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EXAMPLES OF AUSTERE ORBITS OF THE ISOTROPY REPRESENTATIONS FOR SEMISIMPLE PSEUDO-RIEMANNIAN SYMMETRIC SPACES

KURANDO BABA

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Abstract. Harvey-Lawson and Anciaux introduced the notion of austere submanifolds in pseudo-Riemannian geometry. We give an equivalent condition for an orbit of the isotropy representations for semisimple pseudo-Riemannian symmetric space to be an austere submanifold in a pseudo-sphere in terms of restricted root system theory with respect to Cartan subspaces. By using the condition we give examples of austere orbits.

Introduction. In pseudo-Riemannian geometry, the notion of austere submanifolds was introduced by Harvey-Lawson ([4]) and Anciaux ([1]). They defined an austere submanifold as a submanifold such that, for each normal vector, the coefficients of odd degree for the characteristic polynomial of its shape operator vanish. In particular, an austere submanifold is a submanifold with vanishing mean curvature vector. It is well-known that such a submanifold is minimal in Riemannian geometry. Recently, examples of austere submanifolds were given by using the method of orbits on semisimple Riemannian symmetric spaces ([8], [7], [9]). In [8], Ikawa-Sakai-Tasaki classified austere orbits (in a sphere) of the isotropy representation for a semisimple Riemannian symmetric space in terms of restricted root system theory. The aim of this paper is to adapt their method to a pseudo-Riemannian framework and to give examples of austere orbits (in a pseudo-sphere) of the isotropy representation for a semisimple pseudo-Riemannian symmetric space.

Let G/H be a semisimple pseudo-Riemannian symmetric space equipped with the metric induced from the Killing form B of $\mathfrak{g}$ $(:= \mathrm{Lie}(G))$. Let σ be an involution of $\mathfrak{g}$ whose fixed point set coincides with $\mathfrak{h}$ $(:= \mathrm{Lie}(H))$. Denote by $\mathfrak{q}$ the (-1)-eigenspace of σ, which is identified with the tangent space of G/H at the origin. The isotropy representation of G/H is equivalent to the adjoint representation Ad of H on $\mathfrak{q}$. Let M be an $\mathrm{Ad}(H)$-orbit through $X \in \mathfrak{q}$. If X is non-null (i.e., $B(X, X) \neq 0$), then M is contained in the (central) hyperquadrics of $\mathfrak{q}$. In this paper, we assume that M is a pseudo-Riemannian submanifold in the pseudo-hypersphere $\boldsymbol{S}$

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$(:= \{v \in \mathfrak{q} \mid B(v,v) = r(>0)\})$. The nondegeneracy of the induced metric on $M \hookrightarrow \boldsymbol{S}$ implies the following result.

KEY LEMMA. Assume that the $\mathrm{Ad}(H)$-orbit M through $X \in \mathfrak{q}$ is contained in a pseudo-hypersphere $\boldsymbol{S}$ $(\subset \mathfrak{q})$. Then, $M \hookrightarrow \boldsymbol{S}$ is a pseudo-Riemannian submanifold if and only if X is semisimple (i.e., an element of $\mathfrak{q}$ such that $\mathrm{ad}(X) \in \mathrm{End}(\mathfrak{g})$ is diagonalizable over $\boldsymbol{C}$).

A main difficulty in the pseudo-Riemannian case is the situation that the shape operator is not diagonalizable over $\boldsymbol{C}$. Therefore we give the Jordan-Chevalley decomposition of the shape operator of $M \hookrightarrow \boldsymbol{S}$ (see, Proposition 2.2). By using above Key Lemma we describe the semisimple part and the nilpotent part of the shape operator in terms of restricted root system theory with respect to Cartan subspaces (cf. [12], [4] for the notion of restricted root system theory with respect to Cartan subspaces). As its application, we determine the spectrum of the shape operator (see, Corollary 2.6). On the other hand, in the Riemannian case, any maximal abelian subspace is Cartan. This implies that all Cartan subspaces are mutually $\mathrm{Ad}(H)$-conjugate (cf. [6, Lemma 6.3, Chapter V]). However, this conjugacy theorem does not necessarily hold in the pseudo-Riemannian case. Therefore we prove a conjugacy theorem for complexified Cartan subspaces (see, Proposition 3.9). By using these results we give an equivalent condition for $M \hookrightarrow \boldsymbol{S}$ to be austere (see, Proposition 3.2), which is a generalization of Ikawa-Sakai-Tasaki's method ([8]). According to [8], the orbit through a restricted root vector is an austere submanifold in a sphere. In the pseudo-Riemannian case, we need a technical condition for restricted roots (see, Corollary 3.11). The main result of this paper is the following.

THEOREM. For any restricted root α in Table 1, the $\mathrm{Ad}(H)$-orbit through the restricted root vector corresponding to α is an austere submanifold in $\boldsymbol{S}$.

TABLE 1. The real restricted roots of $\boldsymbol{R}$ with respect to a maximally split Cartan subspace.

Type of $(\boldsymbol{R}, \theta)$	Real Restricted Roots
AI	all restricted roots
AIII	$\{\pm(\alpha_i + \cdots + \alpha_{r+1-i}) \mid 1 \le i \le l\}$
BI	$\{\pm(\alpha_i + \cdots + \alpha_{j-1}) \mid 1 \le i < j \le l\} \cup \{\pm(\alpha_i + \cdots + \alpha_r + \alpha_j + \cdots + \alpha_r) \mid 1 \le i < j \le l\}$ $\cup\{\pm(\alpha_i + \cdots + \alpha_r) \mid 1 \le i \le l\}$
BCI	$\{\pm(\alpha_i + \cdots + \alpha_{j-1}) \mid 1 \le i < j \le l\} \cup \{\pm(\alpha_i + \cdots + \alpha_r + \alpha_j + \cdots + \alpha_r) \mid 1 \le i < j \le l\}$ $\cup\{\pm(\alpha_i + \cdots + \alpha_r) \mid 1 \le i \le l\} \cup \{\pm 2(\alpha_i + \cdots + \alpha_r) \mid 1 \le i \le l\}$
BCIII	$\{\pm(\alpha_{2i-1} + 2\alpha_{2i} + \cdots + 2\alpha_r) \mid 1 \le i \le l\}$
CI	$\{\pm(\alpha_i + \cdots + \alpha_{j-1}) \mid 1 \le i < j \le l\} \cup \{\pm(\alpha_i + \cdots + \alpha_{j-1} + \alpha_j + \cdots + \alpha_r) \mid 1 \le i < j \le l\}$ $\cup\{\pm(2\alpha_i + \cdots + 2\alpha_{r-1} + \alpha_r) \mid 1 \le i \le l\}$
CIII	$\{\pm(\alpha_{2i-1} + 2\alpha_{2i} + \cdots + 2\alpha_{r-1} + \alpha_r) \mid 1 \le i \le l\}$
DI	$\{\pm(\alpha_i + \cdots + \alpha_{j-1}) \mid 1 \le i < j \le l\} \cup \{\pm(\alpha_i + \cdots + \alpha_{r-2} + \alpha_j + \cdots + \alpha_r) \mid 1 \le i < j \le l\}$
DIII	$\{\pm(\alpha_{2i-1} + \cdots + \alpha_{r-2} + \alpha_{2i} + \cdots + \alpha_r) \mid 1 \le i \le l\}$
EI	all restricted roots
EII	$\{\pm\alpha_2, \pm\alpha_4, \pm(\alpha_3 + \alpha_4 + \alpha_5), \pm(\alpha_2 + \alpha_4), \pm(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\} \cup$ $\{\pm(\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5), \pm(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6), \pm(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\} \cup$ $\{\pm(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6), \pm(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6)\} \cup$ $\{\pm(\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6), \pm(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6)\}$

TABLE 1. (continued).

Type of (R, θ)	Real Restricted Roots
EIII	$\{\pm(\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6), \pm(\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6)\}$
EV	all restricted roots
EVI	$\{\pm\alpha_1, \pm\alpha_3, \pm(\alpha_1+\alpha_3), \pm(\alpha_2+\alpha_3+2\alpha_4+\alpha_5), \pm(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5)\} \cup \{\pm(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5), \pm(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7)\} \cup \{\pm(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7), \pm(2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7)\} \cup \{\pm(\alpha_1+2\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7), \pm(\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7)\} \cup \{\pm(\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7)\}$
EVII	$\{\pm\alpha_7, \pm(\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7), \pm(2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7)\}$
EVIII	all restricted roots
EIX	$\{\pm\alpha_7, \pm\alpha_8, \pm(\alpha_7+\alpha_8), \pm(\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7)\} \cup \{\pm(\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8), \pm(\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+\alpha_8)\} \cup \{\pm(2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7)\} \cup \{\pm(2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+2\alpha_7+\alpha_8)\} \cup \{\pm(2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+\alpha_8)\} \cup \{\pm(2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+2\alpha_7+\alpha_8)\} \cup \{\pm(2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+3\alpha_7+\alpha_8)\} \cup \{\pm(2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+3\alpha_7+2\alpha_8)\}$
FI	all restricted roots
FII	$\{\pm(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)\}$
FIII	$\{\pm(\alpha_1+\alpha_2+\alpha_3), \pm(\alpha_2+2\alpha_3+2\alpha_4), \pm(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4), \pm(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)\}$
G	all restricted roots

Here we remark on Theorem. Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{q}$ and R $(\subset (\mathfrak{a}^{\boldsymbol{C}})^* \setminus \{0\})$ denote the restricted root system with respect to $\mathfrak{a}$. In the pseudo-Riemannian case, a restricted root vector is in $\mathfrak{a}$ if and only if its restricted root takes real values on $\mathfrak{a}$. In Theorem 3.13, we classify all the real restricted roots when $\mathfrak{a}$ is maximally split and the list is as in Table 1 (see, Section 1 for the definition of a maximally split Cartan subspace). For the determination of the real roots, we use a Satake diagram of $(\mathfrak{g}, \mathfrak{h})$ associated with (R, θ), where θ is a Cartan involution of $\mathfrak{g}$ such that θ commutes with σ and preserves $\mathfrak{a}$ invariantly (cf. [12] for the existence of such a Cartan involution). In Table 1, the types of (R, θ) are as in Table 3, the α_i's are fundamental roots as in Table 3, and r (resp. l) denotes the rank (resp. the split rank) of $(\mathfrak{g}, \mathfrak{h})$. In Table 2, we determine the rank, the split rank and the type of (R, θ) for each irreducible semisimple pseudo-Riemannian symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ which was classified by Berger ([3]). Here a semisimple symmetric pair $(\mathfrak{g}, \mathfrak{h})$ is said to be *irreducible* if $\mathfrak{g}$ has no non-trivial σ-invariant ideals.

The organization of this paper is as follows. In Section 1, we prove Key Lemma, give preliminaries for restricted root system theory with respect to Cartan subspaces, and recall the notion of its Satake diagram. In Section 2, we give the Jordan-Chevalley decomposition for the shape operator of an $\mathrm{Ad}(H)$-orbit. Moreover, we determine the spectrum of the shape operator. In Section 3, we prove Corollary 3.11 and Theorem 3.13, which give the proof of Theorem. In Appendix A, we give a recipe to determine the Satake diagrams associated with the restricted root systems with respect to maximally split Cartan subspaces for all irreducible semisimple pseudo-Riemannian symmetric pairs.

FUTURE DIRECTIONS. We will classify all the austere orbits (in a pseudo-sphere) of the isotropy representation for a semisimple pseudo-Riemannian symmetric space. For this purpose, we need to determine the orbit space. However, the orbit space for general orbits becomes quite complicated in the pseudo-Riemannian case. We expect that any austere orbit is a hyperbolic orbit. In [2], the orbit space for hyperbolic orbits is described in terms of restricted root system theory with respect to maximal split abelian subspaces (cf. [14], [13] for the definition of a maximal split abelian subspace).

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1. Preliminaries. Let G be a connected semisimple noncompact Lie group, and σ be an involution of G. Let H be a closed subgroup of G with $(G_\sigma)_0 \subset H \subset G_\sigma$, where G_σ denotes the fixed point group of σ and $(G_\sigma)_0$ denotes its identity component. Then the coset space G/H equipped with the metric induced from the Killing form B of $\mathfrak{g}$ (:= Lie(G)) is a semisimple pseudo-Riemannian symmetric space. The involution σ of G induces an involution of $\mathfrak{g}$, which is also denoted by the same symbol σ. Then the Lie algebra $\mathfrak{h}$ of H coincides with $\{X \in \mathfrak{g} \mid \sigma(X) = X\}$. The pair $(\mathfrak{g}, \mathfrak{h})$ is called a *semisimple symmetric pair*. Set $\mathfrak{q} = \{X \in \mathfrak{g} \mid \sigma(X) = -X\}$, which is identified with the tangent space of G/H at the origin. It is useful to identify the isotropy representation of G/H with the adjoint representation Ad of H on $\mathfrak{q}$ in the context of symmetric spaces. For each $X \in \mathfrak{g}$, the Jordan-Chevalley (JC) decomposition of X is induced from that of $\mathrm{ad}(X) \in \mathrm{End}(\mathfrak{g})$ (cf. [16, Proposition 1.3.5.1]), where $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is the adjoint representation of $\mathfrak{g}$. Denote by X_s (resp. X_n) the semisimple part (resp. the nilpotent part) of X. Here we note that $[X_s, X_n] = 0$ holds. When X is in $\mathfrak{q}$, a simple calculation shows that $X_s + X_n = X = -\sigma(X) = -\sigma(X_s) - \sigma(X_n)$. By using the uniqueness of the JC decomposition of X we obtain $\sigma(X_s) = -X_s$ and $\sigma(X_n) = -X_n$. Hence X_s, X_n are in $\mathfrak{q}$ (cf. Proposition 2 in [12]). An element $X \in \mathfrak{q}$ is said to be *semisimple* (resp. *nilpotent*) if $X = X_s$ (resp. $X = X_n$) holds. Here, we prove Key Lemma stated in Introduction.

PROOF OF KEY LEMMA. Suppose that $M \hookrightarrow \boldsymbol{S}$ $(:= \{v \in \mathfrak{q} \mid B(v, v) = r(> 0)\})$ is a pseudo-Riemannian submanifold. Then the tangent space $T_X M$ and the normal space $T_X^{\perp} M$ of $M \hookrightarrow \boldsymbol{S}$ at X are given as follows:

(1) $T_X M = [\mathfrak{h}, X]$,

(2) $T_X^{\perp} M = \{\xi \in \mathfrak{q} \mid B(\xi, [\mathfrak{h}, X]) = 0, B(\xi, X) = 0\} = \{\xi \in \mathfrak{q} \mid [X, \xi] = 0, B(\xi, X) = 0\}$.

Let $X = X_s + X_n$ be the JC decomposition of X. Since, for any $\xi \in T_X^{\perp} M$, $[\xi, X] = 0$ holds, we have $[\xi, X_s] = [\xi, X_n] = 0$ by using Proposition 1.3.5.1 in [16]. From Lemma 12 in [12] there exists a $Z \in \mathfrak{h}$ such that $[Z, X_n] = X_n$. This implies that X_n is orthogonal to X by calculating $B(X_n, X) = B([Z, X_n], X) = B([X_n, X], Z) = 0$. Hence we have $X_n \in T_X^{\perp} M$. The nondegeneracy of $M \hookrightarrow \boldsymbol{S}$ implies that the restriction of B on $T_X^{\perp} M$ is nondegenerate. By using the calculation $B(\xi, X_n) = B(\xi, [Z, X_n]) = B(Z, [X_n, \xi]) = 0$ for all $\xi \in T_X^{\perp} M$, we have $X_n = 0$. Hence $X = X_s$ holds.

