RELATIVE ALGEBRO-GEOMETRIC STABILITIES
OF TORIC MANIFOLDS

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Abstract. In this paper we study the relative Chow and K-stability of toric manifolds. First, we give a criterion for relative K-stability and instability of toric Fano manifolds in the toric sense. The reduction of relative Chow stability on toric manifolds will be investigated using the Hibert-Mumford criterion in two ways. One is to consider the maximal torus action and its weight polytope. We obtain a reduction by the strategy of Ono [34], which fits into the relative GIT stability detected by Székelyhidi. The other way relies on $\mathbb{C}^*$-actions and Chow weights associated to toric degenerations following Donaldson and Ross-Thomas [13, 36]. In the end, we determine the relative K-stability of all toric Fano threefolds and present counter-examples which are relatively K-stable in the toric sense but which are asymptotically relatively Chow unstable.

1. Introduction. The well-known Yau-Tian-Donaldson conjecture asserts that a compact complex polarized manifold (X, L) admits canonical metrics (Kähler-Einstein metrics, constant scalar curvature (cscK) metrics, and extremal metrics, etc) in $2\pi c_1(L)$ if and only if (X, L) is stable in the sense of Geometric Invariant Theory. Among various notions of stability, K-stability and Chow stability are the most widely studied. Many authors use the term *polystability* rather than *stability*, since the former agrees better with the notions in GIT. Throughout this paper, we use the latter for simplicity.

The conception of K-stability was first introduced by Tian [41] in the study of the existence of Kähler-Einstein metrics in the first Chern class (if it is positive) on a Kähler manifold. Later, Donaldson extended it to general polarized varieties [13] and made a conjecture on the relation between K-stability and the existence of cscK metrics. More generally, for the existence of extremal metrics, the definition of K-stability was extended by Székelyhidi [40] to Kähler classes with non-vanishing Futaki invariant and was called relative K-stability. Meanwhile, the conception of Chow stability is also significant in Kähler geometry. Let $\mathrm{Aut}(X, L)$ be the automorphism group of (X, L). In [12] Donaldson showed that the existence of a cscK metric in $2\pi c_1(L)$ implies the asymptotic Chow stability of (X, L) if $\mathrm{Aut}(X, L)$ is discrete. Donaldson's result was generalized by Mabuchi [25], with the assumption on $\mathrm{Aut}(X, L)$ replaced by the condition of vanishing higher order Futaki invariants. Very recently, it has been shown that the existence of extremal metrics implies asymptotically Chow stability relative to a maximal torus [27, 37]. With these remarkable progress, the verification of the stabilities is drawing

more and more attention. In general, this is a complicated problem since one has to study an infinite number of possible degenerations of the manifold. In this paper, we shall discuss the stabilities of toric manifolds.

For toric manifolds, a well-understood reduced version of the relative K-stability on the moment polytope is believed to be equivalent to the existence of extremal metrics [13, 48]. This conjecture has been confirmed for toric surfaces [14, 15, 7].

Let (X_Δ, L_Δ) be the polarized toric manifold which corresponds to a lattice polytope Δ

$$(1.1) \qquad\qquad \langle l_i, x \rangle \le \lambda_i, i = 1, \ldots, d,$$

satisfying Delzant's condition, where $\lambda_i \in \mathbb{Z}$, $l_i \in \mathbb{Z}^n$ is primitive. Let θ_Δ be the potential function of the extremal vector field V[18], which is affine linear on Δ, and normalized by $\int_\Delta \theta_\Delta \, dx = 0$ (see Lemma 2.1). In [13], Donaldson reduced K-stability to the positivity of a linear functional defined on Δ. This functional was generalized for relative K-stability and is given by [48]

$$(1.2) \qquad\qquad \mathcal{L}_\Delta(u) = \int_{\partial\Delta} u \, d\sigma - \int_\Delta (\bar{S} + \theta_\Delta) u \, dx$$

where $\bar{S}$ is the average of the scalar curvature, and $d\sigma = |l_i|^{-1} d\sigma_0$ on the face in $\{x \in \mathbb{R}^n : \langle l_i, x \rangle = \lambda_i\}$. Here $d\sigma_0$ is the standard Lebesgue measure on $\partial\Delta$. Note that $\bar{S} = \frac{\text{Vol}(\partial\Delta)}{\text{Vol}(\Delta)}$ and the functional $\mathcal{L}_\Delta$ corresponds to the modified Futaki invariant in [40]. We mention that the potential function θ_Δ is uniquely determined by the condition that $\mathcal{L}_\Delta(u) = 0$ for any affine linear function u, namely, one can solve the $n + 1$-linear system

$$\mathcal{L}_\Delta(1) = 0, \quad \mathcal{L}_\Delta(x_i) = 0 \qquad \text{for} \qquad i = 1, \ldots, n,$$

in order to find $\theta_\Delta = \sum a_i x_i + c$ with a_i and c. See Section 5.1 for more detail.

Recall that a convex function u is *piecewise linear* if there are affine linear functions $f_1, \ldots, f_\ell$ such that $u = \max\{f_1, \ldots, f_\ell\}$. Furthermore u is *simple piecewise linear* if it is of the form $u = \max\{0, f\}$ for a linear function f. In view of [13, 40, 48, 49], we have:

DEFINITION 1.1. A toric manifold (X_Δ, L_Δ) is called *relatively K-semistable in the toric sense* if $\mathcal{L}_\Delta(u) \ge 0$ for all piecewise linear convex functions. Furthermore, it is called *relatively K-stable in the toric sense* when $\mathcal{L}_\Delta(u) = 0$ if and only if u is affine linear.

When proving the existence of cscK metrics on toric surfaces [14, 15] Donaldson introduced a stronger notion called *strong K-stability*. Let Δ^* be the union of the interior of Δ and the interiors of its co-dimension 1 faces. Denote

$$C_1 = \{u \mid u \text{ is convex on } \Delta^* \text{ and } \int_{\partial\Delta} u < \infty\}.$$

The linear functional $\mathcal{L}_\Delta$ is well-defined on C_1.

DEFINITION 1.2. (X_Δ, L_Δ) is called *relatively strongly K-stable in the toric sense* if $\mathcal{L}_\Delta(u) \geqslant 0$ for all convex functions in $\mathcal{C}_1$ and $\mathcal{L}_\Delta(u) = 0$ if and only if u is an affine linear function.

To check the positivity of $\mathcal{L}_\Delta$ in dimension 2, it suffices to consider simple piecewise linear convex functions on Δ [13, 45]. In higher dimensions, a sufficient condition for the relative strong K-stability in the toric sense was given by [48]. When X is a Fano n-fold, L is the anti-canonical line bundle and $\bar{S} = n$ the sufficient condition is

$$(1.3) \qquad \sup_\Delta \theta_\Delta \leqslant 1 \,.$$

Condition (1.3) has been verified for all toric Fano surfaces. In [48], it was also asked whether it holds for higher dimensions or not. Furthermore, Mabuchi proposed the following question when considering the existence of extremal metrics on toric Fano manifolds.

PROBLEM 1.3.   Let (X_Δ, L_Δ) be a polarized toric Fano manifold. Is (X_Δ, L_Δ) always relatively K-stable or not?

If the answer were affirmative, one can expect that any toric Fano manifold admits an extremal metric like the case of Kähler-Ricci solitons [46]. However, we have found counter-examples (Corollary 1.6). By a simple observation and together with (1.3), we have:

THEOREM 1.4.   *Assume (X_Δ, L_Δ) is a toric Fano manifold and $\theta_\Delta = \sum_{i=1}^n a_i x_i + c$, where $a_i, c \in \mathbb{R}$. Let $\Delta^- = \{ x \in \Delta : 1 - \theta_\Delta < 0 \}$.*

   (1) *If $\mathrm{Int}(\Delta^-) = \emptyset$, i.e. $\theta_\Delta \leq 1$ on Δ, then (X_Δ, L_Δ) is relatively strongly K-stable in the toric sense. Here $\mathrm{Int}(\Delta^-)$ is the interior of Δ^-.*
   (2) *If $\mathrm{Int}(\Delta^-) \neq \emptyset$ and satisfies*

$$(1.4) \qquad 1 - c < \frac{\int_{\Delta^-} (1 - \theta_\Delta)^2 \, dx}{\mathrm{Vol}(\Delta^-)},$$

   then there exists a simple piecewise linear function such that $\mathcal{L}_\Delta(u) < 0$.

An application of Theorem 1.4 is to determine relative K-stablities of all toric Fano threefolds. The condition $\theta_\Delta \equiv 0$ is equivalent to the vanishing of the Futaki invariant [18]. Smooth toric Fano threefolds were classified by Batyrev [1, 2] and K. Watanabe and M. Watanabe [44] independently. In this paper, we use the notation of [2]. See Section 2.1 for more detail. Among all of them, $\mathbb{C}P^3$, $\mathcal{B}_4$, $\mathcal{C}_3$, $\mathcal{C}_5$ and $\mathcal{F}_1$ have the vanishing Futaki invariant, so condition (1.3) is true. By computation with Theorem 1.4, we have:

THEOREM 1.5.   *Let X be a toric Fano threefold. We assume that the Futaki invariant of X does not vanish. Then X is relatively strongly K-stable in the toric sense in the anti-canonical class if and only if X is one of the following: $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{C}_1$, $\mathcal{C}_4$, $\mathcal{E}_3$, $\mathcal{E}_4$ and $\mathcal{F}_2$.*

It is known that all toric Fano surfaces admit extremal metrics in the anti-canonical class [6, 8]. The instability tells us that counter-examples appear in dimension 3.

COROLLARY 1.6. *If X is one of $\mathcal{B}_1$, C_2, $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{E}_1$ and $\mathcal{E}_2$, then X does not admit extremal metrics in its first Chern class.*

On the other hand, the reduction of Chow stability is also an interesting problem. A natural idea is to use the Hilbert-Mumford criterion. Ono [34] studied Chow stability of toric manifolds by considering the maximal torus action and its weight polytope. He obtained a reduction by adapting Gelfand-Kapranov-Zelevinsky's theory of Chow polytopes [20, 22]. He also defined a notion of the relative Chow semistability in the toric sense. In this paper, we introduce a refinement of this notion so that it fits naturally into the relative GIT stability detected by Székelyhidi [39, Chapter 1].

Let (X_Δ, L_Δ) be a polarized toric manifold and $N = \dim(H^0(X_\Delta, L_\Delta)) - 1$. We consider the relative Chow stability of $X_\Delta \subset \mathbb{C}P^N$(see Section 3.1 for definitions). Now we assume $G = (\mathbb{C}^*)^{N+1}$ is a subgroup of diagonal matrices in $\mathrm{GL}(N + 1, \mathbb{C})$. Following [34], we only consider the specific maximal torus of $\mathrm{SL}(N + 1, \mathbb{C})$ which is also a subtorus of $(\mathbb{C}^*)^{N+1}$ given by

$$T_\Delta^{\mathbb{C}} = \{ (t_1, \ldots, t_{N+1}) \in (\mathbb{C}^*)^{N+1} \mid \prod_{j=1}^{N+1} t_j = 1 \} .$$

Let β be the $\mathbb{C}^*$-action induced by the extremal vector field V as in Section 2.2.

DEFINITION 1.7. (X_Δ, L_Δ) is *relatively Chow semistable*(*stable, unstable*) *in the toric sense* if the Chow form is $T_\Delta^{\mathbb{C}}$-semistable(stable, unstable) relative to β.

Finally we consider the asymptotic relative Chow stability. Denote the Ehrhart polynomial of Δ by $E_\Delta(t)$. For any $i \in \mathbb{Z}_+$, we replace Δ above by $i\Delta$, $N + 1$ by $E_\Delta(i)$ and $G = (\mathbb{C}^*)^{E_\Delta(i)}$. Then we consider the maximal diagonalized torus $T_{i\Delta}^{\mathbb{C}} := G \cap \mathrm{SL}(E_\Delta(i), \mathbb{C})$.

DEFINITION 1.8. (X_Δ, L_Δ) is *asymptotically relatively Chow semistable* (*stable, unstable*) *in the toric sense* if the Chow form is $T_{i\Delta}^{\mathbb{C}}$-semistable (stable, unstable) relative to β for all sufficiently large i.

