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STEADY STATES OF FITZHUGH-NAGUMO SYSTEM WITH NON-DIFFUSIVE ACTIVATOR AND DIFFUSIVE INHIBITOR

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Abstract. In this paper, we consider a diffusion equation coupled to an ordinary differential equation with FitzHugh-Nagumo type nonlinearity. We construct continuous spatially heterogeneous steady states near, as well as far from, constant steady states and show that they are all unstable. In addition, we construct various types of steady states with jump discontinuities and prove that they are stable in a weak sense defined by Weinberger. The results are quite different from those for classical reaction-diffusion systems where all species diffuse.

1. Introduction. Turing ([30]) considered spontaneous formation of patterns in developmental biology as a result of diffusion-driven-instability (DDI, for short): When two chemicals with different diffusion rates react, the spatially homogeneous state can be destabilized, leading to emergence of nontrivial spatial structures. Since then, the concept has become a paradigm for pattern formation and led to development of numerous theoretical models describing naturally occurring patterns [8, 13, 16, 29]. In this paper we study pattern formation in a model system where only one species diffuses and the other does not diffuse, an extreme situation of "different diffusion rates". Such models have been proposed in developmental biology (e.g., [18] and references therein) and ecology (e.g., [4] and references therein) and so forth. In [10] and [15], we studied a receptor-based model for *hydra* regeneration in which destabilization of a spatially homogeneous steady state by DDI and bi-stable type of nonlinearity coexist. Our finding is, among other things, that there is no stable nonhomogeneous steady states near the constant steady state. However, there are continua of (weakly) stable steady states with jump discontinuity. We would like to show that such a phenomenon (i.e., coexistence of unstable continuous steady states and stable discontinuous steady states) is not limited to a particular system considered in [10, 15], rather it is universal. For this purpose, we take the FitzHugh-Nagumo equations as a reference system and prove that the above-mentioned phenomenon occurs under appropriately chosen parameter values.

The original FitzHugh-Nagumo system

(FHN)
$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon^2 u_{xx} + u(1-u)(u-\beta) - v\,, \\ \dfrac{\partial v}{\partial t} = \sigma u - \gamma v - \rho\,, \end{cases}$$

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was used to model pulse propagation in excitable media, see [7, 21] for model introduction and [14, 27] for overview of its analysis. The parameters $0 < \beta < 1$, γ, ρ and σ are positive.

Later, pattern formation in systems including

$$\text{(DFHN)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon_1^2 u_{xx} + u(1-u)(u-\beta) - v\,, \\ \dfrac{\partial v}{\partial t} = \dfrac{1}{\varepsilon_2} v_{xx} + \sigma u - \gamma v - \rho \end{cases}$$

was studied by [20] in the context of DDI, where ε_1 and ε_2 are sufficiently small positive constants. Subsequent studies have revealed that this reaction-diffusion system admits steady states with transition layers (see, e.g., [23, 5, 24] and references therein), traveling wave solutions and other types of solutions with complex behaviours (see, e.g., [26, 2, 3] and references therein).

In this paper, motivated by the receptor-based model in [10] and [15], we consider a system with $\varepsilon_1 = 0$, i.e., a single reaction-diffusion equation coupled with an ordinary differential equation. For the interval $I = (0, l)$ we consider:

$$\text{(1)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = f_\beta(u) - v & \text{for } x \in \overline{I}, t > 0\,, \\ \dfrac{\partial v}{\partial t} = D\dfrac{\partial^2 v}{\partial x^2} + \sigma u - \gamma v - \rho & \text{for } x \in I, t > 0\,, \\ \dfrac{\partial v}{\partial x}(0) = \dfrac{\partial v}{\partial x}(l) = 0\,, \\ (u(0,x), v(0,x)) = (u_0(x), v_0(x)) & \text{for } x \in I\,, \end{cases}$$

where $f_\beta(u) = u(1-u)(u-\beta)$ with $\beta \in (0,1)$; D, γ, σ and ρ are positive constants. The initial data satisfies $(u_0(x), v_0(x)) \in C^0(\overline{I}) \times (C^2(I) \cap C(\overline{I}))$.

We choose (γ, σ, ρ) so that the kinetic system (i.e., (1) with $D = 0$) has three distinct equilibria and two of them undergo DDI. This setting is different from that for the equations in [15] where only one equilibrium of the kinetic system undergoes DDI. Our results on bifurcation stated in Section 4 cover the case where only one equilibrium undergoes bifurcation. To the best of our knowledge, even for classical reaction-diffusion models such as (DFHN), little is known about pattern formation through DDI in the case where two or more equilibria undergo DDI. Generally speaking, if there exist more than one stable steady states in addition to an unstable steady state of saddle type, long-time-behaviour of the solution to the initial-boundary value problem becomes very complicated. It is a legitimate theoretical question to ask how patterns are selected when two equilibria undergo DDI. We believe that our analysis serves as a first step in this direction.

The initial-boundary value problem (1) has two types of stationary solutions $(u(x), v(x))$. One is *continuous* steady states by which we mean that both $u(x)$ and $v(x)$ are continuous functions of x (as a result they are real analytic). The other type of solution is called *discontinous* since $u(x)$ and $v''(x)$ have a finitely many jump discontinuities.

Main results of the present paper are summarized as follows:

First, we construct (i) one-parameter families of spatially heterogeneous continuous steady states near constant steady states by applying the bifurcation theory (Theorem 4.6) and (ii) those far from the constant steady states under additional assumptions on the choice of parameters (Theorem 5.3). Steady states in family (ii) cannot be found by perturbation techniques such as bifurcation theorems; they constitute an isolated branch of solutions. We study the spectrum of the linearized operator and show that these continuous steady states are unstable (Proposition 4.7, Remark 4.8 and 5.7). Global structure of the set of continuous steady states is also considered in Subsection 5.4.

Second, we prove that (1) has continua of discontinuous steady states (Corollary 6.2) by reducing the problem to finding (weak) solutions of the boundary value problem for a single equation in $v(x)$ and solving the latter by using the approach taken in [18]. We start with construction of monotone increasing solutions of this boundary value problem (31) and then use them to construct various kinds of non-monotone discontinuous steady states in Subsection 6.2. Finally we prove in Theorem 6.8 that all such discontinuous steady states are (ε_0, E)-stable, a notion formulated by Weinberger [32] in order to treat the stability of discontinuous steady states.

In numerical simulations we have observed that solutions starting from initial functions close to a constant steady state develop spatial heterogeneity and eventually approach steady states exhibiting periodic or irregular patterns with sharp transitions. The final form of solutions seems to be affected by the diffusion coefficient D significantly as in simulations shown in [10] and in [9]. Our results provide a partial explanation for such dynamic behaviour; however, we are far from a total understanding of the dynamics of solutions in system (1).

As a matter of fact, for decades such discontinuous steady states have been known to exist (see, e.g., [20]), because they served as a first approximation to continuous steady states, with interior transition layers, of reaction-diffusion systems where both species diffuse. However, there have been not many systematic studies on discontinuous steady states themselves. This seems partly because most models of pattern formation were based on Turing's framework and discontinuous steady states were regarded as intermediate products. In this classical setting, as the diffusion coefficient of the activator tends to 0, stable steady states are known to converge to a particular family of discontinuous steady states (see Section 7 for more details). However, recent researches on models coupling diffusive and non-diffusive components of signalling systems have revealed the significance of discontinuous steady states appearing systems of reaction-diffusion equations coupled with ordinary differential equations [11, 12, 19, 25, 31]. We hope that our results will contribute to further studies in this direction.

This paper is organized as follows. As preliminaries, in Sections 2 and 3 we summarize mostly known results but fundamental to subsequent arguments. Section 4 is devoted to the bifurcation problem around constant steady states and the stability analysis of bifurcating solutions. In Section 5, we reduce the problem of finding continuous steady states of (1) to finding smooth solutions of a boundary value problem for a single equation with a smooth nonlinearity and study the global structure of branches of such continuous solutions. In Section 6, we focus on the boundary value problem for a single equation with a discontinuous nonlinearity. Particular attention is given to the construction of monotone increasing solutions. Using these monotone solutions as building blocks, we obtain different types of non-monotone steady states. Subsection 6.3 gives the proof of the stability of steady states with jump discontinuity. In Section 7, as concluding remarks, we discuss the relationship between (DFHN) and (1). Finally, in appendix we present classification tables for the location of contant steady states of (1) according to ranges of parameter values.

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2. Spatially homogeneous steady states. In this section we study spatially homogeneous steady states of (1), which are equilibria of the kinetic system

$$\begin{cases} \dfrac{du}{dt} = f(u,v), \\ \dfrac{dv}{dt} = g(u,v), \end{cases} \tag{2}$$

where

$$f(u,v) = f_\beta(u) - v \quad \text{with} \quad f_\beta(u) = u(1-u)(u-\beta),$$
$$g(u,v) = \sigma u - \gamma v - \rho\,.$$

We always assume that $0 < \beta < 1$, $\gamma > 0$, $\sigma > 0$, $\rho > 0$. Let $u_L = (1+\beta-\sqrt{1-\beta+\beta^2})/3$ be the unique local minimum point of $f_\beta(u)$ and $u_R = (1+\beta+\sqrt{1-\beta+\beta^2})/3$ be the unique local maximum point of $f_\beta(u)$. Put $v_L = f_\beta(u_L)$ and $v_R = f_\beta(u_R)$.

The zero-level curve $f(u,v) = 0$ in the (u,v)-plane defines three different branches $u = h_i(v)$ defined on the open interval J_i $(i = 0, 1, 2)$, respectively, such that $h_0(v) < h_2(v) < h_1(v)$ holds in the intersection $\bigcap_{i=0}^2 J_i$ $(\neq \emptyset)$. Note that $J_0 = (v_L, +\infty)$, $J_1 = (-\infty, v_R)$ and $J_2 = (v_L, v_R)$. We can extend $h_i(v)$ up to the end points of the intervals, so that $h_0(v_L) = h_2(v_L)$, $h_2(v_R) = h_1(v_R)$. Hence, $f(u,v) = 0$ if and only if

$$u = \begin{cases} h_0(v) & \text{for} \quad v_L \le v < +\infty, \\ h_2(v) & \text{for} \quad v_L \le v \le v_R, \\ h_1(v) & \text{for} \quad -\infty < v \le v_R, \end{cases} \tag{3}$$

and $u = h_0(v)$ and $u = h_1(v)$ are strictly decreasing, whereas $u = h_2(v)$ is strictly increasing. We define $\mathcal{B}_0 = \{(h_0(v), v) \mid v_L \le v < +\infty\}$, $\mathcal{B}_1 = \{(h_1(v), v) \mid -\infty < v \le v_R\}$ and $\mathcal{B}_2 = \{(h_2(v), v) \mid v_L \le v \le v_R\}$.

The number of equilibria of the kinetic system varies from one to three depending on parameters $(\beta, \gamma, \sigma, \rho)$. In Appendix we shall classify them.

In this paper we are particularly interested in the case where three distinct equilibria are on the branch $\mathcal{B}_2$. Hence, in what follows, we always assume that (2) has three equilibria (u_1, v_1), (u_2, v_2) and (u_3, v_3) such that $u_L < u_1 < u_3 < u_2 < u_R$ and hence $v_L < v_1 < v_3 <$

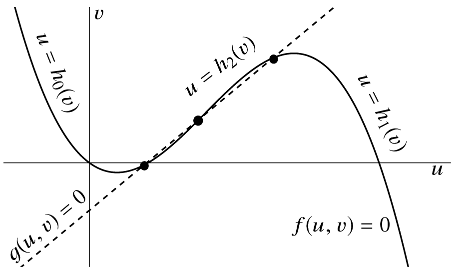

FIGURE 1. Nullclines $f(u, v) = 0$ and $g(u, v) = 0$.

$v_2 < v_R$. See Proposition A.3 (v) and Proposition A.4 (iii) for how to choose parameters. The existence of three spatially homogeneous steady states is illustrated in Figure 1.

3. Boundedness of solutions. We state here the existence and boundedness of solutions of the initial-boundary value problem (1), which are obtained by the standard theory. In this section we assume only that $0 < \beta < 1$, $\gamma > 0$, $\sigma > 0$ and $\rho > 0$.