Conversely, let X be a semisimple element in $\mathfrak{q}$. Then, we have the eigenspace decomposition $\mathfrak{g}^{\boldsymbol{C}} = \sum_{\alpha \in \mathrm{Spec}\,\mathrm{ad}(X)} \mathrm{Ker}(\mathrm{ad}(X) - \alpha\,\mathrm{id})$ of $\mathrm{ad}(X) (\in \mathrm{End}(\mathfrak{g}^{\boldsymbol{C}}))$, where $\mathrm{Spec}\,\mathrm{ad}(X)$ $(\subset \boldsymbol{C})$ denotes the spectrum of $\mathrm{ad}(X)$ and id denotes the identity transformation on $\mathfrak{g}^{\boldsymbol{C}}$. Since

$\sigma(\mathrm{Ker}(\mathrm{ad}(X) - \alpha\,\mathrm{id})) = \mathrm{Ker}(\mathrm{ad}(X) + \alpha\,\mathrm{id})$, we have a decomposition of $\mathfrak{h}^{\boldsymbol{C}}$ as follows:

$$\mathfrak{h}^{\boldsymbol{C}} = \mathrm{Ker}\,\mathrm{ad}(X) \cap \mathfrak{h}^{\boldsymbol{C}} + \sum_{\alpha \in \mathrm{Spec}\,\mathrm{ad}(X) \setminus \{0\}} (\mathrm{Ker}(\mathrm{ad}(X) - \alpha\,\mathrm{id}) + \mathrm{Ker}(\mathrm{ad}(X) + \alpha\,\mathrm{id})) \cap \mathfrak{h}^{\boldsymbol{C}} .$$

Then we have $(T_X M)^{\boldsymbol{C}} = \sum_{\alpha \in \mathrm{Spec}\,\mathrm{ad}(X) \setminus \{0\}} (\mathrm{Ker}(\mathrm{ad}(X) - \alpha\,\mathrm{id}) + \mathrm{Ker}(\mathrm{ad}(X) + \alpha\,\mathrm{id})) \cap \mathfrak{q}^{\boldsymbol{C}}$, where $(T_X M)^{\boldsymbol{C}}$ denotes the complexification of the tangent space of M at X. It can be shown that the following decomposition is orthogonal with respect to B:

$$\mathfrak{q}^{\boldsymbol{C}} = \mathrm{Ker}\,\mathrm{ad}(X) \cap \mathfrak{q}^{\boldsymbol{C}} + \sum_{\alpha \in \mathrm{Spec}\,\mathrm{ad}(X) \setminus \{0\}} (\mathrm{Ker}(\mathrm{ad}(X) - \alpha\,\mathrm{id}) + \mathrm{Ker}(\mathrm{ad}(X) + \alpha\,\mathrm{id})) \cap \mathfrak{q}^{\boldsymbol{C}} .$$

This implies that the restriction of B on $(T_X M)^{\boldsymbol{C}}$ is nondegenerate. Hence $M \hookrightarrow \boldsymbol{S}$ is a pseudo-Riemannian submanifold. □

In the present paper, we investigate the structures of semisimple $\mathrm{Ad}(H)$-orbits in terms of the theory of restricted root systems with respect to Cartan subspaces. First, we review the notion of Cartan subspaces for reductive symmetric pairs.

DEFINITION 1.1. Let $\mathfrak{l}$ be a reductive Lie algebra and τ be an involution of $\mathfrak{l}$. Set $\mathfrak{l}^{\pm} = \{Z \in \mathfrak{l} \mid \tau(Z) = \pm Z\}$. A subspace $\mathfrak{a}$ of $\mathfrak{l}^{-}$ is called a *Cartan subspace* (for $(\mathfrak{l}, \mathfrak{l}^{+})$) if $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{l}^{-}$ and consists only semisimple elements in $\mathfrak{l}$.

In the case when $\mathfrak{l}$ is semisimple, Definition 1.1 coincides with the those of Cartan subspaces defined in [4] or A-subspaces defined in [12]. In the following, we state basic facts on Cartan subspaces for a semisimple symmetric pair $(\mathfrak{g}, \mathfrak{h})$, which are needed later.

LEMMA 1.2 ([4, p. 272]). *A maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{q}$ is Cartan if and only if the bilinear form on $\mathfrak{a}$ induced from the Killing form is nondegenerate.*

Since the proof of Lemma 1.2 is omitted in [4], we prove it for completeness.

PROOF OF LEMMA 1.2. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{q}$. If $\mathfrak{a}$ is Cartan, then there exists a Cartan involution θ of $\mathfrak{g}$ satisfying $\theta \circ \sigma = \sigma \circ \theta$ and $\theta(\mathfrak{a}) = \mathfrak{a}$ (see, Lemma 5 in [12]). Then we have $\mathfrak{a} = \mathfrak{k} \cap \mathfrak{a} + \mathfrak{p} \cap \mathfrak{a}$, where $\mathfrak{k} = \mathrm{Ker}(\theta - \mathrm{id})$ and $\mathfrak{p} = \mathrm{Ker}(\theta + \mathrm{id})$. This implies that the induced bilinear form on $\mathfrak{a}$ is nondegenerate.

Conversely, suppose that B gives a nondegenerate bilinear form on $\mathfrak{a}$. Let $A = A_s + A_n$ be the JC decomposition of $A \in \mathfrak{a}$. Then $\sigma(A_s)$ (resp. $\sigma(A_n)$) is also semisimple (resp. nilpotent). By using the uniqueness of the JC decomposition we have $A_s, A_n \in \mathfrak{q}$. It follows from Proposition 1.3.5.1 in [16] that $[A_s, A'] = [A_n, A'] = 0$ for all $A' \in \mathfrak{a}$. By using the maximality of $\mathfrak{a}$ in $\mathfrak{q}$ we have $A_s, A_n \in \mathfrak{a}$, respectively. Since there exists a $Z \in \mathfrak{h}$ satisfying $[Z, A_n] = A_n$ (see, Lemma 12 in [12]) we have $B(A_n, A') = B([Z, A_n], A') = B([A_n, A'], Z) = 0$ for all $A' \in \mathfrak{a}$. It follows from the nondegeneracy of B on $\mathfrak{a}$ that $A_n = 0$ holds. Therefore $A = A_s$ is semisimple, i.e., $\mathfrak{a}$ is Cartan. □

LEMMA 1.3 ([12, Corollary]). *Any semisimple element in* $\mathfrak{q}$ *is contained in some Cartan subspace of* $\mathfrak{q}$.

Next, we recall the notion of restricted root system theory with respect to Cartan subspaces for a semisimple symmetric pair $(\mathfrak{g}, \mathfrak{h})$ (cf. [12], [4]).

Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{q}$. Suppose that θ is a Cartan involution of $\mathfrak{g}$ satisfying $\theta \circ \sigma = \sigma \circ \theta$ and $\theta(\mathfrak{a}) = \mathfrak{a}$. Then we have $\mathfrak{a} = \mathfrak{k} \cap \mathfrak{a} + \mathfrak{p} \cap \mathfrak{a}$, where $\mathfrak{k} = \mathrm{Ker}(\theta - \mathrm{id})$ and $\mathfrak{p} = \mathrm{Ker}(\theta + \mathrm{id})$. Set, for any $\alpha \in (\mathfrak{a}^{\boldsymbol{C}})^*$,

$$\mathfrak{g}_\alpha^{\boldsymbol{C}} = \left\{X \in \mathfrak{g}^{\boldsymbol{C}} \mid \mathrm{ad}(A)X = \alpha(A)X, \forall A \in \mathfrak{a}^{\boldsymbol{C}}\right\},$$
$$\mathfrak{h}_\alpha^{\boldsymbol{C}} = \left\{Z \in \mathfrak{h}^{\boldsymbol{C}} \mid \mathrm{ad}(A)^2 Z = \alpha(A)^2 Z, \forall A \in \mathfrak{a}^{\boldsymbol{C}}\right\},$$
$$\mathfrak{q}_\alpha^{\boldsymbol{C}} = \left\{Y \in \mathfrak{q}^{\boldsymbol{C}} \mid \mathrm{ad}(A)^2 Y = \alpha(A)^2 Y, \forall A \in \mathfrak{a}^{\boldsymbol{C}}\right\}.$$

Denote by $R = \{\alpha \in (\mathfrak{a}^{\boldsymbol{C}})^* \setminus \{0\} \mid \mathfrak{q}_\alpha^{\boldsymbol{C}} \neq \{0\}\}$, which is called the *restricted root system* of G/H (or $(\mathfrak{g}, \mathfrak{h})$) with respect to $\mathfrak{a}$. It is known that R become a (reduced) root system on the real vector space $\mathfrak{a}_{\boldsymbol{R}} := \sqrt{-1}(\mathfrak{k} \cap \mathfrak{a}) + \mathfrak{p} \cap \mathfrak{a} (\subset \mathfrak{a}^{\boldsymbol{C}})$ equipped with the (positive definite) inner product induced from B. We also have $(\mathfrak{a}_{\boldsymbol{R}})^{\boldsymbol{C}} = \mathfrak{a}^{\boldsymbol{C}}$ and $\mathrm{Span}_{\boldsymbol{C}} R = (\mathfrak{a}^{\boldsymbol{C}})^*$ (cf. Appendix A). It can be shown that $\mathfrak{g}^{\boldsymbol{C}}$, $\mathfrak{h}^{\boldsymbol{C}}$ and $\mathfrak{q}^{\boldsymbol{C}}$ are decomposed as follows:

$$\mathfrak{g}^{\boldsymbol{C}} = \mathfrak{g}_0^{\boldsymbol{C}} + \sum_{\alpha \in R} \mathfrak{g}_\alpha^{\boldsymbol{C}}, \quad \mathfrak{h}^{\boldsymbol{C}} = \mathfrak{h}_0^{\boldsymbol{C}} + \sum_{\alpha \in R_+} \mathfrak{h}_\alpha^{\boldsymbol{C}}, \quad \mathfrak{q}^{\boldsymbol{C}} = \mathfrak{a}^{\boldsymbol{C}} + \sum_{\alpha \in R_+} \mathfrak{q}_\alpha^{\boldsymbol{C}},$$

where $\mathfrak{g}_0^{\boldsymbol{C}} := \{X \in \mathfrak{g}^{\boldsymbol{C}} \mid [X, \mathfrak{a}] = \{0\}\}$, $\mathfrak{h}_0^{\boldsymbol{C}} := \mathfrak{g}_0^{\boldsymbol{C}} \cap \mathfrak{h}^{\boldsymbol{C}}$ and R_+ is a positive root system. The dimension of $\mathfrak{a}$ is called the *rank* of G/H (or $(\mathfrak{g}, \mathfrak{h})$). Note that the type of R (as root system) and the value of $\mathrm{rank}(G/H)$ do not depend on the choice of a Cartan subspace of $\mathfrak{q}$. The following Lemma immediately follows from (1) and (2).

LEMMA 1.4 ([4, 2.1 Proposition]). *Assume that the* $\mathrm{Ad}(H)$*-orbit* M *through* $X \in \mathfrak{a}$ *is contained in* $\boldsymbol{S}$. *Then we have orthogonal decompositions of* $(T_X M)^{\boldsymbol{C}}$ *and the complexification of the normal space of* M *in* $\boldsymbol{S}$ *as follows:*

$$(T_X M)^{\boldsymbol{C}} = \sum_{\alpha \in R_+ :\, \alpha(X) \neq 0} \mathfrak{q}_\alpha^{\boldsymbol{C}},$$
$$(T_X^\perp M)^{\boldsymbol{C}} = (\mathfrak{a} \ominus \boldsymbol{R}X)^{\boldsymbol{C}} + \sum_{\alpha \in R_+ :\, \alpha(X) = 0} \mathfrak{q}_\alpha^{\boldsymbol{C}}.$$

Moreover, the above decompositions are orthogonal with respect to the Killing form of $\mathfrak{g}^{\boldsymbol{C}}$.

For each $\alpha \in R$, we define a vector $A_\alpha \in \mathfrak{a}^{\boldsymbol{C}}$ by $B(A, A_\alpha) = \alpha(A)$ for all $A \in \mathfrak{a}^{\boldsymbol{C}}$, which is called the *restricted root vector* of α. Then we have $A_\alpha \in \mathfrak{a}_{\boldsymbol{R}}$. For each $\alpha \in R$, the dimension of $\mathfrak{q}_\alpha^{\boldsymbol{C}}$ is called the *multiplicity* of α. Since R is a root system, we obtain

$$s_\beta \alpha := \alpha - 2\frac{B(A_\alpha, A_\beta)}{B(A_\beta, A_\beta)}\beta \in R \quad (\alpha, \beta \in R).$$

We also have $\dim \mathfrak{q}_{s_\beta \alpha}^{\boldsymbol{C}} = \dim \mathfrak{q}_\alpha^{\boldsymbol{C}}$ for $\alpha, \beta \in R$. A restricted root $\alpha \in R$ is said to be *real* (resp. *imaginary*) if α takes real (resp. pure imaginary) values on $\mathfrak{a}$. It is clear that $A_\alpha \in \mathfrak{a}$ (resp.

$\sqrt{-1}A_\alpha \in \mathfrak{a}$) if and only if α is real (resp. imaginary). For each semisimple pseudo-Riemannian symmetric space, we will determine all the real restricted roots and all the imaginary restricted roots (see, Section 3). For this purpose, we give a useful condition for $\alpha \in R$ to be real or imaginary by using θ. Then, for each $\alpha \in R$, α takes real values on $\mathfrak{a}_{\boldsymbol{R}}$. Hence α is real (resp. imaginary) if and only if $\theta(\alpha) = -\alpha$ (resp. $\theta(\alpha) = \alpha$). We can make use of a Satake diagram associated with $(R, \theta, \mathfrak{a})$ to determine subsets $\{\alpha \in R \mid \theta(\alpha) = -\alpha\}$ and $\{\alpha \in R \mid \theta(\alpha) = \alpha\}$ $(=: R_0)$ of R. Let $>$ denote the lexicographic ordering in $(\mathfrak{a}_{\boldsymbol{R}})^*$ with respect to an ordered basis $(A_1, \ldots, A_l, A_{l+1}, \ldots, A_r)$ of $\mathfrak{a}_{\boldsymbol{R}}$ such that $(A_1, \ldots, A_l)$ (resp. $(A_{l+1}, \ldots, A_r)$) is a basis of $\mathfrak{p} \cap \mathfrak{a}$ (resp. $\sqrt{-1}(\mathfrak{k} \cap \mathfrak{a})$), where $r = \operatorname{rank} R$ and $l = \dim(\mathfrak{p} \cap \mathfrak{a})$. Then the order $>$ becomes a $(-\theta)$-order in R (cf. [15]). Denote by $\Psi(R)$ the fundamental system of R with respect to $>$. Set $\Psi(R_0) = \Psi(R) \cap R_0$. Then we have the following result.

LEMMA 1.5 ([15, Theorem 5.4]). *There exists a permutation p of $\Psi(R) \setminus \Psi(R_0)$ with order 2 such that, for each $\alpha \in \Psi(R) \setminus \Psi(R_0)$, $(-\theta)(\alpha) \equiv p\alpha \ (\mathrm{mod}\, \mathrm{Span}_{\boldsymbol{Z}}\{\alpha \mid \alpha \in \Psi(R_0)\})$.*

We call the permutation p as in Lemma 1.5 the *Satake involution* of $\Psi(R) \setminus \Psi(R_0)$. From the Dynkin diagram of $\Psi(R)$ we define the Satake diagram associated with $(R, \theta, \mathfrak{a})$ as follows. First, replace a white circle of the Dynkin diagram, which belongs to $\Psi(R_0)$ with a black circle. Next, if restricted roots $\alpha, \beta \in \Psi(R) \setminus \Psi(R_0)$ satisfy $\alpha \neq \beta$ and $p\alpha = \beta$, join α and β with an arrowed segment $\leftrightarrow$. Note that this Satake diagram depends on the choice of a Cartan subspace of $\mathfrak{q}$. A Cartan subspace $\mathfrak{a}$ is said to be *maximally split* (resp. *maximally compact*) if $\mathfrak{p} \cap \mathfrak{a}$ (resp. $\mathfrak{k} \cap \mathfrak{a}$) is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$ (resp. $\mathfrak{k} \cap \mathfrak{q}$). The dimension of the $\mathfrak{p}$-part of a maximally split Cartan subspace (MSCS) is called the *split rank* of G/H (or $(\mathfrak{g}, \mathfrak{h})$). Any two MSCSs are conjugate to each other. Therefore, the definition of split rank does not depend on the choice of an MSCS. We can easily determine the rank and the split rank by using Table 2.5.2 in [13]. In Table 3, we will determine the rank, the split rank and the Satake diagram associated with $(R, \theta, \mathfrak{a})$ when $\mathfrak{a}$ is maximally split for all semisimple pseudo-Riemannian symmetric spaces (see, Appendix A for the determination). Here we will often omit $\mathfrak{a}$ for the notation of the Satake diagram when there is no confusion.

