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QUASI-GALOIS POINTS, I: AUTOMORPHISM GROUPS OF PLANE CURVES

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Abstract. We investigate the automorphism group of a plane curve, introducing the notion of a quasi-Galois point. We show that the automorphism group of several curves, for example, Klein quartic, Wiman sextic and Fermat curves, is generated by the groups associated with quasi-Galois points.

1. Introduction. The automorphism group $\mathrm{Aut}(C)$ of an algebraic curve C over an algebraically closed field K of characteristic $p \geq 0$ is a classical subject in algebraic geometry, and has been studied by many mathematicians. However, determining $\mathrm{Aut}(C)$ in general is difficult. In this article, we introduce the notion of a *quasi-Galois point* so that we represent $\mathrm{Aut}(C)$. We infer that quasi-Galois points play an important role for investigating $\mathrm{Aut}(C)$.

Let $C \subset \mathbf{P}^2$ be an irreducible plane curve of degree $d \geq 4$. Consider a point $P \in \mathbf{P}^2$ and let $\pi_P : C \dashrightarrow \mathbf{P}^1$ be the projection from P. We define the set

$$G[P] := \{\tau \in \mathrm{Bir}(C) \mid \pi_P \circ \tau = \pi_P\}$$

of all birational transformations of C preserving the fibers of the projection π_P.

DEFINITION 1.1. If $|G[P]| \geq 2$, then we say that P is a *quasi-Galois point*.

This is a generalization of the Galois point, which was introduced by Hisao Yoshihara in 1996 (see [5, 12, 15]).

We show that $\mathrm{Aut}(C)$ is generated by associated groups $G[P]$ with quasi-Galois points P, in the case where $\mathrm{Aut}(C)$ of a smooth plane curve C is simple and of even order (Theorem 2.7). The Klein quartic and the Wiman sextic are examples of this kind. For the Fermat curve, we prove a similar result (Theorem 3.5). For these results, we determine the defining equation of a curve with a quasi-Galois point (Theorem 2.3), and describe the number of quasi-Galois points for the Fermat curve (Theorem 3.3).

2. A representation of $\mathrm{Aut}(C)$**.** We introduce the system $(X : Y : Z)$ of homogeneous coordinates on $\mathbf{P}^2$ with local coordinates $x = X/Z$, $y = Y/Z$ for the affine open set $Z \neq 0$. The

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line passing through points P and Q is denoted by $\overline{PQ}$, when $P \neq Q$. Let $G_0[P] \subset G[P]$ be the set of all elements of $G[P]$ which are the restrictions of some linear transformations of $\mathbf{P}^2$.

DEFINITION 2.1. A point P is said to be extendable quasi-Galois if $|G_0[P]| \geq 2$.

REMARK 2.2. If C is smooth, then any automorphism is the restriction of a linear transformation (see [1, Appendix A, 17 and 18] or [3]). Therefore, every quasi-Galois point is extendable and $G[P] = G_0[P]$.

THEOREM 2.3 (cf. [10, 15, 16]). *If $p = 0$ or p does not divide the order $|G_0[P]|$, then the group $G_0[P]$ is a cyclic group. Furthermore, for an integer $n \geq 2$, n divides $|G_0[P]|$ if and only if there exists a linear transformation ϕ such that*

(1) $\phi(P) = (1 : 0 : 0)$,
(2) *there exists an element $\sigma \in G_0[\phi(P)] \subset \mathrm{Bir}(\phi(C))$ which is represented by the matrix*

$$A_\sigma = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where ζ is a primitive n-th roof of unity, and
(3) *$\phi(C)$ is given by*

$$\sum_i G_{d-ni}(Y, Z) X^{ni} = 0,$$

where G_{d-ni} is a homogeneous polynomial of degree $d - ni$ in variables Y, Z.