THEOREM 3.1. *Let $I = (0, l)$ be a bounded interval in $\mathbb{R}$. Let $u_0(x)$, $v_0(x)$ be Hölder continuous functions on $\overline{I}$. Moreover, suppose that $dv_0/dx\,(0) = dv_0/dx\,(l) = 0$ and $v_0 \in C^{2+\omega}(\overline{I})$, $0 < \omega < 1$. Then the initial-boundary value problem* (1) *has a unique classical solution for all $t > 0$.*

PROOF. See Theorem 2 in [18]. Global existence of the solutions $(u(x, t), v(x, t))$ follows from the next theorem. □

THEOREM 3.2. *There exists an invariant rectangle $\mathfrak{R} = \{(u, v) \mid U_0 \le u \le U_1, V_0 \le v \le V_1\}$ such that if the initial data $(u_0(x), v_0(x))$ is included in $\mathfrak{R}$, then the solution of* (1) *remains in $\mathfrak{R}$ for all $t \ge 0$.*

PROOF. See Example 2 in [28] p.209 for details of the framework of invariant rectangles and how to construct one for this particular system. The phase diagram is shown in Figure 2. □

Next, we derive *a priori* bounds on steady states. By a steady state of (1) we mean that $u(x)$ is piecewise continuous on $[0, l]$, $v(x)$ is continuously differentiable on $(0, l)$ with piecewise continuous second order derivative on $(0, l)$, and $(u(x), v(x))$ satisfies (1) expect points of discontinuity of $u(x)$.

THEOREM 3.3. *There exist four constants $U_\star < U^\star$, $V_\star < V^\star$ such that any steady state $(u(x), v(x))$ of* (1) *satisfies the inequalities $U_\star \le u(x) \le U^\star$ and $V_\star \le v(x) \le V^\star$ for all $x \in [0, l]$.*

PROOF. Since $v'(x)$ is continuous, we have $v'(x_M) = 0$ at a maximum point x_M of $v(x)$. Moreover, since $v''(x)$ is piecewise continuous, we have $v''(x_M-) \le 0$ and $v''(x_M+) \le 0$. Indeed, for any $h > 0$ sufficiently small, we have $v(x_M) \ge v(x_M - h) = v(x_M) - v'(x_M)h + \frac{1}{2}v''(x_M - \theta h)h^2$ with $0 < \theta < 1$ since $v(x)$ is twice differentiable in the interval $(x_M - h, x_M)$.

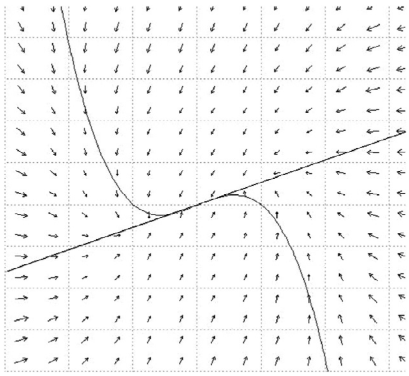

FIGURE 2. Vector field $(f(u,v), g(u,v))$.

Hence, $v''(x_M - \theta h)h^2 \le 0$, so that $v''(x_M-) = \lim_{h\downarrow 0}(x_M - \theta h) \le 0$. Similarly we obtain $v''(x_M+) \le 0$.

In the same way, we see that if x_m is a minimum point of $v(x)$, then $v''(x_m-) \ge 0$ and $v''(x_m+) \ge 0$.

Case 1. First we consider the case where all the constant steady states are on the branch $\mathcal{B}_2$. Suppose that $v(x_M) = \max_{x\in[0,l]} v(x)$. Then $v''(x_M\pm) \le 0$. Hence, $g(u(x_M), v(x_M)) = -Dv''(x_M\pm) \ge 0$, i.e., $(u(x_M), v(x_M))$ is below the straight line $g(u,v) = 0$. Note that $f_\beta(u(x_M)) = v(x_M)$, and hence $v(x_M) \le v_R$. Similarly, if $v(x_m) = \min_{x\in[0,l]} v(x)$, then $v''(x_m\pm) \ge 0$, hence $g(u(x_m), v(x_m)) \le 0$, which means that $(u(x_m), v(x_m))$ is above the line $g = 0$. Since $f_\beta(u(x_m)) = v(x_m)$, we see that $v(x_m) \ge v_L$. Consequently we have $v_L \le v(x) \le v_R$.

Next, we derive bounds on $u(x)$. Suppose that $u(y_M) = \max_{x\in[0,l]} u(x)$ and $u(y_m) = \min_{x\in[0,l]} u(x)$. If we assume that $u(y_m) < u_L$, then $(u(y_m), v(y_m))$ is on the branch $\mathcal{B}_0$. Observe that $v = f_\beta(u)$ is strictly decreasing for $u < u_L$. Therefore, $f_\beta(u) \le f_\beta(u(y_m))$ if $u(y_m) \le u \le u_L$. This means that $v(x) = f_\beta(u(x))$ attains a local maximum at $x = y_m$, hence $v''(x_m\pm) \le 0$. On the other hand, the branch $\mathcal{B}_0$ is above the straight line $g = 0$ by assumption of Case 1. Therefore, $v''(y_m\pm) = -g(u(y_m), v(y_m))/D > 0$, a contradiction. Hence $u(y_m) \ge u_L$.

In a similar fashion, we can prove that $\max_{x\in[0,l]} u(x) \le u_R$. We thus obtain the desired inequalities with $U_\star = u_L$, $U^\star = u_R$, $V_\star = v_L$ and $V^\star = v_R$.

Case 2. Second, we consider the case where (u_1, v_1) is on the branch $\mathcal{B}_0$ and it is the only constant steady state. Let x_m be a minimum point of $v(x)$. Then $v''(x_m\pm) \ge 0$, so that $g(u(x_m), v(x_m)) \le 0$. This implies that $v(x_m) \ge v_1$ since $(u(x_m), v(x_m))$ is on the curve $f(u,v) = 0$. Similarly, at a maximum point x_M of $v(x)$, we must have $g(u(x_M), v(x_M)) \ge 0$, which implies $v(x_M) \le v_R$. Therefore, $V_\star = v_1$ and $V^\star = v_R$ in this case. To obtain bounds on $u(x)$, we notice that $(u(x), v(x))$ must be on one of the branches $\mathcal{B}_0$, $\mathcal{B}_1$ and $\mathcal{B}_2$, and at the same time in the strip $\{(u,v) \mid V_\star \le v \le V^\star\}$. Let $U_\star$ be the smallest root of $f(u,V^\star) = 0$ and $U^\star$ be the largest root of $f(u,V_\star) = 0$. We then obtain $U_\star \le u(x) \le U^\star$.

Case 3. Third, we consider the case where there is only one constant steady state (u_1, v_1) and it is on the branch $\mathcal{B}_1$. This case is treated in exactly the same way as in Case 2, and we see that $V_\star = v_L$, $V^\star = v_1$, while $U_\star$ and $U^\star$ are defined as in Case 2.

Case 4. It remains to consider the cases (a) there is one constant steady state on every branch, (b) there is one constant steady state on $\mathcal{B}_0$ and two (or one) on $\mathcal{B}_2$, and (c) there is one constant steady state on $\mathcal{B}_1$ and two (or one) on $\mathcal{B}_2$. In the same way as above, we can prove that it is sufficient to take $V_\star = v_L$, $V^\star = v_R$ and to define $U_\star$ as the smallest root of $f(u,V^\star) = 0$ and $U^\star$ as the largest root of $f(u,V_\star) = 0$. □

4. Bifurcation analysis. In this section, regarding D as a bifurcation parameter, we consider the bifurcation problem around a constant steady state and the stability of bifurcating solutions for the boundary value problem

$$\begin{cases} f(u,v) = 0 \quad \text{for } 0 \le x \le l\,, \\ Dv'' + g(u,v) = 0 \quad \text{for } 0 < x < l\,, \\ v'(0) = v'(l) = 0\,. \end{cases} \tag{4}$$

4.1. Preliminaries. To begin with, we formulate the problem as an abstract equation in appropriate spaces of functions. Let $X = C^0([0,l]) \times C^2_N([0,l])$ and $Y = C^0([0,l]) \times C^0([0,l])$, where $C^0([0,l])$ denote the Banach space of all continuous functions on $[0,l]$ with the maximum norm and $C^2_N([0,l])$ denote the space of twice continuously differentiable functions satisfying homogeneous Neumann boundary conditions. Let $\mathcal{V}$ be an open set of X defined by $\mathcal{V} = \{V = (u,v) \in X \mid v_L < v < v_R \text{ for all } 0 \le x \le l\}$.

We define a mapping $\mathcal{F}(V,D)$ from $\mathcal{V} \times (0,+\infty)$ into Y by

$$\mathcal{F}(V,D) = \big(f(u,v), Dv'' + g(u,v)\big) \qquad \text{for } V = (u,v)\,. \tag{5}$$

Hence, solving (4) is reduced to finding a pair $(V,D) \in \mathcal{V} \times (0,+\infty)$ satisfying $\mathcal{F}(V,D) = 0$. We observe that if $V_* = (u_*, v_*)$ is an equilibrium of the kinetic system (2) then $\mathcal{F}(V_*,D) = 0$ for any $D > 0$. The set $\{(V_*,D) \mid D > 0\}$ is called the branch of constant steady state V_*. We call (V_*,D_*) a bifurcation point (with respect to $\{(V_*,D) \mid D > 0\}$) if there exists a sequence $\{(V_m,D_m)\}_{m=1}^{\infty} \subset \mathcal{V} \times (0,+\infty)$ such that $\mathcal{F}(V_m,D_m) = 0$, $V_m \ne V_*$, $V_m \to V_*$ and $D_m \to D_*$ as $m \to \infty$.

For a constant steady state $V_* = (u_*, v_*)$ of (1) and for $D > 0$, let $\partial_V \mathcal{F}(V_*,D)$ denote the Fréchet (partial) derivative with respect to $V = (u,v)$ of $\mathcal{F}$ at (V_*,D):

$$\partial_V \mathcal{F}(V_*,D) := \begin{pmatrix} f_u^* & f_v^* \\ g_u^* & Dd^2/dx^2 + g_v^* \end{pmatrix}, \tag{6}$$

where we have defined

$$f_u^* = f_u(u_*,v_*), \quad f_v^* = f_v(u_*,v_*), \quad g_u^* = g_u(u_*,v_*), \quad g_v^* = g_v(u_*,v_*)\,.$$

The Jacobi matrix J_* at V_* of the kinetic system (2) is given by

$$J_* := \begin{pmatrix} f_u^* & f_v^* \\ g_u^* & g_v^* \end{pmatrix} = \begin{pmatrix} f_\beta'(u_*) & -1 \\ \sigma & -\gamma \end{pmatrix}. \tag{7}$$

It is well known that (V_*, D_*) cannot be a bifurcation point if $\partial_V \mathcal{F}(V_*, D_*)$ has a bounded inverse, i.e., 0 is not its eigenvalue (see, e.g., [28, p. 171]).

A complex number λ is an eigenvalue of $\mathcal{L}_D = \partial_V \mathcal{F}(V_*, D)$ if and only if it solves the characteristic equation

$$\lambda^2 - (\operatorname{tr} J_* - D\ell_j)\lambda + \det J_* - D\ell_j f_\beta'(u_*) = 0 \tag{8}$$

for some $j = 0, 1, 2, 3, \dots$, where

$$\ell_j := (\pi j / l)^2 \tag{9}$$

is an eigenvalue of $-d^2/dx^2$ under homogeneous Neumann boundary conditions. The following lemma concerning the spectral structure of $\mathcal{L}_D$ is proved in the same way as Lemma 3.1 in [15]:

LEMMA 4.1. *Suppose that* $V_* = (u_*, v_*)$ *is a constant steady-state of* (1). *Let* $\mathcal{L}_D$ *denote the linearized operator* $\partial_V \mathcal{F}(V_*, D) : X \to Y$. *Then the spectrum of* $\mathcal{L}_D$ *consists of the eigenvalues* $\{\lambda_j\}_{j=0}^\infty \cup \{\mu_j\}_{j=0}^\infty$ *of finite multiplicity, with* $\operatorname{Re} \lambda_j \le \operatorname{Re} \mu_j$, *and the point* $\lambda = f_\beta'(u_*)$ *is in the continuous spectrum. Furthermore,*

$$\begin{aligned} \lambda_j &= -D\ell_j - (f_\beta'(u_*) - \operatorname{tr} J_*) + O(1/\ell_j), \\ \mu_j &= f_\beta'(u_*) + O(1/\ell_j) \end{aligned}$$

as $j \to \infty$.

As to the distribution of eigenvalues we have the following

LEMMA 4.2. *Assume that* (2) *has three equilibria on the branch* $\mathcal{B}_2$. *Let* (u_*, v_*) *be an equilibrium of* (2) *and put*

$$D_j := \frac{\det J_*}{f_\beta'(u_*)\ell_j} \quad \text{for } j = 1, 2, 3, \dots . \tag{10}$$

If $0 < f_\beta'(u_*) < \min\{\gamma, \sigma/\gamma\}$, *then* D_j *is positive for all* $j \ge 1$. *Moreover,*

(I) *if* $D > D_1$, *then* $\lambda_n < 0 < \mu_n$ *for all* $n \ge 0$;

(II) *if* $D = D_j$ *for some* $j \ge 1$, *then* $\operatorname{Re} \lambda_n \le \operatorname{Re} \mu_n < 0$ *for* $1 \le n \le j-1$ *and* $\lambda_j < 0 = \mu_j$. *For* $n \ge j+1$, *we have* $\lambda_n < 0 < \mu_n$;

(III) *if* $D_{j+1} < D < D_j$, *then* $\operatorname{Re} \lambda_n \le \operatorname{Re} \mu_n < 0$ *for* $1 \le n \le j$, *while for any* $n \ge j+1$, $\lambda_n < 0 < \mu_n$.

