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ON THE CURVATURE OF THE FEFFERMAN METRIC OF CONTACT RIEMANNIAN MANIFOLDS

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Abstract. It is known that a contact Riemannian manifold carries a generalized Fefferman metric on a circle bundle over the manifold. We compute the curvature of the metric explicitly in terms of a modified Tanno connection on the underlying manifold. In particular, we show that the scalar curvature descends to the pseudohermitian scalar curvature multiplied by a certain constant. This is an answer to a problem considered by Blair-Dragomir.

1. Introduction. Let (M, θ) be a $(2n+1)$-dimensional contact manifold with a contact form θ. There is a unique vector field ξ such that $\xi \rfloor \theta = 1$ and $\xi \rfloor d\theta = 0$. Let us equip M with a Riemannian metric g and a $(1,1)$-tensor field J which satisfy $g(\xi, X) = \theta(X)$, $g(X, JY) = -d\theta(X, Y)$ and $J^2 X = -X + \theta(X)\xi$ for any vector fields X, Y. We set $H = \ker\theta$, $H_\pm = \{X \in H \otimes \mathbb{C} \mid JX = \pm iX\}$. In this paper we adopt such a notation as $(\omega_1 \wedge \cdots \wedge \omega_k)(X_1, \ldots, X_k) = \det(\omega_i(X_j))$ for 1-forms ω_i and vectors X_j, and, hence, $d\theta(X, Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y])$. To study the contact Riemannian manifold (M, θ, g, J), Tanno ([10]) introduced a generalized Tanaka-Webster connection ${}^*\nabla$, called the Tanno connection in this paper, given by

$$ {}^*\nabla_X Y = \nabla^g_X Y - \frac{1}{2}\theta(X)JY - \theta(Y)\nabla^g_X \xi + (\nabla^g_X \theta)(Y)\xi $$

(∇^g is the Levi-Civita connection of g), whose action does not commute with that of the almost complex structure J in general, however. In fact, he showed

$$ ({}^*\nabla_X J)Y = Q(Y, X) := (\nabla^g_X J)Y + (\nabla^g_X \theta)(JY)\,\xi + \theta(Y) J \nabla^g_X \xi\,. $$

The author ([7]) considered a modified Tanno connection ∇, called the hermitian Tanno connection, defined by

$$ \nabla_X Y = {}^*\nabla_X Y - \frac{1}{2} JQ(Y, X) = \begin{cases} {}^*\nabla_X(f\xi) & : Y = f\xi \quad (f \in C^\infty(M)), \\ \frac{1}{2}\left({}^*\nabla_X Y - J\,{}^*\nabla_X JY\right) & : Y \in \Gamma(H), \end{cases} $$

so that $\nabla J = 0$. This has been profitably employed by the author et al. in investigating the subjects relating to the Kohn-Rossi Laplacian, the CR conformal Laplacian and Bochner type tensors, etc., on contact Riemannian manifolds ([7], [5] (with Imai), [8], [9] (with Sasaki)).

In this paper, our study by means of the connection focuses on a generalized Fefferman metric $G = G_\theta$ (cf. (2.4)) of the contact Riemannian manifold M, i.e., a Lorentz metric on the

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total space of a canonical $U(1)$-bundle $\pi : F(M) \to M$, introduced by Barletta-Dragomir in [1, §6]: recall that the ordinary one ([4], [6]) is restricted to the case where J is integrable, i.e., $[\Gamma(H_+), \Gamma(H_+)] \subset \Gamma(H_+)$. After preliminaries in §2 through §4, we will present an explicit description of the curvature $F(\nabla^G)$ of the Levi-Civita connection ∇^G of G in §5. In particular, the following formula for the scalar curvature will be confirmed in the last paragraph.

THEOREM 1.1. *We have*

$$s(\nabla^G) = \frac{2(2n+1)}{n+1}\,\pi^* s^\nabla\,,$$

where s^∇ is the pseudohermitian scalar curvature of ∇.

If J is integrable, in other words, if the Tanno tensor Q vanishes (cf. [10, Proposition 2.1]), then the connections ${}^*\nabla$, ∇ and the Tanaka-Webster connection coincide (cf. [10, Proposition 3.1], [7, Lemma 1.1], [3, §1.2]), and accordingly the generalized Fefferman metric also coincides with the ordinary one (cf. the comment following (2.4)). The theorem is thus a generalization of Lee's result [6, Theorem 6.2] and is an answer to the problem remaining in Blair-Dragomir's paper [2, Remark 5]. The author is uncertain whether the Chern-Moser normal form theory employed by the easier proof of [6, Theorem 6.2] has improved enough to be applicable to the non-integrable case. In this paper we intend to calculate the curvature directly as Lee did for the proof of [6, Theorem 6.6]. It is rather simplified by considering the concept of hermitian Tanno connection, the formulas (2.7) and a derived connection $\pi_*\nabla^G$ (cf. §3).

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2. The connections ${}^*\nabla$ and ∇, and the (generalized) Fefferman metric of contact Riemannian manifolds. First, let us collect some properties of the connections for quick reference. Refer to [10], [7], [9] for more detailed explanation. We have ${}^*\nabla\theta = \nabla\theta = 0$, ${}^*\nabla g = \nabla g = 0$, $T({}^*\nabla)(Z, W) = 0$, $T({}^*\nabla)(Z, \overline{W}) = ig(Z, \overline{W})\xi$, $T(\nabla)(Z, W) = [J, J](Z, W)/4 := (-[Z, W] + [JZ, JW] - J[JZ, W] - J[Z, JW])/4$, $T(\nabla)(Z, \overline{W}) = ig(Z, \overline{W})\xi$ $(Z, W \in \Gamma(H_+))$, where $T({}^*\nabla)$, etc., are the torsion tensors. If we set ${}^*\tau X = T({}^*\nabla)(\xi, X)$, etc., then ${}^*\tau = \tau$ and $\tau \circ J + J \circ \tau = 0$. In this paper, a local frame $\xi_\bullet = (\xi_0 = \xi, \xi_1, \dots, \xi_n, \xi_{\bar{1}}, \dots, \xi_{\bar{n}})$ $(\xi_{\bar{\alpha}} := \overline{\xi_\alpha} \in H_-)$ of the bundle $TM \otimes \mathbb{C} = \mathbb{C}\xi \oplus H_+ \oplus H_-$ is always assumed to be unitary, i.e., $g(\xi_\alpha, \xi_\beta) = 0$, $g(\xi_\alpha, \xi_{\bar{\beta}}) = \delta_{\alpha\beta}$ $(1 \le \alpha, \beta \le n)$, and its dual frame is denoted by $\theta^\bullet = (\theta^0 = \theta, \theta^1, \dots, \theta^n, \theta^{\bar{1}}, \dots, \theta^{\bar{n}})$. As usual the Greek indices α, β, ... vary from 1 to n, the block Latin indices A, B, ... vary in $\{0, 1, \dots, n, \bar{1}, \dots, \bar{n}\}$, and the summation symbol $\sum$ will be omitted in an unusual manner. We have

$$\tau = \xi_\alpha \otimes \theta^{\bar{\gamma}} \cdot \tau^\alpha_{\bar{\gamma}} + \xi_{\bar{\alpha}} \otimes \theta^\gamma \cdot \tau^{\bar{\alpha}}_\gamma \quad (\tau^{\bar{\alpha}}_\gamma = \tau^{\bar{\gamma}}_\alpha)\,,$$
$$Q = \xi_\alpha \otimes \theta^{\bar{\beta}} \otimes \theta^{\bar{\gamma}} \cdot Q^\alpha_{\bar{\beta}\bar{\gamma}} + \xi_{\bar{\alpha}} \otimes \theta^\beta \otimes \theta^\gamma \cdot Q^{\bar{\alpha}}_{\beta\gamma} \quad (Q^{\bar{\alpha}}_{\beta\gamma} = -Q^{\bar{\beta}}_{\alpha\gamma} = -Q^{\bar{\beta}}_{\gamma\alpha} - Q^{\bar{\gamma}}_{\alpha\beta})\,.$$