PROOF. We can assume that $P = (1 : 0 : 0)$. The projection π_P is given by $(x : y : 1) \mapsto (y : 1)$. We have a field extension $K(x, y)/K(y)$. Let $\sigma \in G_0[P]$ and $A_\sigma = (a_{ij})$ a matrix representing σ, as $\sigma : (X : Y : Z) \mapsto (X : Y : Z)\,{}^t A_\sigma$. Since $\sigma^*(y) = y$,

$$(a_{21}x + a_{22}y + a_{23}) - (a_{31}x + a_{32}y + a_{33})y = 0$$

in $K(C) = K(x, y)$. Since $d \geq 4$, $a_{21} = a_{23} = a_{31} = a_{32} = 0$ and $a_{22} = a_{33}$. We take the representative matrix with $a_{22} = a_{33} = 1$. Let $N = |G_0[P]|$. If $a_{11} = 1$, then, by $A_\sigma^N = 1$, it follows that $Na_{12} = Na_{13} = 0$. Since N is not divisible by p if $p > 0$, $a_{12} = a_{13} = 0$. Then, there exists an injective homomorphism

$$G_0[P] \hookrightarrow K \setminus \{0\};\ \sigma \mapsto a_{11}(\sigma),$$

where $a_{11}(\sigma)$ is the $(1, 1)$-element of A_σ. Therefore, $G_0[P]$ is a cyclic group.

By the assumption, the order of any element of $|G_0[P]|$ is not divisible by p if $p > 0$. Assume that $n \geq 2$ divides $|G_0[P]|$. Since $G_0[P]$ is a cyclic group, there exists an element

$\sigma \in G_0[P]$ of order n and σ is represented by the matrix

$$A_\sigma = \begin{pmatrix} \zeta & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $a, b \in K$ and ζ is a primitive n-th root of unity. If we take

$$B = \begin{pmatrix} 1 & a & b \\ 0 & 1-\zeta & 0 \\ 0 & 0 & 1-\zeta \end{pmatrix},$$

then

$$B^{-1} A_\sigma B = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We take the linear transformation given by $(X : Y : Z) \mapsto (X : Y : Z)\, {}^t B^{-1}$, so that assertion (2) follows. Let $f(x, y) = \sum_i a_i(y) x^i$ be a defining polynomial with $a_0(y) \neq 0$. Then, $\sigma^* f = \sum_i a_i(y)\zeta^i x^i$. There exists $c \in K$ such that $cf = \sigma^* f$. Since $a_0(y) \neq 0$, it follows that $c = 1$. For all i, $a_i(y) = a_i(y)\zeta^i$. If $a_i(y) \neq 0$, then $\zeta^i = 1$. This implies that n divides i. Assertion (3) follows.

The if-part is obvious. □

COROLLARY 2.4. *For $\sigma \in G_0[P] \setminus \{1\}$, we define $F[P] := \{Q \in \mathbf{P}^2 \mid \sigma(Q) = Q\}$. If we use the standard form as in Theorem* 2.3, *$F[P] = \{P\} \cup \{X = 0\}$. In particular, the set $F[P]$ does not depend on σ.*

If C is smooth, we define

$$G_n(C) = \langle G[P] \mid P : \text{ quasi-Galois with } |G[P]| = n \rangle \subset \mathrm{Aut}(C)$$

after Kanazawa–Takahashi–Yoshihara [8] and Miura–Ohbuchi [11]. Similar to [11, Theorem 1], we have the following.

THEOREM 2.5. *Let C be smooth. Then, $G_n(C)$ is a normal subgroup of* $\mathrm{Aut}(C)$.

On the other hand, we have the following.

PROPOSITION 2.6. *Let $p \neq 2$, and let C be smooth. The following conditions are equivalent.*

(1) *The order* $|\mathrm{Aut}(C)|$ *is even.*
(2) *The group* $\mathrm{Aut}(C)$ *contains an involution.*
(3) *There exists a quasi-Galois point P such that $|G[P]|$ is even.*

PROOF. If $|\mathrm{Aut}(C)|$ is even, then a Sylow 2-group contains an element of order two. Therefore, assertion (1) $\Rightarrow$ (2) follows. Since $G[P]$ is a subgroup of $\mathrm{Aut}(C)$, the assertion (3) $\Rightarrow$ (1) is obvious.