PROOF. Since $(u_*, v_*) \in \mathcal{B}_2$, we have $f'_\beta(u_*) > 0$. If $f'_\beta(u_*) < \gamma$, then $\operatorname{tr} J_* = f_u^* + g_v^* = f'_\beta(u_*) - \gamma < 0$. If $f'_\beta(u_*) < \sigma/\gamma$, then $\det J_* = -\gamma f'_\beta(u_*) + \sigma > 0$. Therefore, $D_j = \det J_*/(f'_\beta(u_*)\ell_j) > 0$ for all $j = 1, 2, \ldots$.

Also from (8) it follows that (i) if $\det J_* - Df_u^* \ell_n < 0$ for some $n > 0$, then $\lambda_n < 0 < \mu_n$ and (ii) if $\det J_* - Df_u^* \ell_n > 0$ and $\operatorname{tr} J_* - D\ell_n < 0$ then $\operatorname{Re} \lambda_n \leq \operatorname{Re} \mu_n < 0$. The assertions of the lemma are obtained easily by these observations combined with the definition of D_j. □

Under the assumptions of Lemma 4.2, we observe that if $(u_*, v_*) = (u_1, v_1)$ or (u_2, v_2), then $f'_\beta(u_*) < \sigma/\gamma$ and if $(u_*, v_*) = (u_3, v_3)$ then $f'_\beta(u_*) > \sigma/\gamma$ since $f'_\beta(u_*)$ is the slope of the tangent to $f(u, v) = 0$ at (u_*, v_*). On the other hand, the inequality $f'_\beta(u_*) < \gamma$ is satisfied by replacing $\sigma u - \gamma v - \rho$ with $(\sigma u - \gamma v - \rho)/\tau$ for some $\tau > 0$ sufficiently small.

DEFINITION 4.3. *An equilibrium (u_*, v_*) of the kinetic system* (2) *is said to be double if $g(u, v) = 0$ is tangent to $f(u, v) = 0$ at (u_*, v_*). Otherwise it is said to be simple.*

In the following theorem, by the stability of a steady state $(u_0(x), v_0(x))$ of (1) for $D = D_0 > 0$ it is meant in the linearized sense, i.e., $V_0 = (u_0(x), v_0(x))$ is said to be stable if the spectrum of $\partial_V \mathcal{F}(V_0, D_0)$ is contained in the left half plane $\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda < 0\}$. If the spectrum of $\partial_V \mathcal{F}(V_0, D_0)$ has a point with positive real part, then we say that V_0 is unstable. We remark that by the standard analytic semigroup theory we can prove that the linearized stability implies the asymptotic stability in the nonlinear sense.

THEOREM 4.4. *Let (u_*, v_*) be a constant steady state of the initial-boundary value problem* (1).

- (I) *If (u_*, v_*) is on either the branch $\mathcal{B}_0$ or $\mathcal{B}_1$, i.e., if $u_* \leq u_L$ or $u_* \geq u_R$, then it is stable for all $D > 0$. In particular, bifurcation does not occur at (u_*, v_*);*
- (II) *Assume that (u_*, v_*) is on $\mathcal{B}_2$ and simple.*
  - (IIa) *If either (u_*, v_*) is the only constant steady state or there exist two other constant steady states (u^*, v^*) and (u', v') such that $u_L < u_* < u^* \leq u' \leq u_R$ or $u_L \leq u^* \leq u' < u_* < u_R$, then (u_*, v_*) is asymptotically stable for $D = 0$, but it is unstable for $D > 0$. In fact, the linearized operator around (u_*, v_*) has infinitely many positive eigenvalues for any $D > 0$. Moreover, there exists a strictly decreasing sequence of positive numbers D_j such that $D_j \to 0$ and $\mathcal{L}_{D_j}$ has 0 as an eigenvalue.*
  - (IIb) *If there exist two other constant steady states (u^*, v^*) and (u', v') such that $u^* < u_* < u'$, then (u_*, v_*) is a saddle point. The linearized operator around (u_*, v_*) has infinitely many positive eigenvalues and infinitely many negative eigenvalues for $D > 0$. Moreover, bifurcation does not occur at (u_*, v_*).*
- (III) *Assume that (u_*, v_*) is a double steady state on $\mathcal{B}_2$. Then* (*i*) *for $D = 0$, it is asymptotically stable if $f'_\beta(u_*) < \gamma$, and unstable if $f'_\beta(u_*) > \gamma$,* (*ii*) *for $D > 0$*

it is unstable (the linearized operator has infinitely many positive eigenvalues and infinitely many negative eigenvalues) and bifurcation does not occur at (u_*, v_*).

From (II) of Lemma 4.2 we know that as D increases over D_j, $V_* = (u_*, v_*)$ becomes more unstable in the sense that the stability index decreases, which is defined by

DEFINITION 4.5. Let $V_* = (u_*, v_*)$ be a constant steady state of (1). Define the *stability index* $\mathrm{Ind_S}\,(V_*, D)$ of (V_*, D) by

$$\mathrm{Ind_S}\,(V_*, D) = \#\{\mu_n \mid \mathrm{Re}\,\mu_n < 0\},$$

where $\#N$ stands for the number of distinct elements of a countable set N.

If $V_* = (u_*, v_*)$ with $0 < f'_\beta(u_*) < \min\{\gamma, \sigma/\gamma\}$, then $\mathrm{Ind_S}\,(V_*, D) = j$ for $D_{j+1} < D < D_j$. If we understand $D_0 = +\infty$, this formula is valid for all $j \geq 0$. We observe that under disturbances of wave number $\leq \mathrm{Ind_S}\,(V_*, D)$, V_* is linearly stable. However, under disturbance of wave number $> \mathrm{Ind_S}\,(V_*, D)$, V_* is unstable.

Now let $V_* = (u_*, v_*)$ be a constant steady state of (1) which satisfies the assumption of (IIa) of Theorem 4.4 above. Hence the linearized operator $\mathcal{L}_{D_j}$ has 0 as an eigenvalue. By applying the standard bifurcation theory, we have a one-parameter family of non-constant solutions of $\mathcal{F}(V, D) = 0$. To state the rigorous result, we need some preparation.

From (6) it is straightforward to compute an eigenvector ϕ_0 of $\mathcal{L}_{D_j}$ belonging to 0:

$$\mathcal{L}_{D_j}\phi_0 = 0 \quad \text{if} \quad \phi_0 = \begin{pmatrix} 1 \\ f'_\beta(u_*) \end{pmatrix} \cos\frac{\pi j x}{l}\,.$$

Let $\mathcal{L}_1$ denote the linear operator $\partial_V \partial_D \mathcal{F}(V_*, D_j)$, i.e.,

$$\mathcal{L}_1 = \begin{pmatrix} 0 & 0 \\ 0 & d^2/dx^2 \end{pmatrix}.$$

Then we can prove that $\mathcal{L}_{D_j}$ has 0 as an $\mathcal{L}_1$-simple eigenvalue, which means that (i) $\dim\mathrm{Ker}\,\mathcal{L}_{\mathrm{D_j}} = \mathrm{codim\,range}\,\mathcal{L}_{\mathrm{D_j}} = 1$ and (ii) if $\mathrm{Ker}\,\mathcal{L}_{\mathrm{D_j}} = \mathrm{span}\{\phi_0\}$, then $\mathcal{L}_1\phi_0 \notin \mathrm{range}\,\mathcal{L}_{\mathrm{D_j}}$. (For a proof, see §3.2 of [15].) Hence, all the assumptions of the standard theorem on bifurcation from simple eigenvalues (e.g., Theorem 1.7 and 1.18 of [1]) are satisfied and we obtain the following theorem.

THEOREM 4.6. *Let* $V_* = (u_*, v_*)$*, a constant steady state of* (1) *satisfying* $0 < f'_\beta(u_*) < \min\{\gamma, \sigma/\gamma\}$. *Let* $\phi_{j,+} = \cos(\pi j x/l)$ *and* $\psi_{j,+} = f'_\beta(u_*)\cos(\pi j x/l)$. *Then there exists an* $\varepsilon_0 > 0$ *such that* (4) *has a one-parameter family of nonconstant solutions* $\{(V(\varepsilon), D(\varepsilon))\}_{|\varepsilon|<\varepsilon_0}$ *of the form* $V(\varepsilon) = (u(x,\varepsilon), v(x,\varepsilon))$*,* $D(\varepsilon) = D_j + d(\varepsilon)$ *where*

$$\begin{aligned} u(x,\varepsilon) &= u_* + \varepsilon(\phi_{j,+}(x) + \phi(x,\varepsilon)), \\ v(x,\varepsilon) &= v_* + \varepsilon(\psi_{j,+}(x) + \psi(x,\varepsilon))), \\ \phi(x,0) &\equiv 0,\ \psi(x,0) \equiv 0,\ d(0) = 0\,. \end{aligned}$$

Furthermore, in a small neighborhood of (V_*, D_j) *in* $X \times \mathbb{R}$*, there is no solutions other than* $\{(V(\varepsilon), D(\varepsilon))\}_{|\varepsilon|<\varepsilon_0} \cup \{(V_*, D)\}_{|D-D_j|<\varepsilon_0}$.

4.2. Perturbation of critical eigenvalue. Let $\mathcal{L}(\varepsilon)$ denote the linearized operator $\partial_V \mathcal{F}(V(\varepsilon), D(\varepsilon))$ where $(V(\varepsilon), D(\varepsilon))$ is a bifurcating solution of (4) given by Theorem 4.6. Since $\mathcal{L}_{D_j} = \mathcal{L}(0)$, $\mathcal{L}(0)$ has 0 as an eigenvalue. Hence, $\mathcal{L}(\varepsilon)$ is expected to have an eigenvalue $\mu(\varepsilon)$ such that $\mu(0) = 0$. This is rigorously proved once we know that 0 is an i-simple eigenvalue of $\mathcal{L}_{D_j}$, where i is the inclusion mapping $X \hookrightarrow Y$. Here, i-simplicity means (i) $\dim\ker \mathcal{L}_{D_j} = \operatorname{codim}\operatorname{range} \mathcal{L}_{D_j} = 1$ and (ii) if $\ker \mathcal{L}_{D_j} = \operatorname{span}\{\phi_0\}$, then $i\phi_0 \notin \operatorname{range} \mathcal{L}_{D_j}$. (This fact is shown in the same way as in §3.3 of [15].)

Now we see that $\mathcal{L}_{\mathcal{D}}$ has an i-simple eigenvalue $\mu_j(D)$ near $D = D_j$ and $\mathcal{L}(\varepsilon)$ has an i-simple eigenvalue $\mu(\varepsilon)$ for $|\varepsilon|$ sufficiently small. Moreover, by the well-known theorem of Crandall and Rabinowitz (see, e.g., [28, Theorem 13.8]) we have

$$\lim_{\varepsilon\to 0,\, \mu(\varepsilon)\neq 0} \frac{-\varepsilon D'(\varepsilon)\, \mu_j'(D_j)}{\mu(\varepsilon)} = 1\,. \tag{11}$$

From (8) it follows that $2\mu_j(D)\,\mu_j'(D) + \ell_j\,\mu_j(D) - (\operatorname{tr} J_* - D\,\ell_j)\mu_j'(D) - \ell_j f_u^* = 0$, and hence $\mu_j'(D_j) = -\ell_j f_u^*/(\operatorname{tr} J_* - D_j\ell_j) > 0$. Therefore, $\mu(\varepsilon)$ has the same sign as $-\varepsilon D'(\varepsilon)$.