2. The Jordan-Chevalley decompositions of shape operators. Assume that the $\mathrm{Ad}(H)$-orbit M through an $X \in \mathfrak{q}$ is a pseudo-Riemannian submanifold in a pseudo-hypersphere $\boldsymbol{S}$. It follows from Key Lemma that X is semisimple. In general, the shape operator of $M \hookrightarrow \boldsymbol{S}$ is not necessarily diagonalizable over $\boldsymbol{C}$. In this section, for each $\xi \in T_X^\perp M$, we give the JC decomposition of the shape operator A_ξ in direction ξ, where A denotes the shape tensor of $M \hookrightarrow \boldsymbol{S}$.

LEMMA 2.1. *Let ξ be a normal vector of M at X, and $\xi = \xi_s + \xi_n$ be the JC decomposition of ξ. Then ξ_s, ξ_n are normal vectors of M at X.*

PROOF. By using Proposition 2 in [12] and Proposition 1.3.5.1 in [16] we have $\xi_s, \xi_n \in \{Y \in \mathfrak{q} \mid [Y, X] = 0\}$. From Lemma 12 in [12] there exists a $Z \in \mathfrak{h}$ such that $[Z, \xi_n] = \xi_n$. Then we have $B(\xi_n, X) = B([Z, \xi_n], X) = B([\xi_n, X], Z) = 0$, where B denotes the Killing form of $\mathfrak{g}$. Hence $\xi_n \in T_X^\perp M$ holds. Moreover, we have $\xi_s = \xi - \xi_n \in T_X^\perp M$. □

From above lemma a decomposition $A_\xi = A_{\xi_s} + A_{\xi_n}$ is well-defined.

PROPOSITION 2.2. *Let $\xi = \xi_s + \xi_n$ be the JC decomposition of $\xi \in T_X^\perp M$. Then the decomposition $A_\xi = A_{\xi_s} + A_{\xi_n}$ gives the JC decomposition of the shape operator A_ξ, i.e., A_{ξ_s} is semisimple, A_{ξ_n} is nilpotent, and $A_{\xi_s}A_{\xi_n} = A_{\xi_n}A_{\xi_s}$ hold.*

The proof of Proposition 2.2 requires some preparation. For any $Z \in \mathfrak{h}$, we define a tangent vector field Z^* on M by $Z_p^* = (d/dt)|_{t=0} \operatorname{Ad}(\exp tZ)p = [Z, p]$ for all $p \in M$. Denote by h the second fundamental form of $M = \operatorname{Ad}(H)X \hookrightarrow \boldsymbol{S}$. For any $Z, W \in \mathfrak{h}$, we have

$$h(Z_X^*, W_X^*) = (\tilde{\nabla}_{Z^*} W^*)_X^\perp = \left(\frac{d}{dt} W^*_{\operatorname{Ad}(\exp tZ)X|_{t=0}}\right)^\perp = [W, [Z, X]]^\perp,$$

where $\tilde{\nabla}$ denotes the covariant derivative of $\boldsymbol{S}$. By using this calculation we have

$$\begin{aligned} B(A_\xi Z_X^*, W_X^*) &= B(h(W_X^*, Z_X^*), \xi) = B([Z, [W, X]], \xi) = -B([Z, \xi], [W, X]) \\ &= B(-[Z, \xi], W_X^*). \end{aligned}$$

Hence we have $A_\xi Z_X^* = A_\xi[Z, X] = -[Z, \xi]$ for all $\xi \in T_X^\perp M$.

LEMMA 2.3. *For each semisimple $\xi \in T_X^\perp M$, A_ξ is semisimple. Moreover, if R is the restricted root system with respect to a Cartan subspace of $\mathfrak{q}$ containing X and ξ, we have the spectrum of $A_\xi^{\boldsymbol{C}}$ as follows:*

$$\operatorname{Spec} A_\xi^{\boldsymbol{C}} = \left\{ -\frac{\alpha(\xi)}{\alpha(X)} \;\middle|\; \alpha \in R_+ \textit{ with } \alpha(X) \neq 0 \right\}, \tag{3}$$

where R_+ is a positive root system of R.

PROOF. Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{q}$ containing X and ξ, and R denote the restricted root system with respect to $\mathfrak{a}$. For each $\alpha \in R$, we obtain $\mathfrak{q}_\alpha^{\boldsymbol{C}} = \{Z - \sigma(Z) \mid Z \in \mathfrak{g}_\alpha^{\boldsymbol{C}}\}$. Then, for each $\alpha \in R$ with $\alpha(X) \neq 0$ and $Y = Z - \sigma(Z) \in \mathfrak{q}_\alpha^{\boldsymbol{C}}$, we have

$$A_\xi^{\boldsymbol{C}} Y = -\frac{1}{\alpha(X)} A_\xi^{\boldsymbol{C}} [Z + \sigma(Z), X] = \frac{1}{\alpha(X)} [Z + \sigma(Z), \xi] = -\frac{\alpha(\xi)}{\alpha(X)} Y.$$

Hence $A_\xi^{\boldsymbol{C}} = (-\alpha(\xi)/\alpha(X)) \operatorname{id}$ on $\mathfrak{q}_\alpha^{\boldsymbol{C}}$, where id denotes the identity transformation on $(T_X M)^{\boldsymbol{C}} = \sum_{\alpha \in R_+ : \alpha(X) \neq 0} \mathfrak{q}_\alpha^{\boldsymbol{C}}$. This proves the assertion. □

REMARK 2.4. As we shall see in Section 3, there exists a Cartan subspace satisfying the property stated in Lemma 2.3 (see Lemma 3.5).

LEMMA 2.5. *For each nilpotent $\xi \in T_X^\perp M$, A_ξ is nilpotent.*

PROOF. Denote by R the restricted root system with respect to a Cartan subspace of $\mathfrak{q}$ containing X (cf. Lemma 1.3). For $Y = [Z, X] \in \mathfrak{q}_\alpha^{\boldsymbol{C}}$ ($\alpha \in R$ with $\alpha(X) \neq 0$, $Z \in \mathfrak{h}_\alpha^{\boldsymbol{C}}$) we have

$$A_\xi^{\boldsymbol{C}} Y = A_\xi^{\boldsymbol{C}}[Z, X] = [\xi, Z] = \frac{1}{\alpha(X)^2}[\xi, [X, [X, Z]]] = -\frac{1}{\alpha(X)^2}[\xi, [X, Y]].$$

By induction on $n \in \boldsymbol{N}$, for each $\alpha \in R$ with $\alpha(X) \neq 0$,

$$(A_\xi^{\boldsymbol{C}})^n Y = \begin{cases} \dfrac{1}{\alpha(X)^n} \operatorname{ad}(\xi)^n Y & (n : \text{even}), \\ -\dfrac{1}{\alpha(X)^{n+1}} \operatorname{ad}(\xi)^n \operatorname{ad}(X) Y & (n : \text{odd}), \end{cases}$$

for all $Y \in \mathfrak{q}_\alpha^{\boldsymbol{C}}$. Therefore A_ξ is nilpotent if ξ is nilpotent. □

PROOF OF PROPOSITION 2.2. Let $\xi = \xi_s + \xi_n$ be the JC decomposition of $\xi \in T_X^\perp M$. It follows from Lemmas 2.3 and 2.5 that A_{ξ_s} and A_{ξ_n} are semisimple and nilpotent, respectively. Since $[\xi_s, \xi_n] = 0$ holds, A_{ξ_s} and A_{ξ_n} commute with each other. It follows from the uniqueness of the JC decomposition of A_ξ that A_{ξ_s} and A_{ξ_n} coincide with the semisimple part and the nilpotent part of A_ξ, respectively. □

By using Proposition 2.2 and Lemma 2.3 we have the following result.

COROLLARY 2.6. *Let $\xi = \xi_s + \xi_n$ be the JC decomposition of $\xi \in T_X^\perp M$, and R denote the restricted root system with respect to a Cartan subspace of $\mathfrak{q}$ containing X and ξ_s. The spectrum of $A_\xi^{\boldsymbol{C}}$ coincides with that of $A_{\xi_s}^{\boldsymbol{C}}$ and is given as follows:*

$$\operatorname{Spec} A_\xi^{\boldsymbol{C}} = \left\{ -\frac{\alpha(\xi_s)}{\alpha(X)} \;\middle|\; \alpha \in R_+ \textit{ with } \alpha(X) \neq 0 \right\}.$$

3. Austere orbits. First, we recall the notion of austere submanifolds.

DEFINITION 3.1 ([5, Definition 3.15], [1, p. 27]). Let $\tilde{M}$ be a pseudo-Riemannian manifold. A pseudo-Riemannian submanifold $M \hookrightarrow \tilde{M}$ is said to be *austere* if, for all $x \in M$ and $\xi \in T_x M$, all the coefficients of odd degree for the characteristic polynomial of A_ξ vanish.

We can prove that M is austere if and only if, for all $x \in M$ and $\xi \in T_x^\perp M$, $\operatorname{Spec} A_\xi^{\boldsymbol{C}}$ is invariant (considering multiplicities) under the multiplication by -1. Therefore, it is clear that any austere submanifold has zero mean curvature. In this section, we will give an equivalent condition for the austerity when $M \hookrightarrow \boldsymbol{S}$ ($\subset \mathfrak{q}$) is an orbit of the isotropy representation for a semisimple pseudo-Riemannian symmetric space, which is identified with an $\operatorname{Ad}(H)$-orbit as we mentioned in Section 1. Moreover, we will give examples of austere orbits by using the condition. Assume that the $\operatorname{Ad}(H)$-orbit M through $X \in \mathfrak{q}$ is a pseudo-Riemannian submanifold in $\boldsymbol{S}$. This implies that X is semisimple in $\mathfrak{q}$.

PROPOSITION 3.2. *Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{q}$ containing X, and R denote the restricted root system with respect to $\mathfrak{a}$. Then M is an austere orbit in $\boldsymbol{S}$ if and only if $\{(-1/\alpha(X)) p_X(A_\alpha) \mid \alpha \in R_+ \text{ with } \alpha(X) \neq 0\}$ is invariant (considering multiplicities)* *under the multiplication by* -1, *where* p_X *denotes the orthogonal projection along* X (*i.e.*, $p_X : \mathfrak{a}^{\boldsymbol{C}} \to (\mathfrak{a} \ominus \boldsymbol{R}X)^{\boldsymbol{C}}; Y \mapsto Y - (B(Y,X)/B(X,X))X$).

The proof of Proposition 3.2 requires some preparation. Let $\mathfrak{g}_X$ denote the centralizer of X in $\mathfrak{g}$. By imitating the proof of Proposition 1.3.5.4 in [16] we have the following result.

LEMMA 3.3. *There exists a Cartan involution* θ *of* $\mathfrak{g}$ *satisfying* $\theta \circ \sigma = \sigma \circ \theta$ *and* $\theta(\mathfrak{g}_X) = \mathfrak{g}_X$.

PROOF. Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{q}$ containing X. Then there exists a Cartan involution θ of $\mathfrak{g}$ satisfying $\theta \circ \sigma = \sigma \circ \theta$ and $\theta(\mathfrak{a}) = \mathfrak{a}$ (cf. [12, Lemma 5]). If we put $\mathfrak{k} = \mathrm{Ker}(\theta - \mathrm{id})$ and $\mathfrak{p} = \mathrm{Ker}(\theta + \mathrm{id})$, then $\mathfrak{a} = \mathfrak{k} \cap \mathfrak{a} + \mathfrak{p} \cap \mathfrak{a}$ holds. Denote by R the restricted root system with respect to $\mathfrak{a}$. The restricted root space decomposition of $\mathfrak{g}^{\boldsymbol{C}}$ gives a decomposition $\mathfrak{g}_X^{\boldsymbol{C}} = \mathfrak{g}_0^{\boldsymbol{C}} + \sum_{\alpha \in R_X} \mathfrak{g}_\alpha^{\boldsymbol{C}}$ of $\mathfrak{g}_X^{\boldsymbol{C}}$, where $R_X := \{\alpha \in R \mid \alpha(X) = 0\}$. Then we have $\mathfrak{g}_0^{\boldsymbol{C}}$ is θ-invariant. If we write $X = X_1 + X_2$ ($X_1 \in \mathfrak{k} \cap \mathfrak{a}$, $X_2 \in \mathfrak{p} \cap \mathfrak{a}$), then we have $\alpha(X_i) = 0$ ($i = 1, 2$) for all $\alpha \in R_X$, since $\alpha(\mathfrak{k} \cap \mathfrak{a}) \subset \sqrt{-1}\boldsymbol{R}$ and $\alpha(\mathfrak{p} \cap \mathfrak{a}) \subset \boldsymbol{R}$. The action of θ on R is given by $(\theta \cdot \alpha)(A) = \alpha(\theta(A))$ for all $A \in \mathfrak{a}^{\boldsymbol{C}}$. Then, for any $\alpha \in R_X$, we have $(\theta \cdot \alpha)(X) = \alpha(\theta(X)) = \alpha(X_1) - \alpha(X_2) = 0$. This implies that R_X is θ-invariant. Hence $\mathfrak{g}_X^{\boldsymbol{C}}$ is θ-invariant. □

Fix a Cartan involution θ of $\mathfrak{g}$ as in Lemma 3.3. It follows from Corollary 1.1.5.4 in [16] that $\mathfrak{g}_X$ is reductive in $\mathfrak{g}$. Then we have a decomposition $\mathfrak{g}_X = \mathfrak{c}_X + \mathfrak{s}_X$ of $\mathfrak{g}_X$, where $\mathfrak{c}_X$ (resp. $\mathfrak{s}_X = [\mathfrak{g}_X, \mathfrak{g}_X]$) denotes the center (resp. the derived algebra) of $\mathfrak{g}_X$. Note that $\mathfrak{c}_X$ is θ-invariant, and $\mathfrak{s}_X$ is θ-invariant and semisimple. On the other hand, it is clear that $\mathfrak{g}_X, \mathfrak{c}_X, \mathfrak{s}_X$ are σ-invariant. Set $\mathfrak{h}_X = \mathfrak{h} \cap \mathfrak{g}_X$, $\mathfrak{q}_X = \mathfrak{q} \cap \mathfrak{g}_X$. Then the pair $(\mathfrak{g}_X, \mathfrak{h}_X)$ gives a reductive symmetric pair.

LEMMA 3.4. *A subspace* $\mathfrak{a}$ *of* $\mathfrak{q}$ *is Cartan for* $(\mathfrak{g}, \mathfrak{h})$ *and contains* X *if and only if* $\mathfrak{a}$ *is a Cartan subspace of* $\mathfrak{q}_X$ *for* $(\mathfrak{g}_X, \mathfrak{h}_X)$.

PROOF. Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{q}$ for $(\mathfrak{g}, \mathfrak{h})$ containing X. Since $[A, X] = 0$ holds for all $A \in \mathfrak{a}$ we have $\mathfrak{a} \subset \mathfrak{q}_X$. It follows from the maximality of $\mathfrak{a}$ in $\mathfrak{q}$ that $\mathfrak{a}$ is maximal abelian in $\mathfrak{q}_X (\subset \mathfrak{q})$. Since $\mathfrak{g}_X$ is $\mathrm{ad}(A)$-invariant for any $A \in \mathfrak{a}$, the endomorphism $\mathrm{ad}(A)$ of $\mathfrak{g}_X$ is semisimple. Hence $\mathfrak{a}$ is Cartan for $(\mathfrak{g}_X, \mathfrak{h}_X)$.