In this paper, we will describe the asymptotic relative Chow stability in the toric sense in a combinatorial way. The character group $\chi(G)$ of G is identified with

$$\{i\Delta \cap \mathbb{Z}^n \to \mathbb{Z}\} \cong \{\Delta \cap (\mathbb{Z}/i)^n \to \mathbb{Z}\} \cong \mathbb{Z}^{E_\Delta(i)} .$$

For future convenience, we denote $\chi(G) \otimes \mathbb{R}$ by

(1.5) $$W(i\Delta) := \{i\Delta \cap \mathbb{Z}^n \to \mathbb{R}\} \cong \{\Delta \cap (\mathbb{Z}/i)^n \to \mathbb{R}\} \cong \mathbb{R}^{E_\Delta(i)} .$$

As in [20, p.220], we identify $W(i\Delta)$ with its dual space by the scalar product

$$\langle \varphi, \psi \rangle = \sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} \varphi(\mathbf{a})\psi(\mathbf{a}) .$$

Let $\bar{\theta}_{i\Delta} = \frac{1}{E_\Delta(i)} \sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} \theta_\Delta(\frac{\mathbf{a}}{i})$. We define $d_{i\Delta}, \tilde{\theta}_{i\Delta} \in W(i\Delta)$ by

$$d_{i\Delta}(\mathbf{a}) = 1, \ \tilde{\theta}_{i\Delta}(\mathbf{a}) = \frac{\theta_\Delta(\mathbf{a}) - \bar{\theta}_{i\Delta}}{i}, \quad \mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n .$$

THEOREM 1.9. (X_Δ, L_Δ) *is asymptotically relatively Chow semistable in the toric sense if there is an i_0 such that for each $i \geqslant i_0$, there exists s_i satisfying*

$$(1.6) \qquad \frac{i^n(n+1)!\text{Vol}(\Delta)}{E_\Delta(i)} \left(d_{i\Delta} + s_i\tilde{\theta}_{i\Delta} \right) \in \text{Ch}(i\Delta),$$

where $\text{Ch}(i\Delta) \subset W(i\Delta)$ is the Chow polytope of (X_Δ, L_Δ^i). Furthermore, it is asymptotically relatively Chow stable in the toric sense if

$$(1.7) \qquad \frac{i^n(n+1)!\text{Vol}(\Delta)}{E_\Delta(i)} \left(d_{i\Delta} + s_i\tilde{\theta}_{i\Delta} \right) \in \text{Int}(\text{Ch}(i\Delta)).$$

REMARK 1.10.   As can be shown (Remark 3.12(1)), s_i, if exists, can be explicitly given by

$$(1.8) \qquad s_i = \frac{i\bar{\theta}_{i\Delta}E_\Delta(i)}{\displaystyle\sum_{\mathbf{a}\in\Delta\cap(\mathbb{Z}/i)^n} (\theta_\Delta(\mathbf{a}) - \bar{\theta}_{i\Delta})^2}.$$

In [36], Ross-Thomas gave a description of Chow stability by using the Hilbert-Mumford criterion for the $\mathbb{C}^*$-actions induced by test configurations [13]. Inspired by this idea, we give an alternative reduction of the relative Chow stability of toric manifolds in Section 4. In order to see its relation to relative K-stability, we define

$$(1.9) \qquad Q_\Delta(i, g) = E_\Delta(i) \int_\Delta g\,dx - \text{Vol}(\Delta) \sum_{\mathbf{a}\in\Delta\cap(\mathbb{Z}/i)^n} \left(1 + s_i\tilde{\theta}_{i\Delta}(\mathbf{a}) \right) g(\mathbf{a})$$

for any $g \in PL(\Delta, i)$. Here $PL(\Delta, i)$ is the subset of piecewise linear concave functions (see Section 3.2) and s_i is given by (1.8). Then we have

THEOREM 1.11.   *For any $i \in \mathbb{Z}_+$, (X_Δ, L_Δ^i) is relatively Chow semistable in the toric sense if and only if $Q_\Delta(i, g) \geqslant 0$, for all $g \in PL(\Delta, i)$. In addition, it is relatively Chow stable in the toric sense if the equality holds only if g is an affine linear function.*

We would like to point out that Theorems 1.9, 1.11 also hold for general polarized toric varieties. Concerning on relation between Chow and K-stabilities, we have:

THEOREM 1.12.   *If a polarized toric manifold (X, L) is asymptotically relatively Chow semistable in the toric sense, then it is relatively K-semistable in the toric sense.*

In view of Corollary 1.6, we also have the following.

COROLLARY 1.13.   *If X is one of $\mathcal{B}_1$, $\mathcal{C}_2$, $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{E}_1$ and $\mathcal{E}_2$, then X is asymptotically relatively Chow unstable.*

In general, asymptotic Chow semistability is much stronger than K-semistability. In order to see the direct evidence of the difference between Chow stability and K-stability consider the first counter-example that was discovered in [35]. They used the non-symmetric Kähler-Einstein toric Fano 7-fold of [30]. In the case where X is non-toric, lower dimensional counter-examples were discovered by Odaka [32] and Vedova and Zuddas [43]. In [30], it was

TABLE 1.   Relative stability in the toric sense of toric Fano threefolds.

Notation	Relative K-stability	Asymptotic relative Chow stability (Definition 1.9)
$\mathbb{C}P^3$	<u>stable</u>	<u>stable</u>
$\mathcal{B}_1$	unstable	unstable
$\mathcal{B}_2$	stable	
$\mathcal{B}_3$	stable	
$\mathcal{B}_4$	<u>stable</u>	<u>stable</u>
$\mathcal{C}_1$	stable	
$\mathcal{C}_2$	unstable	unstable
$\mathcal{C}_3$	<u>stable</u>	<u>stable</u>
$\mathcal{C}_4$	stable	
$\mathcal{C}_5$	<u>stable</u>	<u>stable</u>
$\mathcal{D}_1$	unstable	unstable
$\mathcal{D}_2$	unstable	unstable
$\mathcal{E}_1$	unstable	unstable
$\mathcal{E}_2$	unstable	unstable
$\mathcal{E}_3$	stable	
$\mathcal{E}_4$	stable	unstable
$\mathcal{F}_1$	<u>stable</u>	<u>stable</u>
$\mathcal{F}_2$	stable	

(1) All <u>stable</u> in the table are known before. Others are new in the present paper. All <u>stable</u> in relative K-stability follows from [49]. All <u>stable</u> in asymptotic relative Chow stability (Definition 1.9) follows from [25] and [19].

(2) Other relative K-stability/instability follows from Thereom 1.4. (Proposition 5.1).

(3) Asymptotic relative Chow unstability except for $\mathcal{E}_4$ follows from Theorem 1.12.

(4) Asymptotic relative Chow unstability of $\mathcal{E}_4$ follows from Proposition 5.4.

also proved that all toric Fano manifolds with the vanishing Futaki invariant are symmetric if $\dim X \leqslant 6$. Note that if X is a symmetric toric Fano manifold, then $(X, -K_X)$ is asymptotically Chow stable [25, 16]. Hence the lowest dimension for an anti-canonically polarized Kähler-Einstein toric Fano manifold $(X, -K_X)$ to be asymptotically Chow unstable is 7. One aim of this paper is to provide such an example in a lower dimensional toric case. We have found a 3-dimensional toric orbifold example admitting the Kähler-Einstein metric but which is asymptotically Chow unstable in the case where X is $\mathbb{Q}$-Fano (Proposition 5.2). When we consider the relative stabilities, we find the smooth example $\mathcal{E}_4$ which is relatively K-stable but not asymptotically relatively Chow semistable (Proposition 5.4). The asymptotic Chow stability of $\mathbb{C}P^3$, $\mathcal{B}_4$, C_3, C_5, $\mathcal{F}_1$ follows from [19, 25]. Hence, we list all the determined stability of toric Fano threefolds in this paper in Table 1. Note that the stabilities are all in the toric sense. It is an interesting question to complete the table, i.e. to determine the remaining stabilities.

This paper is organized as follows. Section 2 is a brief review of toric varieties and the reduction of relative K-stability on toric manifolds which will be used at later stages in the paper. We also prove Theorem 1.4. In Sections 3 and 4 we shall discuss the two ways of reduction of the relative Chow stability on toric manifolds. In Section 5, we present various examples for the stabilities considered in the paper. We compute normalized potentials on toric Fano threefolds and verify the relative K-stability or instablity in Section 5.1. We also provide an example of K-stable toric Fano orbifold X which is asymptotically Chow unstable in $\dim X = 3$. Finally, we discuss the asymptotic relative Chow stability of toric Fano threefolds. The computational results for θ_Δ and Δ^- are listed in Table 2.

2. Preliminaries.
2.1. Toric varieties.
We review some of notations of toric varieties. Detailed discussion on the general theory can be found in [9]. Let $\mathfrak{M}$ be a lattice of rank n, where $\mathfrak{N} = \mathrm{Hom}(\mathfrak{M}, \mathbb{Z})$ is the $\mathbb{Z}$-dual of $\mathfrak{M}$. We define $\mathfrak{M}_\mathbb{R} := \mathfrak{M} \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}^n$ (resp. $\mathfrak{N}_\mathbb{R} := \mathfrak{N} \otimes_\mathbb{Z} \mathbb{R}$). Let Σ denote a complete fan in $\mathfrak{N}_\mathbb{R}$, i.e. $\bigcup_{\sigma \in \Sigma} \sigma = \mathfrak{N}_\mathbb{R}$. The k-dimensional cones of Σ form a set $\Sigma(k)$. Let σ be a cone in Σ. The associated affine scheme $U_\sigma := \mathrm{Spec}\,\mathbb{C}[\mathfrak{M} \cap \sigma^\vee]$ is called an affine toric variety. Then Σ defines a toric variety $X := X(\mathfrak{N}, \Sigma)$ by constructing the disjoint union of the affine toric varieties U_σ, where one glues U_{σ_1} and U_{σ_2} along the open subvariety $U_{\sigma_1 \cap \sigma_2}$, for $\sigma_1, \sigma_2 \in \Sigma$. We generally define a *toric variety* X as a complex algebraic normal variety containing a torus $T_\mathbb{C}^n = \mathrm{Spec}\,\mathbb{C}[\mathfrak{M}]$ as a Zariski open subset, such that, the action of $T_\mathbb{C}^n$ on itself extends to an algebraic action of $T_\mathbb{C}^n$ on X.

A polytope $\Delta \subseteq \mathfrak{M}_\mathbb{R}$ is called a lattice (resp. rational) polytope if all its vertices are in $\mathfrak{M}$ (resp. $\mathfrak{M}_\mathbb{Q} = \mathfrak{M} \otimes_\mathbb{Z} \mathbb{Q}$). Let $\Delta \subseteq \mathfrak{M}_\mathbb{R}$ be a rational n-dimensional polytope with $0 \in \mathrm{Int}\Delta$. We define the dual polytope $\Delta^\circ \subseteq \mathfrak{N}_\mathbb{R}$ by

$$\varDelta^\circ := \{\, a \in \mathfrak{N}_\mathbb{R} \mid \langle a, b \rangle \geqslant -1 \text{ for all } b \in \varDelta \,\}\,,$$

which is also a rational n-dimensional polytope with $0 \in \mathrm{Int}\varDelta^\circ$. We denote a face F of $\varDelta^\circ$ by $F \prec \varDelta^\circ$. The fan $\mathcal{N}_\varDelta := \{\, \mathrm{pos}(F) \mid F \prec \varDelta^\circ \,\}$ is called the *normal fan* of $\varDelta$, where $\mathrm{pos}(F)$ is the linear positive hull of F. For a rational polytope $\varDelta \subseteq \mathfrak{M}_\mathbb{R}$, we define the associated toric variety by $X_\varDelta := X(\mathfrak{N}, \mathcal{N}_\varDelta)$. In particular, when $\varDelta$ satisfies Delzant's condition, it corresponds to a smooth compact toric manifold. It is well-known that there is the bijective correspondence between n-dimensional lattice polytopes and compact toric varieties with $(\mathbb{C}^*)^n$-equivariant very ample line bundles.

The discussion on examples (Section 5) will focus on toric Fano threefolds. We recall the related notations here. See [11] and [29] for more details.