To know the sign of $\mu(\varepsilon)$, we compute $D'(\varepsilon)$. We expand $u(x,\varepsilon)$, $v(x,\varepsilon)$ and $D(\varepsilon)$ in ε:

$$\begin{aligned}
u &= u_* + \varepsilon w\,, \quad w = w_1 + \varepsilon w_2 + \varepsilon^2 w_3 + \cdots\,,\\
v &= v_* + \varepsilon z\,, \quad z = z_1 + \varepsilon z_2 + \varepsilon^2 z_3 + \cdots\,,\\
D &= D_j + \varepsilon d_1 + \varepsilon^2 d_2 + \varepsilon^3 d_3 + \cdots\,.
\end{aligned}$$

Then, since $f_{uv} \equiv f_{vu} \equiv f_{uuv} \equiv f_{uvu} \equiv f_{vvv} \equiv 0$, we have

$$\begin{aligned}
f(u(x,\varepsilon), v(x,\varepsilon)) &= f(u_*, v_*) + f_u(u_*, v_*)\varepsilon w + f_v(u_*, v_*)\varepsilon z\\
&\quad + \frac{1}{2} f_{uu}(u_*, v_*)(\varepsilon w)^2 + \frac{1}{6} f_{uuu}(u_* + \theta\varepsilon w, v_* + \theta\varepsilon w)(\varepsilon w)^3\\
&= f_u(u_*, v_*)(\varepsilon w_1 + \varepsilon^2 w_2 + \varepsilon^3 w_3 + \cdots)\\
&\quad + f_v(u_*, v_*)(\varepsilon z_1 + \varepsilon^2 z_2 + \varepsilon^3 z_3 + \cdots)\\
&\quad + \frac{1}{2}(f_\beta''(u_*, v_*) + O(\varepsilon))(\varepsilon w_1 + \varepsilon^2 w_2 + \cdots)^2\\
&\quad - (1 + O(\varepsilon))(\varepsilon w_1 + \varepsilon^2 w_2 + \cdots)^3\,.
\end{aligned}$$

Since g is linear in (u, v), we have

$$\begin{aligned}
g(u(x,\varepsilon), v(x,\varepsilon)) &= g(u_*, v_*) + g_u(u_*, v_*)\varepsilon w + g_v(u_*, v_*)\varepsilon z\\
&= g_u(u_*, v_*)(\varepsilon w_1 + \varepsilon^2 w_2 + \varepsilon^3 w_3 + \cdots)\\
&\quad + g_v(u_*, v_*)(\varepsilon z_1 + \varepsilon^2 z_2 + \varepsilon^3 z_3 + \cdots)\,.
\end{aligned}$$

We substitute these expressions into (4). Putting

$$\begin{aligned}
a_{11} &= f_u(u_*, v_*) = f_\beta'(u_*)\,, \quad a_{12} = f_v(u_*, v_*) = -1\,,\\
a_{21} &= g_u(u_*, v_*) = \sigma\,, \quad a_{22} = g_v(u_*, v_*) = -\gamma
\end{aligned}$$

and

$$J_* = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

we obtain the following series of equations by equating the coefficient of each power of ε to zero: from ε^1:

$$\begin{cases} a_{11}w_1 + a_{12}z_1 = 0, \\ D_j\, z_1'' + a_{21}w_1 + a_{22}z_1 = 0; \end{cases} \tag{12}$$

from ε^2:

$$\begin{cases} a_{11}w_2 + a_{12}z_2 + \dfrac{1}{2}f_\beta''(u_*)w_1^2 = 0, \\ D_j\, z_2'' + d_1 z_1'' + a_{21}w_2 + a_{22}z_2 = 0; \end{cases} \tag{13}$$

from ε^3:

$$\begin{cases} a_{11}w_3 + a_{12}z_3 + f_\beta''(u_*)w_1w_2 - w_1^3 = 0, \\ D_j\, z_3'' + d_1 z_2'' + d_2 z_1'' + a_{21}w_3 + a_{22}z_3 = 0. \end{cases} \tag{14}$$

Notice that (12) is equivalent to $\mathcal{L}_{D_j}{}^{\mathrm{T}}(w_1, z_1) = 0$, where ${}^{\mathrm{T}}(w_1, z_1)$ denote the transpose of the vector (w_1, z_1). Hence, in view of $a_{11} = f_\beta'(u_*) > 0$, we see that

$$w_1 = -\frac{a_{12}}{a_{11}}\phi_{j,+}, \quad z_1 = \phi_{j,+}$$

satisfies (12), where $\phi_{j,+} = \cos(\pi j x/l)$.

The first equation of (13) implies

$$w_2 = -\frac{a_{12}}{a_{11}}z_2 - \frac{f_\beta''(u_*)}{2a_{11}}w_1^2.$$

Inserting this into the second equation of (13), we get

$$D_j\, z_2'' + \frac{\det J_*}{a_{11}}z_2 - d_1\ell_j z_1 - \frac{a_{21}}{2a_{11}}f_\beta''(u_*)w_1^2 = 0. \tag{15}$$

Since $D_j\, d^2/dz^2 + \det J_*/a_{11}$ has 0 as an eigenvalue, (15) has a solution if and only if the solvability condition

$$d_1\ell_j \int_0^l z_1^2\, dx + \frac{a_{21}}{2a_{11}}f_\beta''(u_*)\int_0^l w_1^2\, z_1\, dx = 0 \tag{16}$$

is satisfied. Since $w_1 = -(a_{12}/a_{11})z_1$, the second integral on the left-hand side vanishes. Therefore, $d_1 = 0$.

Then (15) reduces to

$$D_j\, z_2'' + \frac{\det J_*}{a_{11}}z_2 = \frac{a_{21}a_{12}^2}{2a_{11}^3}f_\beta''(u_*)z_1^2.$$

In view of $z_1^2 = (1 + \cos(2\pi jx/l))/2$, we are led to

$$D_j\, z_2'' + \frac{\det J_*}{a_{11}} z_2 = \frac{a_{21}a_{12}^2}{4a_{11}^3} f_\beta''(u_*) \left\{1 + \cos\frac{2\pi jx}{l}\right\} .$$

Hence we obtain

$$z_2 = \frac{a_{21}a_{12}^2}{4a_{11}^2 \det J_*} f_\beta''(u_*) - \frac{a_{21}a_{12}^2 f_\beta''(u_*)}{12a_{11}^2 \det J_*} \cos\frac{2\pi jx}{l} + C\cos\frac{\pi jx}{l},$$

where C is an arbitrary constant. If we require further that $\int_0^l z_2\, z_1\, dx = 0$, then $C = 0$ and we conclude that

$$z_2 = \frac{a_{21}a_{12}^2}{4a_{11}^2 \det J_*} f_\beta''(u_*) - \frac{a_{21}a_{12}^2}{12a_{11}^2 \det J_*} f_\beta''(u_*) \cos\frac{2\pi jx}{l},$$

$$w_2 = -\frac{a_{12}^2}{4a_{11}^3} f_\beta''(u_*) \left(\frac{a_{12}a_{21}}{\det J_*} + 1\right) + \frac{a_{12}^2}{4a_{11}^3} f_\beta''(u_*) \left(\frac{a_{12}a_{21}}{3\det J_*} - 1\right) \cos\frac{2\pi jx}{l} .$$

In the computations above, we have used the fact that $D_j\,(\cos(2\pi jx/l))'' = -4\ell_j\, D_j \cos(2\pi jx/l) = -(\det J_*/a_{11}) \cos(2\pi jx/l)$.

Let us turn to (14). The first equation yields

$$w_3 = -\frac{1}{a_{11}} \left(a_{12}z_3 + f_\beta''(u_*) w_1 w_2 - w_1^3\right) .$$

Substituting this in the second equation of (14) gives

$$D_j\, z_3'' + \frac{\det J_*}{a_{11}} z_3 - d_2\, \ell_j\, z_1 - \frac{a_{21}}{a_{11}} \left[f_\beta''(u_*) w_1 w_2 - w_1^3\right] = 0 .$$

The solvability condition for this equation reads

$$d_2 \ell_j \int_0^l z_1^2\, dx + \frac{a_{21}}{a_{11}} \int_0^l \left(f_\beta''(u_*) w_1 w_2 - w_1^3\right) z_1\, dx = 0,$$

which yields

$$\frac{\ell_j}{2} d_2 = -\frac{3a_{21}a_{12}^3}{8a_{11}^4} - \frac{a_{21}a_{12}^3}{8a_{11}^5} [f_\beta''(u_*)]^2 \left(1 + \frac{a_{21}}{\det J_*}\right) + \frac{a_{21}a_{12}^3}{16a_{11}^5} [f_\beta''(u_*)]^2 \left(\frac{a_{21}a_{12}}{3\det J_*} - 1\right) .$$

In deriving this expression, we have used the elementary formulas $\cos^2(2\pi jx/l) = (1 + \cos(4\pi jx/l))/2$ and $\cos^4(\pi jx/l) = \{3 + 4\cos(2\pi jx/l) + \cos(4\pi jx/l)\}/8$.

Recalling that $a_{12} = -1$, $a_{21} = \sigma$ and $a_{22} = -\gamma$, we obtain

(17) $$\frac{\ell_j}{2} d_2 = \frac{3\sigma}{8a_{11}^4} + \frac{\sigma}{8a_{11}^5} [f_\beta''(u_*)]^2 \left(1 + \frac{\sigma}{\det J_*}\right) + \frac{\sigma}{16a_{11}^5} [f_\beta''(u_*)]^2 \left(\frac{\sigma}{3\det J_*} + 1\right) .$$

Since $d_1 = 0$ and $d_2 > 0$, we see that $-\varepsilon D'(\varepsilon) = -2d_2\varepsilon^2 + O(\varepsilon^3) < 0$ as long as $\varepsilon \neq 0$ is sufficiently small. Therefore, by virtue of formula (11) we obtain the following proposition.

PROPOSITION 4.7. *Let* $V_* = (u_*, v_*)$ *be a steady state of* (1) *satisfying* $0 < f_\beta'(u_*) < \min\{\gamma, \sigma/\gamma\}$. *Let* $\{(V(\varepsilon), D(\varepsilon))\}_{|\varepsilon|<\varepsilon_0}$ *be a family of nonconstant solutions bifurcating from*

(V_*, D_j). Then $D(\varepsilon) > D_j$ for $0 < |\varepsilon| < \varepsilon_0$. Moreover, the i-simple eigenvalue $\mu_j(\varepsilon)$ of the linearized operator $\mathcal{L}(\varepsilon) = \partial_V \mathcal{F}(V(\varepsilon), D(\varepsilon))$ is negative for $|\varepsilon| > 0$ sufficiently small.

REMARK 4.8. *For $D > D_j$ with $D - D_j$ sufficiently small, there are exactly two nonconstant stationary solutions near the constant solution $V_* = (u_*, v_*)$. These nonconstant stationary solutions are more stable than V_* in the sense that $\mathrm{Ind_S}\,(V(\varepsilon), D(\varepsilon)) > \mathrm{Ind_S}\,(V_*, D(\varepsilon))$. However, $\mathcal{L}(\varepsilon)$ has infinitely many positive eigenvalues.*

5. Spatially nonhomogeneous continuous steady states. In order to know the global behaviour of the branch of bifurcating solutions obtained in the previous section, we turn to the boundary value problem

$$\begin{cases} D\dfrac{\partial^2 v}{\partial x^2} + g(h_2(v), v) = 0 & \text{for } 0 < x < l\,, \\ \dfrac{\partial v}{\partial x}(0) = \dfrac{\partial v}{\partial x}(l) = 0\,. \end{cases} \tag{18}$$

It is convenient to set

$$g_2(v) = g(h_2(v), v) \qquad \text{for} \quad v_L \le v \le v_R\,. \tag{19}$$

In this section we construct solutions of the boundary problem (18). Since all nonconstant continuous solution can be obtained from monotone increasing solutions, we focus on the monotone increasing solutions of (18). Recall that (u_1, v_1), (u_2, v_2) and (u_3, v_3) are equilibria of (2) and they are on the branch $\mathcal{B}_2 : u = h_2(v)$.