Conversely, suppose that $\mathfrak{a}$ is a Cartan subspace of $\mathfrak{q}_X$ for $(\mathfrak{g}_X, \mathfrak{h}_X)$. It follows from $[X, \mathfrak{a}] = \{0\}$ that $\mathfrak{a} + \boldsymbol{R}X$ is a abelian subspace of $\mathfrak{q}_X$ containing $\mathfrak{a}$. Then, by using the maximality of $\mathfrak{a}$ in $\mathfrak{q}_X$ we obtain $X \in \mathfrak{a}$. Let $\tilde{\mathfrak{a}}$ be an abelian subspace of $\mathfrak{q}$ containing $\mathfrak{a}$. From $X \in \tilde{\mathfrak{a}}$ we obtain $\tilde{\mathfrak{a}} \subset \mathfrak{q}_X$. This implies that $\tilde{\mathfrak{a}} = \mathfrak{a}$ because of the maximality of $\mathfrak{a}$ in $\mathfrak{q}_X$. Hence $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{q}$. In order to prove that any $A \in \mathfrak{a}$ is semisimple in $\mathfrak{g}$, it is sufficient to show that B is nondegenerate on $\mathfrak{a}$ (cf. Lemma 1.2). We can verify that

$$\mathfrak{a} = (\mathfrak{c}_X \cap \mathfrak{a}) + (\mathfrak{s}_X \cap \mathfrak{a}) . \tag{4}$$

Let θ be a Cartan involution of $\mathfrak{g}$ as in Lemma 3.3. Since the subspace $\mathfrak{c}_X \cap \mathfrak{a}$ coincides with $\mathfrak{c}_X \cap \mathfrak{q}$, it is θ-invariant. Hence B is nondegenerate on $\mathfrak{c}_X \cap \mathfrak{a}$. The subspace $\mathfrak{s}_X \cap \mathfrak{a}$ gives a Cartan subspace of $\mathfrak{s}_X \cap \mathfrak{q}$ for the semisimple symmetric pair $(\mathfrak{s}_X, \mathfrak{s}_X \cap \mathfrak{h})$. It follows from Lemma 5 in [11] that θ gives a Cartan involution of $\mathfrak{s}_X$. By using Remark in [12] there exists an $h \in (H_X)_0$ such that $\mathrm{Ad}(h)(\mathfrak{s}_X \cap \mathfrak{a})$ is θ-invariant, where H_X denotes the isotropy subgroup of H at X. This implies that B is nondegenerate on $\mathrm{Ad}(h)(\mathfrak{s}_X \cap \mathfrak{a})$. Therefore B is also nondegenerate on $\mathfrak{s}_X \cap \mathfrak{a}$. Since the decomposition (4) is orthogonal with respect to B, we have the nondegeneracy of B on $\mathfrak{a}$. Hence we have proved the statement. □

LEMMA 3.5. *Let X, Y be semisimple elements in $\mathfrak{q}$ with $[X, Y] = 0$. Then there exists a Cartan subspace of $\mathfrak{q}$ for $(\mathfrak{g}, \mathfrak{h})$ containing X and Y.*

PROOF. From $[X, Y] = 0$ we have $Y \in \mathfrak{q}_X$ and $\mathrm{ad}(Y)(\mathfrak{g}_X) \subset \mathfrak{g}_X$. It follows from the semisimplicity of Y in $\mathfrak{g}$ that Y is also a semisimple element in $\mathfrak{g}_X$. The decomposition $\mathfrak{g}_X = \mathfrak{c}_X + \mathfrak{s}_X$ leads a decomposition $\mathfrak{q}_X = (\mathfrak{c}_X \cap \mathfrak{q}) + (\mathfrak{s}_X \cap \mathfrak{q})$. This allows us to write $Y = Y_1 + Y_2$ for $Y_1 \in \mathfrak{c}_X \cap \mathfrak{q}$ and $Y_2 \in \mathfrak{s}_X \cap \mathfrak{q}$. Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{s}_X \cap \mathfrak{q}$ for $(\mathfrak{s}_X, \mathfrak{s}_X \cap \mathfrak{h})$ containing Y_2. Set $\mathfrak{a}' := (\mathfrak{c}_X \cap \mathfrak{q}) + \mathfrak{a}$. Then it is shown that $\mathfrak{a}'$ is a Cartan subspace of $\mathfrak{q}_X$ for $(\mathfrak{g}_X, \mathfrak{h}_X)$ containing Y. By using Lemma 3.4 $\mathfrak{a}'$ is a Cartan subspace of $\mathfrak{q}$ for $(\mathfrak{g}, \mathfrak{h})$ containing X and Y. □

Let $\mathcal{CS}_X$ denote the set of all Cartan subspaces of $\mathfrak{q}$ containing X. The isotropy subgroup H_X acts naturally on $\mathcal{CS}_X$.

LEMMA 3.6. *The set $(T_X^\perp M)_s$ of all semisimple normal vectors in $T_X^\perp M$ is given as follows*:

$$(T_X^\perp M)_s = \bigcup_{\mathfrak{a} \in \mathcal{CS}_X} (\mathfrak{a} \ominus \boldsymbol{R}X). \tag{5}$$

PROOF. Let ξ be a semisimple normal vector in $T_X^\perp M$. From Lemma 3.5 there exists a Cartan subspace $\mathfrak{a}$ of $\mathfrak{q}$ containing X and ξ. Since $B(\xi, X) = 0$ holds, we have $\xi \in \mathfrak{a} \ominus \boldsymbol{R}X$. The converse is trivial from the definition of Cartan subspaces. □

LEMMA 3.7. *Let $\mathfrak{a}, \mathfrak{a}'$ be Cartan subspaces of $\mathfrak{q}$ containing X. Suppose that there exists a $k \in H_X$ satisfying $\mathfrak{a}' = \mathrm{Ad}(k)\mathfrak{a}$. For any $\xi \in \mathfrak{a} \ominus \boldsymbol{R}X$ and $\xi' := \mathrm{Ad}(k)\xi \in \mathfrak{a}' \ominus \boldsymbol{R}X$, we have* $\mathrm{Spec}\, A_\xi^{\boldsymbol{C}} = \mathrm{Spec}\, A_{\xi'}^{\boldsymbol{C}}$.

PROOF. Denote by R (resp. R') the restricted root system of $(\mathfrak{g}, \mathfrak{h})$ with respect to $\mathfrak{a}$ (resp. $\mathfrak{a}'$). For $\alpha \in (\mathfrak{a}^{\boldsymbol{C}})^*$, we define $\alpha^k \in (\mathfrak{a}'^{\boldsymbol{C}})^*$ by $\alpha^k(A') = \alpha(\mathrm{Ad}(k)^{-1}A')$ for all $A' \in \mathfrak{a}'^{\boldsymbol{C}}$. Then we obtain $R' = \{\alpha^k \mid \alpha \in R\}$ and $\alpha^k(\xi') = \alpha(\xi)$ for $\alpha \in R$. In addition, it is readily verified that $\alpha^k(X) = \alpha(X)$ for $\alpha \in R$. Hence a direct calculation by means of (3) shows

$$\begin{aligned}\mathrm{Spec}\, A_{\xi'}^{\boldsymbol{C}} &= \left\{ -\frac{\alpha'(\xi')}{\alpha'(X)} \;\middle|\; \alpha' \in R' \text{ with } \alpha'(X) \neq 0 \right\} \\ &= \left\{ -\frac{\alpha^k(\xi')}{\alpha^k(X)} \;\middle|\; \alpha \in R \text{ with } \alpha(X) \neq 0 \right\} \\ &= \mathrm{Spec}\, A_\xi^{\boldsymbol{C}}.\end{aligned}$$

□

LEMMA 3.8. *Let θ be a Cartan involution of $\mathfrak{g}$ as in Lemma 3.3. For any $\mathfrak{a} \in \mathcal{CS}_X$, there exists an $h \in H_X$ satisfying $\theta(\mathrm{Ad}(h)\mathfrak{a}) = \mathrm{Ad}(h)\mathfrak{a}$.*

PROOF. Let $\mathfrak{a} \in \mathcal{CS}_X$. From Lemma 3.4 $\mathfrak{a}$ is a Cartan subspace of $\mathfrak{q}_X$. Then by a similar argument in the proof of Lemma 3.4 we can verify that $\mathfrak{a} = (\mathfrak{c}_X \cap \mathfrak{q}) + (\mathfrak{s}_X \cap \mathfrak{a})$ and $\mathrm{Ad}(h)(\mathfrak{s}_X \cap \mathfrak{a})$ is θ-invariant for some $h \in (H_X)_0$. Moreover, we have $\mathrm{Ad}(h)Y = Y$ for all $Y \in \mathfrak{c}_X \cap \mathfrak{q}$. Hence $\mathrm{Ad}(h)\mathfrak{a}$ is θ-invariant. □

PROPOSITION 3.9. *Let θ be a Cartan involution of $\mathfrak{g}$ as in Lemma 3.3. For any θ-invariant Cartan subspaces $\mathfrak{a}$, $\mathfrak{a}'$ of $\mathfrak{q}$ containing X, there exists an isomorphism ψ of $\mathfrak{g}^{\boldsymbol{C}}$ satisfying $\psi \circ \sigma = \sigma \circ \psi$, $\psi(\mathfrak{a}^{\boldsymbol{C}} \ominus \boldsymbol{C}X) = \mathfrak{a}'^{\boldsymbol{C}} \ominus \boldsymbol{C}X$ and $\psi(X) = X$.*

PROOF. Set $\mathfrak{k} = \mathrm{Ker}(\theta - \mathrm{id}), \mathfrak{p} = \mathrm{Ker}(\theta + \mathrm{id}), \mathfrak{k}_X = \mathfrak{k} \cap \mathfrak{g}_X$ and $\mathfrak{p}_X = \mathfrak{p} \cap \mathfrak{g}_X$. Then we have $\mathfrak{a} = \mathfrak{k} \cap \mathfrak{a} + \mathfrak{p} \cap \mathfrak{a}$ and $\mathfrak{a}' = \mathfrak{k} \cap \mathfrak{a}' + \mathfrak{p} \cap \mathfrak{a}'$, and the simultaneous decomposition $\mathfrak{g}_X = \mathfrak{k}_X \cap \mathfrak{h}_X + \mathfrak{p}_X \cap \mathfrak{h}_X + \mathfrak{k}_X \cap \mathfrak{q}_X + \mathfrak{p}_X \cap \mathfrak{q}_X$ of σ and θ. Set $\mathfrak{g}_X^d = \mathfrak{k}_X \cap \mathfrak{h}_X + \sqrt{-1}(\mathfrak{p}_X \cap \mathfrak{h}_X) + \sqrt{-1}(\mathfrak{k}_X \cap \mathfrak{q}_X) + \mathfrak{p}_X \cap \mathfrak{q}_X (\subset \mathfrak{g}_X^{\boldsymbol{C}})$. Then σ gives a Cartan involution of $\mathfrak{g}_X^d$, so that $\mathfrak{g}_X^d = \mathfrak{k}_X^d + \mathfrak{p}_X^d$ is the Cartan decomposition for σ, where $\mathfrak{k}_X^d := \mathfrak{k}_X \cap \mathfrak{h}_X + \sqrt{-1}(\mathfrak{p}_X \cap \mathfrak{h}_X)$ and $\mathfrak{p}_X^d := \sqrt{-1}(\mathfrak{k}_X \cap \mathfrak{q}_X) + \mathfrak{p}_X \cap \mathfrak{q}_X$. By the maximality of $\mathfrak{a}$ (resp. $\mathfrak{a}'$) in $\mathfrak{q}_X$ $\mathfrak{a}^d := \sqrt{-1}(\mathfrak{k} \cap \mathfrak{a}) + \mathfrak{p} \cap \mathfrak{a}$ (resp. $\mathfrak{a}'^d := \sqrt{-1}(\mathfrak{k} \cap \mathfrak{a}') + \mathfrak{p} \cap \mathfrak{a}'$) is a maximal abelian subspace of $\mathfrak{p}_X^d$. This implies that there exists a $Z \in \mathfrak{k}_X^d$ satisfying $\mathfrak{a}'^d = e^{\mathrm{ad}(Z)}\mathfrak{a}^d$. If we put $\psi = e^{\mathrm{ad}(Z)}$, then we have $\psi \circ \sigma = \sigma \circ \psi$ and $\psi(X) = X$. Hence $\psi(\mathfrak{a}^{\boldsymbol{C}} \ominus \boldsymbol{C}X) = \mathfrak{a}'^{\boldsymbol{C}} \ominus \boldsymbol{C}X$ holds. □

By imitating the argument in pp. 459–460, [8] we have the following result.

LEMMA 3.10. *Let V be a vector space over $\boldsymbol{R}$ and B be a nondegenerate bilinear form on V. For any finite subset $\mathcal{A} \subset V^{\boldsymbol{C}}$, the set $\{B(a, v) \mid a \in \mathcal{A}\}$ $(=: \mathcal{A}(v) \subset \boldsymbol{C})$ is invariant by multiplication of -1 for all $v \in V$ if and only if $\mathcal{A}$ is invariant by multiplication of -1.*

PROOF. Suppose that $\mathcal{A}(v)$ is invariant by multiplication of -1 for all $v \in V$. Take an $a \in \mathcal{A}$. Then we have

$$V = \bigcup_{b \in \mathcal{A}} \{v \in V \mid B(a, v) = -B(b, v)\}.$$

For each $b \in \mathcal{A}$, $\{v \in V \mid B(a, v) = -B(b, v)\}$ is a subspace of V. For $b \in \mathcal{A}$ with $b \neq -a$, this subspace does not coincide with V. Since $\mathcal{A}$ is a finite set, we have $-a \in \mathcal{A}$. □

PROOF OF PROPOSITION 3.2. Let $\mathfrak{a} \in \mathcal{CS}_X$ and R denote the restricted root system of $(\mathfrak{g}, \mathfrak{h})$ with respect to $\mathfrak{a}$. Let $\{\mathfrak{a}_\lambda\}_{\lambda \in \Lambda}$ be a complete representatives of $\mathcal{CS}_X/H_X$ containing $\mathfrak{a}$. Without loss of generality, we can assume that $\mathfrak{a}_\lambda$ $(\lambda \in \Lambda)$ is θ-invariant (see, Lemmas 3.7 and 3.8). By using Lemma 3.8 we have

$$(T_X^\perp M)_s = \bigcup_{\substack{h \in H_X \\ \lambda \in \Lambda}} \mathrm{Ad}(h)(\mathfrak{a}_\lambda \ominus \boldsymbol{R}X).$$

It follows from Corollary 2.6 and Lemma 3.7 that M is austere if and only if, for any $\lambda \in \Lambda$ and $\xi \in \mathfrak{a}_\lambda \ominus \boldsymbol{R}X$, $\mathrm{Spec}\, A_\xi^{\boldsymbol{C}}$ is invariant (considering multiplicities) under the multiplication by -1. Let R_λ denote the restricted root system of $(\mathfrak{g}, \mathfrak{h})$ with respect to $\mathfrak{a}_\lambda$ and $(R_\lambda)_+$ be a positive root system of R_λ. Then we have

$$\mathrm{Spec}\, A_\xi^{\boldsymbol{C}} = \left\{ B\left(-\frac{1}{\alpha(X)}(p_\lambda)_X(A_\alpha), \xi\right) \,\middle|\, \alpha \in (R_\lambda)_+ \text{ with } \alpha(X) \neq 0 \right\}$$

for $\xi \in \mathfrak{a}_\lambda \ominus \boldsymbol{R}X$, where $(p_\lambda)_X : \mathfrak{a}_\lambda^{\boldsymbol{C}} \to (\mathfrak{a}_\lambda \ominus \boldsymbol{R}X)^{\boldsymbol{C}}$ denotes the orthogonal projection along X. By applying Lemma 3.10 to $\{(-1/\alpha(X))((p_\lambda)_X(A_\alpha)) \mid \alpha \in (R_\lambda)_+ \text{ with } \alpha(X) \neq 0\}(=: \mathcal{A}_\lambda)$ it is verified that M is austere if and only if, for any $\lambda \in \Lambda$, $\mathcal{A}_\lambda$ is invariant (considering multiplicities) by multiplication of -1. Therefore it is sufficient to show that if $\{(-1/\alpha(X))(p_X(A_\alpha)) \mid \alpha \in R_+ \text{ with } \alpha(X) \neq 0\}$ is invariant (considering multiplicities) by multiplication of -1 then the same property holds for $\mathcal{A}_\lambda$ ($\lambda \in \Lambda$).