If we set ${}^*\nabla\xi_B = \xi_A \cdot \omega({}^*\nabla)^A_B$, $\nabla\xi_B = \xi_A \cdot \omega(\nabla)^A_B$, then

$$\omega({}^*\nabla)^\alpha_\beta = \omega(\nabla)^\alpha_\beta,\ \ \omega({}^*\nabla)^{\bar{\alpha}}_{\bar{\beta}} = \omega(\nabla)^{\bar{\alpha}}_{\bar{\beta}},\ \ \omega({}^*\nabla)^{\bar{\alpha}}_\beta(\xi_\gamma) = -\frac{i}{2}Q^{\bar{\alpha}}_{\beta\gamma},\ \ \omega({}^*\nabla)^\alpha_{\bar{\beta}}(\xi_{\bar{\gamma}}) = \frac{i}{2}Q^\alpha_{\bar{\beta}\bar{\gamma}}$$

and the others vanish. Let us mention briefly also the pseudohermitian Ricci curvature $\mathrm{Ric}^\nabla(X,Y) := \sum g(F(\nabla)(X,Y)\xi_\nu, \xi_{\bar\nu})$, the pseudohermitian scalar curvature $s^\nabla := \sum \mathrm{Ric}^\nabla(\xi_\alpha, \xi_{\bar\alpha})$ and the ordinary ones $\mathrm{Ric}(\nabla)(X,Y) := \mathrm{tr}_{TM}(Z \mapsto F(\nabla)(Z,Y)X) = \sum g(F(\nabla)(\xi_\nu, Y)X, \xi_{\bar\nu}) + \sum g(F(\nabla)(\xi_{\bar\nu}, Y)X, \xi_\nu)$, etc.

PROPOSITION 2.1 (cf. [9, Proposition 1.1 and 1.2]). *We have*

$$\mathrm{Ric}^\nabla(\xi_\alpha, \xi_{\bar\beta}) = F(\nabla)^\nu_{\nu\alpha\bar\beta} = F(\nabla^g)^\nu_{\nu\alpha\bar\beta} - \left\{\frac{1}{4}Q^{\bar\mu}_{\nu\alpha}Q^{\mu}_{\bar\nu\bar\beta} + \tau^{\bar\nu}_{\alpha}\tau^{\nu}_{\bar\beta}\right\} + \frac{2n+1}{4}\delta_{\alpha\beta},$$
$$\mathrm{Ric}^\nabla(\xi_\alpha, \xi_\beta) = \frac{i}{2}(\nabla_{\xi_{\bar\nu}} Q)^{\bar\beta}_{\alpha\nu}, \quad \mathrm{Ric}^\nabla(\xi_\alpha, \xi) = (\nabla_{\xi_{\bar\mu}}\tau)^{\bar\alpha}_{\mu} + \frac{i}{2}\tau^{\nu}_{\bar\mu}Q^{\bar\alpha}_{\nu\mu},$$
$$\mathrm{Ric}^{*\nabla}(\xi_\alpha, \xi_{\bar\beta}) = \mathrm{Ric}^\nabla(\xi_\alpha, \xi_{\bar\beta}) + \frac{1}{4}Q^{\bar\mu}_{\nu\alpha}Q^{\mu}_{\bar\nu\bar\beta},$$
$$\mathrm{Ric}^{*\nabla}(\xi_\alpha, \xi_\beta) = \mathrm{Ric}^\nabla(\xi_\alpha, \xi_\beta), \quad \mathrm{Ric}^{*\nabla}(\xi_\alpha, \xi) = \mathrm{Ric}^\nabla(\xi_\alpha, \xi),$$
$$s^{*\nabla} = s^\nabla + \frac{1}{4}\sum \left|Q^{\bar\mu}_{\nu\alpha}\right|^2$$

and $\mathrm{Ric}^\nabla(\overline{X}, \overline{Y}) = -\overline{\mathrm{Ric}^\nabla(X,Y)}$, *etc. In addition, we have*

$$\mathrm{Ric}(\nabla)(\xi_\alpha, \xi_{\bar\beta}) = \mathrm{Ric}^\nabla(\xi_\alpha, \xi_{\bar\beta}) - \frac{1}{4}Q^{\bar\alpha}_{\nu\mu}Q^{\beta}_{\bar\mu\bar\nu},$$
$$\mathrm{Ric}(\nabla)(\xi_\alpha, \xi_\beta) = \frac{i}{2}(\nabla_{\xi_{\bar\nu}} Q)^{\bar\beta}_{\nu\alpha} + i(n-1)\tau^{\bar\alpha}_{\beta}, \quad \mathrm{Ric}(\nabla)(\xi_\alpha, \xi) = (\nabla_{\xi_{\bar\mu}}\tau)^{\bar\alpha}_{\mu},$$
$$\mathrm{Ric}({}^*\nabla)(\xi_\alpha, \xi_{\bar\beta}) = \mathrm{Ric}(\nabla)(\xi_\alpha, \xi_{\bar\beta}) + \frac{1}{4}\left(Q^{\bar\mu}_{\nu\alpha}Q^{\mu}_{\bar\nu\bar\beta} - Q^{\bar\alpha}_{\nu\mu}Q^{\beta}_{\bar\mu\bar\nu}\right),$$
$$\mathrm{Ric}({}^*\nabla)(\xi_\alpha, \xi_\beta) = \mathrm{Ric}(\nabla)(\xi_\alpha, \xi_\beta) + \frac{i}{2}(\nabla_{\xi_{\bar\nu}} Q)^{\bar\alpha}_{\nu\beta},$$
$$\mathrm{Ric}({}^*\nabla)(\xi_\alpha, \xi) = \mathrm{Ric}(\nabla)(\xi_\alpha, \xi) - \frac{i}{2}\tau^{\mu}_{\bar\nu}Q^{\bar\alpha}_{\mu\nu},$$
$$s({}^*\nabla) = s(\nabla) = 2s^\nabla$$

and $\mathrm{Ric}(\nabla)(\xi, Y) = 0$, $\mathrm{Ric}(\nabla)(\overline{X}, \overline{Y}) = \overline{\mathrm{Ric}(\nabla)(X,Y)}$, *etc.*

Next, let us recall the definition of a generalized Fefferman metric introduced by Barletta-Dragomir [1, §6]. The canonical bundle $\pi_0 : K(M) := \{\omega \in \wedge^{n+1}T^*M \otimes \mathbb{C} \mid X\rfloor\omega = 0 \ (X \in H_-)\} \to M$ carries a natural tautologous $(n+1)$-form Υ on $K(M)$, whose value at $\omega \in K(M)$ is the lift to $K(M)$ of ω itself. We set $K^0(M) = \{\omega \in K(M) \mid \omega \neq 0\}$ and consider the canonical $U(1)$-bundle $\pi : F(M) := K^0(M)/\mathbb{R}_+ \to M$. There is a natural embedding

$$\iota_\theta : F(M) \to K(M), \quad \iota_\theta([\omega]) = \frac{1}{\sqrt{\lambda}}\,\omega,$$

where $\lambda \in C^\infty(M, \mathbb{R}_+)$ is uniquely defined by $i^{n^2} n!\,\theta \wedge (\xi\rfloor\omega) \wedge (\xi\rfloor\overline{\omega}) = \lambda\,\theta \wedge (d\theta)^n$. This induces a differential form

$$\mathcal{Z} = \iota_\theta^*\Upsilon \ \in \Gamma(\wedge^{n+1}T^*F(M) \otimes \mathbb{C}).$$

Given a local unitary frame $\theta^\bullet$ of $T^*M \otimes \mathbb{C}$ over an open set U, there is a local trivialization

$$F(M)|U \cong U \times [0, 2\pi), \quad [\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n](p) \cdot e^{i\varphi} \leftrightarrow (p, \varphi), \tag{2.1}$$

via which $\mathcal{Z}_{[\omega]} = e^{i\varphi([\omega])}\pi^*(\theta\wedge\theta^1\wedge\cdots\wedge\theta^n)_{[\omega]}$ (cf. [1, Lemma 4]). The local forms $\rho_{[\omega]} := e^{i\varphi([\omega])}\pi^*(\theta^1\wedge\cdots\wedge\theta^n)_{[\omega]}$ on $F(M)|U$ determine all together a global n-form ρ on $F(M)$, which satisfies $\mathcal{Z} = \pi^*\theta\wedge\rho$ and $V\rfloor\rho = 0$ for any lift V of ξ to $F(M)$. [1, Lemma 5] indicates that a global n-form ρ satisfying the conditions is actually unique.