5.1. Monotone increasing solutions. Multiplying both sides of the first equation of (18) by v', we obtain

$$\left(\frac{D}{2}(v')^2 + G_2(v)\right)' = 0,$$

where

$$G_2(v) = \int_{v_2}^{v} g(h_2(s), s)\, ds \quad \text{for } v_L \le v \le v_R\,. \tag{20}$$

Notice that $G_2'(v) = g(h_2(v), v) < 0$ if $v_L < v < v_1$, $G_2'(v) = 0$ if $v = v_L$, $G_2'(v) > 0$ if $v_1 < v < v_3$, $G_2'(v) = 0$ if $v = v_3$, $G_2'(v) < 0$ if $v_3 < v < v_2$, $G_2'(v) = 0$ if $v = v_2$ and $G_2'(v) > 0$ if $v_2 < v < v_R$. Therefore, G_2 is monotone decreasing in the intervals (v_L, v_1) and (v_3, v_2), monotone increasing in the intervals (v_1, v_3) and (v_2, v_R). Moreover, $G_2(v)$ achieves a local minimum at $v = v_1, v_2$ and a local maximum at $v = v_3$. Let $a = v(0)$. Then we have, in view of $v'(0) = 0$,

$$\frac{D}{2}v'(x)^2 + G_2(v(x)) = G_2(a)\,.$$

Therefore

$$v'(x)^2 = \frac{2}{D}(G_2(a) - G_2(v(x)))\,. \tag{21}$$

Since $v(x)$ is monotone increasing,

$$v'(x) = \frac{1}{\sqrt{D}}\sqrt{2(G_2(a) - G_2(v(x)))}\,. \tag{22}$$

Due to the boundary condition $v'(l) = 0$, we see that $b = v(l)$ must satisfy $G_2(a) = G_2(b)$. Integrating (22), we obtain

$$\frac{x}{\sqrt{D}} = \int_a^v \frac{dw}{\sqrt{2(G_2(a) - G_2(w))}}, \tag{23}$$

and by putting $x = l$,

$$D = \left(l \Bigg/ \int_a^{b(a)} \frac{dw}{\sqrt{2(G_2(a) - G_2(w))}} \right)^2 . \tag{24}$$

Expressions (23) and (24) are meaningful as long as (i) the algebraic equation $G_2(v) = G_2(a)$ has a solution $b > a$ and (ii) $G_2(v) < G_2(a)$ for $a < v < b$. This requirement restricts the range of a. Since the potential $G_2(v)$ is of double-well type, we can take a near the bottom of each well. Moreover, if $G_2(v_L) > G_2(v_3)$ and $G_2(v_R) > G_2(v_3)$ then a may be taken far from v_1.

First we consider the neighbourhood of v_2. There are two cases to be distinguished.

(I) $G_2(v_3) \le G_2(v_R)$. In this case, there exists a unique $b^{(2)} = b^{(2)}(a^{(2)})$ for each $a^{(2)} \in [v_3, v_2]$ such that $G_2(b^{(2)}) = G_2(a^{(2)})$. We can define $D(a^{(2)})$ and $v(x, a^{(2)})$ for $a^{(2)} \in (v_3, v_2]$.

(II) $G_2(v_3) > G_2(v_R)$. In this case, $G_2(b^{(2)}) = G_2(a^{(2)})$ defines a unique solution $b^{(2)} = b^{(2)}(a^{(2)}) \ge a^{(2)}$ if $a^{(2)} \in [v_0^{(2)}, v_2]$, where $v_0^{(2)} \in [v_3, v_2]$ is uniquely determined by $G_2(v_0^{(2)}) = G_2(v_R)$. Therefore, we can define $D(a^{(2)})$ and $v(x, a^{(2)})$ for $a^{(2)} \in (v_0^{(2)}, v_2]$.

Now, we claim that both case (I) and case (II) can occur under some parameter values. We choose a value of $\beta \in (0, 1)$ and fix the curve $v = f_\beta(u)$. First, we fix the point (u_1, v_1) on $u = h_2(v)$. Then we adjust the slope σ/γ of the straight line $g(u, v) = 0$ so that $(h_2(v_2), v_2)$ is close to $(h_2(v_R), v_R)$. Under this situation $G_2(v_R)$ can be made arbitrarily small. Therefore we have case (II). Next, we choose (u_3, v_3) close to (u_R, v_R). Draw the tangent of $v = f_\beta(u)$ at (u_3, v_3). Then, make the slope of $g(u, v) = 0$ slightly smaller than that of the tangent. Then (u_2, v_2) is close to (u_3, v_3). This makes $G_2(v_3)$ smaller than $G_2(v_R)$ and we have case (I). We show the two cases in the Figures 3–5.

Concerning solutions satisfying $v(x) \in [v_L, v_3]$, we have by a similar argument the following classification:

(III) $G_2(v_3) \le G_2(v_L)$. In this case, $G_2(b^{(1)}) = G_2(a^{(1)})$ defines a unique solution $b^{(1)} = b^{(1)}(a^{(1)})$ if $a^{(1)} \in [v_0^{(1)}, v_1]$, where $v_0^{(1)} \in [v_L, v_1]$ is uniquely determined by $G_2(v_0^{(1)}) = G_2(v_3)$. Therefore, we can define $D(a^{(1)})$ and $v(x, a^{(1)})$ for $a^{(1)} \in (v_0^{(1)}, v_1]$.

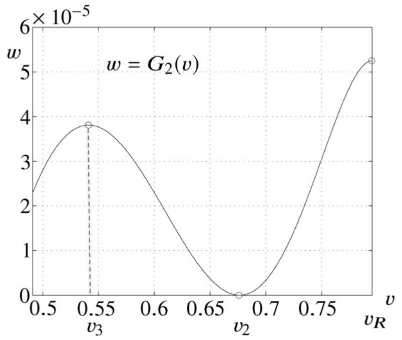

FIGURE 3. Case (I) with $G_2(v_3) < G_2(v_R)$.

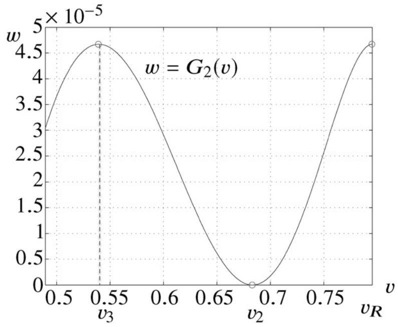

FIGURE 4. Case (I) with $G_2(v_3) = G_2(v_R)$.

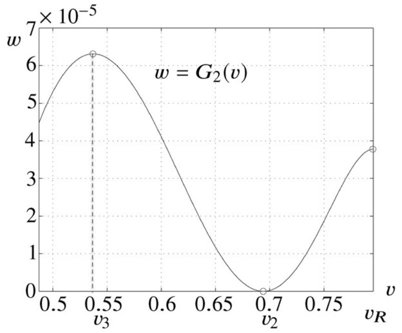

FIGURE 5. Case (II) $G_2(v_3) > G_2(v_R)$.

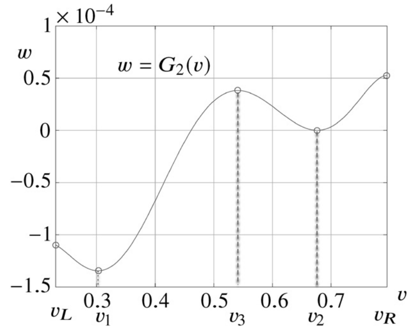

FIGURE 6. Case (III) with $G_2(v_3) > G_2(v_L)$.

(IV) $G_2(v_3) > G_2(v_L)$. In this case, there exists a unique $b^{(1)} = b^{(1)}(a^{(1)})$ for each $a^{(1)} \in [v_L, v_1]$ such that $G_2(b^{(1)}) = G_2(a^{(1)})$. We can define $D(a^{(1)})$ and $v(x, a^{(1)})$ for $a^{(1)} \in (v_L, v_1]$. This case is shown in Figure 6.

In addition to these two types of monotone solutions, there exists another family of monotoe solutions when $G_2(v_3)$ is small:

(V) $G_2(v_3) < \min\{G_2(v_L), G_2(v_R)\}$. In this case, there exists a unique $v_0^{(3)} \in (v_L, v_3)$ such that $G_2(v_0^{(3)}) = G_2(v_3)$. There are two possibilities:

(V.1) If $G_2(v_L) \le G_2(v_R)$, then for each $a^{(3)} \in [v_L, v_0^{(3)})$ there is a unique $b^{(3)} = b^{(3)}(a^{(3)})$ such that $G_2(b^{(3)}) = G_2(a^{(3)})$. We can define $D(a^{(3)})$ and $v(x, a^{(3)})$ for $a^{(3)} \in (v_L, v_0^{(3)})$.

(V.2) If $G_2(v_L) > G_2(v_R)$, then there exists a unique $v_m^{(3)} \in (v_L, v_0^{(3)})$ such that $G_2(v_m^{(3)}) = G_2(v_R)$. For each $a^{(3)} \in [v_m^{(3)}, v_0^{(3)})$, there is a unique $b^{(3)} = b^{(3)}(a^{(3)}) > a^{(3)}$ such that $G_2(b^{(3)}) = G_2(a^{(3)})$. We can define $D(a^{(3)})$ and $v(x, a^{(3)})$ for $a^{(3)} \in (v_m^{(3)}, v_0^{(3)})$.

We introduce the following notation:

$$a_0^{(1)} = \begin{cases} v_0^{(1)} & \text{if } G_2(v_3) \le G_2(v_L), \\ v_L & \text{if } G_2(v_3) > G_2(v_L), \end{cases} \qquad a_0^{(2)} = \begin{cases} v_3 & \text{if } G_2(v_3) \le G_2(v_R), \\ v_0^{(2)} & \text{if } G_2(v_3) > G_2(v_R), \end{cases}$$

and

$$a_0^{(3)} = \begin{cases} v_L & \text{if } G_2(v_L) \le G_2(v_R), \\ v_m^{(3)} & \text{if } G_2(v_L) > G_2(v_R). \end{cases}$$

We define $\mathscr{C}_{1,+}^{(\nu)} = \{(v(x,a^{(\nu)}), D(a^{(\nu)})) \mid a_0^{(\nu)} < a^{(\nu)} \le v_\nu\}$ for $\nu = 1,2$, which are the branches of monotone increasing solutions; $\mathscr{C}_{1,-}^{(\nu)} = \{(v(l-x,a^{(\nu)}), D(a^{(\nu)})) \mid a_0^{(\nu)} < a^{(\nu)} \le v_\nu\}$ for $\nu = 1,2$, which are the branches of monotone decreasing solutions; and $\mathscr{C}_1^{(\nu)} = \mathscr{C}_{1,+}^{(\nu)} \cup \mathscr{C}_{1,-}^{(\nu)}$, which are the branches of monotone solutions ($\nu = 1,2$). In the case $G_2(v_3) < \min\{G_2(v_L), G_2(v_R)\}$ we obtain one more set of branches of monotone solutions: $\mathscr{C}_{1,+}^{(3)} = \{(v(x,a^{(3)}), D(a^{(3)})) \mid a_0^{(3)} < a^{(3)} \le v_0^{(3)}\}$, $\mathscr{C}_{1,-}^{(3)} = \{(v(l-x,a^{(3)}), D(a^{(3)})) \mid a_0^{(3)} < a^{(3)} \le v_0^{(3)}\}$ and $\mathscr{C}_1^{(3)} = \mathscr{C}_{1,+}^{(3)} \cup \mathscr{C}_{1,-}^{(3)}$.

5.2. Limiting behaviour of the branch of monotone solutions. In this subsection we investigate the behaviour of the increasing solution $(v_\nu(x,a^{(\nu)}), D^{(\nu)}(a^{(\nu)}))$ as $a^{(\nu)} \to v_\nu$ ($\nu = 1,2$) or $a^{(3)} \to v_0^{(3)}$, and as $a^{(\nu)} \to a_0^{(\nu)}$ ($\nu = 1,2,3$).

THEOREM 5.1. *For $\nu = 1,2$, as $a^{(\nu)} \uparrow v_\nu$, $v_\nu(x,a^{(\nu)}) \to v_\nu$ uniformly on $[0,l]$, and $D^\nu(a^{(\nu)}) \to D_1^{(\nu)} = g_2'(v_\nu)/(\pi/l)^2$.*

THEOREM 5.2. *It holds that*

1) *if $G_2(v_3) \le G_2(v_R)$, then $v(x,a^{(2)})$ develops a boundary layer at $x = l$ as $a^{(2)} \downarrow v_3$, namely, $v(x,a^{(2)}) \to v_3$ locally uniformly in $[0,l)$, whereas $v(l,a^{(2)}) \to b^{(2)}(v_3)$ and $D(a^{(2)}) \to 0$;*
2) *if $G_2(v_3) > G_2(v_R)$, then, as $a^{(2)} \downarrow v_0^{(2)}$, $v(x,a^{(2)}) \to v(x,v_0^{(2)})$ uniformly on $[0,l]$ and $D(a^{(2)}) \to D_C^{(2)}$, where $D_C^{(2)}$ is a positive number;*
3) *if $G_2(v_3) \le G_2(v_L)$, then $v(x,a^{(1)})$ develops a boundary layer at $x = l$ as $a^{(1)} \downarrow v_0^{(1)}$, namely, $v(x,a^{(1)}) \to v_3$ locally uniformly in $[0,l)$, whereas $v(l,a^{(1)}) \to b^{(1)}(v_3)$ and $D(a^{(1)}) \to 0$;*
4) *if $G_2(v_3) > G_2(v_L)$, then as $a^{(1)} \downarrow v_L$, $v(x,a^{(1)}) \to v(x,v_L)$ uniformly on $[0,l]$ and $D(a^{(1)}) \to D_C^{(1)}$, where $D_C^{(1)}$ is a positive number.*

PROOF OF THEOREMS 5.1 AND 5.2. Since the properties of $G_2(v)$ are identical with those of $H(v)$ stated in Section 4 of [15], the assertions of these theorems are proved in the same way as Theorem 4.1 and 4.2 in [15]. □