By using Proposition 3.9, for $\lambda \in \Lambda$, there exists an isomorphism ψ_λ of $\mathfrak{g}^{\boldsymbol{C}}$ satisfying

$$\psi_\lambda \circ \sigma = \sigma \circ \psi_\lambda, \quad \psi_\lambda(\mathfrak{a}^{\boldsymbol{C}} \ominus \boldsymbol{C}X) = \mathfrak{a}_\lambda^{\boldsymbol{C}} \ominus \boldsymbol{C}X, \quad \psi_\lambda(X) = X. \tag{6}$$

This implies that $R_\lambda = \{\alpha^{\psi_\lambda} \mid \alpha \in R\}$, where $\alpha^{\psi_\lambda} \in (\mathfrak{a}_\lambda^{\boldsymbol{C}})^*$ is defined by $\alpha^{\psi_\lambda}(A) = \alpha(\psi_\lambda^{-1}(A))$ for all $A \in \mathfrak{a}_\lambda^{\boldsymbol{C}}$. We also have $A_{\alpha^{\psi_\lambda}} = \psi_\lambda(A_\alpha)$ for $\alpha \in R$. A simple calculation by means of (6) shows that

$$\begin{aligned}
-\frac{1}{\alpha^{\psi_\lambda}(X)}(p_\lambda)_X(A_{\alpha^{\psi_\lambda}}) &= -\frac{1}{\alpha(X)}(p_\lambda)_X(\psi_\lambda(A_\alpha)) \\
&= -\frac{1}{\alpha(X)}\left(\psi_\lambda(A_\alpha) - \frac{B(\psi_\lambda(A_\alpha), X)}{B(X,X)}X\right) \\
&= \psi_\lambda\left(-\frac{1}{\alpha(X)}p_X(A_\alpha)\right).
\end{aligned}$$

Hence we obtain $\mathcal{A}_\lambda = \psi_\lambda(\{(-1/\alpha(X))(p_X(A_\alpha)) \mid \alpha \in R_+ \text{ with } \alpha(X) \neq 0\})$. This proves the statement. □

COROLLARY 3.11. *The orbit through the restricted root vector corresponding to any real restricted root is an austere submanifold in* $\boldsymbol{S}$.

PROOF. Let α be a real restricted root. Then the restricted root vector A_α is in $\mathfrak{p} \cap \mathfrak{q}$ and $B(A_\alpha, A_\alpha) > 0$. If we put $X = A_\alpha$, then $\{(-1/\beta(X))p_X(A_\beta) \mid \beta \in R_+ \text{ with } \beta(X) \neq 0\}$ $(=: \mathcal{A})$ is invariant (considering multiplicities) under the multiplication by -1. Indeed, for any $v = (-1/\beta(X))p_X(A_\beta) \in \mathcal{A}$, we have $s_\alpha(\beta)(X) = -\beta(X) \neq 0$ and $-v = (-1/s_\alpha(\beta)(X))p_X(s_\alpha(A_\beta)) \in \mathcal{A}$. Moreover, the multiplicity of any restricted root is invariant under the action of s_α. This implies that, for each $\xi \in \mathfrak{a} \ominus \boldsymbol{R}X$, the multiplicities of $A_\xi^{\boldsymbol{C}}$ associated with $B(\pm v, \xi)$ coincide with each other. □

REMARK 3.12. Ikawa-Sakai-Tasaki proved Corollary 3.11 in the case when G/H is a Riemannian symmetric space (cf. [8, Proposition 4.4]). In fact, they classified austere orbits (cf. [8, Theorem 5.1]).

In the sequel, we give all the real restricted roots in the restricted root system with respect to an MSCS. Let θ be a Cartan involution of $\mathfrak{g}$ commuting with σ (cf. [10, Theorem 2.1, Chapter IV]). Denote by R the restricted root system with respect to a θ-invariant MSCS $\mathfrak{a}$. As we mentioned in Section 1, a restricted root α is real if and only if $\theta(\alpha) = -\alpha$. On the other hand, we can determine the action of θ on R in terms of the Satake diagram associated with (R,θ). Then we have the following result.

THEOREM 3.13. *All the real restricted roots in the restricted root system with respect to an MSCS for all semisimple pseudo-Riemannian symmetric spaces are as in Table 1.*

The proof of Theorem 3.13 is given by Lemmas 3.14–3.32 as shown in the following. Set $\tilde{\theta} = -\theta$ and $\alpha^{\tilde{\theta}} = -\theta(\alpha)$. Then $\alpha \in R$ is real if and only if $\alpha^{\tilde{\theta}} = \alpha$ holds.

LEMMA 3.14. *In the case where (R,θ) is of type AI, DI*(rank = s-rank)*, EI, EV, EVIII, FI, or G, all restricted roots are real.*

PROOF. From the Satake diagram of (R,θ) the Satake involution is trivial and $\Psi(R_0) = \emptyset$. This implies that $\alpha^{\tilde{\theta}} = \alpha$ for all $\alpha \in \Psi(R)$. Therefore all restricted roots are real. □

In the sequel, for each root $\alpha \in R$, we give the form $\alpha = \sum n_i\alpha_i$, where the α_i's are fundamental roots as in Table 3 and the n_i's are integers which are either all nonnegative or all nonpositive.

LEMMA 3.15. *In the case where (R,θ) is of type AII, there exists no real restricted root.*

PROOF. Without loss of generality, we assume that rank $R = 2r - 1$. From the Satake diagram of (R,θ) the Satake involution is trivial and $\Psi(R_0) = \{\alpha_{2i-1} \mid 1 \leq i \leq r\}$. Note that any restricted root α is the form $\pm(\alpha_i + \cdots + \alpha_{j-1})$ for $1 \leq i < j \leq 2r$. Therefore, for each $1 \leq i \leq r$, the possibility of the form $\alpha_{2i}^{\tilde{\theta}}$ is either α_{2i}, $\alpha_{2i-1} + \alpha_{2i}$, $\alpha_{2i} + \alpha_{2i+1}$ or $\alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}$. If $\alpha_{2i}^{\tilde{\theta}} = \alpha_{2i}$, we have $(\alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1})^{\tilde{\theta}} = -\alpha_{2i-1} + \alpha_{2i} - \alpha_{2i+1}$. But this contradicts that $(\alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1})^{\tilde{\theta}}$ is a restricted root. Hence we have $\alpha_{2i}^{\tilde{\theta}} \neq \alpha_{2i}$. By the same argument we have $\alpha_{2i}^{\tilde{\theta}} = \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}$ for $1 \leq i \leq r-1$. Moreover, we have

$$(\alpha_i + \cdots + \alpha_{j-1})^{\tilde{\theta}} = \begin{cases} \alpha_{i+1} + \cdots + \alpha_j & (i : \text{odd},\ j : \text{odd}), \\ \alpha_{i+1} + \cdots + \alpha_{j-2} & (i : \text{odd},\ j : \text{even}), \\ \alpha_{i-1} + \cdots + \alpha_j & (i : \text{even},\ j : \text{odd}), \\ \alpha_{i-1} + \cdots + \alpha_{j-2} & (i : \text{even},\ j : \text{even}). \end{cases}$$

Hence there exists no real restricted root. □

LEMMA 3.16. *In the case where (R,θ) is of type AIII*(rank $= r$, s-rank $= l$)*, the set of all real restricted roots of R coincides with $\{\pm(\alpha_i + \cdots + \alpha_{r+1-i}) \mid 1 \leq i \leq l\}$.*

PROOF. In this case we have $p\alpha_i = \alpha_{r+1-i}$ for $i = 1,\ldots,l,r-l+1,\ldots,r$. First, we consider the case of $r = 2l-1$ or $2l$. Then, from the Satake diagram of (R,θ) we have $\Psi(R_0) = \emptyset$. This implies that $\alpha_i^{\tilde{\theta}} = p\alpha_i$ for $i = 1,\ldots,l,r-l+1,\ldots,r$. Therefore, for each $\alpha = \alpha_i + \cdots + \alpha_{j-1}$, $\alpha^{\tilde{\theta}} = \alpha$ holds if and only if $i + j = r + 2$ holds. Next, we consider the case of $r > 2l$. From the Satake diagram of (R,θ) we have $\Psi(R_0) = \{\alpha_i \mid l+1 \leq i \leq r-l\}$.

Since $\tilde{\theta}$ leaves R invariant, we have

$$\alpha_i^{\tilde{\theta}} = \begin{cases} \alpha_{r-i+1} & (1 \leq i \leq l-1, r-l+2 \leq i \leq r), \\ \alpha_{l+1} + \cdots + \alpha_{r-l+1} & (i = l), \\ \alpha_l + \cdots + \alpha_{r-l} & (i = r-l+1), \\ -\alpha_i & (l+1 \leq i \leq r-l). \end{cases}$$

Hence $\alpha^{\tilde{\theta}} = \alpha$ holds if and only if α has the form $\alpha = \pm(\alpha_i + \cdots + \alpha_{r+1-i})$. □

LEMMA 3.17. *In the case where (R, θ) is of type BI(rank $= r$, s-rank $= l$), the set of all real restricted roots of R coincides with*

$$\{ \pm(\alpha_i + \cdots + \alpha_{j-1}) \mid 1 \leq i < j \leq l\}$$
$$\cup\{\pm(\alpha_i + \cdots + \alpha_r + \alpha_j + \cdots + \alpha_r) \mid 1 \leq i < j \leq l\} \cup \{\pm(\alpha_i + \cdots + \alpha_r) \mid 1 \leq i \leq l\}.$$

PROOF. From the Satake diagram of (R, θ) the Satake involution is trivial and $\Psi(R_0) = \{\alpha_{l+k} \mid 1 \leq k \leq r-l\}$. Since any positive root has the form $\alpha_i + \cdots + \alpha_{j-1}, \alpha_i + \cdots + \alpha_r + \alpha_j + \cdots + \alpha_r$ or $\alpha_i + \cdots + \alpha_r$, we have $\alpha_i^{\tilde{\theta}} = \alpha_i$ for $1 \leq i \leq l-1$. Moreover, $\alpha_l^{\tilde{\theta}} = \alpha_l + \cdots + \alpha_r + \alpha_{l+1} + \cdots + \alpha_r$ holds because $\tilde{\theta}$ leaves the root system R invariant. Therefore, by direct calculation we can explicitly determine the set $\{\alpha \in R \mid \alpha = \alpha^{\tilde{\theta}}\}$ as in the assertion. □

By imitating the proof of Lemma 3.17 we have the following two facts.

LEMMA 3.18. *In the case where (R, θ) is of type BCI(rank $= r$, s-rank $= l$), the set of all real restricted roots of R coincides with*

$$\{ \pm(\alpha_i + \cdots + \alpha_{j-1}) \mid 1 \leq i < j \leq l\}$$
$$\cup\{\pm(\alpha_i + \cdots + \alpha_r + \alpha_j + \cdots + \alpha_r) \mid 1 \leq i < j \leq l\}$$
$$\cup\{\pm(\alpha_i + \cdots + \alpha_r) \mid 1 \leq i \leq l\} \cup \{\pm 2(\alpha_i + \cdots + \alpha_r) \mid 1 \leq i \leq l\}.$$

LEMMA 3.19. *In the case where (R, θ) is of type CI(rank $= r$, s-rank $= l$), the set of all real restricted roots coincides with*

$$\{ \pm(\alpha_i + \cdots + \alpha_{j-1}) \mid 1 \leq i < j \leq l\}$$
$$\cup\{\pm(\alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{r-1} + \alpha_r) \mid 1 \leq i < j \leq l\}$$
$$\cup\{\pm(2\alpha_i + \cdots + 2\alpha_{r-1} + \alpha_r) \mid 1 \leq i \leq l\}.$$

LEMMA 3.20. *In the case where (R, θ) is of type CIII(rank $= r$, s-rank $= l$), the set of all real restricted roots of R coincides with $\{\pm(\alpha_{2i-1} + 2\alpha_{2i} + \cdots + 2\alpha_{r-1} + \alpha_r) \mid 1 \leq i \leq l\}$.*

PROOF. First, we consider the case of $r = 2l$. From the Satake diagram of (R, θ) the Satake involution is trivial and $\Psi(R_0) = \{\alpha_{2i-1} \mid 1 \leq i \leq l\}$. Since $\tilde{\theta}$ leaves the root system R invariant and any positive root has the form $\alpha_i + \cdots + \alpha_{j-1}, \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + \alpha_{2l-1} + \alpha_{2l}$

or $2\alpha_i + \cdots + 2\alpha_{2l-1} + \alpha_{2l}$, we have

$$\alpha_{2i}^{\tilde{\theta}} = \begin{cases} \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} & (1 \leq i \leq l-1), \\ 2\alpha_{2l-1} + \alpha_{2l} & (i = l). \end{cases}$$

Therefore, by direct calculation we can explicitly determine the set $\{\alpha \in R \mid \alpha = \alpha^{\tilde{\theta}}\}$ as in the assertion. Next, we consider the case of $r > 2l$. By the same argument as above we have $\alpha_{2i-1}^{\tilde{\theta}} = -\alpha_{2i-1}(1 \leq i \leq l)$, $\alpha_{2l+k}^{\tilde{\theta}} = -\alpha_{2l+k}(1 \leq k \leq r-2l)$ and

$$\alpha_{2i}^{\tilde{\theta}} = \begin{cases} \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} & (1 \leq i \leq l-1), \\ \alpha_{2l-1} + \alpha_{2l} + 2\alpha_{2l+1} + \cdots + 2\alpha_{r-1} + \alpha_r & (i = l). \end{cases}$$

Therefore, by direct calculation we can explicitly determine the set $\{\alpha \in R \mid \alpha = \alpha^{\tilde{\theta}}\}$ as in the assertion. □

By imitating the proof of Lemma 3.20 we have the following fact.

LEMMA 3.21. *In the case where (R, θ) is of type BCIII*(rank $= r$, s-rank $= l$)*, the set of all real restricted roots of R coincides with* $\{\pm(\alpha_{2i-1} + 2\alpha_{2i} + \cdots + 2\alpha_r) \mid 1 \leq i \leq l\}$.

LEMMA 3.22. *In the case where (R, θ) is of type DI*(rank $= r$, s-rank $= l$) $(r > l)$*, the set of all real restricted roots of R coincides with*

$$\{\pm(\alpha_i + \cdots + \alpha_{j-1}) \mid 1 \leq i < j \leq l\} \cup \{\pm(\alpha_i + \cdots + \alpha_{r-2} + \alpha_j + \cdots + \alpha_r) \mid 1 \leq i < j \leq l\}.$$

PROOF. First, we consider the case of $r = l + 1$. From the Satake diagram of (R, θ) we have $\Psi(R_0) = \emptyset$ and

$$\alpha_i^{\tilde{\theta}} = p\alpha_i = \begin{cases} \alpha_i & (1 \leq i \leq r-2), \\ \alpha_r & (i = r-1), \\ \alpha_{r-1} & (i = r). \end{cases}$$

By direct calculation we have

$$(\alpha_i + \cdots + \alpha_{j-1})^{\tilde{\theta}} = \begin{cases} \alpha_i + \cdots + \alpha_{j-1} & (1 \leq i < j \leq r-1), \\ \alpha_i + \cdots + \alpha_{r-2} + \alpha_r & (1 \leq i < j = r), \end{cases}$$

$$(\alpha_i + \cdots + \alpha_{r-2} + \alpha_j + \cdots + \alpha_r)^{\tilde{\theta}} = \begin{cases} \alpha_i + \cdots + \alpha_{r-2} + \alpha_j + \cdots + \alpha_r & (1 \leq i < j \leq r-1), \\ \alpha_i + \cdots + \alpha_{r-1} & (1 \leq i < j = r). \end{cases}$$

This proves the statement. Next, we consider the case of $r > l + 1$. From the Satake diagram of (R, θ) the Satake involution is trivial and $\Psi(R_0) = \{\alpha_{l+k} \mid 1 \leq i \leq r - l\}$. Since $\tilde{\theta}$ leaves the root system R invariant, we have

$$\alpha_i^{\tilde{\theta}} = \begin{cases} \alpha_i & (1 \leq i \leq l-1), \\ \alpha_l + \cdots + \alpha_{r-2} + \alpha_{l+1} + \cdots + \alpha_r & (i = l). \end{cases}$$

Therefore, by direct calculation we can explicitly determine the set $\{\alpha \in R \mid \alpha = \alpha^{\tilde{\theta}}\}$ as in the assertion. □

By imitating the proof of Lemma 3.22 we have the following fact.

