Einstein Fano manifolds, Amer. J. Math. 140 (2018), no. 2, 391–414.
[8] T. HISAMOTO, Stability and coersivity for toric polarizations, arXiv:1610.07998v1.
[9] T. MABUCHI, Kähler Einstein metrics for manifolds with non vanishing Futaki character, Tohoku Math. J. (2) 53 (2001), 171–182.
[10] G. TIAN, Canonical Metrics on Kähler Manifolds, Birkhäuser, 1999.
[11] Y. YAO, Mabuchi metrics and relative Ding stability of toric Fano varieties, arXiv:1701.04016v2.
[12] B. ZHOU AND X. ZHU, Relative K-stability and modified K-energy on toric manifolds, Adv. Math. 219 (2008), no. 4, 1327–1362.

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