The branch $\mathscr{C}_1^{(3)}$ did not appear in the nonlinearity considered in [15] and exhibits a different type of limiting behaviour:

THEOREM 5.3. *If $G_2(v_3) < \min\{G_2(v_L), G_2(v_R)\}$, then we have*

1) *as $a^{(3)} \uparrow v_0^{(3)}$, $v(x,a^{(3)})$ develops a boundary layer at each of the boundary points $x = 0$ and $x = l$, namely, $v(x,a^{(3)}) \to v_3$ locally uniformly in $(0,l)$, whereas $v(0,a^{(3)}) \to$*

$v_0^{(3)}(< v_3)$ and $v(l, a^{(3)}) \to v_1^{(3)}(> v_3)$. Here $v_1^{(3)}$ is a unique solution of $G_2(v) = G_2(v_3)$ satisfying $v_1^{(3)} > v_3$. Moreover, $D(a^{(3)}) \to 0$ as $a^{(3)} \uparrow v_0^{(3)}$;

2) *if $G_2(v_L) \le G_2(v_R)$, then $v(x, a^{(3)})$ develops a boundary layer at $x = l$ as $a^{(3)} \uparrow v_3$, namely, $v(x, a^{(3)}) \to v_3$ locally uniformly in $[0, l)$, whereas $v(l, a^{(3)}) \to b^{(3)}(v_3)$ and $D(a^{(3)}) \to 0$;*

3) *if $G_2(v_L) > G_2(v_R)$, then, as $a^{(3)} \downarrow v_m^{(3)}$, $v(x, a^{(3)}) \to v(x, v_m^{(3)})$ uniformly on $[0, l]$ and $D(a^{(3)}) \to D_C^{(3)}$, where $D_C^{(3)}$ is a positive number.*

PROOF. Assertions 2) and 3) are proved in the same way as those of Theorem 5.2. To prove 1) we observe that $G_2(a) = G_2(v_0^{(3)}) + (G_2'(v_0^{(3)}) + O(|a - v_0^{(3)}|))(a - v_0^{(3)})$, $G_2(w) = G_2(v_3) + G_2'(v_3)(w - v_3) + 2^{-1}(G_2''(v_3) + O(w - v_3))(w - v_3)^2$, $G_2(v_0^{(3)}) = G_2(v_3)$ and $G_2'(v_3) = 0$. Hence,

$$\begin{aligned} &2(G_2(a) - G_2(w)) \\ &= - G_2''(v_3) \left\{ (1 + O(w - v_3))(w - v_3)^2 - \frac{2G_2'(v_0^{(3)})(1 + O(a - v_0^{(3)}))}{G_2''(v_3)}(a - v_0^{(3)}) \right\} \end{aligned}$$

as $a \uparrow v_0^{(3)}$ and $w \to v_3$. Using this expression it is not hard to prove that for any positive constant δ sufficiently small we have

$$I_1(a) = \int_{v_3-\delta}^{v_3} \frac{dw}{\sqrt{2(G_2(a) - G_2(w))}} = \frac{1}{2\sqrt{-G_2''(v_3)}} \log \frac{1}{v_0^{(3)} - a} + O(1),$$

$$J_1(a) = \int_{v_3}^{v_3+\delta} \frac{dw}{\sqrt{2(G_2(a) - G_2(w))}} = \frac{1}{2\sqrt{-G_2''(v_3)}} \log \frac{1}{v_0^{(3)} - a} + O(1),$$

as $a \uparrow v_0^{(3)}$. On the other hand,

$$I_2(a) = \int_{a}^{v_3-\delta} \frac{dw}{\sqrt{2(G_2(a) - G_2(w))}} = O(1),$$

$$J_2(a) = \int_{v_3+\delta}^{b(a)} \frac{dw}{\sqrt{2(G_2(a) - G_2(w))}} = O(1),$$

as $a \uparrow v_0^{(3)}$ (see, e.g., Lemmas 16 and 17 in [18]). Therefore,

$$\int_{a}^{b(a)} \frac{dw}{\sqrt{2(G_2(a) - G_2(w))}} = \frac{1}{\sqrt{-G_2''(v_3)}} \log \frac{1}{v_0^{(3)} - a} + O(1) \quad \text{as } a \uparrow v_0^{(3)}.$$

Hence, $D(a^{(3)}) \to 0$ as $a^{(3)} \uparrow v_0^{(3)}$. Moreover,

$$\frac{x}{l} = \int_{a^{(3)}}^{v} \frac{dw}{\sqrt{2(G_2(a^{(3)}) - G_2(w))}} \Bigg/ \int_{a^{(3)}}^{b(a^{(3)})} \frac{dw}{\sqrt{2(G_2(a^{(3)}) - G_2(w))}}$$

implies that if $v(x_\kappa) = v_3 - \kappa$, then $x_\kappa/l \to 0$ for $\kappa > 0$ and $x_\kappa/l \to 1$ for $\kappa < 0$ as $a^{(3)} \uparrow v_0^{(3)}$. This completes the proof of Assertion 1). □

5.3. Symmetric continuous steady states. We can construct solutions of the boundary value problem (18) which are not monotone increasing. Let $v_*(x)$ be any monotone increasing solution of (18) with $D = D_*$, given by (23). For each integer $k \geq 2$, define a function $v_+^k(x)$ on $0 \leq x \leq l$ by

$$v_+^k(x) = \begin{cases} v_*(kx - 2jl) & \text{for } 2jl/k \leq x \leq (2j+1)l/k\,, \\ v_*(2(j+1)l - kx) & \text{for } (2j+1)l/k \leq x \leq 2(j+1)l/k\,, \end{cases}$$

where $j = 0, 1, 2, \ldots, [k/2]$. Also we define

$$v_-^k(x) = \begin{cases} v_*(2(j+1)l - kx) & \text{for } 2jl/k \leq x \leq (2j+1)l/k\,, \\ v_*(kx - (2j+1)l) & \text{for } (2j+1)l/k \leq x \leq 2(j+1)l/k\,. \end{cases}$$

Then $v_\pm^k(x)$ is a solution of (18) for $D = D_*/k^2$. Moreover, by making use of the uniqueness of solution of the initial value problem for $v'' = -g_2(v)$, we see easily that all the solutions of (18) are obtained from monotone increasing solutions by this method.

We denote this continuation mapping by $E_{k,\pm}$:

$$E_{k,+}v(x) = v_+^k(x)\,, \quad E_{k,-}v(x) = v_-^k(x)\,. \tag{25}$$

In this way we can define the branch of k-mode solutions of (18) as follows:

$$\mathscr{C}_{k,\pm}^{(\nu)} = \left\{ \left(E_{k,\pm}v(x, a^{(\nu)}), D(a^{(\nu)})/k^2 \right) \mid a_0^{(\nu)} < a^{(\nu)} \leq v_\nu \right\} \quad \text{and} \quad \mathscr{C}_k^{(\nu)} = \mathscr{C}_{k,+}^{(\nu)} \cup \mathscr{C}_{k,-}^{(\nu)}\,, \tag{26}$$

where $a_0^{(\nu)} < a^{(\nu)} \leq v_\nu$ is replaced with $a_0^{(3)} < a^{(3)} \leq v_0^{(3)}$ for $\nu = 3$.

5.4. Global behaviour of bifurcating branches. Let $\mathscr{S}$ denote the set of all nonconstant solutions of the single equation (18). Let $\mathscr{C}_j^{(\nu)}$ $(\nu = 1, 2)$ be the connected component of $\overline{\mathscr{S}}$, the closure of $\mathscr{S}$ in $C^0([0, l]) \times (0, +\infty)$, which contains the bifurcation point (v_ν, D_j) $(\nu = 1, 2)$. Notice that $\mathscr{C}_j^{(\nu)}$ coincides with that defined by (26) at the end of Subsection 5.3.

The following two propositions are proved in the same way as Lemma 4.1 and Proposition 4.2 in [15]:

PROPOSITION 5.4. *If $m \neq n$ then $\mathscr{C}_m^{(\nu)} \bigcap \mathscr{C}_n^{(\nu)} = \emptyset$ $(\nu = 1, 2)$. Moreover $\mathscr{C}_j^{(\nu)}$ is not compact in $\mathfrak{D} \times (0, +\infty)$, where $\mathfrak{D} = \{v \in C^2([0, l]) \mid v_L < v(x) < v_R \text{ for all } x \in [0, l]\}$.*

Therefore, we are interested in how the branch $\mathscr{C}_j$ approaches the boundary $\partial(\mathfrak{D} \times (0, +\infty))$. We consider the case when D is sufficiently large.

PROPOSITION 5.5. *Assume the condition of Lemma 4.2 is satisfied. Then the boundary value problem (18) has only constant solutions if $D > D^\star$, where $D^\star$ is a positive constant depending only on the function $g_2(v)$ and l.*

We make here an obvious

REMARK 5.6. *For j, $k \geq 1$, it holds that (a) $\mathscr{C}_j^{(1)} \cap \mathscr{C}_k^{(2)} = \emptyset$, and (b) $\mathscr{C}_j^{(\nu)} \cap \mathscr{C}_k^{(3)} = \emptyset$ for $\nu = 1$, 2 in the case $G_2(v_3) < \min\{G_1(v_L), G_2(v_R)\}$.*

Therefore, combined with Theorem 5.2, we conclude that the projection of the branch $\mathscr{C}_n^{(\nu)}$ on $\mathbb{R}$ forms (i) an interval $(0, D_M]$ if $G_2(v_3) \leq G_2(v_R)$ or $G_2(v_3) \leq G_2(v_L)$ and (ii) an interval

$[d_\star, D_M]$ if $G_2(v_3) > G_2(v_R)$ or $G_2(v_3) > G_2(v_L)$, where $0 < d_\star < D_M$. Furthermore, we have $D_M > D_1^{(\nu)} = g_2'(v_\nu)/(\pi/l)^2$ by virtue of Proposition 4.7.

On the other hand, in the case where $G_2(v_3) < \min\{G_2(v_L), G_2(v_R)\}$, the projection on $\mathbb{R}$ of the connected component $\mathscr{C}_1^{(3)} = \{(v(x, a^{(3)}), D(a^{(3)})) \mid a_0^{(3)} < a^{(3)} < v_0^{(3)}\}$ always forms an interval $(0, D_M]$ for some positive constant D_M.

We close this section with the following remark:

REMARK 5.7. *Each solution $(v_*(x), D_*)$ of the boundary value problem for the single equation* (18) *gives rise to a steady state $(h_2(v_*(x)), v_*(x))$ of* (1) *for $D = D_*$. All the continuous steady states thus obtained are unstable. For, $f_u(u_*(x), v_*(x)) = f_\beta'(u_*(x)) > 0$ is satisfied whenever $(u_*(x), v_*(x))$ is on the branch $\mathcal{B}_2$. Hence, the interval* $[\min_{0\le x\le l} f_\beta'(u_*(x)), \max_{0\le x\le l} f_\beta'(u_*(x))]$ *is a continuous spectrum of the linearized operator $\mathcal{L}_*$ around (u_*, v_*) by Theorem* 4.5 *of* [17].

6. Spatially discontinuous steady states. In this section we construct various discontinuous steady states of (1), where $u(x)$ has finitely many jump discontinuities and $v(x)$ has jump discontinuities in the second order derivative.

Let $h_0(v) < h_2(v) < h_1(v)$ be the three branches of solutions of $f_\beta(u) - v = 0$ (see (3)). Put

$$g_j(v) = g(h_j(v), v), \quad j = 0, 1\,. \tag{27}$$

Moreover, we define

$$G_0(v) = \int_{v_L}^{v} g_0(s)\,ds, \quad G_1(v) = \int_{v_R}^{v} g_1(s)\,ds \tag{28}$$

for $v_L \le v \le v_R$.