We prove (2) $\Rightarrow$ (3). Let $\sigma \in \mathrm{Aut}(C)$ be an involution and A_σ a matrix representing σ. Since $A_\sigma^2 = \lambda I_3$ for some $\lambda \neq 0$, where I_3 is the identity matrix, we can assume that $A_\sigma^2 = I_3$. The eigenvalues of A_σ are 1 or -1. For any vector $x \in K^3$, $(A - E)x$ and $(A + E)x$ are contained in the direct sum of eigenspaces, since $(A+E)(A-E) = (A-E)(A+E) = 0$. Then, $2x = -(A-E)x + (A+E)x$ also. By the assumption $p \neq 2$, the direct sum of eigenspaces spans the vector space K^3. We find that A_σ is diagonalizable. For a suitable system of coordinates, we can assume that A_σ is one of the following matrices:

$$\pm\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \pm\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \pm\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

It follows from Theorem 2.3 that $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$ are quasi-Galois, for each case. Then, $|G[P]|$ is even. □

Combining Theorem 2.5 and Proposition 2.6, we have the following.

THEOREM 2.7. *Let $p \neq 2$, and let C be smooth. If* $\mathrm{Aut}(C)$ *is simple and* $|\mathrm{Aut}(C)|$ *is even, then* $\mathrm{Aut}(C) = G_n(C)$ *for some even integer $n \geq 2$. For example, if $p = 0$, and C is the Klein quartic or the Wiman sextic (see* [4, 14] *and* [7, Remark 2.4]*), then* $\mathrm{Aut}(C) = G_2(C)$.

PROOF. By Proposition 2.6, if $|\mathrm{Aut}(C)|$ is even, then there exists a quasi-Galois point P with $|G[P]| = n$, where n is an even integer. Since $G_n(C) \neq \{1\}$ is a normal subgroup by Theorem 2.5, by the assumption that $\mathrm{Aut}(C)$ is simple, it follows that $G_n(C) = \mathrm{Aut}(C)$.

For the Klein quartic and the Wiman sextic, $\mathrm{Aut}(C)$ is a simple group of even order (see [2, Section 232], [6, pp.348–349], [9] for the Klein quartic and see [14] for the Wiman sextic). Since if $G_n(C) \neq \{1\}$ then the number n must divide $d = 4$ (resp., $d = 6$), n is equal to two or four (resp., two or six) for the Klein quartic (resp., the Wiman sextic). It follows from [12, Example 4.8] that there does not exist a point P with $|G[P]| = 4$ for the Klein quartic. For the Wiman sextic, if there exists a point P with $|G[P]| = 6$, then there exists an element of order six, by Theorem 2.3. Since $\mathrm{Aut}(C) \cong A_6$ for the Wiman sextic, there does not exist an element of order six. Therefore, $n = 2$. □

3. Fermat curves.

In this section, we assume that C is the Fermat curve $X^d + Y^d + Z^d = 0$ of degree $d \geq 4$ in $p = 0$. Let $P_1 = (1:0:0)$, $P_2 = (0:1:0)$, and let $P_3 = (0:0:1)$. It follows from Yoshihara–Miura's theorem [12, 15] that $|G[P_i]| = d$ (i.e. P_i is outer Galois) for each i, and the number of points $P \in \mathbf{P}^2 \setminus C$ with $|G[P]| = d$ is three.