Let X be a complex projective normal variety. We call X a $\mathbb{Q}$-*Fano variety* if the anti-canonical divisor $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor. Furthermore, X is *Fano variety* if $-K_X$ is an ample Cartier divisor. Let $P \subseteq \mathfrak{N}_\mathbb{R}$ be an n-dimensional lattice polytope which contains the origin in its interior. Then P is called a *canonical Fano polytope* if $\mathrm{Int}P \cap \mathfrak{N} = \{\, 0 \,\}$. Furthermore, P is called a *Fano polytope* if the vertices of any facet of P form a $\mathbb{Z}$-basis of the lattice $\mathfrak{N}$. By [29, Proposition 2.3.7], there is a bijective correspondence between isomorphism classes of Fano polytopes (resp. canonical Fano polytopes) and smooth toric Fano varieties (resp. toric Fano varieties with canonical singularities). Here and hereafter we assume that all singularities are at worst orbifold singularities. Hence they are associated to complete simplicial fans [9, Theorem 3.1.19].

Let X be a complex normal variety. Recall that X is called $\mathbb{Q}$-*factorial* if any Weil divisor is $\mathbb{Q}$-Cartier. For the toric case, we have a well-known description in terms of a Fano polytope. A polytope is called *simplicial* if any facet is a simplex. It was shown that simplicial Fano polytopes correspond uniquely up to isomorphism to $\mathbb{Q}$-factorial toric Fano varieties (see Proposition 2.3.12 in [29]). Since a toric variety X has only finite quotient singularities (that is, X is an orbifold) if and only if the associated fan Σ is simplicial, $\mathbb{Q}$-factorial toric Fano varieties are toric Fano orbifolds.

In [23], Kasprzyk found that there are $12,190$ $\mathbb{Q}$-factorial toric Fano varieties up to isomorphism. In the case when X is smooth, toric Fano threefolds are classified in [1, 44]. There are 18 toric Fano threefolds, that is, $\mathbb{C}P^3$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4 = \mathbb{C}P^2 \times \mathbb{C}P^1$, C_1, C_2, $C_3 = \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, $C_4 = \mathbb{C}P^1 \times (\mathbb{C}P^2 \,\#\, \overline{\mathbb{C}P^2})$, $C_5 = \mathbb{C}P_{\mathbb{C}P^1 \times \mathbb{C}P^1}(O \oplus O(1,-1))$, $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3 = \mathbb{C}P^1 \times (\mathbb{C}P^2 \,\#\, 2\overline{\mathbb{C}P^2})$, $\mathcal{E}_4$, $\mathcal{F}_1 = \mathbb{C}P^1 \times (\mathbb{C}P^2 \,\#\, 3\overline{\mathbb{C}P^2})$ and $\mathcal{F}_2$. Note that we use the same notation as in [2]. These classification results are available online at [50, 31].

2.2. Reduction of relative K-stability. In this subsection, we recall the reduction of the Futaki invariant on toric manifolds. We also present the formulae to determine the normalized potential $\theta_\varDelta$ of the extremal vector field in symplectic coordinate and criterions for relative K-stability and instablity.

First we recall the Futaki invariant and the extremal vector field. Let (X, ω_g) be a compact Kähler manifold. Let $\mathrm{Aut}^0(X)$ be the identity component of the biholomorphisms group of X. Then $\mathrm{Aut}^0(X)$ has a semidirect decomposition

$$\mathrm{Aut}^0(X) = \mathrm{Aut}_r(X) \ltimes R_u,$$

where $\mathrm{Aut}_r(X)$ is a reductive algebraic subgroup of $\mathrm{Aut}^0(X)$, which is the complexification of a maximal compact subgroup K, and R_u is the nilpotent radical of $\mathrm{Aut}^0(X)$. We denote the Lie algebra of $\mathrm{Aut}_r(X)$ by $\eta_r(X)$. Let $v \in \eta_r(X)$ so that its imaginary part generates a one-parameter compact subgroup of K. Then if the Kähler form ω_g is K-invariant, there exists a unique real-valued function $\theta_v(\omega_g)$ (called normalized potential of v) such that

$$(2.1) \qquad i_v\omega_g = \sqrt{-1}\bar{\partial}\theta_v(\omega_g) \ \text{ and } \ \int_X \theta_v(\omega_g)\frac{\omega_g^n}{n!} = 0.$$

For simplicity, we denote the set of such potentials θ_v by Ξ_{ω_g}. Then the Futaki invariant on η_r can be written as

$$(2.2) \qquad F(v) = -\int_X \theta_v(\omega_g)(S(\omega_g) - \bar{S})\frac{\omega_g^n}{n!},$$

where $S(\omega_g)$ is the scalar curvature of ω_g. In [18], Futaki and Mabuchi defined the *extremal vector field*, $V = g^{i\bar{j}}(\mathrm{proj}(S(\omega_g)))_{\bar{j}}\frac{\partial}{\partial z_i}$ in $\eta_r(X)$ for the Kähler class $[\omega_g]$, where $\mathrm{proj}(S(\omega_g))$ is the L^2-inner projection of the scalar curvature of ω_g to Ξ_{ω_g}. They showed that V is independent of the choice of K-invariant metrics in $[\omega_g]$, and its potential is uniquely determined as the dual of the Futaki invariant with respect to the L^2 bilinear form

$$(2.3) \qquad F(v) = -\int_X \theta_v(\omega_g)\theta_V(\omega_g)\frac{\omega_g^n}{n!}, \ \ \forall \, v \in \eta_r(X).$$

Now we consider the reduction on a polarized toric manifolds (X, L). Choose an $(S^1)^n$-invariant Kähler metric g with $\omega_g \in 2\pi c_1(L)$. By choosing a base point, we identify the open dense orbit of the complex torus action on X with $(\mathbb{C}^*)^n$ and use the coordinates $(z_1, \ldots, z_n)$ induced from $(\mathbb{C}^*)^n$. Denote the affine logarithmic coordinates $w_i = \log z_i = y_i + \sqrt{-1}\eta_i$. Then ω_g is determined by a smooth convex function φ which depends only on $y_1, \ldots, y_n \in \mathbb{R}^n$ in the coordinates $(w_1, \ldots, w_n)$, namely

$$(2.4) \qquad \omega_g = 2\sqrt{-1}\partial\bar{\partial}\varphi$$

on $(\mathbb{C}^*)^n$. As is well-known, the moment map can be given by $D\varphi$ and the image $\varDelta = D\varphi(\mathbb{R}^n)$ is a polytope. We denote by $x_i = \frac{\partial\varphi}{\partial y_i}, i = 1, \ldots, n$, the symplectic coordinates. Note that in the affine logarithm coordinates $(w_1, \ldots, w_n)$, $\{\frac{\partial}{\partial w_i}, \ i = 1, \ldots, n\}$ is a basis of the Lie algebra of $T_{\mathbb{C}}^n$ as a complex subalgebra of the Lie algebra of holomorphic vector fields on X. In particular, the Futaki invariant is given by the formula

$$F\Big(\frac{\partial}{\partial w_i}\Big) = \int_{\partial\varDelta} x_i \, d\sigma - \bar{S}\int_\varDelta x_i \, dx$$

and this leads the fact that $\mathcal{L}_\varDelta(u) = 0$ for any affine linear function u as mentioned in (1.2). The following lemma was given in [48] on how to determine $\theta_V(\omega_g)$ in symplectic coordinates through the Futaki invariant.

LEMMA 2.1.   *Let g be an $(S^1)^n$-invariant metric on X. Assume V is the extremal vector field and $\theta_V(\omega_g)$ is the normalized potential associated to ω_g by (2.1). Then there are $2n$-numbers a_i and c_i such that*

$$\theta_V(\omega_g) = \sum_{i=1}^{n} a_i(x_i + c_i) =: \theta_\Delta,$$

where $x = (x_i) = D\varphi \in \Delta$. Moreover a_i and c_i are determined uniquely by $2n$-equations,

$$(2.5) \qquad \frac{1}{(2\pi)^n} F\left(\frac{\partial}{\partial w_i}\right) = -\int_\Delta \left(\sum_{j=1}^{n} a_j(x_j + c_j)\right)(x_i + c_i)dx, \ i = 1, \ldots, n,$$

$$(2.6) \qquad \int_\Delta (x_i + c_i)dx = 0, \ i = 1, \ldots, n.$$

As mentioned in the introduction, the relative K-stability in the toric sense refers to the positivity of the linear functional (1.2) for convex functions. Note that $\mathcal{L}_\Delta(u)$ is invariant when adding an affine linear function to u. Without loss of generality, we assume $\mathbf{0}$ lies in the interior of Δ. Hence, it suffices to consider convex functions normalized at $\mathbf{0}$ in the sense that $\inf_{x \in \Delta} u(x) = u(\mathbf{0}) = 0$. When X is a toric Fano manifold, it is observed in [48] that

$$(2.7) \qquad \mathcal{L}_\Delta(u) = \int_\Delta \left(\sum_{i=1}^{n} x_i u_i - u\right) + (1 - \theta_\Delta)u \, dx$$

for C^1 functions by an integration by parts from (1.2). Here $u_i = \frac{\partial u}{\partial x_i}$. By approximation, it is easy to see that (2.7) can also be used for the computation of $\mathcal{L}_\Delta(u)$ for piecewise C^1 functions. As can be seen from (2.7), the positivity of $\mathcal{L}_\Delta$ relies heavily on the positivity of $1 - \theta_\Delta$. Assume $\theta_\Delta = \sum_{i=1}^{n} a_i x_i + c$. Then we shall prove Theorem 1.4.

PROOF OF THEOREM 1.4.   (1) is obvious [48]. We only need to consider (2). If $1 - \theta_\Delta < 0$, i.e. $\Delta^- = \Delta$, it is obvious that all simple piecewise linear convex functions of the form $\max\{\sum_{i=1}^{n} b_i x_i, 0\}$ will destabilize Δ. So we assume $1 - \theta_\Delta = 0$ intersects the interior of Δ. Let

$$u = \max\{-(1 - \theta_\Delta), 0\}.$$

Then

$$\sum_{i=1}^{n} x_i u_i - u = \begin{cases} 1 - c, & x \in \Delta^-; \\ 0, & x \in \Delta \setminus \Delta^-. \end{cases}$$

Hence,

$$\mathcal{L}_\Delta(u) = (1 - c)\mathrm{Vol}(\Delta^-) - \int_{\Delta^-} (1 - \theta_\Delta)^2 \, dx.$$

The theorem follows.   □

3.   Relative Chow stability of toric manifolds.   In this section, we consider relative Chow stability of polarized toric manifolds.

3.1. Notions of Chow stabilities. We first recall various notions of Chow stabilities. We refer to the monograph [17] by Futaki for a more general concept of Chow stability in Kähler geometry. In [24, 26], Mabuchi defined the notion of relative Chow stability in order to consider the existence problem of extremal Kähler metrics. A historical background of relative GIT stability is given by Székelyhidi [39, Chapter 1].

Let G be a connected reductive complex algebraic group with Lie algebra $\mathfrak{g}$. Let $\mathbf{V}$ be a finite dimensional complex vector space. Suppose that G acts linearly on $\mathbf{V}$. Assume v^* is a nonzero vector in $\mathbf{V}$ which is a representative of $v = [v^*] \in \mathbb{P}(\mathbf{V})$. According to GIT, v^* is *G-semistable* if the closure of the G-orbit $O_G(v^*)$ does not contain the origin. Furthermore, v^* is *G-stable* if $O_G(v^*)$ is closed. We call v^* *G-unstable* if v^* is not G-semistable. Analogously, $v \in \mathbb{P}(\mathbf{V})$ is said to be *G-semistable* (resp. stable, unstable) if any representative of v is *G-semistable* (resp. stable, unstable).

To feature relative stability, following [39, Chapter 1], we consider a torus T in G, and denote its Lie algebra by $\mathfrak{t}$. Then we define subalgebras of $\mathfrak{g}$ by

$$\mathfrak{g}_T = \{\, \alpha \in \mathfrak{g} \mid [\alpha, \beta] = 0 \text{ for all } \beta \in \mathfrak{t} \,\},$$

$$\mathfrak{g}_{T^\perp} = \{\, \alpha \in \mathfrak{g}_T \mid \langle \alpha, \beta \rangle = 0 \text{ for all } \beta \in \mathfrak{t} \,\},$$

where $\langle \cdot, \cdot \rangle$ is a rational invariant inner product. We denote the image of $\mathfrak{g}_T$ (resp. $\mathfrak{g}_{T^\perp}$) under the exponential map by G_T (resp. $G_{T^\perp}$).

DEFINITION 3.1 ([39]). Let T be a torus in G fixing the point v. Then v is said to be *semistable* (*resp. stable, unstable*) *relative to T* if it is $G_{T^\perp}$-semistable (resp. stable, unstable).