LEMMA 3.23. *In the case where (R, θ) is of type DIII*(rank $= r$, s-rank $= l$)*, the set of all real restricted roots of R coincides with*

$$\{\pm(\alpha_{2i-1} + \cdots + \alpha_{r-2} + \alpha_{2i} + \cdots + \alpha_r) \mid 1 \leq i \leq l\}.$$

LEMMA 3.24. *In the case where (R, θ) is of type EII, the set of all real restricted roots of R coincides with*

$$\begin{aligned} &\{ \pm\alpha_2, \pm\alpha_4, \pm(\alpha_3 + \alpha_4 + \alpha_5), \pm(\alpha_2 + \alpha_4), \pm(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\} \\ &\cup \{\pm(\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5), \pm(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\} \\ &\cup \{\pm(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\} \\ &\cup \{\pm(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6), \pm(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6)\} \\ &\cup \{\pm(\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6), \pm(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6)\}. \end{aligned}$$

PROOF. From the Satake diagram of (R, θ) the Satake involution p satisfies $p\alpha_1 = \alpha_6, p\alpha_2 = \alpha_2, p\alpha_3 = \alpha_5$ and $p\alpha_4 = \alpha_4$, and $\Psi(R_0) = \emptyset$. Therefore we have $\alpha_1^{\tilde{\theta}} = \alpha_6, \alpha_2^{\tilde{\theta}} = \alpha_2, \alpha_3^{\tilde{\theta}} = \alpha_5$ and $\alpha_4^{\tilde{\theta}} = \alpha_4$. If we put $\alpha = \sum_{i=1}^{6} n_i\alpha_i \in R$ then, $\alpha = \alpha^{\tilde{\theta}}$ holds if and only if α satisfies $n_1 = n_6$ and $n_3 = n_5$. Therefore, by direct calculation we can explicitly determine the set $\{\alpha \in R \mid \alpha = \alpha^{\tilde{\theta}}\}$ as in the assertion. □

LEMMA 3.25. *In the case where (R, θ) is of type EIII, the set of all real restricted roots of R coincides with* $\{\pm(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6), \pm(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6)\}$.

PROOF. From the Satake diagram of (R, θ) the Satake involution p satisfies $p\alpha_1 = \alpha_6$ and $p\alpha_2 = \alpha_2$, and $\Psi(R_0) = \{\alpha_3, \alpha_4, \alpha_5\}$. Therefore the possibility of the form $\alpha_1^{\tilde{\theta}}$ is either $\alpha_6, \alpha_5 + \alpha_6, \alpha_4 + \alpha_5 + \alpha_6$ or $\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$. Since $\tilde{\theta}$ leaves the root system R invariant, we have $\alpha_1^{\tilde{\theta}} = \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$. The same argument shows $\alpha_2^{\tilde{\theta}} = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$. Since $\tilde{\theta}$ is involutive, we have $\alpha_6^{\tilde{\theta}} = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5$. If we put $\alpha = \sum_{i=1}^{6} n_i\alpha_i \in R$ then, $\alpha = \alpha^{\tilde{\theta}}$ holds if and only if α satisfies $n_1 = n_6, n_1 + n_2 + n_6 = 2n_3, n_1 + 2n_2 + n_6 = 2n_4$ and $n_1 + n_2 + n_6 = 2n_5$. Therefore, by direct calculation we can explicitly determine the set $\{\alpha \in R \mid \alpha = \alpha^{\tilde{\theta}}\}$ as in the assertion. □

LEMMA 3.26. *In the case where (R, θ) is of type EIV, there exists no real restricted root in R.*

PROOF. From the Satake diagram of (R,θ) the Satake involution is trivial and $\Psi(R_0) = \{\alpha_2,\alpha_3,\alpha_4,\alpha_5\}$. Therefore the possibility of the form $\alpha_1^{\tilde\theta}$ is either $\alpha_1, \alpha_1+\alpha_3, \alpha_1+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_3+\alpha_4+\alpha_5, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5$ or $\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5$. Since $\tilde\theta$ leaves the root system R invariant, we have $\alpha_1^{\tilde\theta} = \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5$. The same argument shows $\alpha_6^{\tilde\theta} = \alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6$. If we put $\alpha = \sum_{i=1}^{6} n_i\alpha_i \in R$ then, $\alpha = \alpha^{\tilde\theta}$ holds if and only if α satisfies $2n_2 = n_1+n_6, 2n_3 = 2n_1+n_6, n_4 = n_1+n_6$ and $2n_5 = n_1+2n_6$. Therefore, by direct calculation we have $\{\alpha \in R \mid \alpha = \alpha^{\tilde\theta}\} = \emptyset$. □

By imitating the proof of Lemma 3.26, we have the following five facts.

LEMMA 3.27. *In the case where (R,θ) is of type EVI, the set of all real restricted roots of R coincides with*

$$\begin{aligned}&\{\pm\alpha_1, \pm\alpha_3, \pm(\alpha_1+\alpha_3), \pm(\alpha_2+\alpha_3+2\alpha_4+\alpha_5), \pm(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5)\}\\ &\cup\{\pm(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5), \pm(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7)\}\\ &\cup\{\pm(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7)\}\\ &\cup\{\pm(2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7)\}\\ &\cup\{\pm(\alpha_1+2\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7)\}\\ &\cup\{\pm(\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7)\}\\ &\cup\{\pm(\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7)\}\,.\end{aligned}$$

LEMMA 3.28. *In the case where (R,θ) is of type EVII, the set of all real restricted roots of R coincides with*

$$\{\pm\alpha_7, \pm(\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7), \pm(2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7)\}.$$

LEMMA 3.29. *In the case where (R,θ) is of type EIX, the set of all real restricted roots of R coincides with*

$$\begin{aligned}&\{\pm\alpha_7, \pm\alpha_8, \pm(\alpha_7+\alpha_8), \pm(\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7)\}\\ &\cup\{\pm(\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8)\}\\ &\cup\{\pm(\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+\alpha_8)\}\\ &\cup\{\pm(2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7)\}\\ &\cup\{\pm(2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+2\alpha_7+\alpha_8)\}\\ &\cup\{\pm(2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+\alpha_8)\}\\ &\cup\{\pm(2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+2\alpha_7+\alpha_8)\}\\ &\cup\{\pm(2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+3\alpha_7+\alpha_8)\}\\ &\cup\{\pm(2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+3\alpha_7+2\alpha_8)\}\,.\end{aligned}$$

LEMMA 3.30. *In the case where (R,θ) is of type FII, the set of all real restricted roots of R coincides with* $\{\pm(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)\}$.

LEMMA 3.31. *In the case where (R,θ) is of type FIII, the set of all real restricted roots of R coincides with*

$$\{\pm(\alpha_1+\alpha_2+\alpha_3), \pm(\alpha_2+2\alpha_3+2\alpha_4), \pm(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4), \pm(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)\}.$$

LEMMA 3.32. *In the case where $(R,\tilde{\theta})$ is of type A+A, B+B, C+C, D+D, BC+BC, EI+EI, EV+EV, EVIII+EVIII, FI+FI or G+G, there exists no real restricted root in R.*

PROOF. The restricted root system R has two irreducible components R^1, R^2, which are isomorphic to each other. Set $\Psi(R^j) = \{\alpha_1^j, \dots, \alpha_r^j\}(r = \operatorname{rank} R^j, j = 1,2)$. Renumbering α_i^j, if necessary, we assume that $\alpha_1^j > \cdots > \alpha_r^j$. From the Satake diagram of $(R,\tilde{\theta})$ we have $\Psi(R_0^j) = \emptyset$ and $p\alpha_i^2 = \alpha_i^1 (1 \leq i \leq r)$. This implies that $(\alpha_i^2)^{\tilde{\theta}} = \alpha_i^1 (1 \leq i \leq r)$. Since any restricted root in R is a linear combination of either $\{\alpha_1^1, \dots, \alpha_r^1\}$ or $\{\alpha_1^2, \dots, \alpha_r^2\}$, there exists no real restricted root. □

By using Corollary 3.11 and Theorem 3.13 we have Theorem stated in Introduction.

REMARK 3.33. By imitating our method we can give examples of austere orbits in a pseudo-hyperbolic space $\boldsymbol{H}$ $(:= \{v \in \mathfrak{q} \mid B(v,v) = r(<0)\})$. In fact, for any imaginary root α, the orbit through $\sqrt{-1}A_\alpha$ is an austere orbit in $\boldsymbol{H}$. Moreover, the Dynkin diagram of the subsystem $\{\alpha \in R \mid \theta(\alpha) = \alpha\}$ can be determined by the black circles in the Satake diagram associated with (R,θ).

Appendix A. Satake diagram of (R,θ)**.** Let $(\mathfrak{g},\mathfrak{h})$ be a semisimple symmetric pair, σ be an involution of $\mathfrak{g}$ with $\operatorname{Ker}(\sigma - \mathrm{id}) = \mathfrak{h}$, and θ be a Cartan involution θ commuting with σ. Denote by R the restricted root system of $(\mathfrak{g},\mathfrak{h})$ with respect to a θ-invariant MSCS $\mathfrak{a}$ of $\mathfrak{q}$. In this appendix, we determine the Satake diagram of $(R,\theta,\mathfrak{a})$. Set $\mathfrak{g}^d = \mathfrak{k}\cap\mathfrak{h} + \sqrt{-1}(\mathfrak{p}\cap\mathfrak{h}) + \sqrt{-1}(\mathfrak{k}\cap\mathfrak{q}) + \mathfrak{p}\cap\mathfrak{q}$, which is a subalgebra of $\mathfrak{g}^{\boldsymbol{C}}$. The involutions σ and θ induce involutions of $\mathfrak{g}^d$, which are also denoted by the same symbol σ and θ, respectively. In particular, σ is a Cartan involution of $\mathfrak{g}^d$. Denoted by $\mathfrak{k}^d$ (resp. $\mathfrak{p}^d$) the $(+1)$-eigenspace (resp. the (-1)-eigenspace) of σ in $\mathfrak{g}^d$. Then we have $\mathfrak{a}_{\boldsymbol{R}}$ $(:= \sqrt{-1}(\mathfrak{k}\cap\mathfrak{a}) + \mathfrak{p}\cap\mathfrak{a})$ is a maximal abelian subspace of $\mathfrak{p}^d$ $(= \sqrt{-1}(\mathfrak{k}\cap\mathfrak{q}) + \mathfrak{p}\cap\mathfrak{q})$. Note that R give also the restricted root system of the Riemannian symmetric pair $(\mathfrak{g}^d,\mathfrak{k}^d)$ with respect to $\mathfrak{a}_{\boldsymbol{R}}$. Let $\mathfrak{a}_\mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$ containing $\mathfrak{p}\cap\mathfrak{a}$. Since $\mathfrak{p}\cap\mathfrak{a}$ is maximal in $\mathfrak{p}\cap\mathfrak{q}$, we have $[\mathfrak{a},\mathfrak{a}_\mathfrak{p}] = \{0\}$ (cf. [13, Lemma 2.4]). If $\tilde{\mathfrak{a}}$ is a maximal abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$ and $\mathfrak{a}_\mathfrak{p}$, then $\tilde{\mathfrak{a}}$ is a Cartan subalgebra of $\mathfrak{g}$. Denote by Σ the root system of $\mathfrak{g}^{\boldsymbol{C}}$ with respect to $\tilde{\mathfrak{a}}^{\boldsymbol{C}}$. We can give a (θ,σ)-fundamental system Ψ of Σ (cf. [13] for the definition of a (θ,σ)-fundamental system). Therefore, Ψ gives the Satake diagram of Riemannian symmetric pairs $(\mathfrak{g},\mathfrak{k})$ and $(\mathfrak{g}^d,\mathfrak{k}^d)$, which are denoted by $S(\mathfrak{g},\mathfrak{k})$ and $S(\mathfrak{g}^d,\mathfrak{k}^d)$, respectively. Then we give a recipe to determine the Satake diagram of $(R,\theta,\mathfrak{a})$ by using $S(\mathfrak{g},\mathfrak{k})$ and $S(\mathfrak{g}^d,\mathfrak{k}^d)$ as follows.

RECIPE A.1. Denote by $S(R,\theta,\mathfrak{a})$ the Satake diagram of $(R,\theta,\mathfrak{a})$.

(Step 1) For each $\alpha \in \Psi$, we determine $\sigma(\alpha)$ (resp. $\theta(\alpha)$) by using $S(\mathfrak{g},\mathfrak{k})$ (resp. $S(\mathfrak{g}^d,\mathfrak{k}^d)$).

(Step 2) We give the set $\{\alpha \in \Psi \mid \alpha|_{\mathfrak{a}^{\boldsymbol{C}}} = 0\}$ $(=: \Psi_0)$, and determine the Dynkin diagram of R by investigating $\{\alpha|_{\mathfrak{a}^{\boldsymbol{C}}} \mid \alpha \in \Psi \setminus \Psi_0\}$ $(=: \overline{\Psi})$. In fact, we calculate $(\alpha - \sigma(\alpha))/2$ as $\alpha|_{\mathfrak{a}^{\boldsymbol{C}}}$ for each $\alpha \in \Psi$.

(Step 3) We determine $\{\lambda \in \overline{\Psi} \mid \lambda|_{\mathfrak{p}\cap\mathfrak{a}} = 0\}$ $(=: \overline{\Psi}_0)$ by investigating $\{\alpha \in \Psi \mid \alpha|_{\mathfrak{a}_{\mathfrak{p}}} = 0\}$. In fact, for each $\lambda = (\alpha - \sigma(\alpha))/2$ $(\alpha \in \Psi \setminus \Psi_0)$, we determine whether or not $\alpha|_{\mathfrak{a}_{\mathfrak{p}}} = 0$ holds, that is, α is a black circle in $S(\mathfrak{g}, \mathfrak{k})$. Then the elements in $\overline{\Psi}_0$ are black circles in $S(R, \theta, \mathfrak{a})$.

(Step 4) For any $\lambda_1, \lambda_2 \in \overline{\Psi} \setminus \overline{\Psi}_0, \lambda_1 \neq \lambda_2$, we determine whether or not $\lambda_1|_{\mathfrak{p}\cap\mathfrak{a}} = \lambda_2|_{\mathfrak{p}\cap\mathfrak{a}}$ holds by calculating $\lambda_i - \theta(\lambda_i) (i = 1, 2)$. In fact, we calculate $(\alpha - \sigma(\alpha) - \theta(\alpha) + \sigma(\theta(\alpha))/4$ as $\alpha|_{\mathfrak{a}^{\boldsymbol{C}}} - \theta(\alpha|_{\mathfrak{a}^{\boldsymbol{C}}})$ $(\alpha \in \Psi \setminus \Psi_0)$. If $\lambda_1|_{\mathfrak{p}\cap\mathfrak{a}} = \lambda_2|_{\mathfrak{p}\cap\mathfrak{a}}$ holds, then $\lambda_1|_{\mathfrak{p}\cap\mathfrak{a}}, \lambda_2|_{\mathfrak{p}\cap\mathfrak{a}}$ are joined with $\leftrightarrow$.