The first goal of this section is to prove

THEOREM 6.1. *For each $\alpha \in (v_L, v_R)$, let*

$$T_c(\alpha) = \int_{\underline{v}}^{\alpha} \frac{dv}{\sqrt{2[(G_0(\alpha) - G_1(\alpha))_+ - G_0(v)]}} + \int_{\alpha}^{\overline{v}} \frac{dv}{\sqrt{2[(G_0(\alpha) - G_1(\alpha))_+ - G_1(v)]}} \tag{29}$$

where $(G_0(\alpha) - G_1(\alpha))_+ = \max\{0, G_0(\alpha) - G_1(\alpha)\}$. Let

$$D_c(\alpha) = (\ell/T_c(\alpha))^2\,. \tag{30}$$

Then for each $D > D_c(\alpha)$, there exists a unique solution of the boundary value problem

$$\begin{cases} D\dfrac{d^2v}{dx^2} + g_0(v) = 0 & \textit{for} \quad 0 < x < x_*\,, \\ D\dfrac{d^2v}{dx^2} + g_1(v) = 0 & \textit{for} \quad x_* < x < l\,, \\ \dfrac{dv}{dx}(0) = \dfrac{dv}{dx}(l) = 0\,, \end{cases} \tag{31}$$

such that

$$v(x_*) = \alpha\,, \quad \lim_{x\uparrow x_*} \frac{dv}{dx}(x) = \lim_{x\downarrow x_*} \frac{dv}{dx}(x)\,, \tag{32}$$
$$\frac{dv}{dx}(x) > 0 \quad \textit{for} \quad 0 < x < l\,.$$

Let $v_{1,+}(x;\alpha,D)$ *denote this monotone increasing solution. Then* $v_{1,-}(x;\alpha,D) = v_{1,+}(l-x;\alpha,D)$ *is a unique solution of* (31) *with* x_* *replaced by* $l - x_*$ *satisfying*

$$v(l-x_*) = \alpha\,, \quad \lim_{x\uparrow l-x_*} \frac{dv}{dx}(x) = \lim_{x\downarrow l-x_*} \frac{dv}{dx}(x)\,, \tag{33}$$
$$\frac{dv}{dx}(x) < 0 \quad \textit{for} \quad 0 < x < l\,.$$

The following is an immediate consequence of this theorem:

COROLLARY 6.2. *For each* $\alpha \in (v_L, v_R)$ *and* $D > D_c(\alpha)$, *let*

$$u_{1,+}(x;\alpha,D) = \begin{cases} h_0(v_{1,+}(x;\alpha,D)) & \textit{for}\ 0 \le x \le x_*\,,\\ h_1(v_{1,+}(x;\alpha,D)) & \textit{for}\ x_* < x \le l\,. \end{cases}$$

Then $(u_{1,+}(x;\alpha,D), v_{1,+}(x;\alpha,D))$ *is a solution of the boundary value problem* (4) *with the properties* (i) $u_{1,+}$ *has a jump discontinuity at* $x = x_*$ *and* (ii) $v_{1,+}(x_*;\alpha,D) = \alpha$. *Let*

$$u_{1,-}(x;\alpha,D) = \begin{cases} h_0(v_{1,-}(x;\alpha,D)) & \textit{for}\ 0 \le x \le l - x_*\,,\\ h_1(v_{1,-}(x;\alpha,D)) & \textit{for}\ l - x_* < x \le l\,. \end{cases}$$

Then $(u_{1,-}(x;\alpha,D), v_{1,-}(x;\alpha,D))$ *is a solution of* (4) *with properties* (a) $u_{1,-}$ *has a jump discontinuity at* $x = l - x_*$ *and* (b) $v_{1,-}(l-x_*;\alpha,D) = \alpha$.

Solutions with non-monotone v are considered in Subsection 6.2.

6.1. Monotone increasing solutions. In this subsection we prove Theorem 6.1, following the approach in [18]: First we solve the initial value problems

$$\begin{cases} w_{yy} + g_0(w) = 0\,,\\ w(0) = \alpha\,, \quad w'(0) = m\,, \end{cases} \tag{34}$$

and

$$\begin{cases} w_{yy} + g_1(w) = 0\,,\\ w(0) = \alpha\,, \quad w'(0) = m\,, \end{cases} \tag{35}$$

where m is a given positive number. Then we glue the two solutions together and scale the spatial variable appropriately to obtain a solution of (31).

LEMMA 6.3. *Given a pair* (m,α) *such that* $v_L < \alpha < v_R$, *and* $0 < m < \min\{\sqrt{-2G_0(\alpha)}, \sqrt{-2G_1(\alpha)}\}$, *there exists a unique* $\underline{v} \in (v_L,\alpha)$ *and* $\overline{v} \in (\alpha, v_R)$ *satisfying*

$$G_0(\underline{v}) = \frac{1}{2}m^2 + G_0(\alpha) \quad \textit{and} \quad G_1(\overline{v}) = \frac{1}{2}m^2 + G_1(\alpha)\,. \tag{36}$$

Let

$$Y(m,\alpha) = \frac{1}{\sqrt{2}} \int_{\underline{v}}^{\alpha} \frac{dv}{\sqrt{G_0(\underline{v}) - G_0(v)}}, \quad X(m,\alpha) = \frac{1}{\sqrt{2}} \int_{\alpha}^{\overline{v}} \frac{dv}{\sqrt{G_1(\overline{v}) - G_1(v)}} . \tag{37}$$

Then the unique solution $w_0(y)$ of (34) *satisfies $w_0'(y) > 0$ for $-Y(m,\alpha) < y < 0$, $w_0'(-Y(m,\alpha)) = 0$ and $w_0(-Y(m,\alpha)) = \underline{v}$. The unique solution $w_1(y)$ of* (35) *satisfies $w_1'(y) > 0$ for $0 < y < X(m,\alpha)$, $w_1'(X(m,\alpha)) = 0$ and $w_1(X(m,\alpha)) = \overline{v}$.*

PROOF. Since the method of proof is the same, we treat (35) only. Multiple the first equation of (35) by w' and then integrate the resulting equation from 0 to y. We hence obtain

$$\frac{1}{2} w'(y)^2 - \frac{1}{2} m^2 + G_1(w(y)) - G_1(\alpha) = 0,$$

where we have used the initial conditions $w(0) = \alpha$ and $w'(0) = m$. Put

$$\Psi(w) = m^2 + 2[G_1(\alpha) - G_1(w)] . \tag{38}$$

Then

$$w'(y)^2 = \Psi(w(y)) .$$

Since we are interested in monotone increasing solutions, we require $w'(y) > 0$, and obtain

$$\frac{dw}{dy} = \sqrt{\Psi(w(y))} . \tag{39}$$

The solution is well-defined as long as $\Psi(w(y)) \geq 0$. From (38) we see that $\Psi(\alpha) = m^2 > 0$, $\Psi'(w) = -2g_1(w) < 0$ and $\Psi''(w) = -2[g_u(h_1(w), w)h_1'(w) + g_v(h_1(w), w)] = -2[\sigma h_1'(w) - \gamma] > 0$ because $h_1'(w) < 0$ and $\sigma, \gamma > 0$. Moreover, $\Psi(v_R) = m^2 + 2G_1(\alpha) < 0$ due to $G_1(v_R) = 0$ and the assumption on m. Therefore, there is a unique $\overline{v} = \overline{v}(m,\alpha)$ in the interval (α, v_R) such that $\Psi(\overline{v}) = 0$.

Now, $w'(y)$ and $\Psi(w(y))$ are positive for $y > 0$ sufficiently small, but $\Psi(w(y))$ decreases as y increases. Hence, $w(y)$ is increasing until it reaches the value $\overline{v}$ for which $\Psi(\overline{v}) = 0$ is satisfied.

We note that $\Psi(w) = 2[G_1(\overline{v}) - G_1(w)]$ since $m^2 + 2G_1(\alpha) = 2G_1(\overline{v})$. Hence, (39) is rewritten as

$$\frac{dw}{dy} = \sqrt{2[G_1(\overline{v}) - G_1(w)]} .$$

We integrate this equation to get $w(y)$ as the inverse function of

$$y = \frac{1}{\sqrt{2}} \int_{\alpha}^{w(y)} \frac{dw}{\sqrt{[G_1(\overline{v}) - G_1(w)]}} . \tag{40}$$

The integral on the right-hand side is convergent as $w \uparrow \overline{v}$, since

$$G_1(w) = G_1(\overline{v}) + [g_1(\overline{v}) + o(1)](w - \overline{v}) ,$$

and $g_1(\overline{v}) > 0$. Therefore, we can define $X(m,\alpha)$ by

$$X(m,\alpha) = \frac{1}{\sqrt{2}} \int_{\alpha}^{\overline{v}} \frac{dw}{\sqrt{G_1(\overline{v}) - G_1(w)}} .$$

The dependence of $X(m,\alpha)$ on m is by way of $\overline{v} = \overline{v}(m,\alpha)$. This shows that $w(X(m,\alpha)) = \overline{v}$ and $w'(X(m,\alpha)) = 0$. Calling thus obtained solution $w_1(y) = w_1(y; m, \alpha)$, we finish the proof of assertions concerning $w_1(y)$.

The proof of the assertions on $w_0(y)$ is the same and we omit the detail. □

LEMMA 6.4. *For each $\alpha \in (v_L, v_R)$ fixed, there exists a one-parameter family of solutions*

$$\left\{ v(x; m, \alpha) \mid 0 < m < \min\{\sqrt{-2G_0(\alpha)}, \sqrt{-2G_1(\alpha)}\} \right\}$$

of the boundary value problem (31) *with*

$$\begin{cases} x_* = x_*(m,\alpha) = \dfrac{lY(m,\alpha)}{X(m,\alpha) + Y(m,\alpha)}, \\ D = D(m,\alpha) = \dfrac{l^2}{[X(m,\alpha) + Y(m,\alpha)]^2} . \end{cases} \tag{41}$$

Moreover, the solution $v(x; m, \alpha)$ is given as the inverse functions of the following indefinite integrals:

$$\frac{1}{\sqrt{2}} \int_{v(x)}^{\alpha} \frac{dv}{\sqrt{G_0(\underline{v}) - G_0(v)}} = \frac{1}{\sqrt{D}}(x_* - x) \qquad \textit{for} \quad 0 < x < x_* ,$$

$$\frac{1}{\sqrt{2}} \int_{\alpha}^{v(x)} \frac{dv}{\sqrt{G_1(\overline{v}) - G_1(v)}} = \frac{1}{\sqrt{D}}(x - x_*) \qquad \textit{for} \quad x_* < x < l .$$

PROOF. We define $x_*(m,\alpha)$ and $D(m,\alpha)$ by (41) and put

$$v(x; m, \alpha) = \begin{cases} w_0\left(\dfrac{x_* - x}{\sqrt{D(m,\alpha)}}\right) & \text{for} \quad 0 < x < x_* , \\ w_1\left(\dfrac{x - x_*}{\sqrt{D(m,\alpha)}}\right) & \text{for} \quad x_* < x < l . \end{cases}$$

Then it is straightforward to check that $v(x; m, \alpha)$ satisfies (31) for $D = D(m,\alpha)$. □

Next we consider the behaviour of $v(x; m, \alpha)$ as $m \downarrow 0$ and $m \uparrow \min\left\{\sqrt{-2G_0(\alpha)}, \sqrt{-2G_1(\alpha)}\right\}$.

LEMMA 6.5. *Let $x_*(m,\alpha)$ and $D(m,\alpha)$ be the functions defined in Lemma* 6.4.

(a) *As $m \downarrow 0$,*

$$D(m,\alpha) \to +\infty , \quad x_*(m,\alpha) \to \frac{lg(h_1(\alpha),\alpha)}{g(h_1(\alpha),\alpha) - g(h_0(\alpha),\alpha)}$$

and $v(x; m, \alpha) \to \alpha$ uniformly on $[0, l]$.

(b) *Let* $(G_0(\alpha) - G_1(\alpha))_- = -\min\{G_0(\alpha) - G_1(\alpha)), 0\}$. *Let* $v_R^*(\alpha)$ *and* $v_L^*(\alpha)$ *be defined as a unique solution of*

$$G_1(v_R^*) = -(G_0(\alpha) - G_1(\alpha))_-, \quad G_0(v_L^*) = -(G_0(\alpha) - G_1(\alpha))_-,$$

respectively. Then as $m \uparrow \min\left\{\sqrt{-2G_0(\alpha)}, \sqrt{-2G_1(\alpha)}\right\}$,

$$X(m,\alpha) \to \frac{1}{\sqrt{2}} \int_\alpha^{v_R^*} \frac{dv}{\sqrt{-(G_0(\alpha) - G_1(\alpha))_- - G_1(v)}},$$
$$Y(m,\alpha) \to \frac{1}{\sqrt{2}} \int_{v_L^*}^{\alpha} \frac{dv}{\sqrt{-(G_0(\alpha) - G_1(\alpha))_- - G_0(v)}}.$$

PROOF. (a) From $\Psi(\overline{v}) = 0$ it follows that $\Psi(v) = \Psi'(\overline{v} + \theta(v - \overline{v}))(v - \overline{v})$ for some $\theta \in (0,1)$. On the other hand, $\Psi''(v) > 0$ implies that $\Psi'(\alpha) < \Psi'(\overline{v} + \theta(v - \overline{v})) < \Psi'(\overline{v})$ for $\alpha < v < \overline{v}$. Hence, $\Psi(v) > -\Psi'(\alpha)(\overline{v} - v)$ for $\alpha < v < \overline{v}$. Therefore,

$$X(m,\alpha) = \int_\alpha^{\overline{v}} \frac{dv}{\sqrt{\Psi(v)}} < \frac{1}{\sqrt{|\Psi'(\alpha)|}} \int_\alpha^{\overline{v}} \frac{dv}{\sqrt{\overline{v} - v}} = \frac{2\sqrt{\overline{v} - \alpha}}{\sqrt{|\Psi'(\alpha)|}}.$$

Observe, however, that $m^2 + 2[G_1(\alpha) - G_1(\overline{v})] = 0$ implies $\overline{v} \to \alpha$ as $m \to 0$ since $G_1(v)$ is strictly increasing. Consequently, we have $X(m,\alpha) \to 0$ as $m \downarrow 0$. Similarly, we obtain $Y(m,\alpha) \to 0$ as $m \downarrow 0$. Therefore, $D(m,\alpha) \to +\infty$ as $m \downarrow 0$. Since $\underline{v} \le v(x; m, \alpha) \le \overline{v}$ and $\underline{v} \to \alpha$, $\overline{v} \to \alpha$, we see that $v(x; m, \alpha)$ converges to α uniformly as $m \downarrow 0$.