The Hilbert-Mumford criterion says that $v \in \mathbb{P}(\mathbf{V})$ is G-semistable if and only if v is H-semistable for any maximal algebraic torus $H \subset G$ [10, p.137]. When G itself is isomorphic to an algebraic torus, the above stabilities can be described by the weight polytopes of the actions as follows. Let $\chi(G)$ denote the character group of G. Then $\chi(G)$ consists of algebraic homomorphisms $\chi : G \longrightarrow \mathbb{C}^*$. If we fix an isomorphism $G \cong (\mathbb{C}^*)^{N+1}$, we may express each χ as a Laurent monomial

$$\chi(t_1, \ldots, t_{N+1}) = t_1^{a_1} \cdots t_{N+1}^{a_{N+1}}, \quad t_i \in \mathbb{C}^*, \ a_i \in \mathbb{Z}.$$

Thus, there is the identification between $\chi(G)$ and $\mathbb{Z}^{N+1}$ by $\chi = (a_1, \ldots, a_{N+1}) \in \mathbb{Z}^{N+1}$. Then it is well-known that $\mathbf{V}$ decomposes under the action of G into weight spaces

$$\mathbf{V} = \bigoplus_{\chi \in \chi(G)} \mathbf{V}_\chi, \qquad \mathbf{V}_\chi := \{\, v^* \in \mathbf{V} \mid t \cdot v^* = \chi(t) \cdot v^*, \ t \in G \,\}.$$

DEFINITION 3.2. Let v^* be a nonzero vector in $\mathbf{V}$ with $v^* = \sum_{\chi \in \chi(G)} v_\chi$, $v_\chi \in \mathbf{V}_\chi$. The *weight polytope* of v^* (with respect to G-action) is the convex lattice polytope in $\chi(G) \otimes \mathbb{R} \cong \mathbb{R}^{N+1}$ defined by

$$\mathcal{N}_G(v^*) := \mathrm{Conv}\{\, \chi \in \chi(G) \mid v_\chi \neq 0 \,\} \subseteq \mathbb{R}^{N+1}.$$

According to [10, Theorem 9.2], v^* is G-semistable (resp. stable) if and only if $0 \in \mathcal{N}_G(v^*)$ (resp. $0 \in \text{Int } \mathcal{N}_G(v^*)$). In the relative stability setting, we also have the following.

PROPOSITION 3.3 ([39, Theorem 1.5.2]). *Let T be a torus in G fixing the point v. v is semistable (resp. stable) relative to T if and only if the orthogonal projection of the origin onto the minimal affine subspace containing $\mathcal{N}_G(v^*)$ is in $\mathcal{N}_G(v^*)$ (resp, relint $\mathcal{N}_G(v^*)$).*

Next we define Chow form and Chow stability of irreducible projective varieties. See [20, 47] for more details. Let $X \subset \mathbb{C}P^N$ be an n-dimensional irreducible complex projective variety of degree d. Recall that the Grassmann variety $\mathbb{G}(k, \mathbb{C}P^N)$ parameterizes k-dimensional projective linear subspaces of $\mathbb{C}P^N$. The *associated hypersurface* of $X \subset \mathbb{C}P^N$ is the subvariety in $\mathbb{G}(N - n - 1, \mathbb{C}P^N)$ which is given by

$$Z_X := \{ W \in \mathbb{G}(N - n - 1, \mathbb{C}P^N) \mid W \cap X \neq \emptyset \} \ .$$

It is known that is Z_X is an irreducible hypersurface with deg $Z_X = d$ in the Plücker coordinates. In particular, Z_X is given by the vanishing of a section $R_X^* \in H^0(\mathbb{G}(N - n - 1, \mathbb{C}P^N), O(d))$. We call R_X^* the *Chow form* of X. Note that R_X^* is well defined up to a multiplicative constant. Let $\mathbf{V} := H^0(\mathbb{G}(N - n - 1, \mathbb{C}P^N), O(d))$ and $R_X \in \mathbb{P}(\mathbf{V})$ be the projectivization of R_X^*. We call R_X the *Chow point* of X. The weight polytope of $R_X^* \in \mathbf{V}$ with respect to the action $(\mathbb{C}^*)^{N+1} \subset \text{GL}(N + 1, \mathbb{C})$ of diagonal matrices is called *Chow polytope* of X, and is denoted by $\text{Ch}(X)$. See [20, Chapter 6] for more details. Since we have the natural action of $G = \text{SL}(N + 1, \mathbb{C})$ into $\mathbb{P}(\mathbf{V})$, we can define stabilities of R_X as follows.

DEFINITION 3.4.   Let $X \subset \mathbb{C}P^N$ be an irreducible, n-dimensional complex projective variety. Then X is said to be *Chow semistable* (*resp. stable, unstable*) if the Chow point R_X of X is $\text{SL}(N + 1, \mathbb{C})$-semistable (resp. stable, unstable).

We consider relative Chow stability when the Futaki invariant does not vanish. Choose $T = \beta$ to be the $\mathbb{C}^*$- action induced by the extremal vector field V. T also acts on $\mathbb{P}(\mathbf{V})$.

DEFINITION 3.5.   Let $X \subset \mathbb{C}P^N$ be an irreducible, n-dimensional complex projective variety. Then X is said to be *relatively Chow semistable* (*resp. stable, unstable*) if the Chow point R_X of X is $\text{SL}(N + 1, \mathbb{C})$-semistable (resp. stable, unstable) relative to T.

DEFINITION 3.6.   Let (X, L) be a polarized variety. For $i \gg 0$, let $\Psi_i : X \longrightarrow \mathbb{P}(H^0(X, L^i)^*)$ be the Kodaira embedding.

(1) Suppose that L is very ample. (X, L) is said to be *relatively Chow semistable* (*resp. stable, unstable*) if $\Psi_1(X) \subset \mathbb{P}(H^0(X, L)^*)$ is relatively Chow semistable (resp. stable, unstable).

(2) (X, L) is called *asymptotically relatively Chow semistable* (*resp. stable*) if there is an i_0 such that $\Psi_i(X)$ is relatively Chow semistable (resp. stable) for each $i \geqslant i_0$.

We say that (X, L) is *asymptotically relatively Chow unstable* if it is not asymptotically relatively Chow semistable.

3.2. Reduction on toric manifolds. We reduce the relative Chow stability of polarized toric manifolds by developing on an idea in [33, 34].

Recall that $T_\Delta^\mathbb{C}$ of $SL(N + 1, \mathbb{C})$ is given by

$$T_\Delta^\mathbb{C} \hookrightarrow G = (\mathbb{C}^*)^{N+1} \cap SL(N + 1, \mathbb{C})$$

$$(t_1, \ldots, t_N) \longmapsto (t_1, \ldots, t_N, (t_1 \cdots t_N)^{-1}).$$

In particular, $T_\Delta^\mathbb{C} \cong (\mathbb{C}^*)^N$. We view the Lie algebra of $T_\Delta^\mathbb{C}$ as a subalgebra of $sl(N + 1, \mathbb{C})$ by considering the traceless part. The inner product $\langle, \rangle$ on $sl(N + 1, \mathbb{C})$ is given by $\langle A, B \rangle = Tr(AB)$. Let $\{ \mathbf{a}_1, \ldots, \mathbf{a}_{N+1} \}$ be all the lattice points in Δ. We define

$$\bar{\theta}_\Delta = \frac{1}{N + 1} \sum_{j=1}^{N+1} \theta_\Delta(\mathbf{a}_j).$$

Let θ_Δ be the potential function as in Section 2, and $T = \beta$ be the $\mathbb{C}^*$- action induced by V. Then T is given in $T_\Delta^\mathbb{C}$ by

(3.1)
$$T : \mathbb{C}^* \hookrightarrow G$$

$$t \longmapsto \left(t^{(\theta_\Delta(\mathbf{a}_1) - \bar{\theta}_\Delta)}, \ldots, t^{(\theta_\Delta(\mathbf{a}_{N+1}) - \bar{\theta}_\Delta)} \right).$$

Let $\mathrm{Ch}(\Delta)$ be the Chow polytope of $X_\Delta \subset \mathbb{C}P^N$. In the literature of Gelfand-Kapranov-Zelevinsky's theory, $\mathrm{Ch}(\Delta)$ coincides with the secondary polytope [22]. In particular, it is known that the affine span of the secondary polytope is given by the following.

PROPOSITION 3.7 ([20, Chapter 7, Proposition 1.11]). *Let $\varphi = (\varphi_1, \ldots, \varphi_{N+1})$ be a point in the affine hull of* $\mathrm{Ch}(\Delta)$ *in* $\chi(G) \otimes \mathbb{R} \cong \mathbb{R}^{N+1}$*. Then*

(3.2)
$$\sum_{j=1}^{N+1} \varphi_j = (n + 1)!\mathrm{Vol}(\Delta), \quad \sum_{j=1}^{N+1} \varphi_j \mathbf{a}_j = (n + 1)! \int_\Delta \mathbf{x} \, dx.$$

Here $\mathbf{x} = (x_1, \ldots, x_n)$ and $\Delta \cap M = \{ \mathbf{a}_1, \ldots, \mathbf{a}_{N+1} \}$ is all the lattice points in Δ.

Denote

$$d_\Delta = (1, \ldots, 1), \quad \tilde{\theta}_\Delta = ((\theta_\Delta(\mathbf{a}_1) - \bar{\theta}_\Delta), \ldots, (\theta_\Delta(\mathbf{a}_{N+1}) - \bar{\theta}_\Delta))$$

in $\chi(G) \otimes \mathbb{R}$. Then we have the following.

THEOREM 3.8. *(X_Δ, L_Δ) is relatively Chow semistable in the toric sense if and only if there exists $s \in \mathbb{R}$ such that*

(3.3)
$$\frac{(n + 1)!\mathrm{Vol}(\Delta)}{N + 1} \left(d_\Delta + s\tilde{\theta}_\Delta \right) \in \mathrm{Ch}(\Delta).$$

Furthermore, it is relatively Chow stable in the toric sense if

(3.4)
$$\frac{(n + 1)!\mathrm{Vol}(\Delta)}{N + 1} \left(d_\Delta + s\tilde{\theta}_\Delta \right) \in \mathrm{Int}(\mathrm{Ch}(\Delta)).$$

PROOF.   We define a two dimensional subspace in $\mathbb{R}^{N+1}$ by $\mathbf{W} := \mathrm{Span}_{\mathbb{R}}\{d_\varDelta, \tilde{\theta}_\varDelta\}$.
Let $\beta_1, \ldots, \beta_{N-1} \in \mathbb{R}^{N+1}$ be a basis of the subspace perpendicular to $\mathbf{W}$. Note that $G_{T^\perp}$ is
isomorphic to $(\mathbb{C}^*)^{N-1}$. Considering the projection

$$\pi_{G_{T^\perp}} : \chi(G) \otimes \mathbb{R} \cong \mathbb{R}^{N+1} \longrightarrow \chi(G_{T^\perp}) \otimes \mathbb{R} \cong \mathbb{R}^{N-1}$$

$$\varphi = (\varphi_1, \ldots, \varphi_{N+1}) \longmapsto (\langle \varphi, \beta_1 \rangle, \ldots, \langle \varphi, \beta_{N-1} \rangle),$$

we observe that $\mathcal{N}_{G_{T^\perp}}(R_{X_\varDelta}) = \pi_{G_{T^\perp}}(\mathcal{N}_G(R_{X_\varDelta})) \subset \mathbb{R}^{N-1}$.

By definition, $R_{X_\varDelta}$ is $G_{T^\perp}$-semistable if and only if $0 \in \mathcal{N}_{G_{T^\perp}}(R_{X_\varDelta})$. By the projection
above, it is equivalent to $\mathbf{W} \cap \mathrm{Ch}(\varDelta) \neq \emptyset$, that is, there exist $s_1, s_2 \in \mathbb{R}$ such that

$$(3.5) \qquad\qquad\qquad s_1 d_\varDelta + s_2 \tilde{\theta}_\varDelta \in \mathrm{Ch}(\varDelta).$$

By (3.2) and the fact $\sum_{j=1}^{N+1} \tilde{\theta}_\varDelta(\mathbf{a}_j) = 0$, we have $s_1 = \frac{(n+1)!\mathrm{Vol}(\varDelta)}{N+1}$. $\square$

REMARK 3.9.   The reader should notice that (3.3) implies

$$(3.6) \qquad \sum_{j=1}^{N+1} \mathbf{a}_j + s \sum_{j=1}^{N+1} \tilde{\theta}_\varDelta(\mathbf{a})\mathbf{a}_j = \frac{N+1}{\mathrm{Vol}(\varDelta)} \int_\varDelta \mathbf{x}\, dx,$$

by Proposition 3.7. This theorem extends Ono's description of Chow semistability [33, 34] to
relative case.