PROPOSITION 2.2 (cf. Barletta-Dragomir [1, Proposition 3], Blair-Dragomir [2, §4.2]). (1) *There is a unique real* 1*-form* σ *on* $F(M)$ *such that*

$$d\mathcal{Z} = i\,(n+2)\sigma\wedge\mathcal{Z} + e^{i\varphi}\,\pi^*\mathcal{W}\,, \tag{2.2}$$

$$\sigma\wedge d\rho\wedge\overline{\rho} = \mathrm{tr}(d\sigma)\,i\sigma\wedge(\pi^*\theta)\wedge\rho\wedge\overline{\rho}, \tag{2.3}$$

where $\mathcal{W}$ *is the* $(n+2)$*-form on* M *given by*

$$\mathcal{W} = \frac{i}{2}\,\theta\wedge\sum(-1)^\alpha\theta^1\wedge\cdots\wedge\left(Q^\alpha_{\bar\beta\bar\gamma}\theta^{\bar\beta}\wedge\theta^{\bar\gamma}\right)\wedge\cdots\wedge\theta^n \quad (on\ U)$$

and, for a 2*-form* $\Phi = i\,\Phi_{\alpha\bar\beta}\,\pi^*\theta^\alpha\wedge\pi^*\theta^{\bar\beta} + \cdots$ *on* $F(M)$*, we set* $\mathrm{tr}(\Phi) = \Phi_{\alpha\bar\alpha}$.

(2) *On* $F(M)|U$*, the* 1*-form* σ *is expressed as*

$$\sigma = \frac{1}{n+2}\Big\{d\varphi + \pi^*\Big(i\omega(\nabla)^\alpha_\alpha - \frac{s^\nabla}{2(n+1)}\theta\Big)\Big\}\,.$$

PROOF. Let us verify (2). We set $\Upsilon_0 = \theta\wedge\theta^1\wedge\cdots\wedge\theta^n$. Since $d\theta = i\theta^\alpha\wedge\theta^{\bar\alpha}$ and $d\theta^\alpha = \theta^\beta\wedge\omega({}^*\nabla)^\alpha_\beta + \theta^{\bar\beta}\wedge\omega({}^*\nabla)^\alpha_{\bar\beta} + \theta\wedge\tau^\alpha = \omega(\nabla)^\alpha_\beta(\xi_{\bar\gamma})\,\theta^\beta\wedge\theta^{\bar\gamma} + \frac{i}{2}Q^\alpha_{\bar\beta\bar\gamma}\,\theta^{\bar\beta}\wedge\theta^{\bar\gamma} + \cdots$,

$$\begin{aligned} d\,\Upsilon_0 &= d\theta\wedge\theta^1\wedge\cdots\wedge\theta^n + \theta\wedge\sum(-1)^\alpha\theta^1\wedge\cdots\wedge d\theta^\alpha\wedge\cdots\wedge\theta^n \\ &= \theta\wedge\sum(-1)^\alpha\theta^1\wedge\cdots\wedge\Big\{\omega(\nabla)^\alpha_\alpha(\xi_{\bar\gamma})\,\theta^\alpha\wedge\theta^{\bar\gamma} + \frac{i}{2}Q^\alpha_{\bar\beta\bar\gamma}\,\theta^{\bar\beta}\wedge\theta^{\bar\gamma}\Big\}\wedge\cdots\wedge\theta^n \\ &= -\,\omega(\nabla)^\alpha_\alpha\wedge\Upsilon_0 + \mathcal{W}\,. \end{aligned}$$

Hence, for any $f\in C^\infty(M,\mathbb{R})$, the global real 1-form σ on $F(M)$ defined by

$$\sigma = \frac{1}{n+2}\Big\{d\varphi + \pi^*\,i\omega(\nabla)^\alpha_\alpha\Big\} + \pi^*(f\theta) \quad (\text{on } F(M)|U)$$

satisfies (2.2). In addition, we have

$$\begin{aligned} \sigma\wedge d\rho\wedge\overline{\rho} &= \sigma\wedge\Big(id\varphi\wedge\rho - \pi^*\omega(\nabla)^\alpha_\alpha(\xi)\pi^*\theta\wedge\rho\Big)\wedge\overline{\rho} \\ &= -\,i\pi^*f\cdot d\varphi\wedge(\pi^*\theta)\wedge\rho\wedge\overline{\rho} \end{aligned}$$

and

$$\begin{aligned} \mathrm{tr}(d\sigma) &= \pi^*\Big\{-i\Big(\frac{i}{n+2}\,d\,\omega(\nabla)^\alpha_\alpha + df\wedge\theta + f\,d\theta\Big)(\xi_\gamma,\xi_{\bar\gamma})\Big\} \\ &= \pi^*\Big(\frac{1}{n+2}d\,\omega(\nabla)^\alpha_\alpha(\xi_\gamma,\xi_{\bar\gamma}) + nf\Big) = \pi^*\Big(\frac{s^\nabla}{n+2} + nf\Big), \\ \mathrm{tr}(d\sigma)\,i\sigma\wedge(\pi^*\theta)\wedge\rho\wedge\overline{\rho} &= \frac{i}{n+2}\pi^*\Big(\frac{s^\nabla}{n+2} + nf\Big)\cdot d\varphi\wedge(\pi^*\theta)\wedge\rho\wedge\overline{\rho}\,. \end{aligned}$$

Consequently, (2.3) also holds for σ with $f = -s^\nabla/2(n+1)(n+2)$. □

Now, the (*generalized*) *Fefferman metric* of the contact Riemannian manifold (M, θ, g, J) is the Lorentz metric G_θ on $F(M)$ (cf. [1, (60)]) given by

$$G_\theta = \frac{1}{2}(\pi^*\theta^\alpha \otimes \pi^*\theta^{\bar\alpha} + \pi^*\theta^{\bar\alpha} \otimes \pi^*\theta^\alpha) + (\pi^*\theta \otimes \sigma + \sigma \otimes \pi^*\theta), \tag{2.4}$$

which certainly coincides with the ordinary one (cf. [4], [6]) in the case J is integrable (i.e., $Q = 0$). One finds its systematic study in [1], [2]. For example, it is invariant of weight -2 under the CR conformal change $\theta \Rightarrow e^{2f}\theta$ (together with canonical changes of unitary frames $\xi_\bullet$ and $\theta^\bullet$), i.e., $G_{e^{2f}\theta} = e^{2f} G_\theta$ ([2, Theorem 11]), which is the contact Riemannian analogue of Lee's result [6, Theorem 3.8]. As stated in the introduction, Theorem 1.1 is that of his another result [6, Theorem 6.2].

Last, let us introduce an assertion, which is obvious but plays an important role in the study of the curvature.