PROPOSITION 3.1. (1) *Let $d = 2n$, and let ζ satisfy $\zeta^{2n} = 1$. Then, points $(\zeta : 0 : 1)$, $(1 : \zeta : 0)$ and $(0 : 1 : \zeta)$ are quasi-Galois points in $\mathbf{P}^2 \setminus C$.*

(2) *Let $d = 2n + 1$, and let η satisfy $\eta^{2n+1} = -1$. Then, points $(\eta : 0 : 1)$, $(1 : \eta : 0)$ and $(0 : 1 : \eta)$ are quasi-Galois points in C.*

PROOF. Let P be one of the points given in (1). Since C is invariant under the linear transformations $(X : Y : Z) \mapsto (Z : X : Y)$ and $(X : Y : Z) \mapsto (\zeta^{-1}X : Y : Z)$, we can assume that $P = (1 : 0 : 1)$. We take the linear transformation $\phi : (X : Y : Z) \mapsto (X + Z : Y : X - Z)$.

Then, $\phi^{-1}(P) = (1 : 0 : 0)$ and $\phi^{-1}(C)$ is given by

$$(X+Z)^{2n} + (X-Z)^{2n} + Y^{2n} = 2\sum_{k=0}^{n}\binom{2n}{2k}X^{2n-2k}Z^{2k} + Y^{2n} = 0\,.$$

By Theorem 2.3, $\phi^{-1}(P)$ is quasi-Galois. Therefore, P is quasi-Galois.

Let P be one of the points given in (2). Since C is invariant under the linear transformations $(X : Y : Z) \mapsto (Z : X : Y)$ and $(X : Y : Z) \mapsto (-\eta^{-1}X : Y : Z)$, we can assume that $P = (-1 : 0 : 1)$. We take the linear transformation $\phi : (X : Y : Z) \mapsto (X+Z : Y : X-Z)$. Then, $\phi^{-1}(P) = (0 : 0 : 1)$ and $\phi^{-1}(C)$ is given by

$$(X+Z)^{2n+1} + (X-Z)^{2n+1} + Y^{2n+1} = 2\sum_{k=0}^{n}\binom{2n}{2k}X^{2n+1-2k}Z^{2k} + Y^{2n+1} = 0\,.$$

By Theorem 2.3, $\phi^{-1}(P)$ is quasi-Galois. Therefore, P is quasi-Galois. □

PROPOSITION 3.2. *If a point P satisfies $1 < |G[P]| < d$, then $|G[P]| = 2$ and $P \in \{XYZ = 0\}$. Furthermore, the number of such points is at most $3d$.*

PROOF. We prove that $P \in \{XYZ = 0\}$. Assume for contradiction that $P \notin \{XYZ = 0\}$. Let $\sigma \in G[P]\setminus\{1\}$. The point P_1 is the only outer Galois point on the line $\overline{PP_1}$. It follows from Theorem 2.3 that $\sigma(\overline{PP_1}) = \overline{PP_1}$. Then, $\sigma(P_1) = P_1$. Similarly, $\sigma(P_2) = P_2$ and $\sigma(P_3) = P_3$. By Corollary 2.4, $\sigma(P) = P$. Since σ fixes non-collinear three points P_1, P_2, P_3 and the point $P \notin \bigcup_{i\neq j}\overline{P_iP_j}$, σ is identity on $\mathbf{P}^2$. This is a contradiction.

We can assume that $P \in \overline{P_1P_2}$. Let $\sigma \in G[P]$ be a generator. Since P_1 and P_2 are the only outer Galois points on the line $\overline{P_1P_2}$, σ acts on the set $\{P_1, P_2\}$. Note that $\sigma(P) = P$. Then, the restriction $\sigma|_{\overline{P_1P_2}}$ is of order two. By Theorem 2.3, the order of σ is equal to the order of $\sigma|_\ell$ for each line $\ell \ni P$. Therefore, $|G[P]| = 2$.