Next, we consider asymptotic relative Chow semistability. Denote the Ehrhart polynomial
of $\varDelta$ by $E_\varDelta(t)$. It has degree $n = \dim \varDelta$ and satisfies

$$E_\varDelta(i) = \mathrm{Card}(i\varDelta \cap \mathbb{Z}^n) = \mathrm{Card}(\varDelta \cap (\mathbb{Z}/i)^n)$$

for any positive integer $i \in \mathbb{Z}_+$. Moreover, $E_\varDelta(t)$ has the form

$$E_\varDelta(t) = \mathrm{Vol}(\varDelta)t^n + \frac{\mathrm{Vol}(\partial \varDelta)}{2} t^{n-1} + \cdots + 1,$$

by theorem of Ehrhart. Note that $\int_{i\varDelta} \mathbf{x}\, dx = i^{n+1} \int_\varDelta \mathbf{x}\, dx$, hence Theorem 1.9 follows from the
same argument in the proof of Theorem 3.8.

The asymptotic relative Chow semistability can be related to relative K-semistability in
the toric sense. For this purpose, we recall some notations. For a fixed $\varphi \in W(i\varDelta)$(see (1.5)
for the definition of $W(i\varDelta)$), let

$$G_\varphi = \text{the convex hull of} \bigcup_{\mathbf{a} \in \varDelta \cap (\mathbb{Z}/i)^n} \{(\mathbf{a}, t) \mid t \leqslant \varphi(\mathbf{a})\} \subset \mathfrak{M}_{\mathbb{R}} \times \mathbb{R} \cong \mathbb{R}^{n+1}.$$

Then we define a piecewise linear function $g_\varphi : \varDelta \longrightarrow \mathbb{R}$ by

$$g_\varphi(\mathbf{x}) = \max\{t \mid (\mathbf{x}, t) \in G_\varphi\}.$$

The upper boundary of G_φ can be regarded as the graph of g_φ. Furthermore, g_φ has the
following properties.

LEMMA 3.10 ([20] p.221, Lemma 1.9).   *For any $\varphi \in W(i\varDelta)$,*

(a) *the function g_φ is concave,*

(b) *we have the equality*

$$\max\{\langle\varphi,\psi\rangle \mid \psi \in \mathrm{Ch}(i\Delta)\} = i^n (n+1)! \int_\Delta g_\varphi \, dx \,.$$

Denote $PL(\Delta, i) = \{g_\varphi \mid \varphi \in W(i\Delta)\}$. The proof of Theorem 1.11 is similar to [34]. One can see that the condition (1.6) holds if and only if the following condition holds:

$$(3.7) \qquad \max\{\langle\varphi,\psi\rangle \mid \psi \in \mathrm{Ch}(i\Delta)\} \geqslant \frac{i^n (n+1)!\mathrm{Vol}(\Delta)}{E_\Delta(i)}\langle\varphi, 1 + s_i\tilde{\theta}_{i\Delta}\rangle$$

for all $\varphi \in W(i\Delta)$. By applying Lemma 3.10 to (3.7), we obtain (1.9).

COROLLARY 3.11. *If (X_Δ, L_Δ) is asymptotically relatively Chow semistable, then for any $i \in \mathbb{Z}_+$, there exists s_i such that*

$$(3.8) \qquad \sum_{\mathbf{a}\in\Delta\cap(\mathbb{Z}/i)^n} i\mathbf{a} + s_i \sum_{\mathbf{a}\in\Delta\cap(\mathbb{Z}/i)^n} \tilde{\theta}_{i\Delta}(\mathbf{a})\mathbf{a} = \frac{iE_\Delta(i)}{\mathrm{Vol}(\Delta)}\int_\Delta \mathbf{x}\,dx \,.$$

PROOF. Since (X_Δ, L_Δ) is asymptotically relatively Chow semistable, (X_Δ, L_Δ^i) is relatively Chow semistable for $i \gg 0$. By Theorem 1.11, $Q_\Delta(i,g) \geqslant 0$ for all $g \in PL(\Delta, i)$. Taking g to affine linear functions, one can see that $Q_\Delta(i,g) = 0$. Note that $Q_\Delta(i,g)$ can be written as a fractional polynomial in i as in (3.10). We prove the assertion by contradiction. If (3.8) does not hold for an integer $i_0 \in \mathbb{Z}_+$, (i.e. $Q_\Delta(i_0, \mathbf{x}) \neq 0$), then the identity theorem for polynomial functions implies that there is an integer $i_1 \in \mathbb{Z}_+$ such that $Q_\Delta(i, \mathbf{x}) \neq 0$ for any $i > i_1$. This means (X_Δ, L_Δ^i) is relatively Chow unstable for $i > i_1$, that is, (X_Δ, L_Δ) is asymptotically relatively Chow unstable. The assertion is proved. □

REMARK 3.12. (1) One can see that if s_i exists, then (1.8) follows from (3.8). (2) It is clear that (3.8) is an over-determined linear system since there is only one parameter s_i, but n equations. Hence, one can expect to find counter-examples from polytopes which are not symmetric with respect to $x_1, \ldots, x_n$. (Cf. Proposition 5.4). (3) When $V = 0$, (3.8) becomes

$$(3.9) \qquad \mathrm{Vol}(\Delta)s_\Delta(i) - E_\Delta(i)\int_\Delta \mathbf{x}\,dx = \mathbf{0}$$

which is the necessary condition for (X_Δ, L_Δ^i) to be Chow semistable proved by Ono [33, Theorem 1.4]. From the argument in the proof of Corollary 3.11, one can see that the following: let $\Delta \subseteq \mathfrak{M}_\mathbb{R}$ be a simple lattice polytope and (X_Δ, L_Δ) be the associated polarized toric orbifold. If (3.9) does not hold for a positive integer $i_0 \in \mathbb{Z}_+$, then there is a positive integer $i_1 \in \mathbb{Z}_+$, such that (X_Δ, L_Δ^i) is Chow unstable for any $i > i_1$.

Finally we show asymptotical relative Chow semistability implies relative K-semistability in the toric case. It is known that asymptotic Chow semistability implies K-semistability in general [36]. First, we need the following lemma.

LEMMA 3.13. s_i *in* (3.8) *satisfies* $s_i = -\frac{1}{2} + O(i^{-1})$.

PROOF.   By $\int_\Delta \theta_\Delta\, dx = 0$ and Lemma 3.3 of [48], we have

$$\bar{\theta}_{i\Delta} = \frac{i^{n-1}}{2E_\Delta(i)} \int_{\partial\Delta} \theta_\Delta d\sigma + O(i^{n-2}),$$

$$\sum_{\mathbf{a}\in\Delta\cap(\mathbb{Z}/i)^n} i\mathbf{a} = i^{n+1} \int_\Delta \mathbf{x}\, dx + \frac{i^n}{2} \int_{\partial\Delta} \mathbf{x}\, d\sigma + O(i^{n-1}), \qquad \text{and}$$

$$\sum_{\mathbf{a}\in\Delta\cap(\mathbb{Z}/i)^n} \theta_\Delta(\mathbf{a})\mathbf{a} = i^n \int_\Delta \theta_\Delta \mathbf{x}\, dx + O(i^{n-1}).$$

Then (3.8) is written as

$$iE_\Delta(i) \int_\Delta \mathbf{x}\, dx = \text{Vol}(\Delta)\left(i^{n+1}\int_\Delta \mathbf{x}\, dx + \frac{i^n}{2}\int_{\partial\Delta}\mathbf{x}\, d\sigma + O(i^{n-1})\right) + s_i\text{Vol}(\Delta)$$

$$\left[i^n \int_\Delta \theta_\Delta \mathbf{x}\, dx + O(i^{n-1}) - \frac{i^n\int_\Delta \mathbf{x}\, dx + O(i^{n-1})}{E_\Delta(i)}\left(\frac{i^{n-1}}{2}\int_{\partial\Delta}\theta_\Delta\, d\sigma + O(i^{n-2})\right)\right].$$

Comparing the coefficient of i^n, we conclude that

$$\text{Vol}(\Delta)\left[\frac{1}{2}\int_{\partial\Delta}\mathbf{x}\, d\sigma + s_i\int_\Delta \theta_\Delta \mathbf{x}\, dx\right] = \frac{1}{2}\text{Vol}(\partial\Delta)\int_\Delta \mathbf{x}\, dx.$$

Since θ_Δ is the potential function of the extremal vector field, it holds

$$\int_{\partial\Delta} x_k\, d\sigma - \int_\Delta \left(\frac{\text{Vol}(\partial\Delta)}{\text{Vol}(\Delta)} + \theta_\Delta\right) x_k\, dx = 0 \quad \text{for} \quad k = 1, \ldots, n.$$

Hence we have $s_i = -\frac{1}{2} + O(i^{-1})$.   □

Now we prove Theorem 1.12.

PROOF OF THEOREM 1.12.   For any $i \in \mathbb{Z}_+$ and $g \in PL(\Delta, i)$, by Lemma 3.13,

$$Q_\Delta(i, g)$$

$$=E_\Delta(i)\int_\Delta g\, dx - \text{Vol}(\Delta)\sum_{\mathbf{a}\in\Delta\cap(\mathbb{Z}/i)^n}\left(-\frac{\theta_\Delta(\mathbf{a}) - \bar{\theta}_{i\Delta}}{2i} + 1 + O(i^{-2})\right)g(\mathbf{a})$$

$$=\left(\text{Vol}(\Delta)i^n + \frac{\text{Vol}(\partial\Delta)}{2}i^{n-1} + O(i^{n-2})\right)\int_\Delta g\, dx + i^{n-1}\text{Vol}(\Delta)\int_\Delta \frac{\theta_\Delta - \bar{\theta}_{i\Delta}}{2}g\, dx$$

$$+ O(i^{n-2}) - \left(i^n\text{Vol}(\Delta)\int_\Delta g\, dx + \frac{i^{n-1}\text{Vol}(\Delta)}{2}\int_{\partial\Delta}g\, d\sigma + O(i^{n-2})\right).$$

Note that $\bar{\theta}_{i\Delta} = O(i^{-1})$, then

$$(3.10) \qquad Q_\Delta(i, g) = -\frac{\text{Vol}(\Delta)}{2}\left[\int_{\partial\Delta}g\, d\sigma - \int_\Delta\left(\frac{\text{Vol}(\partial\Delta)}{\text{Vol}(\Delta)} + \theta_\Delta\right)g\, dx\right]i^{n-1} + O(i^{n-2})$$

$$= -\frac{\text{Vol}(\Delta)}{2}\mathcal{L}_\Delta(g)i^{n-1} + O(i^{n-2}).$$

By the assumption, $Q_\Delta(i, g) \geq 0$ when i is sufficiently large. This implies $\mathcal{L}_\Delta(g) \leq 0$. Hence, $\mathcal{L}_\Delta(u) \geq 0$ for any piecewise linear convex function u. The theorem is proved. □

4. Reduction of relative Chow stability: an alternative approach. The asymptotic Chow stability can also be described through the Chow weight of $\mathbb{C}^*$-actions induced by test configurations [13, 36]. In this section, we derive an alternative reduction of relative Chow stability of toric manifolds by investigating the normalized weight in [36] for toric degenerations.

First, we recall the notion of a test configuration [13]. A *test configuration* for (X, L) is a polarized scheme $(\mathfrak{X}, \mathfrak{L})$ with:

- a $\mathbb{C}^*$-action and a proper flat morphism $\pi : \mathfrak{X} \to \mathbb{C}$ which is $\mathbb{C}^*$-equivariant for the usual action on $\mathbb{C}$,
- a $\mathbb{C}^*$-equivariant line bundle $\mathfrak{L} \to \mathfrak{X}$ which is ample over all fibers of π such that for $z \neq 0$, (X, L) is isomorphic to $(\mathfrak{X}_z, \mathfrak{L}_z)$, $\mathfrak{L}_z = \mathfrak{L}|_{\mathfrak{X}_z}$.

$(\mathfrak{X}, \mathfrak{L})$ is called *product* if $\mathfrak{X} = X \times \mathbb{C}$, and *trivial* if in addition $\mathbb{C}^*$ acts only on $\mathbb{C}$.