PROPOSITION 2.3. *The* $\mathfrak{u}(1)$*-valued* 1*-form* $i(n+2)\sigma \in \Gamma(\mathfrak{u}(1) \otimes T^*F(M))$ *is an Ehresmann-type connection on the* $U(1)$*-bundle* $F(M)$ *over* M*. That is, it satisfies the invariance conditions* $i(n+2)\sigma\left(\frac{d(R_{e^{it\varphi}}([\omega]))}{dt}\Big|_{t=0}\right) = i\varphi$ *and* $R^*_{e^{i\varphi}}\sigma = \sigma$ $(= \mathrm{Ad}(e^{-i\varphi})\sigma)$*, where* $R_{e^{i\varphi}}$ *is the right action of* $e^{i\varphi} \in U(1)$ *on* $F(M)$*.*

Via the trivialization (2.1), the horizontal lift of $X \in TM$ is written as

$$\pi^*_{\mathcal{H}}X = X - i\Big\{\omega(\nabla)^\alpha_\alpha(X) + \frac{i\,s^\nabla\,\theta(X)}{2(n+1)}\Big\}\partial/\partial\varphi$$

and the dual frame of the local frame $(\pi^*\theta, \pi^*\theta^1, \ldots, \pi^*\theta^n, \pi^*\theta^{\bar 1}, \ldots, \pi^*\theta^{\bar n}, \sigma)$ is

$$(N := \pi^*_{\mathcal{H}}\xi, \pi^*_{\mathcal{H}}\xi_1, \ldots, \pi^*_{\mathcal{H}}\xi_n, \pi^*_{\mathcal{H}}\xi_{\bar 1}, \ldots, \pi^*_{\mathcal{H}}\xi_{\bar n}, \Sigma := (n+2)\partial/\partial\varphi). \tag{2.5}$$

The curvature 2-form $F(i(n+2)\sigma) \in \Gamma(\mathfrak{u}(1) \otimes \wedge^2 T^*F(M))$ is expressed as

$$F(i(n+2)\sigma) = d\big(i(n+2)\sigma\big) = i(n+2)\,\pi^*\mathcal{F}(\sigma), \tag{2.6}$$

$$\mathcal{F}(\sigma) := \frac{i}{n+2}\Big(\mathrm{Ric}^\nabla + \frac{i\,d(s^\nabla\theta)}{2(n+1)}\Big) = \overline{\mathcal{F}(\sigma)} \in \Gamma(\wedge^2 T^*M)$$

and it will be obvious that the horizontal and vertical components of the bracket $[\pi^*_{\mathcal{H}}X, \pi^*_{\mathcal{H}}Y]$ are expressed as

$$[\pi^*_{\mathcal{H}}X, \pi^*_{\mathcal{H}}Y]_{\mathcal{H}} = \pi^*_{\mathcal{H}}[X,Y], \quad [\pi^*_{\mathcal{H}}X, \pi^*_{\mathcal{H}}Y]_{\mathcal{V}} = -\mathcal{F}(\sigma)(X,Y)\,\Sigma. \tag{2.7}$$

3. The Levi-Civita connection ∇^G and a derived connection $\pi_*\nabla^G$. In this section, we will offer an explicit expression of the connection form of ∇^G. Since ∇^G is invariant under $U(1)$-action, it descends to a connection $\pi_*\nabla^G$ on M, which is well defined by $(\pi_*\nabla^G)_X Y = \pi_*(\nabla^G_{\pi^*_{\mathcal{H}}X}\pi^*_{\mathcal{H}}Y)$.

PROPOSITION 3.1. *The torsion tensor of* $\pi_*\nabla^G$ *vanishes and we have*

$$(\pi_*\nabla^G)_X Y = {}^*\nabla_X Y + \frac{1}{2}g(X, JY)\xi + \theta(Y)\tau X \tag{3.1}$$

$$- \ \theta(X)\Big(\mathcal{F}^{\sigma}(Y)+\mathcal{F}(\sigma)(Y,\xi)\,\xi\Big)-\theta(Y)\Big(\mathcal{F}^{\sigma}(X)+\mathcal{F}(\sigma)(X,\xi)\,\xi\Big),$$

where $\mathcal{F}^{\sigma}(Y)$ is the vector defined by $g(Z,\mathcal{F}^{\sigma}(Y))=\mathcal{F}(\sigma)(Z,Y)$ for any vector Z.

PROOF. By definition,

$$\begin{aligned} g({}^*\nabla_X Y,Z) &= g(\nabla^g_X Y,Z)-g(\theta(Y)\,\tau X,Z)+g(g(\tau X,Y)\xi,Z)\\ &\quad + \frac{1}{2}\Big\{-g(\theta(X)JY,Z)-g(\theta(Y)JX,Z)-g(g(X,JY)\xi,Z)\Big\}, \end{aligned}$$

which, together with (2.7), produces the formula (3.1). Indeed, for a vector Z with $Z_0 := \theta(Z)\xi = 0$,

$$\begin{aligned} &g((\pi_*\nabla^G)_X Y,Z) = 2G(\nabla^G_{\pi^*_{\mathcal{H}}X}\pi^*_{\mathcal{H}}Y,\pi^*_{\mathcal{H}}Z)\\ &= \pi^*_{\mathcal{H}}XG(\pi^*_{\mathcal{H}}Y,\pi^*_{\mathcal{H}}Z)+\pi^*_{\mathcal{H}}YG(\pi^*_{\mathcal{H}}X,\pi^*_{\mathcal{H}}Z)-\pi^*_{\mathcal{H}}ZG(\pi^*_{\mathcal{H}}X,\pi^*_{\mathcal{H}}Y)\\ &\qquad + G([\pi^*_{\mathcal{H}}X,\pi^*_{\mathcal{H}}Y],\pi^*_{\mathcal{H}}Z)+G([\pi^*_{\mathcal{H}}Z,\pi^*_{\mathcal{H}}X],\pi^*_{\mathcal{H}}Y)-G(\pi^*_{\mathcal{H}}X,[\pi^*_{\mathcal{H}}Y,\pi^*_{\mathcal{H}}Z])\\ &= g(\nabla^g_X Y,Z)+\frac{1}{2}\Big\{Zg(X_0,Y_0)-g([Z,X],Y_0)-g(X_0,[Z,Y])\Big\}\\ &\qquad - \mathcal{F}(\sigma)(Z,X)\theta(Y)-\mathcal{F}(\sigma)(Z,Y)\theta(X)\\ &= g(\nabla^g_X Y,Z)+\frac{1}{2}\Big\{-g(\theta(X)JY,Z)-g(\theta(Y)JX,Z)\Big\}\\ &\qquad - \theta(X)\mathcal{F}(\sigma)(Z,Y)-\theta(Y)\mathcal{F}(\sigma)(Z,X)\\ &= g({}^*\nabla_X Y,Z)+\theta(Y)g(\tau X,Z)-\theta(X)\mathcal{F}(\sigma)(Z,Y)-\theta(Y)\mathcal{F}(\sigma)(Z,X), \end{aligned}$$

and

$$\begin{aligned} (3.2)\qquad & 2g((\pi_*\nabla^G)_X Y,\xi) = 2G(\nabla^G_{\pi^*_{\mathcal{H}}X}\pi^*_{\mathcal{H}}Y,\Sigma)\\ &= \pi^*_{\mathcal{H}}XG(\pi^*_{\mathcal{H}}Y,\Sigma)+\pi^*_{\mathcal{H}}YG(\pi^*_{\mathcal{H}}X,\Sigma)-\Sigma G(\pi^*_{\mathcal{H}}X,\pi^*_{\mathcal{H}}Y)\\ &\qquad + G([\pi^*_{\mathcal{H}}X,\pi^*_{\mathcal{H}}Y],\Sigma)+G([\Sigma,\pi^*_{\mathcal{H}}X],\pi^*_{\mathcal{H}}Y)-G(\pi^*_{\mathcal{H}}X,[\pi^*_{\mathcal{H}}Y,\Sigma])\\ &= X\theta(Y)+Y\theta(X)+\theta([X,Y])\\ &= 2g({}^*\nabla_X Y,\xi)-g(T({}^*\nabla)(X-X_0,Y),\xi)=2g({}^*\nabla_X Y,\xi)-g(JX,Y)\,. \end{aligned}$$