We prove the latter assertion. Since the fixed field $K(C)^{G[P]}$ is an intermediate field of $K(C)/\pi_P^*K(\mathbf{P}^1)$, if d is odd (resp., d is even), then $P \in C$ (resp., $P \in \mathbf{P}^2 \setminus C$). If d is odd, then the assertion is obvious, since the number of points in $C \cap \{XYZ = 0\}$ is $3d$. Assume that d is even. Then, $P \in \mathbf{P}^2 \setminus C$. Let $P \in \overline{P_1P_2}$ with $|G[P]| = 2$ and $\sigma \in G[P]$ a generator. Since $\sigma^2 = 1$, $\sigma(P_3) = P_3$ and σ acts on the set $\{P_1, P_2\}$, σ is represented by the matrix

$$\begin{pmatrix} 0 & \xi & 0 \\ \xi^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for some $\xi \in K$. Considering the action on the defining equation $X^d + Y^d + Z^d = 0$, $\xi^d = 1$ follows. Assume that

$$\begin{pmatrix} x \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & \xi & 0 \\ \xi^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \xi \\ \xi^{-1}x \\ 0 \end{pmatrix},$$

up to a constant. Then, $x = \pm\xi$. The only fixed points of σ on the line $\overline{P_1P_2}$ are $(\xi : 1 : 0)$ and $(-\xi : 1 : 0) \in \mathbf{P}^2 \setminus C$. It follows from Corollary 2.4 that P must be $(\xi : 1 : 0)$ or $(-\xi : 1 : 0)$, and hence, $P \in \{(\zeta : 1 : 0) \mid \zeta^d = 1\}$. Therefore, there exist at most d quasi-Galois points on the line $\overline{P_1P_2}$. □

The number of quasi-Galois points $P \in C$ with $|G[P]| = n$ (resp., $|G[P]| \geq n$) is denoted by $\delta[n]$ (resp., $\delta[\geq n]$). Similarly, we define $\delta'[n]$ and $\delta'[\geq n]$, when we consider the case $P \in \mathbf{P}^2 \setminus C$.

THEOREM 3.3. *Let C be the Fermat curve of degree d. Then:*

(1) $\delta'[d] = 3$.
(2) *If d is even, then* $\delta'[\geq 2] = \delta'[d] + \delta'[2] = 3 + 3d$ *and* $\delta[\geq 2] = 0$.
(3) *If d is odd, then* $\delta'[\geq 2] = \delta'[d] = 3$ *and* $\delta[\geq 2] = \delta[2] = 3d$.

PROOF. Assertion (1) is nothing but Yoshihara–Miura's theorem [12, 15]. By Propositions 3.1 and 3.2, assertions (2) and (3) follow. □

REMARK 3.4. For $d = 4, 5$, this theorem was obtained by Miura–Yoshihara [12, 13, 15].

THEOREM 3.5. *Let C be the Fermat curve of degree d. Then:*

(1)
$$\langle G_2(C), G_d(C) \rangle = \mathrm{Aut}(C)\,.$$

(2) *The assertion $G_2(C) \neq \langle G_2(C), G_d(C) \rangle$ holds if and only if d is divisible by* 3. *In this case, there exists a split exact sequence*
$$0 \to (\mathbf{Z}/(d/3)\mathbf{Z}) \oplus (\mathbf{Z}/d\mathbf{Z}) \to G_2(C) \to S_3 \to 1$$
of groups, where S_3 is the symmetric group of degree three, and $|G_2(C)| = 2d^2$.