It is shown in [36] that, for any $i \in \mathbb{Z}_+$, the data of a test configuration for (X, L^i) gives a $\mathbb{C}^*$-action of $GL(d_i, \mathbb{C})$ and vice versa, where $d_i = \dim H^0(X, L^i)$. Let $(\mathfrak{X}_0, \mathfrak{L}_0)$ be the central fiber. For any $r \in \mathbb{Z}_+$, let $k = ri$. Let $w(k) = w(ri)$ be the total weight of the induced $\mathbb{C}^*$-action on $H^0(\mathfrak{X}_0, \mathfrak{L}_0^r)$. As in [36], we define the normalized weight $\tilde{w}_{i,k}$ by

$$(4.1) \qquad \tilde{w}_{i,k} = w(k)id_i - w(i)kd_k .$$

By general algebraic theory, $\tilde{w}_{i,k}$ is a polynomial of degree $n + 1$ in k, for $k \gg 0$. Write

$$\tilde{w}_{i,k} = \sum_{j=1}^{n+1} e_j(i)k^j .$$

Then the leading coefficient $e_{n+1}(i)$ is the *Chow weight*. Then using the Hibert-Mumford criterion for the $\mathbb{C}^*$-actions, Chow stability is described as follows.

THEOREM 4.1 ([36]). *A polarized variety (X, L) is Chow stable with respect to i if for any nontrivial test configuration for (X, L^i), $e_{n+1}(i) > 0$. (X, L) is asymptotically Chow stable if there exists i_0 such that for $i \geq i_0$, any nontrivial test configuration for (X, L^i) has $e_{n+1}(i) > 0$.*

Now we consider toric manifolds. Recall that a piecewise linear convex function $u = \max\{f_1, \ldots, f_\ell\}$ is called *rational* if $f_k = \sum a_{k,i}x_i + c_k$, $k = 1, \ldots, \ell$, for some vectors $(a_{k,1}, \ldots, a_{k,n}) \in \mathbb{R}^n$ and some numbers $c_k \in \mathbb{R}$ such that all $a_{k,i}$ and c_k are rational. According to [13], a *toric degeneration* for (X_Δ, L_Δ^i) is a test configuration induced by a rational piecewise linear convex function u on Δ, such that iQ is a lattice polytope in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$. Here R is an integer such that $u \leq R$ and

$$Q = \{ (\mathbf{x}, t) \mid \mathbf{x} \in \Delta, \ 0 < t < R - u(\mathbf{x}) \} .$$

Denote the set of all rational piecewise linear convex functions satisfying the above condition by $PL_i(\Delta)_{\mathbb{Q}}$.

LEMMA 4.2. *The Chow weight of the toric degeneration for* (X_Δ, L_Δ^i) *induced by u is given by* $e_{n+1}(i) = -i\mathcal{P}_\Delta(i, u)$, *where*

$$(4.2) \qquad \mathcal{P}_\Delta(i, u) = E_\Delta(i) \int_\Delta u \, dx - \mathrm{Vol}(\Delta) \sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} u(\mathbf{a}) \,.$$

PROOF.   By computation [13, 48],

$$d_i = E_\Delta(i) \,,$$

$$d_k = E_\Delta(k) = \mathrm{Vol}(\Delta)k^n + \frac{\mathrm{Vol}(\partial\Delta)}{2}k^{n-1} + \cdots + 1 \,,$$

$$w(i) = i \sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} (R - u)(\mathbf{a}) \,,$$

$$w(k) = k^{n+1} \int_\Delta (R - u) \, dx + \frac{k^n}{2} \int_{\partial\Delta} (R - u) \, d\sigma + \cdots \,.$$

Substituting them into (4.1), we get

$$e_{n+1}(i) = iE_\Delta(i) \int_\Delta (R - u) \, dx - \mathrm{Vol}(\Delta)i \sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} (R - u)(\mathbf{a})$$

$$= i \left[E_\Delta(i) \int_\Delta (-u) \, dx - \mathrm{Vol}(\Delta) \sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} (-u)(\mathbf{a}) \right] \,.$$

$$\square$$

More generally, for the purpose of relative stability, i.e. when the extremal vector field $V \neq 0$, we consider the toric degenerations perpendicular to the $\mathbb{C}^*$-action β induced by V. We use the notations of toric data in Section 3. Following [26], we view the Lie algebra of $T_{i\Delta}^{\mathbb{C}}$ as a subalgebra of $sl(E_\Delta(i), \mathbb{C})$ by considering the traceless part. The inner product $\langle, \rangle_i$ on $sl(E_\Delta(i), \mathbb{C})$ is given by

$$(4.3) \qquad\qquad\qquad \langle A, B \rangle_i = \frac{Tr(AB)}{i^{n+2}} \,.$$

Now let α be the $\mathbb{C}^*$-action on the central fiber induced by the toric degeneration. According to [48], the infinitesimal generators of α and β are

$$\mathrm{diag}(i(R - u)(\mathbf{a}_1), \ldots, i(R - u)(\mathbf{a}_{E_\Delta(i)})), \ \mathrm{diag}(i\theta_\Delta(\mathbf{a}_1), \ldots, i\theta_\Delta(\mathbf{a}_{E_\Delta(i)}))$$

respectively. By considering the traceless parts of them with (4.3), we call the toric degeneration is *perpendicular* to β if

$$(4.4) \qquad \sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} (R - u)(\mathbf{a})(\theta_\Delta(\mathbf{a}) - \bar{\theta}_{i\Delta}) = \sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} u(\mathbf{a})(\theta_\Delta(\mathbf{a}) - \bar{\theta}_{i\Delta}) = 0 \,.$$

In view of (4.2), we have:

DEFINITION 4.3. (X_Δ, L_Δ) is called *asymptotically relatively Chow semistable for toric degenerations* if there exists $i_0 \in \mathbb{Z}_+$, such that when $i \geqslant i_0$, the Chow weight of any toric degeneration for (X_Δ, L_Δ^i) which is perpendicular to β is nonnegative. Furthermore, it is called *asymptotically relatively Chow stable for toric degenerations* if the Chow weight is positive for any nontrivial toric degeneration for (X_Δ, L_Δ^i).

PROPOSITION 4.4. (X_Δ, L_Δ) *is asymptotically relatively Chow semistable for toric degenerations if and only if* (3.8) *holds and there exists* $i_0 \in \mathbb{Z}_+$, *such that when* $i \geqslant i_0$, *for any* $u \in PL_i(\Delta)_\mathbb{Q}$, $Q_\Delta(i, u) \leqslant 0$.

PROOF.   We consider the projection of u onto the perpendicular space. Let

$$\tilde{u} = u - \frac{\displaystyle\sum_{\mathbf{a}\in\Delta\cap(\mathbb{Z}/i)^n} u(\mathbf{a})(\theta_\Delta(\mathbf{a}) - \bar{\theta}_{i\Delta})}{\displaystyle\sum_{\mathbf{a}\in\Delta\cap(\mathbb{Z}/i)^n} (\theta_\Delta(\mathbf{a}) - \bar{\theta}_{i\Delta})^2}(\theta_\Delta - \bar{\theta}_{i\Delta}) \,.$$

Then there exists $r \in \mathbb{Z}_+$ such that $r\tilde{u}$ induces a toric degeneration for (X_Δ, L_Δ^i) perpendicular to β. By (4.2), we have

$$\mathcal{P}_\Delta(i, r\tilde{u}) = E_\Delta(i) \int_\Delta r\tilde{u}\, dx - \mathrm{Vol}(\Delta) \sum_{\mathbf{a}\in\Delta\cap(\mathbb{Z}/i)^n} r\tilde{u}(\mathbf{a})$$

$$= E_\Delta(i) \int_\Delta ru\, dx - \mathrm{Vol}(\Delta) \sum_{\mathbf{a}\in\Delta\cap(\mathbb{Z}/i)^n} (1 + s_i\tilde{\theta}_{i\Delta}(\mathbf{a}))ru(\mathbf{a}) = rQ_\Delta(i, u)\,,$$

where s_i is given by (1.8), and the normalization condition $\int_\Delta \theta_\Delta\, dx = 0$ is used.   □

REMARK 4.5.   The sign differs from (1.9) because we consider convex functions here, while in (1.9) g is a concave function. Futhermore, one can see that if the nonpositivity of $Q_\Delta(i, \cdot)$ is strengthened for all function in $PL(\Delta, i)$, it corresponds to the asymptotic relative Chow semistability in the toric sense. The necessary condition (3.8) for asymptotic relative Chow semistability in the toric sense in the last section can also be recovered by substituting $u = x_j$ and $-x_j$ into the above for $j = 1, \ldots, n$.

5.   Examples.   Finally we provide many interesting examples which will support our understanding of stabilities. Here we will mainly concentrate on toric Fano manifolds.

5.1.   Relative K-stability of toric Fano threefolds.   In this section, we shall determine the potential θ_Δ of the extremal vector field V of toric Fano threefolds in symplectic coordinates and verify the relative K-stability or instability by Theorem 1.4.

If V is given by $V = \sum_{i=1}^n a_i \frac{\partial}{\partial w_i}$ in the affine logarithm coordinates $(w_1, \ldots, w_n)$, then the potential function θ_Δ is given by $\theta_\Delta = \sum_{i=1}^n a_i x_i + c$ for some constant c. There are several ways to compute θ_Δ. The most general one is to use the linear functional (1.2). In order to determine constants a_i and c, one can solve the $n + 1$-linear system

$$(5.1) \qquad\qquad \mathcal{L}_\Delta(1) = 0,\ \mathcal{L}_\Delta(x_i) = 0 \qquad \text{for} \qquad i = 1, \ldots, n \,.$$

In Fano case, we have a more efficient algorithm. By [3], φ in (2.4) can be given by

$$(5.2) \qquad \varphi = \log\left(\sum_{i=1}^{m} e^{\langle p^{(i)}, y\rangle}\right),$$

where $p^{(1)}, \ldots, p^{(m)}$ are the vertices of Δ. The Fano assumption implies

$$(5.3) \qquad |\varphi + \log\det(\varphi_{ij})| < \infty.$$

We recall another normalization on the potentials of holomorphic vector fields. Let v be a holomorphic vector field on X and $\theta'_v(\omega_g)$ be the potential function determined by

$$(5.4) \qquad i_v\omega_g = \sqrt{-1}\bar{\partial}\theta'_v(\omega_g), \quad \text{and} \quad \int_X \theta'_v(\omega_g)e^{h_g}\frac{\omega_g^n}{n!} = 0,$$

where h_g is a Ricci potential of ω_g. According to [42], the above θ'_v satisfies

$$(5.5) \qquad \theta'_v = -\Delta_g\theta'_v - v(h_g),$$

where Δ_g is the Laplacian. Assume $v = \sum_{i=1}^{n} a_i z_i \frac{\partial}{\partial z_i} = \sum_{i=1}^{n} a_i \frac{\partial}{\partial w_i}$ on the torus orbit, where $a_i \in \mathbb{R}$. Then we have $\theta'_v(\omega_g) = \sum_{i=1}^{n} a_i x_i$ in the symplectic coordinates by a simple observation from (5.2). In particular, $\theta'_{\frac{\partial}{\partial w_i}}(\omega_g) = x_i$ for $i = 1, \ldots, n$. This simple fact has been used in [48]. Then by (5.5), we can compute the Futaki invariant by

$$(5.6) \qquad F(v) = \int_X v(h_g)\omega^n = -\int_X \theta'_v\omega^n = -(2\pi)^2\int_\Delta \theta'_v dx.$$

The first step of the algorithm is to determine $a_1, \ldots, a_n$ by (5.6) and Lemma 2.1. Then

$$(5.7) \qquad c = -\frac{1}{\text{Vol}(\Delta)}\int_\Delta \sum_i a_i x_i \, dx$$

by the normalization condition. An alternative method to compute $a_1, \ldots, a_n$ was also given by Nakagawa. He gave a combinatorial formula for Futaki invariant and the generalized Killing form of toric Fano orbifolds in [28]. As an application, he computed the extremal vector field in the anti-canonical class on a toric Fano manifold X with $\dim X \leqslant 4$. In order to prove Theorem 1.5, we shall use his result on toric Fano threefolds directly.

The main goal of this section is to prove the following proposition.

PROPOSITION 5.1. *Let X be a toric Fano threefold with anti-canonical polarization.*

(a) *If $X = \mathcal{B}_2$, then X is relatively (strongly) K-stable in the toric sense in the anti-canonical class.*

(b) *If $X = \mathcal{B}_1$, then X is relatively K-unstable in its first Chern class.*

Once Proposition 5.1 has been proved, other cases are similar and further details are left to the reader[1]. In Table 1 and Table 2 we give the list of all results proved in Theorem 1.5.