Thus we obtain (3.1). It is easy to show $T(\pi_*\nabla^G)=0$. □

PROPOSITION 3.2.
(1) *Set $(\pi_*\nabla^G)\xi_B = \xi_A\cdot\omega(\pi_*\nabla^G)^A_B$. Then $\omega(\pi_*\nabla^G)^{\bar{A}}_{\bar{B}} = \overline{\omega(\pi_*\nabla^G)^A_B}$ and*

$$\omega(\pi_*\nabla^G)^{\alpha}_{\beta} = \omega({}^*\nabla)^{\alpha}_{\beta}+\mathcal{F}(\sigma)(\xi_\beta,\xi_{\bar{\alpha}})\theta,\quad \omega(\pi_*\nabla^G)^{\bar{\alpha}}_{\beta} = \omega({}^*\nabla)^{\bar{\alpha}}_{\beta}+\mathcal{F}(\sigma)(\xi_\beta,\xi_\alpha)\theta,$$

$$\omega(\pi_*\nabla^G)^{0}_{\beta} = \frac{i}{2}\,\theta^{\bar{\beta}},$$

$$\omega(\pi_*\nabla^G)^{\alpha}_{0} = -\mathcal{F}(\sigma)(\xi_{\bar{\alpha}},\xi_\gamma)\theta^\gamma-\Big(\mathcal{F}(\sigma)(\xi_{\bar{\alpha}},\xi_{\bar{\gamma}})-\tau^{\alpha}_{\bar{\gamma}}\Big)\theta^{\bar{\gamma}}-2\mathcal{F}(\sigma)(\xi_{\bar{\alpha}},\xi)\theta,$$

$$\omega(\pi_*\nabla^G)^{0}_{0} = 0\,.$$

(2) *Denote* (2.5) *by* $(W_0, W_1, \ldots, W_{\bar{1}}, \ldots, W_{2n+1})$ *and set* $\nabla^G W_B = W_A \cdot \omega(\nabla^G)^A_B$. *Then* $\omega(\nabla^G)^{\bar{A}}_{\bar{B}} = \overline{\omega(\nabla^G)^A_B}$ $(\bar{0} := 0,\ \overline{2n+1} := 2n+1)$ *and*

$$\omega(\nabla^G)^\alpha_\beta = \pi^*\omega(\pi_*\nabla^G)^\alpha_\beta + i\,\delta_{\alpha\beta}\,\sigma, \qquad \omega(\nabla^G)^{\bar\alpha}_\beta = \pi^*\omega(\pi_*\nabla^G)^{\bar\alpha}_\beta,$$
$$\omega(\nabla^G)^{(N)}_\beta = \pi^*\omega(\pi_*\nabla^G)^0_\beta, \qquad \omega(\nabla^G)^\alpha_{(N)} = \pi^*\omega(\pi_*\nabla^G)^\alpha_0,$$
$$\omega(\nabla^G)^{(\Sigma)}_\beta = -\frac{1}{2}\,\overline{\pi^*\omega(\pi_*\nabla^G)^\beta_0}, \qquad \omega(\nabla^G)^\alpha_{(\Sigma)} = -2\,\overline{\pi^*\omega(\pi_*\nabla^G)^0_\alpha},$$
$$\omega(\nabla^G)^{(N)}_{(N)} = \omega(\nabla^G)^{(\Sigma)}_{(N)} = \omega(\nabla^G)^{(N)}_{(\Sigma)} = \omega(\nabla^G)^{(\Sigma)}_{(\Sigma)} = 0,$$

where we put $\omega(\nabla^G)^{(N)}_B = \omega(\nabla^G)^0_B$, $\omega(\nabla^G)^{(\Sigma)}_B = \omega(\nabla^G)^{2n+1}_B$, *etc.*

REMARK. *The formulas in* (2) *agree with those of* [6, Proposition 6.5] *in the case* J *is integrable.*

PROOF. (1) follows from Proposition 3.1. As for (2): We have

$$\begin{aligned}\omega(\nabla^G)^\alpha_\beta(\pi^*_{\mathcal{H}}\xi_C) &= 2G(\nabla^G_{\pi^*_{\mathcal{H}}\xi_C}\pi^*_{\mathcal{H}}\xi_\beta, \pi^*_{\mathcal{H}}\xi_{\bar\alpha}) \\ &= \pi^* g((\pi_*\nabla^G)_{\xi_C}\xi_\beta, \xi_{\bar\alpha}) = \pi^*\omega(\pi_*\nabla^G)^\alpha_\beta(\xi_C), \\ \omega(\nabla^G)^\alpha_\beta(\Sigma) &= 2G(\nabla^G_\Sigma \pi^*_{\mathcal{H}}\xi_\beta, \pi^*_{\mathcal{H}}\xi_{\bar\alpha}) = 2G(\nabla^G_{\pi^*_{\mathcal{H}}\xi_\beta}\Sigma, \pi^*_{\mathcal{H}}\xi_{\bar\alpha}) \\ &= -2G(\nabla^G_{\pi^*_{\mathcal{H}}\xi_\beta}\pi^*_{\mathcal{H}}\xi_{\bar\alpha}, \Sigma) = g(J\xi_\beta, \xi_{\bar\alpha}) = i\delta_{\alpha\beta}\,.\end{aligned}$$

In the last line, (3.2) was applied. These yield the formula for $\omega(\nabla^G)^\alpha_\beta$. The others can be shown similarly. □

4. The curvature $F(\pi_*\nabla^G)$.

A straightforward computation based on Proposition 3.1 leads to the following formula.

PROPOSITION 4.1. *We have*

$$\begin{aligned}F(\pi_*\nabla^G)(X,Y)Z &= F({}^*\nabla)(X,Y)Z \\ &+ g(X,JZ)\Big\{\frac{1}{2}\mathcal{F}^\sigma(Y) + \frac{1}{2}\theta(Y)\mathcal{F}^\sigma(\xi) + \frac{1}{2}\mathcal{F}(\sigma)(Y,\xi)\,\xi - \frac{1}{2}\tau Y\Big\} \\ &- g(Y,JZ)\Big\{\frac{1}{2}\mathcal{F}^\sigma(X) + \frac{1}{2}\theta(X)\mathcal{F}^\sigma(\xi) + \frac{1}{2}\mathcal{F}(\sigma)(X,\xi)\,\xi - \frac{1}{2}\tau X\Big\} \\ &- \theta(T({}^*\nabla)(X,Y))\Big\{\mathcal{F}^\sigma(Z) + \mathcal{F}(\sigma)(Z,\xi)\,\xi\Big\} \\ &- \frac{1}{2}\,g(X,Q(Z,Y))\,\xi + \frac{1}{2}\,g(Y,Q(Z,X))\,\xi - \frac{1}{2}\,g(Z,JT({}^*\nabla)(X,Y))\,\xi \\ &+ \theta(X)\Big\{({}^*\nabla_Y\mathcal{F}^\sigma)(Z) + ({}^*\nabla_Y\mathcal{F}(\sigma))(Z,\xi)\,\xi - \frac{1}{2}\,\mathcal{F}(\sigma)(JY,Z)\,\xi\Big\} \\ &- \theta(Y)\Big\{({}^*\nabla_X\mathcal{F}^\sigma)(Z) + ({}^*\nabla_X\mathcal{F}(\sigma))(Z,\xi)\,\xi - \frac{1}{2}\,\mathcal{F}(\sigma)(JX,Z)\,\xi\Big\} \\ &+ \theta(Z)\Big\{({}^*\nabla_Y\mathcal{F}^\sigma)(X) + ({}^*\nabla_Y\mathcal{F}(\sigma))(X,\xi)\,\xi - \frac{1}{2}\,\mathcal{F}(\sigma)(JY,X)\,\xi\end{aligned}$$

$$
\begin{aligned}
&- ({}^*\nabla_X \mathcal{F}^\sigma)(Y) - ({}^*\nabla_X \mathcal{F}(\sigma))(Y,\xi)\,\xi + \frac{1}{2}\,\mathcal{F}(\sigma)(JX,Y)\,\xi \\
&+ ({}^*\nabla_X \tau)Y - ({}^*\nabla_Y \tau)X + \tau T({}^*\nabla)(X,Y) \\
&- \mathcal{F}^\sigma(T({}^*\nabla)(X,Y)) - \mathcal{F}(\sigma)(T({}^*\nabla)(X,Y),\xi)\,\xi\Big\} \\
&+ \theta(X)\theta(Z)\Big\{\mathcal{F}^\sigma(\mathcal{F}^\sigma(Y)) + \mathcal{F}(\sigma)(\mathcal{F}^\sigma(Y),\xi)\,\xi + \mathcal{F}(\sigma)(Y,\xi)\,\mathcal{F}^\sigma(\xi) \\
&\qquad - \mathcal{F}^\sigma(\tau Y) - \mathcal{F}(\sigma)(\tau Y,\xi)\,\xi\Big\} \\
&- \theta(Y)\theta(Z)\Big\{\mathcal{F}^\sigma(\mathcal{F}^\sigma(X)) + \mathcal{F}(\sigma)(\mathcal{F}^\sigma(X),\xi)\,\xi + \mathcal{F}(\sigma)(X,\xi)\,\mathcal{F}^\sigma(\xi) \\
&\qquad - \mathcal{F}^\sigma(\tau X) - \mathcal{F}(\sigma)(\tau X,\xi)\,\xi\Big\}\,.
\end{aligned}
$$