By using Recipe A.1 we shall list up the Satake diagrams of (R, θ) associated with MSCSs for all irreducible semisimple pseudo-Riemannian symmetric pairs. In Table 2, we give the list of the irreducible semisimple pseudo-Riemannian symmetric pairs and their types of (R, θ) associated with MSCSs. In Table 3, we describe the Satake diagrams of (R, θ).

TABLE 2. The Type of (R, θ).

(i-a) $\mathfrak{g}$ is classical and $\mathfrak{g}$ is noncompact simple with no complex structure.

$(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{su}(n, m), \mathfrak{su}(i, j) + \mathfrak{su}(n-i, m-j) + \mathfrak{so}(2))$

Type of (R, θ)	rank	s-rank	Remarks
CI	$\min(i+j, m+n-(i+j))$	$\min(i, m-j) + \min(j, n-i)$	$m+n = 2(i+j)$
BI			$m+n \neq 2(i+j)$

$(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{so}(n, m), \mathfrak{so}(i, j) + \mathfrak{so}(n-i, m-j))$

Type of (R, θ)	rank	s-rank	Remarks
DI	$\min(i+j, m+n-(i+j))$	$\min(i, m-j) + \min(j, n-i)$	$m+n = 2(i+j)$
BI			$m+n \neq 2(i+j)$

$(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{sp}(n, m), \mathfrak{sp}(i, j) + \mathfrak{sp}(n-i, m-j))$

Type of (R, θ)	rank	s-rank	Remarks
CI	$\min(i+j, m+n-(i+j))$	$\min(i, m-j) + \min(j, n-i)$	$m+n = 2(i+j)$
BCI			$m+n \neq 2(i+j)$

Symmetric pair $(\mathfrak{g}, \mathfrak{h})$	Type of (R, θ)	rank	s-rank	Remarks
$(\mathfrak{sl}(n, \boldsymbol{R}), \mathfrak{so}(p, n-p))$	AI	$n-1$	$n-1$	
$(\mathfrak{su}(p, n-p), \mathfrak{so}(p, n-p))$	AIII	$n-1$	$\min(p, n-p)$	
$(\mathfrak{sl}(n, \boldsymbol{R}), \mathfrak{sl}(p, \boldsymbol{R}) + \mathfrak{sl}(n-p, \boldsymbol{R}) + \boldsymbol{R})$	CI	p	p	$n = 2p$
	BCI			$n > 2p$
$(\mathfrak{sl}(2n, \boldsymbol{R}), \mathfrak{sp}(n, \boldsymbol{R}))$	AI	$n-1$	$n-1$	
$(\mathfrak{su}^*(2n), \mathfrak{so}^*(2n))$	AII	$2n-1$	$n-1$	
$(\mathfrak{su}(n, n), \mathfrak{so}^*(2n))$	AIII	$2n-1$	n	
$(\mathfrak{sl}(2n, \boldsymbol{R}), \mathfrak{sl}(n, \boldsymbol{C}) + \mathfrak{so}(2))$	CI	n	n	
$(\mathfrak{su}^*(2n), \mathfrak{sl}(n, \boldsymbol{C}) + \mathfrak{so}(2))$	CIII	n	$[n/2]$	
$(\mathfrak{su}(n, n), \mathfrak{sp}(n, \boldsymbol{R}))$	AIII	$n-1$	$[n/2]$	
$(\mathfrak{su}(n, n), \mathfrak{sl}(n, \boldsymbol{C}) + \boldsymbol{R})$	CI	n	n	
$(\mathfrak{su}^*(2n), \mathfrak{sp}(p, n-p))$	AI	$n-1$	$n-1$	
$(\mathfrak{su}(2p, 2(n-p)), \mathfrak{sp}(p, n-p))$	AIII	$n-1$	$\min(p, n-p)$	
$(\mathfrak{su}^*(2n), \mathfrak{su}^*(2p) + \mathfrak{su}^*(2(n-p)) + \boldsymbol{R})$	CIII	$2p$	p	$n = 2p$
	BCIII			$n > 2p$
$(\mathfrak{so}^*(2n), \mathfrak{su}(p, n-p) + \mathfrak{so}(2))$	CI	$[n/2]$	$[n/2]$	n: even
	BCI			n: odd
$(\mathfrak{so}(2p, 2(n-p)), \mathfrak{su}(p, n-p) + \mathfrak{so}(2))$	CI	$[n/2]$	$\min(p, n-p)$	n: even
	BCI			n: odd
$(\mathfrak{so}^*(2n), \mathfrak{so}^*(2p) + \mathfrak{so}^*(2(n-p)))$	DIII	$2p$	p	$n = 2p$
	BI			$n > 2p$
$(\mathfrak{so}(n, n), \mathfrak{so}(n, \boldsymbol{C}))$	DI	n	n	
$(\mathfrak{so}^*(2n), \mathfrak{so}(n, \boldsymbol{C}))$	DIII	n	$[n/2]$	
$(\mathfrak{so}(n, n), \mathfrak{sl}(n, \boldsymbol{R}) + \boldsymbol{R})$	CI	$[n/2]$	$[n/2]$	n: even
	BCI			n: odd
$(\mathfrak{so}^*(4n), \mathfrak{su}^*(2n) + \boldsymbol{R})$	CI	n	n	
$(\mathfrak{sp}(n, \boldsymbol{R}), \mathfrak{su}(p, n-p) + \mathfrak{so}(2))$	CI	n	n	
$(\mathfrak{sp}(p, n-p), \mathfrak{su}(p, n-p) + \mathfrak{so}(2))$	CIII	n	$\min(p, n-p)$	
$(\mathfrak{sp}(n, \boldsymbol{R}), \mathfrak{sp}(p, \boldsymbol{R}) + \mathfrak{sp}(n-p, \boldsymbol{R}))$	CI	p	p	$n = 2p$
	BI			$n > 2p$
$(\mathfrak{sp}(n, \boldsymbol{R}), \mathfrak{sl}(n, \boldsymbol{R}) + \boldsymbol{R})$	CI	n	n	
$(\mathfrak{sp}(n, n), \mathfrak{sp}(n, \boldsymbol{C}))$	CI	n	n	
$(\mathfrak{sp}(2n, \boldsymbol{R}), \mathfrak{sp}(n, \boldsymbol{C}))$	CI	n	n	
$(\mathfrak{sp}(n, n), \mathfrak{su}^*(2n) + \boldsymbol{R})$	CIII	$2n$	n	

(i-b) $\mathfrak{g}$ is classical and $\mathfrak{g}$ is simple with a complex structure or the direct sum of two noncompact simple Lie algebras with no complex structure.

Symmetric pair $(\mathfrak{g}, \mathfrak{h})$	Type of (R, θ)	rank	s-rank	Remarks
$(\mathfrak{sl}(n, \boldsymbol{C}), \mathfrak{sl}(n, \boldsymbol{R}))$	AIII	$n-1$	$[n/2]$	
$(\mathfrak{sl}(n, \boldsymbol{R})^2, \mathfrak{sl}(n, \boldsymbol{R}))$	AI	$n-1$	$n-1$	
$(\mathfrak{sl}(n, \boldsymbol{C}), \mathfrak{so}(n, \boldsymbol{C}))$	A+A	$2(n-1)$	$n-1$	
$(\mathfrak{sl}(2n, \boldsymbol{C}), \mathfrak{su}^*(2n))$	AI	$2n-1$	n	
$(\mathfrak{su}^*(2n)^2, \mathfrak{su}^*(2n))$	AII	$2n-1$	$n-1$	
$(\mathfrak{sl}(2n, \boldsymbol{C}), \mathfrak{sp}(n, \boldsymbol{C}))$	A+A	$2(n-1)$	$n-1$	
$(\mathfrak{sl}(n, \boldsymbol{C}), \mathfrak{su}(p, n-p))$	AI	$n-1$	$n-1$	
$(\mathfrak{su}(p, n-p)^2, \mathfrak{su}(p, n-p))$	AIII	$n-1$	$\min(p, n-p)$	
$(\mathfrak{sl}(n, \boldsymbol{C}), \mathfrak{sl}(p, \boldsymbol{C}) + \mathfrak{sl}(n-p, \boldsymbol{C}) + \boldsymbol{C})$	C+C	$2p$	p	$n = 2p$
	BC+BC			$n > 2p$
$(\mathfrak{so}(2n, \boldsymbol{C}), \mathfrak{so}^*(2n))$	DI	n	n	
$(\mathfrak{so}^*(2n)^2, \mathfrak{so}^*(2n))$	DIII	n	$[n/2]$	
$(\mathfrak{so}(2n, \boldsymbol{C}), \mathfrak{sl}(n, \boldsymbol{C}) + \boldsymbol{C})$	C+C	$2[n/2]$	$[n/2]$	n: even
	BC+BC			n: odd
$(\mathfrak{so}(n, \boldsymbol{C}), \mathfrak{so}(p, n-p))$	DI	$[n/2]$	$[n/2]$	n: even
	BI			n: odd
$(\mathfrak{so}(p, n-p)^2, \mathfrak{so}(p, n-p))$	DI	$[n/2]$	$\min(p, n-p)$	n: even
	BI			n: odd
$(\mathfrak{so}(n, \boldsymbol{C}), \mathfrak{so}(p, \boldsymbol{C}) + \mathfrak{so}(n-p, \boldsymbol{C}))$	D+D	$2p$	p	$n = 2p$
	B+B			$n > 2p$
$(\mathfrak{sp}(n, \boldsymbol{C}), \mathfrak{sp}(n, \boldsymbol{R}))$	CI	n	n	
$(\mathfrak{sp}(n, \boldsymbol{R})^2, \mathfrak{sp}(n, \boldsymbol{R}))$	CI	n	n	
$(\mathfrak{sp}(n, \boldsymbol{C}), \mathfrak{sl}(n, \boldsymbol{C}) + \boldsymbol{C})$	C+C	$2n$	n	
$(\mathfrak{sp}(n, \boldsymbol{C}), \mathfrak{sp}(p, n-p))$	CI	n	n	
$(\mathfrak{sp}(p, n-p)^2, \mathfrak{sp}(p, n-p))$	CIII	n	$\min(p, n-p)$	
$(\mathfrak{sp}(n, \boldsymbol{C}), \mathfrak{sp}(p, \boldsymbol{C}) + \mathfrak{sp}(n-p, \boldsymbol{C}))$	C+C	$2p$	p	$n = 2p$
	BC+BC			$n > 2p$

TABLE 2. (continued).

(ii-a) $\mathfrak{g}$ is exceptional and $\mathfrak{g}$ is noncompact simple with no complex structure.

Symmetric pair $(\mathfrak{g}, \mathfrak{h})$	Type of (R, θ)	rank	s-rank
$(\mathfrak{e}_{6(6)}, \mathfrak{sp}(4))$	EI	6	6
$(\mathfrak{e}_{6(6)}, \mathfrak{sp}(4, \boldsymbol{R}))$	EI	6	6
$(\mathfrak{e}_{6(6)}, \mathfrak{sl}(6, \boldsymbol{R}) + \mathfrak{sl}(2, \boldsymbol{R}))$	FI	4	4
$(\mathfrak{e}_{6(2)}, \mathfrak{sp}(4, \boldsymbol{R}))$	EII	6	4
$(\mathfrak{e}_{6(6)}, \mathfrak{sp}(2, 2))$	EI	6	6
$(\mathfrak{e}_{6(6)}, \mathfrak{so}(5, 5) + \boldsymbol{R})$	BCI	2	2
$(\mathfrak{e}_{6(-14)}, \mathfrak{sp}(2, 2))$	EIII	6	2
$(\mathfrak{e}_{6(2)}, \mathfrak{su}(6) + \mathfrak{su}(2))$	FI	4	4
$(\mathfrak{e}_{6(2)}, \mathfrak{su}(3, 3) + \mathfrak{sl}(2, \boldsymbol{R}))$	FI	4	4
$(\mathfrak{e}_{6(2)}, \mathfrak{su}(4, 2) + \mathfrak{su}(2))$	FI	4	4
$(\mathfrak{e}_{6(2)}, \mathfrak{so}(6, 4) + \mathfrak{so}(2))$	BCI	2	2
$(\mathfrak{e}_{6(-14)}, \mathfrak{su}(4, 2) + \mathfrak{su}(2))$	FIII	4	2
$(\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) + \mathfrak{so}(2))$	BCI	2	2
$(\mathfrak{e}_{6(-14)}, \mathfrak{so}^*(10) + \mathfrak{so}(2))$	BCI	2	2
$(\mathfrak{e}_{6(-14)}, \mathfrak{su}(5, 1) + \mathfrak{sl}(2, \boldsymbol{R}))$	FIII	4	2
$(\mathfrak{e}_{6(2)}, \mathfrak{so}^*(10) + \mathfrak{so}(2))$	BCI	2	2
$(\mathfrak{e}_{6(-14)}, \mathfrak{so}(8, 2) + \mathfrak{so}(2))$	BCI	2	2
$(\mathfrak{e}_{6(6)}, \mathfrak{f}_{4(4)})$	AI	2	2
$(\mathfrak{e}_{6(-26)}, \mathfrak{sp}(3, 1))$	EIV	6	2
$(\mathfrak{e}_{6(6)}, \mathfrak{su}^*(6) + \mathfrak{su}(2))$	FI	4	4
$(\mathfrak{e}_{6(2)}, \mathfrak{sp}(3, 1))$	EII	6	4
$(\mathfrak{e}_{6(2)}, \mathfrak{f}_{4(4)})$	AIII	2	1
$(\mathfrak{e}_{6(-26)}, \mathfrak{su}^*(6) + \mathfrak{su}(2))$	FII	6	1

Symmetric pair $(\mathfrak{g}, \mathfrak{h})$	Type of (R, θ)	rank	s-rank
$(\mathfrak{e}_{6(-26)}, \mathfrak{f}_4)$	AI	2	2
$(\mathfrak{e}_{6(-26)}, \mathfrak{f}_{4(-20)})$	AI	2	2
$(\mathfrak{e}_{6(-26)}, \mathfrak{so}(9, 1) + \boldsymbol{R})$	BCIII	2	1
$(\mathfrak{e}_{6(-14)}, \mathfrak{f}_{4(-20)})$	AIII	2	1
$(\mathfrak{e}_{7(7)}, \mathfrak{su}(8))$	EV	7	7
$(\mathfrak{e}_{7(7)}, \mathfrak{sl}(8, \boldsymbol{R}))$	EV	7	7
$(\mathfrak{e}_{7(7)}, \mathfrak{su}(4, 4))$	EV	7	7
$(\mathfrak{e}_{7(7)}, \mathfrak{so}(6, 6) + \mathfrak{sl}(2, \boldsymbol{R}))$	FI	4	4
$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(4, 4))$	EVI	7	4
$(\mathfrak{e}_{7(7)}, \mathfrak{su}^*(8))$	EV	7	7
$(\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(6)} + \boldsymbol{R})$	CI	3	3
$(\mathfrak{e}_{7(-25)}, \mathfrak{su}^*(8))$	EVII	7	3
$(\mathfrak{e}_{7(-5)}, \mathfrak{so}(12) + \mathfrak{su}(2))$	FI	4	4
$(\mathfrak{e}_{7(-5)}, \mathfrak{so}^*(12) + \mathfrak{sl}(2, \boldsymbol{R}))$	FI	4	4
$(\mathfrak{e}_{7(-5)}, \mathfrak{so}(8, 4) + \mathfrak{su}(2))$	FI	4	4
$(\mathfrak{e}_{7(-25)}, \mathfrak{e}_6 + \mathfrak{so}(2))$	CI	3	3
$(\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-26)} + \boldsymbol{R})$	CI	3	3
$(\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(2)} + \mathfrak{so}(2))$	CI	3	3
$(\mathfrak{e}_{7(-25)}, \mathfrak{su}(6, 2))$	EVII	7	3
$(\mathfrak{e}_{7(7)}, \mathfrak{so}^*(12) + \mathfrak{su}(2))$	FI	4	4
$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(6, 2))$	EVI	7	4
$(\mathfrak{e}_{7(-5)}, \mathfrak{e}_{6(2)} + \mathfrak{so}(2))$	CI	7	2
$(\mathfrak{e}_{7(-25)}, \mathfrak{so}^*(12) + \mathfrak{su}(2))$	FIII	4	2