---

[1]In the practical computation we used packages (i) `Normaliz` and (ii) `Polymake`. These packages are available at [5] and [21] respectively.

PROOF.   (a) Let e_i $(i = 1, 2, 3)$ be the standard basis of $\mathfrak{N} \cong \mathbb{Z}^3$. Let Σ be the complete fan in $\mathfrak{N}_\mathbb{R} \cong \mathbb{R}^3$ whose 1-dimensional cones are given by $\Sigma(1) = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ where

$$\sigma_1 = \text{Cone}(e_1), \qquad \sigma_2 = \text{Cone}(e_2), \qquad \sigma_3 = \text{Cone}(e_3),$$
$$\sigma_4 = \text{Cone}(-e_3), \quad \text{and} \quad \sigma_5 = \text{Cone}(-e_1 - e_2 - e_3).$$

Then the associated toric manifold X is $\mathbb{C}P(O_{\mathbb{C}P^2} \oplus O_{\mathbb{C}P^2}(1))$ and

$$\Delta = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \geqslant -1, \ x_2 \geqslant -1, \ x_3 \geqslant -1, \ x_3 \leqslant 1, x_1 + x_2 + x_3 \leqslant 1 \right\}.$$

Hence $\text{Vol}(\Delta) = \frac{28}{3}$. Let $V \in \eta_c(X)$ be the extremal vector field in the anti-canonical class and θ_Δ be the potential function of V. Then $\theta_\Delta = -\frac{70}{97}x_3 + c$ for some constant c. (See [28], Section 6, Table 2). Since we have $\int_\Delta x_3 = -2$, we conclude that $c = -\frac{15}{97}$ by (5.7). Thus $\theta_\Delta = -\frac{70}{97}x_3 - \frac{15}{97}$ and

$$\Delta^- = \left\{ (x_1, x_2, x_3) \in \Delta \ \middle| \ -\frac{112}{97} - \frac{70}{97}x_3 \geqslant 0 \right\} = \emptyset.$$

X satisfies the condition (1.3). The assertion is verified.

(b) Let Σ be the complete fan in $N_\mathbb{R} \cong \mathbb{R}^3$ whose 1-dimensional cones are given by $\Sigma(1) = \{\sigma_1, \ldots, \sigma_5\}$ where

$$\sigma_1 = \text{Cone}(e_1), \qquad \sigma_2 = \text{Cone}(e_2), \qquad \sigma_3 = \text{Cone}(e_3),$$
$$\sigma_4 = \text{Cone}(-e_3), \quad \text{and} \quad \sigma_5 = \text{Cone}(-e_1 - e_2 - 2e_3).$$

Then we readily see that $X = \mathbb{C}P(O_{\mathbb{C}P^2} \oplus O_{\mathbb{C}P^2}(2))$ and

$$\Delta = \text{Conv}\left\{ e_3^* - e_2^*, \ 4e_1^* - e_2^* - e_3^*, \ -e_1^* - e_2^* - e_3^*, \ e_3^* - e_1^* - e_2^*, \ 4e_2^* - e_1^* - e_3^*, \ e_3^* - e_1^* \right\},$$

where e_i^* is the dual basis of e_i. Then $\text{Vol}(\Delta) = \frac{31}{3}$ and $\theta_\Delta = -\frac{620}{349}x_3 + c$ for some constant c. Since $\int_\Delta x_3 = -4$, we obtain $c = -\frac{240}{349}$. Hence we conclude $\theta_\Delta = -\frac{620}{349}x_3 - \frac{240}{349}$ and

$$\Delta^- = \{(x_1, x_2, x_3) \in \Delta \mid -\frac{589}{349} - \frac{620}{349}x_3 \geqslant 0\}$$
$$= \text{Conv}\left\{ \begin{array}{lll} 4e_1^* - e_2^* - e_3^*, & \frac{39}{10}e_1^* - e_2^* - \frac{19}{20}e_3^*, & -e_1^* - e_2^* - \frac{19}{20}e_3^* \\[2mm] -e_1^* + \frac{39}{10}e_2^* - \frac{19}{20}e_3^*, & -e_1^* - e_2^* - e_3^*, & -e_1^* + 4e_2^* - e_3^* \end{array} \right\}.$$

Thus

$$(5.8) \qquad\qquad\qquad \text{Vol}(\Delta^-) = \frac{7351}{12000}.$$

Now we shall verify Condition (1.4). First we note that $\int_{\Delta^-} x_3 \, dx = -\frac{96197}{4}$ and $\int_{\Delta^-} x_3^2 \, dx = \frac{1828273}{5}$. Hence one can see that

$$\int_{\Delta^-} (1 - \theta_\Delta)^2 \, dx = \int_{\Delta^-} \left(\frac{589}{349} + \frac{620}{349}x_3 \right)^2 dx = \frac{1475918766336271}{1461612000}.$$

Plugging this and (5.8) into (1.4), we obtain the desired result because $1 - c = \frac{589}{349}$. $\square$

5.2. Relative Chow stability. We study relatively Chow unstable examples of toric Fano manifolds. First, we recall the example found by Nill and Paffenholz [30] which is isomorphic to $\mathbb{P}(\mathcal{O}_W \oplus \mathcal{O}_W(-1,-1,-1,2)) =: X_{NP}$ where $W = (\mathbb{C}P^1)^3 \times \mathbb{C}P^3$. X_{NP} is a non-symmetric Kähler-Einstein toric Fano 7-fold. In [35], Ono, Sano and Yotsutani showed that $(X_{NP}, -K_{X_{NP}})$ is asymptotically Chow unstable. Later Yotsutani observed that X_{NP} is Chow unstable w.r.t. $-K_{X_{NP}}$ (i.e. $i = 1$) using (3.9). Meanwhile, Nill and Paffenholz [30] also proved that all toric Kähler-Einstein Fano manifolds are symmetric if $\dim X \leqslant 6$. It is known that if X is a symmetric toric Fano manifold, then $(X, -K_X)$ is asymptotically Chow stable. Hence, the lowest dimension for an anti-canonically polarized Kähler-Einstein toric Fano manifold $(X, -K_X)$ to be (asymptotically) Chow unstable is 7. However, in the case where X is a Fano orbifold, such an example appears in $\dim X = 3$.

PROPOSITION 5.2. *There is a Kähler-Einstein toric Fano orbifold X with $\dim X = 3$ which is Chow unstable w.r.t. $-2K_X$.*

Our strategy is the following. For any toric Fano orbifold X with the associated simplicial Fano polytope $\varDelta^\circ \subseteq \mathfrak{N}_{\mathbb{R}}$, X admits Kähler-Einstein metric if and only if the Futaki invariant of X vanishes by Shi-Zhu's result [38]. Let $\mathcal{W}(X)$ be the Weyl group of $\mathrm{Aut}(X)$ with respect to the maximal torus and $\mathfrak{N}_{\mathbb{R}}^{\mathcal{W}(X)}$ be the $\mathcal{W}(X)$-invariant subspace of $\mathfrak{N}_{\mathbb{R}}$. It is known that $\sum_{v \in \mathcal{V}(\varDelta^\circ)} v \in \mathfrak{N}_{\mathbb{R}}^{\mathcal{W}(X)}$ (see [29, Chapter 5]). Here $\mathcal{V}(\varDelta^\circ)$ denotes the set of vertices of $\varDelta^\circ$. Note that X is symmetric if and only if $\mathfrak{N}_{\mathbb{R}}^{\mathcal{W}(X)} = \{\mathbf{0}\}$. Hence we may consider toric Fano orbifolds with the vanishing Futaki invariant satisfying $\sum_{v \in \mathcal{V}(\varDelta^\circ)} v \neq \mathbf{0}$. Among all $12,190$ 3-dimensional canonical toric Fano orbifolds (i.e. $\mathbb{Q}$-factorial toric Fano varieties), there are 42 toric Fano orbifolds with the vanishing Futaki invariant. Of these 42 toric Fano orbifolds, there is the only one example satisfying the above conditions. Then it suffices to check (3.9) for the dual moment polytope $\varDelta \subseteq \mathfrak{M}_{\mathbb{R}}$ of this one. Note that the Gorenstein index j_X is given by minimal k such that $k\varDelta$ is a lattice polytope for a fixed canonical Fano polytope $\varDelta^\circ \subseteq \mathfrak{N}_{\mathbb{R}}$ [29, Proposition 2.3.2].

PROOF OF PROPOSITION 5.2. Again we use the same notations as in the proof of Proposition 5.1. We consider the 3-dimensional simplicial canonical Fano polytope[2]

$$\varDelta^\circ := \mathrm{Conv}\{e_1 - e_2 - 2e_3, e_2 + 3e_3, e_1 + e_2 + 3e_3, e_1 + 2e_2 + 4e_3, e_2, -2e_1 - 2e_2 - 3e_3\}.$$

The vertices of the dual polytope $\varDelta$ are

$$-\frac{1}{2}e_1^* + \frac{5}{2}e_2^* - e_3^*, e_1^* - e_2^*, 2e_2^* - e_3^*, \frac{1}{2}e_2^* - \frac{1}{2}e_3^*, -e_1^*, -e_1^* - e_2^* + \frac{1}{2}e_3^*, \frac{3}{2}e_1^* - e_2^*, -e_2^* + e_3^*.$$

Thus $\varDelta$ is a simple polytope and $j_{X_\varDelta} = 2$. Setting $\widetilde{\varDelta} := 2\varDelta$, we compute the Chow weight of $\widetilde{\varDelta}$. We readily see that

$$E_{\widetilde{\varDelta}}(i) = 12i^3 + 9i^2 + 3i + 1, \quad \mathrm{Vol}(\widetilde{\varDelta}) = 12, \quad \int_{\widetilde{\varDelta}} \mathbf{x}\, dx = (0,0,0), \quad \int_{\partial\widetilde{\varDelta}} \mathbf{x}\, d\sigma = (0,0,0)$$

---

[2]ID number in the database [50] is 530571.

and

$$\sum_{\mathbf{a}\in\widetilde{\varDelta}\cap\mathbb{Z}^3} \mathbf{a} = (0, 1, -1), \qquad \sum_{\mathbf{a}\in 2\widetilde{\varDelta}\cap\mathbb{Z}^3} \mathbf{a} = (0, 3, -3), \qquad \sum_{\mathbf{a}\in 3\widetilde{\varDelta}\cap\mathbb{Z}^3} \mathbf{a} = (0, 6, -6)$$

hold. This implies that

$$\int_{\partial\widetilde{\varDelta}} \mathbf{x}\, d\sigma - \frac{\mathrm{Vol}(\partial\widetilde{\varDelta})}{\mathrm{Vol}(\widetilde{\varDelta})} \int_{\widetilde{\varDelta}} \mathbf{x}\, dx = (0, 0, 0)$$

i.e. the Futaki invariant vanishes. However obviously,

$$\sum_{\mathbf{a}\in\widetilde{\varDelta}\cap\mathbb{Z}^3} \mathbf{a} \neq \frac{E_{\widetilde{\varDelta}}(1)}{\mathrm{Vol}(\widetilde{\varDelta})} \int_{\widetilde{\varDelta}} \mathbf{x}\, dx \,.$$

By (3.9), the 2-Gorenstein toric Fano variety $(X_\varDelta, -2K_{X_\varDelta})$ is Chow unstable. $\qquad\square$

In [4], Berman proved that a $\mathbb{Q}$-Fano variety X admitting Kähler-Einstein metrics is K-stable. Hence Remark 3.12 (2) gives the following.

COROLLARY 5.3. *The example in Proposition* 5.2 *is K-stable but asymptotically Chow unstable.*

Next, we consider the general case with the nontrivial Futaki invariant. We see that a smooth counter-example appears in dim $X = 3$.