PROOF. We consider (1). Let $\sigma \in \mathrm{Aut}(C) \subset \mathrm{Aut}(\mathbf{P}^2)$. Then, σ acts on the set $\{P_1, P_2, P_3\}$ of outer Galois points. If $\sigma(P_1) = P_2$, then there exists $\phi_1 \in G_2(C)$ such that $\phi_1\sigma(P_1) = P_1$, by using an action associated with a quasi-Galois point on the line $\overline{P_1P_2}$. If $\phi_1\sigma(P_2) = P_3$, then we take $\phi_2 \in G_2(C)$ such that $\phi_2(P_1) = P_1$ and $\phi_2(P_3) = P_2$, which comes from an action associated with a quasi-Galois point on the line $\overline{P_2P_3}$. Therefore, there exists $\phi \in G_2(C)$ such that $\phi\sigma(P_i) = P_i$ for $i = 1, 2, 3$. Then, $\phi\sigma$ is represented by the matrix of the form
$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
for some $\alpha, \beta \in K$. By considering the action on the defining equation $X^d + Y^d + Z^d = 0$, it follows that $\alpha^d = 1$ and $\beta^d = 1$. If we take
$$\phi_3 = \begin{pmatrix} \alpha^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G[P_1],\ \phi_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G[P_2]\,,$$
then $\phi_4\phi_3\phi\sigma = 1$ on $\mathbf{P}^2$. Therefore, $\sigma = \phi^{-1}\phi_3^{-1}\phi_4^{-1} \in \langle G_2(C), G_d(C) \rangle$.

We consider (2). Let ζ be a primitive d-th root of unity. Note that $3d$ involutions of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \zeta^i \\ 0 & \zeta^{-i} & 0 \end{pmatrix} (=: \sigma_i), \begin{pmatrix} 0 & 0 & \zeta^{-i} \\ 0 & 1 & 0 \\ \zeta^i & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \zeta^i & 0 \\ \zeta^{-i} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for $i = 1, \dots, d$ act on C. By Proposition 2.6 and Theorem 3.3, they generate $G_2(C)$. Let τ_1 (resp., τ_2) be the involution given by $(X : Y : Z) \mapsto (X : Z : Y)$ (resp., $(X : Y : Z) \mapsto (Z : Y : X)$). It follows that

$$\eta_1 := \sigma_1\tau_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^{-1} \end{pmatrix} \text{ and } \eta_2 := \tau_2^{-1}\eta_1\tau_2 = \begin{pmatrix} \zeta^{-1} & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta \end{pmatrix},$$

and $\eta_1, \eta_2 \in G_2(C)$. Then,

$$\eta_1\eta_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G[P_2].$$

If d is not divisible by 3, then $\eta_1\eta_2$ generates $G[P_2]$ and hence, $G[P_2] \subset G_2(C)$. Considering actions of $\tau_1, \tau_2 \in G_2(C)$, it follows that $G_d(C) \subset G_2(C)$. The only-if part of (2) follows. Assume that d is divisible by 3. Let $H_1 := \langle \eta_1, \eta_1\eta_2 \rangle$ and $H_2 := \langle \tau_1, \tau_2 \rangle$. Then, $\langle H_1, H_2 \rangle = G_2(C)$, H_1 is an abelian group isomorphic to $(\mathbf{Z}/(d/3)\mathbf{Z}) \oplus \mathbf{Z}/d\mathbf{Z}$, and $H_2 \cong S_3$. Since $\tau_i^{-1}\eta_j\tau_i \in H_1$ for $i, j = 1, 2$ (for example, $\tau_1^{-1}\eta_2\tau_1 = \eta_2\eta_1^{-1} \in H_1$), H_1 is a normal subgroup of $G_2(C)$. Note that any element of $G_2(C)$ is given by $\eta\tau$ for some $\eta \in H_1$ and $\tau \in H_2$. Then, there exists a split exact sequence

$$0 \to (\mathbf{Z}/(d/3)\mathbf{Z}) \oplus (\mathbf{Z}/d\mathbf{Z}) \to G_2(C) \to S_3 \to 1$$

of groups, and $|G_2(C)| = (d/3)d \times 6 = 2d^2$. Since

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G[P_2] - G_2(C),$$

it follows that $G_2(C) \neq \langle G_2(C), G_d(C) \rangle$. □

REMARK 3.6. According to [11, Example 2], $G_d(C) \neq \mathrm{Aut}(C)$.

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