COROLLARY 4.2. *We have*

$$
\begin{aligned}
\mathrm{Ric}(\pi_*\nabla^G)(\xi_\alpha,\xi_{\bar\beta}) &= \mathrm{Ric}({}^*\nabla)(\xi_\alpha,\xi_{\bar\beta}) + i\mathcal{F}(\sigma)(\xi_\alpha,\xi_{\bar\beta})\,, \\
\mathrm{Ric}(\pi_*\nabla^G)(\xi_\alpha,\xi_\beta) &= \mathrm{Ric}({}^*\nabla)(\xi_\alpha,\xi_\beta)\,, \\
\mathrm{Ric}(\pi_*\nabla^G)(\xi_\alpha,\xi) &= \mathrm{Ric}({}^*\nabla)(\xi_\alpha,\xi) - ({}^*\nabla_{\xi_\nu}\mathcal{F}(\sigma))(\xi_{\bar\nu},\xi_\alpha) \\
&\quad - ({}^*\nabla_{\xi_{\bar\nu}}\mathcal{F}(\sigma))(\xi_\nu,\xi_\alpha) + i\mathcal{F}(\sigma)(\xi_\alpha,\xi)\,, \\
\mathrm{Ric}(\pi_*\nabla^G)(\xi,\xi_\beta) &= \mathrm{Ric}({}^*\nabla)(\xi,\xi_\beta) + i\mathcal{F}(\sigma)(\xi_\beta,\xi) - ({}^*\nabla_{\xi_\nu}\mathcal{F}(\sigma))(\xi_{\bar\nu},\xi_\beta) \\
&\quad - ({}^*\nabla_{\xi_{\bar\nu}}\mathcal{F}(\sigma))(\xi_\nu,\xi_\beta) + ({}^*\nabla_{\xi_\nu}\tau)^\nu_\beta + ({}^*\nabla_{\xi_{\bar\nu}}\tau)^{\bar\nu}_\beta - 2({}^*\nabla_{\xi_\beta}\tau)^\nu_\nu\,, \\
\mathrm{Ric}(\pi_*\nabla^G)(\xi,\xi) &= \mathrm{Ric}({}^*\nabla)(\xi,\xi) - 2({}^*\nabla_{\xi_\nu}\mathcal{F}(\sigma))(\xi_{\bar\nu},\xi) - 2({}^*\nabla_{\xi_{\bar\nu}}\mathcal{F}(\sigma))(\xi_\nu,\xi) \\
&\quad - 2\mathcal{F}(\sigma)(\xi_\nu,\xi_\mu)\mathcal{F}(\sigma)(\xi_{\bar\mu},\xi_{\bar\nu}) - 2\mathcal{F}(\sigma)(\xi_\nu,\xi_{\bar\mu})\mathcal{F}(\sigma)(\xi_\mu,\xi_{\bar\nu}) \\
&\quad + 2\mathcal{F}(\sigma)(\xi_{\bar\nu},\tau_\nu) + 2\mathcal{F}(\sigma)(\xi_\nu,\tau_{\bar\nu}) - 2g(\tau\xi_\nu,\tau\xi_{\bar\nu})
\end{aligned}
$$

and $\mathrm{Ric}(\pi_*\nabla^G)(\overline{X},\overline{Y}) = \overline{\mathrm{Ric}(\pi_*\nabla^G)(X,Y)}$. (*Note that* $\mathrm{Ric}({}^*\nabla)(\xi,\xi_\beta) = \mathrm{Ric}({}^*\nabla)(\xi,\xi) = 0$.) *The scalar curvature of* $\pi_*\nabla^G$ *is*

$$
s(\pi_*\nabla^G) = \frac{2n+1}{n+1}\,s^\nabla + \mathrm{Ric}(\pi_*\nabla^G)(\xi,\xi)\,. \tag{4.1}
$$

PROOF. The formulas for the Ricci curvatures follow from Proposition 4.1 (or Proposition 3.2(1)). As for (4.1): Referring also to Proposition 2.1 and (2.6), we have

$$
\begin{aligned}
s(\pi_*\nabla^G) &= \mathrm{Ric}(\pi_*\nabla^G)(\xi_\alpha,\xi_{\bar\alpha}) + \mathrm{Ric}(\pi_*\nabla^G)(\xi_{\bar\alpha},\xi_\alpha) + \mathrm{Ric}(\pi_*\nabla^G)(\xi,\xi) \\
&= s({}^*\nabla) + 2i\mathcal{F}(\sigma)(\xi_\alpha,\xi_{\bar\alpha}) + \mathrm{Ric}(\pi_*\nabla^G)(\xi,\xi) \\
&= 2s^\nabla - \frac{s^\nabla}{n+1} + \mathrm{Ric}(\pi_*\nabla^G)(\xi,\xi)\,.
\end{aligned}
$$

□

5. The curvature $F(\nabla^G)$ and the proof of Theorem 1.1. Since $F(\nabla^G)$ is also invariant under $U(1)$-action, it descends to a tensor $\pi_*F(\nabla^G) \in \Gamma(TM \otimes T^*M \otimes T^*M \otimes T^*M)$, which is well defined by $(\pi_*F(\nabla^G))(X,Y)Z = \pi_*(F(\nabla^G)(\pi^*_{\mathcal{H}}X, \pi^*_{\mathcal{H}}Y)\pi^*_{\mathcal{H}}Z)$.

THEOREM 5.1. *We have*

$$F(\nabla^G)(\pi^*_{\mathcal{H}}X, \pi^*_{\mathcal{H}}Y)\pi^*_{\mathcal{H}}Z = \pi^*_{\mathcal{H}}\Big((\pi_* F(\nabla^G))(X,Y)Z\Big) + \Big(F(\nabla^G)(\pi^*_{\mathcal{H}}X, \pi^*_{\mathcal{H}}Y)\pi^*_{\mathcal{H}}Z\Big)_{\mathcal{V}},$$

(5.1)

$$\begin{aligned}(\pi_* F(\nabla^G))(X,Y)Z &= F(\pi_*\nabla^G)(X,Y)Z + \mathcal{F}(\sigma)(X,Y)JZ \\ &+ \frac{1}{2}\Big\{\mathcal{F}(\sigma)(Z,Y) - g(\tau Z,Y)\Big\}JX - \frac{1}{2}\Big\{\mathcal{F}(\sigma)(Z,X) - g(\tau Z,X)\Big\}JY \\ &+ \frac{1}{2}\mathcal{F}(\sigma)(Z,\xi)\Big\{\theta(Y)JX - \theta(X)JY\Big\} \\ &+ \frac{1}{2}\theta(Z)\Big\{\mathcal{F}(\sigma)(Y,\xi)JX - \mathcal{F}(\sigma)(X,\xi)JY\Big\},\end{aligned}$$