PROPOSITION 5.4. *Let X be a toric Fano threefold which is isomorphic to $\mathcal{E}_4$. Then $(X, -K_X)$ is relatively K-stable but it is asymptotically relatively Chow unstable.*

PROOF. It suffices to see that (3.8) is not satisfied. The corresponding 3-dimensional moment polytope is listed in Table 3. Thus we have

$$E_\varDelta(i) = \frac{20}{3}i^3 + 10i^2 + \frac{16}{3}i + 1 \quad \text{and} \quad \int_\varDelta \mathbf{x}\, dx = \left(-\frac{7}{8}, \frac{5}{12}, \frac{5}{24}\right).$$

In particular,

$$\sum_{\mathbf{a}\in\varDelta\cap\mathbb{Z}^3} \mathbf{a} = (-4, 2, 1)$$

and

$$s_1 \sum_{\mathbf{a}\in\varDelta\cap\mathbb{Z}^3} (\theta_\varDelta(\mathbf{a}) - \bar{\theta}_\varDelta)\mathbf{a} = s_1 \left(-\frac{11134272}{1816885}, \frac{1079424}{363377}, \frac{539712}{363377}\right)$$

holds. Thus there is no s_1 satisfying (3.8). The assertion follows from Corollary 3.11. $\qquad\square$

TABLE 2. θ_Δ and Δ^- of toric Fano threefolds.

Notation	$\theta_\Delta = \sum_{i=1}^{3} a_i x_i + c$	Δ^-
$\mathbb{C}P^3$	$\equiv 0$	$\emptyset$
$\mathcal{B}_1$	$-\frac{620}{349}x_3 - \frac{240}{349}$	$\mathrm{Conv}\left\{ \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} \frac{39}{10} \\ -1 \\ -\frac{19}{20} \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -\frac{19}{20} \end{pmatrix}, \begin{pmatrix} -1 \\ \frac{39}{10} \\ -\frac{19}{20} \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix} \right\}$
$\mathcal{B}_2$	$-\frac{70}{97}x_3 - \frac{15}{97}$	$\emptyset$
$\mathcal{B}_3$	$-\frac{20}{43}x_1 - \frac{20}{43}x_2 - \frac{5}{43}$	$\emptyset$
$\mathcal{B}_4$	$\equiv 0$	$\emptyset$
$\mathcal{C}_1$	$-\frac{260}{219}x_3 - \frac{80}{219}$	$\emptyset$
$\mathcal{C}_2$	$-\frac{7600}{17787}x_1 - \frac{17750}{17787}x_3 - \frac{4868}{17787}$	$\mathrm{Conv}\left\{ \begin{pmatrix} -\frac{981}{1520} \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -\frac{981}{1520} \\ \frac{4021}{1520} \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ \frac{10111}{3550} \\ -\frac{3011}{3550} \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -\frac{3011}{3550} \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \right\}$
$\mathcal{C}_3$	$\equiv 0$	$\emptyset$
$\mathcal{C}_4$	$-\frac{6}{11}x_2 - \frac{1}{11}$	$\emptyset$
$\mathcal{C}_5$	$\equiv 0$	$\emptyset$
$\mathcal{D}_1$	$\frac{99600}{467581}x_1 - \frac{627000}{467581}x_2 - \frac{213939}{467581}$	$\mathrm{Conv}\left\{ \begin{pmatrix} \frac{48388}{18165} \\ -\frac{12058}{18165} \\ -1 \end{pmatrix}, \begin{pmatrix} \frac{1363}{2490} \\ -1 \\ \frac{3617}{2490} \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} \frac{1363}{2490} \\ -1 \\ -1 \end{pmatrix} \right\}$
$\mathcal{D}_2$	$\frac{219420}{650251}x_1 - \frac{318320}{650251}x_2 - \frac{62565}{650251}$	$\mathrm{Conv}\left\{ \begin{pmatrix} 2 \\ -\frac{1489}{1730} \\ -1 \end{pmatrix}, \begin{pmatrix} \frac{4288}{2385} \\ -1 \\ -\frac{1903}{2385} \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} \frac{4288}{2385} \\ -1 \\ -1 \end{pmatrix} \right\}$
$\mathcal{E}_1$	$-\frac{17020}{19651}x_1 - \frac{17020}{19651}x_2 - \frac{6845}{19651}$	$\mathrm{Conv}\left\{ \begin{pmatrix} -\frac{103}{185} \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -\frac{103}{185} \\ -1 \\ -\frac{473}{185} \end{pmatrix}, \begin{pmatrix} -1 \\ -\frac{103}{185} \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -\frac{103}{185} \\ \frac{473}{185} \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \right\}$

Notation	θ_Δ	Δ^-
$\mathcal{E}_2$	$-\frac{2646160}{2735927}x_1 - \frac{982960}{2735927}x_2 - \frac{692905}{2735927}$	$\mathrm{Conv}\left\{ \begin{pmatrix} -\frac{13897}{15035} \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -\frac{13897}{15035} \\ -1 \\ -\frac{28932}{15035} \end{pmatrix}, \begin{pmatrix} -1 \\ -\frac{4447}{5585} \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -\frac{4447}{5585} \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \right\}$
$\mathcal{E}_3$	$-\frac{168}{409}x_1 - \frac{168}{409}x_2 - \frac{32}{409}$	$\emptyset$
$\mathcal{E}_4$	$-\frac{34208}{78995}x_1 + \frac{7936}{78995}x_2 - \frac{24929}{394975}$	$\emptyset$
$\mathcal{F}_1$	$\equiv 0$	$\emptyset$
$\mathcal{F}_2$	$\frac{36}{67}x_2 - \frac{5}{67}$	$\emptyset$

TABLE 3.   The moment polytopes of toric Fano threefolds.

Notation	The moment polytope Δ
$\mathbb{C}P^3$	$\mathrm{Conv}\left\{\begin{pmatrix}-1\\-1\\-1\end{pmatrix},\begin{pmatrix}-1\\-1\\3\end{pmatrix},\begin{pmatrix}-1\\3\\-1\end{pmatrix},\begin{pmatrix}3\\-1\\-1\end{pmatrix}\right\}$
$\mathcal{B}_1$	$\mathrm{Conv}\left\{\begin{pmatrix}0\\-1\\1\end{pmatrix},\begin{pmatrix}4\\-1\\-1\end{pmatrix},\begin{pmatrix}-1\\-1\\-1\end{pmatrix},\begin{pmatrix}-1\\-1\\1\end{pmatrix},\begin{pmatrix}-1\\4\\-1\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$
$\mathcal{B}_2$	$\mathrm{Conv}\left\{\begin{pmatrix}1\\-1\\1\end{pmatrix},\begin{pmatrix}3\\-1\\-1\end{pmatrix},\begin{pmatrix}-1\\-1\\-1\end{pmatrix},\begin{pmatrix}-1\\-1\\1\end{pmatrix},\begin{pmatrix}-1\\3\\-1\end{pmatrix},\begin{pmatrix}-1\\1\\1\end{pmatrix}\right\}$
$\mathcal{B}_3$	$\mathrm{Conv}\left\{\begin{pmatrix}2\\-1\\0\end{pmatrix},\begin{pmatrix}2\\-1\\-1\end{pmatrix},\begin{pmatrix}-1\\2\\-1\end{pmatrix},\begin{pmatrix}-1\\-1\\-1\end{pmatrix},\begin{pmatrix}-1\\2\\0\end{pmatrix},\begin{pmatrix}-1\\-1\\3\end{pmatrix}\right\}$
$\mathcal{B}_4$	$\mathrm{Conv}\left\{\begin{pmatrix}2\\-1\\1\end{pmatrix},\begin{pmatrix}2\\-1\\-1\end{pmatrix},\begin{pmatrix}-1\\-1\\-1\end{pmatrix},\begin{pmatrix}-1\\-1\\1\end{pmatrix},\begin{pmatrix}-1\\2\\-1\end{pmatrix},\begin{pmatrix}-1\\2\\1\end{pmatrix}\right\}$
$\mathcal{C}_1$	$\mathrm{Conv}\left\{\begin{pmatrix}0\\-1\\1\end{pmatrix},\begin{pmatrix}2\\-1\\-1\end{pmatrix},\begin{pmatrix}2\\2\\-1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}-1\\2\\-1\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix},\begin{pmatrix}-1\\-1\\-1\end{pmatrix},\begin{pmatrix}-1\\-1\\1\end{pmatrix}\right\}$
$\mathcal{C}_2$	$\mathrm{Conv}\left\{\begin{pmatrix}0\\-1\\1\end{pmatrix},\begin{pmatrix}2\\-1\\-1\end{pmatrix},\begin{pmatrix}2\\0\\-1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}-1\\3\\-1\end{pmatrix},\begin{pmatrix}-1\\1\\1\end{pmatrix},\begin{pmatrix}-1\\-1\\-1\end{pmatrix},\begin{pmatrix}-1\\-1\\1\end{pmatrix}\right\}$
$\mathcal{C}_3$	$\mathrm{Conv}\left\{\begin{pmatrix}1\\-1\\1\end{pmatrix},\begin{pmatrix}1\\-1\\-1\end{pmatrix},\begin{pmatrix}1\\1\\-1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}-1\\1\\-1\end{pmatrix},\begin{pmatrix}-1\\1\\1\end{pmatrix},\begin{pmatrix}-1\\-1\\-1\end{pmatrix},\begin{pmatrix}-1\\-1\\1\end{pmatrix}\right\}$
$\mathcal{C}_4$	$\mathrm{Conv}\left\{\begin{pmatrix}2\\-1\\1\end{pmatrix},\begin{pmatrix}2\\-1\\-1\end{pmatrix},\begin{pmatrix}0\\1\\-1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}-1\\1\\-1\end{pmatrix},\begin{pmatrix}-1\\1\\1\end{pmatrix},\begin{pmatrix}-1\\-1\\-1\end{pmatrix},\begin{pmatrix}-1\\-1\\1\end{pmatrix}\right\}$
$\mathcal{C}_5$	$\mathrm{Conv}\left\{\begin{pmatrix}0\\-1\\1\end{pmatrix},\begin{pmatrix}2\\-1\\-1\end{pmatrix},\begin{pmatrix}2\\0\\-1\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix},\begin{pmatrix}-1\\2\\1\end{pmatrix},\begin{pmatrix}-1\\-1\\-1\end{pmatrix},\begin{pmatrix}-1\\-1\\1\end{pmatrix}\right\}$
$\mathcal{D}_1$	$\mathrm{Conv}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\-1\end{pmatrix},\begin{pmatrix}3\\-1\\-1\end{pmatrix},\begin{pmatrix}1\\1\\-1\end{pmatrix},\begin{pmatrix}-1\\-1\\3\end{pmatrix},\begin{pmatrix}-1\\-1\\-1\end{pmatrix},\begin{pmatrix}-1\\0\\-1\end{pmatrix},\begin{pmatrix}-1\\0\\2\end{pmatrix}\right\}$
$\mathcal{D}_2$	$\mathrm{Conv}\left\{\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\-1\end{pmatrix},\begin{pmatrix}2\\-1\\-1\end{pmatrix},\begin{pmatrix}2\\1\\-1\end{pmatrix},\begin{pmatrix}-1\\-1\\2\end{pmatrix},\begin{pmatrix}-1\\-1\\-1\end{pmatrix},\begin{pmatrix}-1\\0\\-1\end{pmatrix},\begin{pmatrix}-1\\0\\2\end{pmatrix}\right\}$

Notation	The moment polytope Δ
$\mathcal{E}_1$	$\mathrm{Conv}\left\{\begin{pmatrix}1\\-1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \right.$ $\left. \begin{pmatrix}0\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\3\end{pmatrix}, \begin{pmatrix}-1\\-1\\-1\end{pmatrix}\right\}$
$\mathcal{E}_2$	$\mathrm{Conv}\left\{\begin{pmatrix}1\\-1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \right.$ $\left. \begin{pmatrix}0\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\\2\end{pmatrix}, \begin{pmatrix}-1\\-1\\2\end{pmatrix}, \begin{pmatrix}-1\\-1\\-1\end{pmatrix}\right\}$
$\mathcal{E}_3$	$\mathrm{Conv}\left\{\begin{pmatrix}1\\-1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \right.$ $\left. \begin{pmatrix}0\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\-1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\-1\end{pmatrix}\right\}$
$\mathcal{E}_4$	$\mathrm{Conv}\left\{\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \right.$ $\left. \begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}-1\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\\2\end{pmatrix}, \begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\-1\\0\end{pmatrix}\right\}$
$\mathcal{F}_1$	$\mathrm{Conv}\left\{\begin{pmatrix}0\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \right.$ $\left. \begin{pmatrix}0\\-1\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\-1\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\-1\end{pmatrix}\right\}$
$\mathcal{F}_2$	$\mathrm{Conv}\left\{\begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}, \right.$ $\left. \begin{pmatrix}0\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\-1\\0\end{pmatrix}, \begin{pmatrix}-1\\-1\\0\end{pmatrix}, \begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$

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