Symmetric pair $(\mathfrak{g}, \mathfrak{h})$	Type of (R, θ)	rank	s-rank
$(\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-14)} + \mathfrak{so}(2))$	CI	3	3
$(\mathfrak{e}_{7(-25)}, \mathfrak{so}(10, 2) + \mathfrak{sl}(2, \boldsymbol{R}))$	FIII	4	2
$(\mathfrak{e}_{7(-5)}, \mathfrak{e}_{6(-14)} + \mathfrak{so}(2))$	CI	3	2
$(\mathfrak{e}_{8(8)}, \mathfrak{so}(16))$	EVIII	8	8
$(\mathfrak{e}_{8(8)}, \mathfrak{so}^*(16))$	EVIII	8	8
$(\mathfrak{e}_{8(8)}, \mathfrak{e}_{7(7)} + \mathfrak{sl}(2, \boldsymbol{R}))$	FI	4	4
$(\mathfrak{e}_{8(-24)}, \mathfrak{so}^*(16))$	EIX	8	4
$(\mathfrak{e}_{8(8)}, \mathfrak{so}(8, 8))$	EVIII	8	8
$(\mathfrak{e}_{8(-24)}, \mathfrak{e}_7 + \mathfrak{su}(2))$	FI	4	4
$(\mathfrak{e}_{8(-24)}, \mathfrak{e}_{7(-5)} + \mathfrak{su}(2))$	FI	4	4
$(\mathfrak{e}_{8(-24)}, \mathfrak{so}(12, 4))$	EIX	8	4
$(\mathfrak{e}_{8(8)}, \mathfrak{e}_{7(-5)} + \mathfrak{su}(2))$	FI	4	4
$(\mathfrak{e}_{8(-24)}, \mathfrak{e}_{7(-25)} + \mathfrak{sl}(2, \boldsymbol{R}))$	FI	4	4
$(\mathfrak{f}_{4(4)}, \mathfrak{sp}(3) + \mathfrak{su}(2))$	FI	4	4
$(\mathfrak{f}_{4(4)}, \mathfrak{sp}(3, \boldsymbol{R}) + \mathfrak{sl}(2, \boldsymbol{R}))$	FI	4	4
$(\mathfrak{f}_{4(4)}, \mathfrak{sp}(2, 1) + \mathfrak{su}(2))$	FI	4	4
$(\mathfrak{f}_{4(4)}, \mathfrak{so}(5, 4))$	BCI	1	1
$(\mathfrak{f}_{4(-20)}, \mathfrak{sp}(2, 1) + \mathfrak{su}(2))$	FII	4	1
$(\mathfrak{f}_{4(-20)}, \mathfrak{so}(9))$	BCI	1	1
$(\mathfrak{f}_{4(-20)}, \mathfrak{so}(8, 1))$	BCI	1	1
$(\mathfrak{g}_{2(2)}, \mathfrak{su}(2) + \mathfrak{su}(2))$	G	2	2
$(\mathfrak{g}_{2(2)}, \mathfrak{sl}(2, \boldsymbol{R}) + \mathfrak{sl}(2, \boldsymbol{R}))$	G	2	2

TABLE 2. (continued).

(ii-b) $\mathfrak{g}$ is exceptional and $\mathfrak{g}$ is simple with a complex structure or the direct sum of two noncompact simple Lie algebras with no complex structure.

Symmetric pair $(\mathfrak{g}, \mathfrak{h})$	Type of (R, θ)	rank	s-rank
$(\mathfrak{e}_6^{\boldsymbol{C}}, \mathfrak{e}_{6(-78)})$	EI	6	6
$(\mathfrak{e}_6^{\boldsymbol{C}}, \mathfrak{e}_{6(6)})$	EII	6	4
$(\mathfrak{e}_{6(6)} + \mathfrak{e}_{6(6)}, \mathfrak{e}_{6(6)})$	EI	6	6
$(\mathfrak{e}_6^{\boldsymbol{C}}, \mathfrak{sp}(4, \boldsymbol{C}))$	EI+EI	12	6
$(\mathfrak{e}_6^{\boldsymbol{C}}, \mathfrak{e}_{6(2)})$	EI	6	6
$(\mathfrak{e}_{6(2)} + \mathfrak{e}_{6(2)}, \mathfrak{e}_{6(2)})$	EII	6	4
$(\mathfrak{e}_6^{\boldsymbol{C}}, \mathfrak{sl}(6, \boldsymbol{C}) + \mathfrak{sl}(2, \boldsymbol{C}))$	FI+FI	8	4
$(\mathfrak{e}_6^{\boldsymbol{C}}, \mathfrak{e}_{6(-14)})$	EI	6	6
$(\mathfrak{e}_{6(-14)} + \mathfrak{e}_{6(-14)}, \mathfrak{e}_{6(-14)})$	EIII	6	2
$(\mathfrak{e}_6^{\boldsymbol{C}}, \mathfrak{so}(10, \boldsymbol{C}) + \boldsymbol{C})$	BC+BC	4	2
$(\mathfrak{e}_6^{\boldsymbol{C}}, \mathfrak{e}_{6(-26)})$	EII	6	4
$(\mathfrak{e}_{6(-26)} + \mathfrak{e}_{6(-26)}, \mathfrak{e}_{6(-26)})$	EIV	6	2
$(\mathfrak{e}_6^{\boldsymbol{C}}, \mathfrak{f}_4^{\boldsymbol{C}})$	A+A	4	2
$(\mathfrak{e}_7^{\boldsymbol{C}}, \mathfrak{e}_{7(-33)})$	EV	7	7
$(\mathfrak{e}_7^{\boldsymbol{C}}, \mathfrak{e}_{7(7)})$	EV	7	7
$(\mathfrak{e}_{7(7)} + \mathfrak{e}_{7(7)}, \mathfrak{e}_{7(7)})$	EV	7	7
$(\mathfrak{e}_7^{\boldsymbol{C}}, \mathfrak{sl}(8, \boldsymbol{C}))$	EV+EV	14	7
$(\mathfrak{e}_7^{\boldsymbol{C}}, \mathfrak{e}_{7(-5)})$	EV	7	7
$(\mathfrak{e}_{7(-5)} + \mathfrak{e}_{7(-5)}, \mathfrak{e}_{7(-5)})$	EVI	7	4
$(\mathfrak{e}_7^{\boldsymbol{C}}, \mathfrak{so}(12, \boldsymbol{C}) + \mathfrak{sl}(2, \boldsymbol{C}))$	FI+FI	8	4

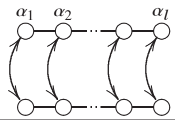
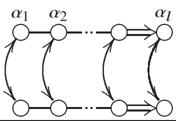
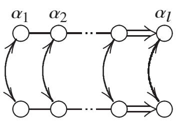
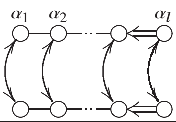
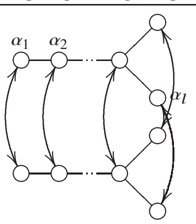
Symmetric pair $(\mathfrak{g}, \mathfrak{h})$	Type of (R, θ)	rank	s-rank
$(\mathfrak{e}_7^{\boldsymbol{C}}, \mathfrak{e}_{7(-25)})$	FI	7	7
$(\mathfrak{e}_{7(-25)} + \mathfrak{e}_{7(-25)}, \mathfrak{e}_{7(-25)})$	EVII	7	3
$(\mathfrak{e}_7^{\boldsymbol{C}}, \mathfrak{e}_6^{\boldsymbol{C}} + \boldsymbol{C})$	C+C	6	3
$(\mathfrak{e}_8^{\boldsymbol{C}}, \mathfrak{e}_{8(-248)})$	EVIII	8	8
$(\mathfrak{e}_8^{\boldsymbol{C}}, \mathfrak{e}_{8(8)})$	EVIII	8	8
$(\mathfrak{e}_{8(8)} + \mathfrak{e}_{8(8)}, \mathfrak{e}_{8(8)})$	EVIII	8	8
$(\mathfrak{e}_8^{\boldsymbol{C}}, \mathfrak{so}(16, \boldsymbol{C}))$	EVIII+EVIII	16	8
$(\mathfrak{e}_8^{\boldsymbol{C}}, \mathfrak{e}_{8(-24)})$	EVIII	8	8
$(\mathfrak{e}_{8(-24)} + \mathfrak{e}_{8(-24)}, \mathfrak{e}_{8(-24)})$	EIX	8	4
$(\mathfrak{e}_8^{\boldsymbol{C}}, \mathfrak{e}_7^{\boldsymbol{C}} + \mathfrak{sl}(2, \boldsymbol{C}))$	FI+FI	8	4
$(\mathfrak{f}_4^{\boldsymbol{C}}, \mathfrak{f}_{4(-52)})$	FI	4	4
$(\mathfrak{f}_4^{\boldsymbol{C}}, \mathfrak{f}_{4(4)})$	FI	4	4
$(\mathfrak{f}_{4(4)} + \mathfrak{f}_{4(4)}, \mathfrak{f}_{4(4)})$	FI	4	4
$(\mathfrak{f}_4^{\boldsymbol{C}}, \mathfrak{sp}(3, \boldsymbol{C}) + \mathfrak{sl}(2, \boldsymbol{C}))$	FI+FI	8	4
$(\mathfrak{f}_4^{\boldsymbol{C}}, \mathfrak{f}_{4(-20)})$	FI	4	4
$(\mathfrak{f}_{4(-20)} + \mathfrak{f}_{4(-20)}, \mathfrak{f}_{4(-20)})$	FII	4	1
$(\mathfrak{f}_4^{\boldsymbol{C}}, \mathfrak{so}(9, \boldsymbol{C}))$	BC+BC	2	1
$(\mathfrak{g}_2^{\boldsymbol{C}}, \mathfrak{g}_{2(-14)})$	G	2	2
$(\mathfrak{g}_2^{\boldsymbol{C}}, \mathfrak{g}_{2(2)})$	G	2	2
$(\mathfrak{g}_{2(2)} + \mathfrak{g}_{2(2)}, \mathfrak{g}_{2(2)})$	G	2	2
$(\mathfrak{g}_2^{\boldsymbol{C}}, \mathfrak{sl}(2, \boldsymbol{C}) + \mathfrak{sl}(2, \boldsymbol{C}))$	G+G	4	2

TABLE 2. (continued).

(iii) $\mathfrak{g}$ is the direct sum of two noncompact simple Lie algebras with complex structure.

Symmetric pair $(\mathfrak{g}, \mathfrak{h})$	Type of (R, θ)	rank	s-rank	
$(\mathfrak{e}_6^{\boldsymbol{C}} + \mathfrak{e}_6^{\boldsymbol{C}}, \mathfrak{e}_6^{\boldsymbol{C}})$	EI+EI	12	6	
$(\mathfrak{e}_7^{\boldsymbol{C}} + \mathfrak{e}_7^{\boldsymbol{C}}, \mathfrak{e}_7^{\boldsymbol{C}})$	EV+EV	14	7	
$(\mathfrak{e}_8^{\boldsymbol{C}} + \mathfrak{e}_8^{\boldsymbol{C}}, \mathfrak{e}_8^{\boldsymbol{C}})$	EVIII+EVIII	16	8	
$(\mathfrak{f}_4^{\boldsymbol{C}} + \mathfrak{f}_4^{\boldsymbol{C}}, \mathfrak{f}_4^{\boldsymbol{C}})$	FI+FI	8	4	
$(\mathfrak{g}_2^{\boldsymbol{C}} + \mathfrak{g}_2^{\boldsymbol{C}}, \mathfrak{g}_2^{\boldsymbol{C}})$	G+G	4	2	
$(\mathfrak{sl}(n, \boldsymbol{C}) + \mathfrak{sl}(n, \boldsymbol{C}), \mathfrak{sl}(n, \boldsymbol{C}))$	A+A	$2(n-1)$	$n-1$	
$(\mathfrak{so}(n, \boldsymbol{C}) + \mathfrak{so}(n, \boldsymbol{C}), \mathfrak{so}(n, \boldsymbol{C}))$	D+D	$2p$	p	$n = 2p$
	B+B			$n = 2p + 1$
$(\mathfrak{sp}(n, \boldsymbol{C}) + \mathfrak{sp}(n, \boldsymbol{C}), \mathfrak{sp}(n, \boldsymbol{C}))$	C+C	$2n$	n	

TABLE 3. The Satake diagram of (R, θ).

Type of (R, θ)	Satake diagram
A+A	
B+B	
BC+BC	
C+C	
D+D	

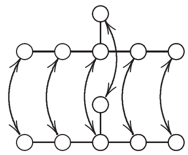
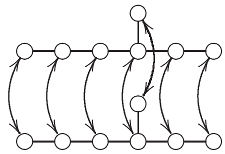
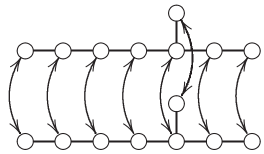
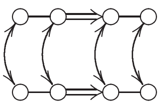
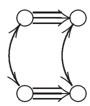
Type of (R, θ)	Satake diagram
EI+EI	
EV+EV	
EVIII+EVIII	
FI+FI	
G+G	

TABLE 3. (continued).

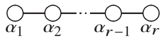
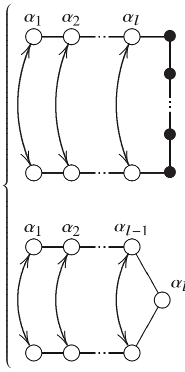
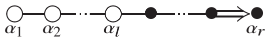
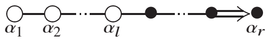
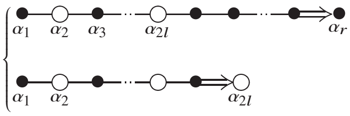
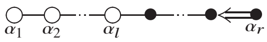
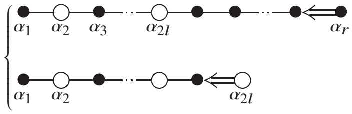
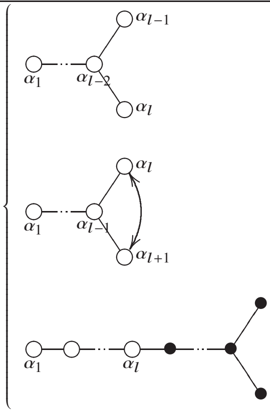
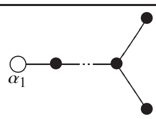
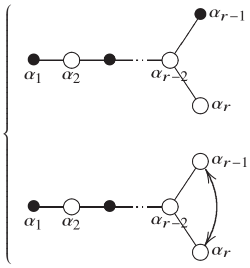
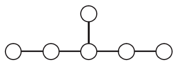
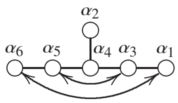
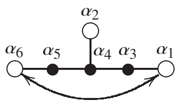
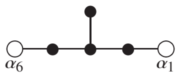
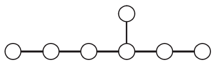
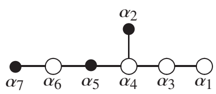
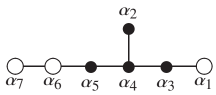
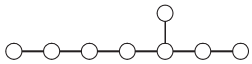
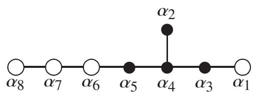
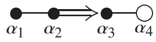
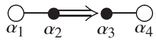
Type of (R,θ)	Satake diagram
AI	
AII	
AIII	
BI	
BCI	
BCII	
BCIII	
CI	
CII	
CIII	
DI	
DII	
DIII	

TABLE 3. (continued).

Type of (R, θ)	Satake diagram
EI	
EII	
EIII	
EIV	
EV	
EVI	
EVII	
EVIII	
EIX	
FI	
FII	
FIII	
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DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND TECHNOLOGY
TOKYO UNIVERSITY OF SCIENCE
NODA, CHIBA 278–8510
JAPAN

E-mail address: baba_kurando@ma.noda.tus.ac.jp