(5.2)

$$\begin{aligned}\Big(F(\nabla^G)(\pi^*_{\mathcal{H}}X, \pi^*_{\mathcal{H}}Y)\pi^*_{\mathcal{H}}Z\Big)_{\mathcal{V}} = \frac{1}{2}\Big\{&((\pi_*\nabla^G)_X\mathcal{F}(\sigma))(Z,Y) - ((\pi_*\nabla^G)_Y\mathcal{F}(\sigma))(Z,X) \\ &+ ((\pi_*\nabla^G)_X\mathcal{F}(\sigma))(Z,Y_0) - ((\pi_*\nabla^G)_X\mathcal{F}(\sigma))(Z_0,Y) \\ &- ((\pi_*\nabla^G)_Y\mathcal{F}(\sigma))(Z,X_0) + ((\pi_*\nabla^G)_Y\mathcal{F}(\sigma))(Z_0,X) \\ &+ \mathcal{F}(\sigma)(\xi,X)((\pi_*\nabla^G)_Y\theta)(Z) - \mathcal{F}(\sigma)(\xi,Y)((\pi_*\nabla^G)_X\theta)(Z) \\ &+ \mathcal{F}(\sigma)(Z,\xi)\Big\{((\pi_*\nabla^G)_X\theta)(Y) - ((\pi_*\nabla^G)_Y\theta)(X)\Big\} \\ &- \theta(X)\,\mathcal{F}(\sigma)(Z,(\pi_*\nabla^G)_Y\xi) + \theta(Y)\,\mathcal{F}(\sigma)(Z,(\pi_*\nabla^G)_X\xi) \\ &- \theta(Z)\Big\{\mathcal{F}(\sigma)((\pi_*\nabla^G)_X\xi,Y) - \mathcal{F}(\sigma)((\pi_*\nabla^G)_Y\xi,X)\Big\} \\ &- g(((\pi_*\nabla^G)_X\tau)Z,Y) + g(((\pi_*\nabla^G)_Y\tau)Z,X)\Big\}\varSigma\end{aligned}$$

and

$$\begin{aligned}F(\nabla^G)(\pi^*_{\mathcal{H}}X, \pi^*_{\mathcal{H}}Y)\varSigma &= \pi^*_{\mathcal{H}}\Big\{((\pi_*\nabla^G)_X J)Y - ((\pi_*\nabla^G)_Y J)X\Big\} \\ &+ \frac{1}{2}\Big\{\mathcal{F}(\sigma)(JY,X) - \mathcal{F}(\sigma)(JX,Y) \\ &\quad + \theta(X)\,\mathcal{F}(\sigma)(JY,\xi) - \theta(Y)\,\mathcal{F}(\sigma)(JX,\xi)\Big\}\varSigma, \\ F(\nabla^G)(\varSigma, \pi^*_{\mathcal{H}}Y)\pi^*_{\mathcal{H}}Z &= \pi^*_{\mathcal{H}}\Big\{-((\pi_*\nabla^G)_Y J)Z\Big\} \\ &+ \frac{1}{2}\Big\{\mathcal{F}(\sigma)(Y,JZ) + \mathcal{F}(\sigma)(Y_0,JZ) + g(\tau Y,JZ)\Big\}\varSigma, \\ F(\nabla^G)(\varSigma, \pi^*_{\mathcal{H}}Y)\varSigma &= \pi^*_{\mathcal{H}}\Big\{-Y + \theta(Y)\xi\Big\},\end{aligned}$$

where we set $Y_0 = \theta(Y)\xi$ *as before.*

PROOF. By Proposition 3.2,

$$\nabla^G_{\pi^*_{\mathcal{H}}Y}\pi^*_{\mathcal{H}}Z = \pi^*_{\mathcal{H}}\Big((\pi_*\nabla^G)_Y Z\Big) + \sigma\Big(\nabla^G_{\pi^*_{\mathcal{H}}Y}\pi^*_{\mathcal{H}}Z\Big)\varSigma,$$

$$\sigma\Big(\nabla^G_{\pi^*_{\mathcal H}Y}\pi^*_{\mathcal H}Z\Big)=G(\nabla^G_{\pi^*_{\mathcal H}Y}\pi^*_{\mathcal H}Z,N)=-G(\pi^*_{\mathcal H}Z,\nabla^G_{\pi^*_{\mathcal H}Y}N) \tag{5.3}$$
$$=-\frac12\theta^\alpha(Z)\,\omega(\nabla^G)^{\bar\alpha}_{(N)}(\pi^*_{\mathcal H}Y)-\frac12\theta^{\bar\alpha}(Z)\,\omega(\nabla^G)^{\alpha}_{(N)}(\pi^*_{\mathcal H}Y)$$
$$=\frac12\mathcal F(\sigma)(Z,Y)+\frac12\Big\{\mathcal F(\sigma)(Z,Y_0)-\mathcal F(\sigma)(Z_0,Y)\Big\}-\frac12 g(\tau Z,Y)$$

and

$$\begin{aligned}\nabla^G_{\pi^*_{\mathcal H}X}\nabla^G_{\pi^*_{\mathcal H}Y}\pi^*_{\mathcal H}Z&=\nabla^G_{\pi^*_{\mathcal H}X}\pi^*_{\mathcal H}\Big((\pi_*\nabla^G)_YZ\Big)+\nabla^G_{\pi^*_{\mathcal H}X}\sigma\Big(\nabla^G_{\pi^*_{\mathcal H}Y}\pi^*_{\mathcal H}Z\Big)\Sigma\\&=\pi^*_{\mathcal H}\Big((\pi_*\nabla^G)_X(\pi_*\nabla^G)_YZ\Big)+\sigma\Big(\nabla^G_{\pi^*_{\mathcal H}Y}\pi^*_{\mathcal H}Z\Big)\pi^*_{\mathcal H}JX\\&\quad+\ \Big\{\sigma\Big(\nabla^G_{\pi^*_{\mathcal H}X}\pi^*_{\mathcal H}\Big((\pi_*\nabla^G)_YZ\Big)\Big)+(\pi^*_{\mathcal H}X)\sigma\Big(\nabla^G_{\pi^*_{\mathcal H}Y}\pi^*_{\mathcal H}Z\Big)\Big\}\Sigma\,,\\\nabla^G_{[\pi^*_{\mathcal H}X,\pi^*_{\mathcal H}Y]}\pi^*_{\mathcal H}Z&=\nabla^G_{\pi^*_{\mathcal H}[X,Y]}\pi^*_{\mathcal H}Z-\mathcal F(\sigma)(X,Y)\nabla^G_\Sigma\pi^*_{\mathcal H}Z\\&=\pi^*_{\mathcal H}\Big\{(\pi_*\nabla^G)_{[X,Y]}Z-\mathcal F(\sigma)(X,Y)JZ\Big\}+\sigma\Big(\nabla^G_{\pi^*_{\mathcal H}[X,Y]}\pi^*_{\mathcal H}Z\Big)\Sigma\,.\end{aligned}$$

Hence, we have

$$\begin{aligned}(\pi_*F(\nabla^G))(X,Y)Z&=F(\pi_*\nabla^G)(X,Y)Z+\mathcal F(\sigma)(X,Y)JZ\\&\quad+\ \sigma\Big(\nabla^G_{\pi^*_{\mathcal H}Y}\pi^*_{\mathcal H}Z\Big)JX-\sigma\Big(\nabla^G_{\pi^*_{\mathcal H}X}\pi^*_{\mathcal H}Z\Big)JY\,,\end{aligned}$$
$$\begin{aligned}&\Big(F(\nabla^G)(\pi^*_{\mathcal H}X,\pi^*_{\mathcal H}Y)\pi^*_{\mathcal H}Z\Big)_{\mathcal V}\\&=\Big\{\sigma\Big(\nabla^G_{\pi^*_{\mathcal H}X}\pi^*_{\mathcal H}\Big((\pi_*\nabla^G)_YZ\Big)\Big)-\sigma\Big(\nabla^G_{\pi^*_{\mathcal H}Y}\pi^*_{\mathcal H}\Big((\pi_*\nabla^G)_XZ\Big)\Big)\\&\quad+\ (\pi^*_{\mathcal H}X)\sigma\Big(\nabla^G_{\pi^*_{\mathcal H}Y}\pi^*_{\mathcal H}Z\Big)-(\pi^*_{\mathcal H}Y)\sigma\Big(\nabla^G_{\pi^*_{\mathcal H}X}\pi^*_{\mathcal H}Z\Big)-\sigma\Big(\nabla^G_{\pi^*_{\mathcal H}[X,Y]}\pi^*_{\mathcal H}Z\Big)\Big\}\Sigma\,,\end{aligned}$$

which, together with (5.3), imply (5.1) and (5.2). Since

$$\begin{aligned}&\nabla^G_{\pi^*_{\mathcal H}X}\nabla^G_{\pi^*_{\mathcal H}Y}\Sigma=\pi^*_{\mathcal H}\Big((\pi_*\nabla^G)_XJY\Big)+\sigma\Big(\nabla^G_{\pi^*_{\mathcal H}X}\pi^*_{\mathcal H}JY\Big)\Sigma\,,\\&\nabla^G_{[\pi^*_{\mathcal H}X,\pi^*_{\mathcal H}Y]}\Sigma=\nabla^G_{\pi^*_{\mathcal H}[X,Y]}\Sigma-\mathcal F(\sigma)(X,Y)\nabla^G_\Sigma\Sigma=\pi^*_{\mathcal H}(J[X,Y])\,,\\&\nabla^G_\Sigma\nabla^G_{\pi^*_{\mathcal H}Y}\pi^*_{\mathcal H}Z=\nabla^G_\Sigma\pi^*_{\mathcal H}\Big((\pi_*\nabla^G)_YZ\Big)+\nabla^G_\Sigma\sigma\Big(\nabla^G_{\pi^*_{\mathcal H}Y}\pi^*_{\mathcal H}Z\Big)\Sigma=\pi^*_{\mathcal H}\Big(J(\pi_*\nabla^G)_YZ\Big),\\&\nabla^G_{\pi^*_{\mathcal H}Y}\nabla^G_\Sigma\pi^*_{\mathcal H}Z=\pi^*_{\mathcal H}\Big((\pi_*\nabla^G)_YJZ\Big)+\sigma\Big(\nabla^G_{\pi^*_{\mathcal H}Y}\pi^*_{\mathcal H}JZ\Big)\Sigma\\&\qquad=\pi^*_{\mathcal H}\Big((\pi_*\nabla^G)_YJZ\Big)+\frac12\Big\{\mathcal F(\sigma)(JZ,Y)+\mathcal F(\sigma)(JZ,Y_0)-g(\tau JZ,Y)\Big\}\Sigma\,,\\&\nabla^G_{[\Sigma,\pi^*_{\mathcal H}Y]}\pi^*_{\mathcal H}Z=0\,,\\&\nabla^G_\Sigma\nabla^G_{\pi^*_{\mathcal H}Y}\Sigma=\nabla^G_\Sigma\pi^*_{\mathcal H}JY=\pi^*_{\mathcal H}J^2Y=-\pi^*_{\mathcal H}Y+\pi^*_{\mathcal H}\theta(Y)\xi\,,\\&\nabla^G_{\pi^*_{\mathcal H}Y}\nabla^G_\Sigma\Sigma=\nabla^G_{[\Sigma,\pi^*_{\mathcal H}Y]}\Sigma=0\,,\end{aligned}$$

the others can be shown similarly. □

COROLLARY 5.2. *We have*

$$\mathrm{Ric}(\nabla^G)(\pi_{\mathcal{H}}^* Z, \pi_{\mathcal{H}}^* Y) = \pi^* \Big\{ \mathrm{Ric}(\pi_* \nabla^G)(Z,Y) + \frac{1}{2}\Big(g(\tau Z, JY) + g(\tau Y, JZ) \Big) + \frac{1}{2}\Big(\mathcal{F}(\sigma)(JZ,Y) + \mathcal{F}(\sigma)(JY,Z) + \mathcal{F}(\sigma)(Z_0, JY) + \mathcal{F}(\sigma)(Y_0, JZ) \Big) \Big\},$$

$$\mathrm{Ric}(\nabla^G)(\pi_{\mathcal{H}}^* Z, \Sigma) = \pi^* \Big\{ -2i\,\theta(Z)\mathcal{F}(\sigma)(\xi_\alpha, \xi_{\bar\alpha}) \Big\},$$

$$\mathrm{Ric}(\nabla^G)(\Sigma, \Sigma) = 2n\,.$$

PROOF. We have

$$\begin{aligned}
\mathrm{Ric}(\nabla^G)(\pi_{\mathcal{H}}^* Z, \pi_{\mathcal{H}}^* Y) &= 2G(F(\nabla^G)(\pi_{\mathcal{H}}^*\xi_\alpha, \pi_{\mathcal{H}}^* Y)\pi_{\mathcal{H}}^* Z, \pi_{\mathcal{H}}^*\xi_{\bar\alpha}) \\
&\quad + 2G(F(\nabla^G)(\pi_{\mathcal{H}}^*\xi_{\bar\alpha}, \pi_{\mathcal{H}}^* Y)\pi_{\mathcal{H}}^* Z, \pi_{\mathcal{H}}^*\xi_\alpha) + G(F(\nabla^G)(\pi_{\mathcal{H}}^*\xi, \pi_{\mathcal{H}}^* Y)\pi_{\mathcal{H}}^* Z, \Sigma) \\
&\quad + G(F(\nabla^G)(\Sigma, \pi_{\mathcal{H}}^* Y)\pi_{\mathcal{H}}^* Z, \pi_{\mathcal{H}}^*\xi)\,, \\
\mathrm{Ric}(\nabla^G)(\pi_{\mathcal{H}}^* Z, \Sigma) &= -2G(F(\nabla^G)(\Sigma, \pi_{\mathcal{H}}^*\xi_\alpha)\pi_{\mathcal{H}}^* Z, \pi_{\mathcal{H}}^*\xi_{\bar\alpha}) \\
&\quad - 2G(F(\nabla^G)(\Sigma, \pi_{\mathcal{H}}^*\xi_{\bar\alpha})\pi_{\mathcal{H}}^* Z, \pi_{\mathcal{H}}^*\xi_\alpha) - G(F(\nabla^G)(\Sigma, \pi_{\mathcal{H}}^*\xi, \Sigma)\pi_{\mathcal{H}}^* Z)\,, \\
\mathrm{Ric}(\nabla^G)(\Sigma, \Sigma) &= -2G(F(\nabla^G)(\Sigma, \pi_{\mathcal{H}}^*\xi_\alpha)\Sigma, \pi_{\mathcal{H}}^*\xi_{\bar\alpha}) - 2G(F(\nabla^G)(\Sigma, \pi_{\mathcal{H}}^*\xi_{\bar\alpha})\Sigma, \pi_{\mathcal{H}}^*\xi_\alpha)\,.
\end{aligned}$$

Hence, by Theorem 5.1, we obtain the formulas. □

Last, Corollary 5.2 implies Theorem 1.1 as follows.

PROOF OF THEOREM 1.1. We have

$$\begin{aligned}
\mathrm{Ric}(\nabla^G)(\pi_{\mathcal{H}}^*\xi_\alpha, \pi_{\mathcal{H}}^*\xi_{\bar\alpha}) &= \pi^* \Big\{ \mathrm{Ric}(\pi_*\nabla^G)(\xi_\alpha, \xi_{\bar\alpha}) + i\,\mathcal{F}(\sigma)(\xi_\alpha, \xi_{\bar\alpha}) \Big\}, \\
\mathrm{Ric}(\nabla^G)(\pi_{\mathcal{H}}^*\xi_{\bar\alpha}, \pi_{\mathcal{H}}^*\xi_\alpha) &= \pi^* \Big\{ \mathrm{Ric}(\pi_*\nabla^G)(\xi_{\bar\alpha}, \xi_\alpha) + i\,\mathcal{F}(\sigma)(\xi_\alpha, \xi_{\bar\alpha}) \Big\}, \\
\mathrm{Ric}(\nabla^G)(\Sigma, N) = \mathrm{Ric}(\nabla^G)(N, \Sigma) &= \pi^* \Big\{ -2i\,\mathcal{F}(\sigma)(\xi_\alpha, \xi_{\bar\alpha}) \Big\}\,.
\end{aligned}$$

Referring also to (4.1), we know

$$\begin{aligned}
s(\nabla^G) &= 2\,\mathrm{Ric}(\nabla^G)(\pi_{\mathcal{H}}^*\xi_\alpha, \pi_{\mathcal{H}}^*\xi_{\bar\alpha}) + 2\,\mathrm{Ric}(\nabla^G)(\pi_{\mathcal{H}}^*\xi_{\bar\alpha}, \pi_{\mathcal{H}}^*\xi_\alpha) \\
&\quad + \mathrm{Ric}(\nabla^G)(N, \Sigma) + \mathrm{Ric}(\nabla^G)(\Sigma, N) \\
&= 2\,\pi^* \Big\{ s(\pi_*\nabla^G) - \mathrm{Ric}(\pi_*\nabla^G)(\xi, \xi) \Big\} = \frac{2(2n+1)}{n+1}\,\pi^* s^\nabla\,.
\end{aligned}$$

□

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