To know the behaviour of $x_*(m,\alpha)$ as $m \downarrow 0$, we need precise asymptotic formulas for $X(m,\alpha)$ and $Y(m,\alpha)$. From $m^2/2 = G_1(\overline{v}) - G_1(\alpha)$ it follows that

$$m^2/2 = G_1'(\alpha + \theta(\overline{v} - \alpha))(\overline{v} - \alpha) = (g_1(\alpha) + O(\overline{v} - \alpha))(\overline{v} - \alpha),$$

so that

$$\overline{v} - \alpha = \frac{m^2(1 + o(1))}{2g_1(\alpha)} \quad \text{as} \quad m \downarrow 0. \tag{42}$$

Similarly, we have

$$\alpha - \underline{v} = -\frac{m^2(1 + o(1))}{2g_0(\alpha)} \quad \text{as} \quad m \downarrow 0. \tag{43}$$

Also, for $v \in (\alpha, \overline{v})$ it holds that $\Psi(v) = (\Psi'(\overline{v}) + O(v - \overline{v}))(v - \overline{v})$. Hence, for $m > 0$ sufficiently small, we obtain

$$\begin{aligned} X(m,\alpha) &= \int_\alpha^{\overline{v}} \frac{dv}{\sqrt{\Psi(v)}} \\ &= \int_\alpha^{\overline{v}} \frac{dv}{\sqrt{-\Psi'(\overline{v}) + O(v - \overline{v})}\sqrt{\overline{v} - v}} \\ &= \frac{1}{\sqrt{|\Psi'(\overline{v})|}} \int_\alpha^{\overline{v}} \frac{1 + O(\overline{v} - v)}{\sqrt{\overline{v} - v}}\, dv. \end{aligned}$$

Therefore, in view of $\Psi'(\overline{v}) = \Psi'(\alpha) + O(\overline{v} - \alpha) = -2g_1(\alpha) + O(\overline{v} - \alpha)$, we are led to

$$X(m, \alpha) = \frac{2}{\sqrt{2|g_1(\alpha)|}}(1 + O(\overline{v} - \alpha))\sqrt{\overline{v} - \alpha}\,. \tag{44}$$

In the same way, we obtain

$$Y(m, \alpha) = \frac{2}{\sqrt{2|g_0(\alpha)|}}(1 + O(\alpha - \underline{v}))\sqrt{\alpha - \underline{v}}\,. \tag{45}$$

In (41), we substitute (44), (45), (42) and (43) to see

$$\begin{aligned} x_*(m, \alpha) &= \frac{lY(m, \alpha)}{X(m, \alpha) + Y(m, \alpha)} \\ &= \frac{l/|g_0(\alpha)|}{1/|g_1(\alpha)| + 1/|g_0(\alpha)|}(1 + O(m^2)) \\ &= \frac{l|g_1(\alpha)|}{|g_0(\alpha)| + |g_1(\alpha)|}(1 + O(m^2)) \quad \text{as } m \downarrow 0\,. \end{aligned}$$

Since $g_0(\alpha) = g(h_0(\alpha), \alpha) < 0$ and $g_1(\alpha) = g(h_1(\alpha), \alpha) > 0$, we obtain the formula

$$x_*(m, \alpha) \to lg(h_1(\alpha), \alpha)/(g(h_1(\alpha), \alpha) - g(h_0(\alpha), \alpha)) \quad \text{as } m \downarrow 0\,.$$

Now we turn to the proof of (b). First, we consider the case $G_0(\alpha) \le G_1(\alpha)$. Hence, $0 < m < \sqrt{-2G_1(\alpha)}$. Let $m_1^* = \sqrt{-2G_1(\alpha)}$. Then $\overline{v} = v_R$ for $m = m_1^*$. Since $\Psi(v) = \Psi(\overline{v}) + \Psi'(\overline{v} + \theta(v - \overline{v}))(v - \overline{v}) = \Psi'(v_R + \theta(v - v_R))(v - v_R)$ and $\Psi'(\overline{v}) = -2g(h_1(v_R), v_R) < 0$ for $m = m_1^*$, we see that the integral

$$X(m_1^*, \alpha) = \int_\alpha^{v_R} \frac{dv}{\sqrt{-2G_1(v)}}$$

is also convergent. Clearly, $X(m, \alpha) \to X(m_1^*, \alpha)$ as $m \uparrow m_1^*$.

On the other hand, if $m = m_1^*$, then $\underline{v}$ satisfies $(m_1^*)^2 + 2(G_0(\alpha) - G_0(\underline{v})) = 0$, that is , $G_0(\underline{v}) = G_0(\alpha) - G_1(\alpha)$. We denote this $\underline{v}$ for m_1^* by $v_L^*(\alpha)$. Obviously $v_L^*(\alpha) \ge v_L$ and the equality holds if and only if $G_0(\alpha) = G_1(\alpha)$. As in the case $X(m_1^*, \alpha)$, we see that

$$Y(m_1^*, \alpha) = \int_{v_L^*(\alpha)}^{\alpha} \frac{dv}{\sqrt{2[G_0(\alpha) - G_1(\alpha) - G_0(v)]}}$$

is convergent and $Y(m, \alpha) \to Y(m_1^*, \alpha)$ as $m \uparrow m_1^*$.

The case $G_0(\alpha) > G_1(\alpha)$ is treated in the same way and we omit the proof. □

LEMMA 6.6. *Let α be any number such that $v_L < \alpha < v_R$. For $0 < m < \min\left\{\sqrt{-2G_0(\alpha)}, \sqrt{-2G_1(\alpha)}\right\}$, put*

$$T(m) = X(m, \alpha) + Y(m, \alpha)\,.$$

Then $T'(m) < 0$ for all m in the interval $\left(0, \min\{\sqrt{-2G_0(\alpha)}, \sqrt{-2G_1(\alpha)}\}\right)$.

PROOF. It is convenient to introduce the notation

$$Y(w) = \int_w^\alpha \frac{dv}{\sqrt{2[G_0(\alpha) - G_0(v)]}} \quad \text{for } v_L < w < \alpha ,$$

$$X(w) = \int_\alpha^w \frac{dv}{\sqrt{2[G_1(\alpha) - G_1(v)]}} \quad \text{for } \alpha < w < v_R .$$

For each $v_L < w < \alpha$, let $v(w)$ denote the unique value of v such that

$$G_1(v(w)) = G_0(w) - G_0(\alpha) + G_1(\alpha) .$$

Then $G_1'(v(w))v'(w) = G_0'(w)$, that is,

$$\frac{d\,v(w)}{dw} = \frac{g(h_0(w), w)}{g(h_1(w), w)} < 0 . \tag{46}$$

We define

$$S(w) = X(w) + Y(v(w))$$

and prove $S'(w) > 0$. We observe that the relation

$$G_0(w) = \frac{1}{2}m^2 + G_0(\alpha)$$

determines a positive number w uniquely as a function of m, and $g(h_0(w), w)\frac{dw}{dm} = m$, i.e.,

$$\frac{dw}{dm} = \frac{m}{g(h_0(w), w)} .$$

Hence, $S'(w) > 0$ implies

$$\frac{dT}{dm} = \frac{d}{dm}S(w(m)) = S'(w(m)) \cdot \frac{m}{g(h_0(w), w)} < 0 .$$

Now we turn to the proof of $S'(w) > 0$. Set $v = \alpha + (w - \alpha)t$. Then we have

$$\sqrt{2}X(w) = \int_0^1 \frac{(w-\alpha)\,dt}{[G_1(w) - G_1(\alpha + (w-\alpha)t)]^{1/2}} .$$

We differentiate both sides to obtain

$$2\sqrt{2}X'(w) = \frac{1}{w-\alpha}\int_\alpha^w \frac{[\Theta_1(w) - \Theta_1(v)]}{[G_1(w) - G_1(v)]^{3/2}}\,dv ,$$

where we have defined

$$\Theta_1(z) = 2G_1(z) - (z-\alpha)g(h_1(z), z) .$$

On the other hand, we see

$$\Theta_1'(z) = g(h_1(z), z) - (z-\alpha)\frac{d}{dz}g(h_1(z), z) .$$

Note that

$$\frac{d}{dz}g(h_1(z), z) = \frac{d}{dz}(\sigma h_1(z) - \gamma z - \rho) = \sigma h_1'(z) - \gamma ,$$

therefore, $\frac{d}{dz}g(h_1(z),z) < 0$. Also,we know that $g(h_1(z),z) > 0$ for $z < v_R$, which yields that

$$\Theta_1'(z) > 0 \quad \text{for } \alpha < z < w\,.$$

We thus conclude that

$$X'(w) > 0 \quad \text{for } \alpha < w\,.$$

Second, we consider $Y'(w)$. By a similar argument, we obtain

$$\sqrt{2}Y(w) = \int_0^1 \frac{(\alpha - w)dt}{[G_0(w) - G_0(w + (\alpha - w)t)]^{1/2}}$$

and

$$2\sqrt{2}Y'(w) = -\frac{1}{\alpha - w}\int_w^\alpha \frac{[\Theta_0(w) - \Theta_0(v)]}{[G_0(w) - G_0(v)]^{3/2}}\,dv$$

where

$$\Theta_0(z) = 2G_0(z) - (z - \alpha)g(h_0(z),z)\,.$$

Hence, $\Theta_0'(z) = g(h_0(z),z) + (\alpha - z)\frac{d}{dz}g(h_0(z),z)$. However, we see that

$$\frac{d}{dz}g(h_0(z),z) = \sigma h_0'(z) - \gamma < 0 \quad \text{and} \quad g(h_0(z),z) < 0 \quad \text{for } z > v_L\,.$$

We therefore conclude that $\Theta_0'(z) > 0$ for $w < z < \alpha$. Hence, $Y'(w) < 0$ for $w < \alpha$.

Now, recalling (46), we obtain $S'(w) = X'(w) + Y'(v(w))v'(w) > 0$. □

6.2. Non-monotone steady states with jump discontinuity.

6.2.1. Periodic steady states. Starting from monotone increasing solutions given by Theorem 6.1 we can construct symmetric (periodic) solutions of the boundary value problem (4) by the method explained in Subsection 5.3. Therfore, for each integer $k \geq 1$, (1) has (symmetric) steady states $(u(x),v(x))$ such that $u(x)$ has k jump discontinuities.

6.2.2. Non-periodic steady-states. To find non-periodic steady states with jump discontinuities (in u), we use the following two types of solutions. For $r > v_L$, let $V_0(x;r)$ be the unique solution of

$$\begin{cases} v_{xx} + g(h_0(v),v) = 0 & \text{for } -\infty < x < +\infty\,, \\ v(0) = r\,, \\ v'(0) = 0\,. \end{cases} \tag{47}$$

For $s < v_R$, let $V_1(x;s)$ denote the unique solution of

$$\begin{cases} v_{xx} + g(h_1(v),v) = 0 & \text{for } -\infty < x < +\infty\,, \\ v(0) = s\,, \\ v'(0) = 0\,. \end{cases} \tag{48}$$

Since $g(h_0(v),v) < 0$ for $v > v_L$ and $g(h_0(v),v)/v \to -\gamma$ as $v \to +\infty$, we see that $V_0(x;r)$ exists for all $x \in \mathbb{R}$ and satisfies $V_0''(x;r) > 0$, $r \leq V_0(x;r)$ for all $x \in \mathbb{R}$. Similarly, $V_1(x;s)$ exists for all $x \in \mathbb{R}$ and satisfies $V_1''(x;s) < 0$, $s \geq V_1(x;s)$ for all $x \in \mathbb{R}$.

Step 1. We would like to glue $V_0(x;r)$ and $V_1(L_1-x;s)$ together to obtain a monotone increasing solution on $0 \le x \le L_1$. We say that $V_0(x;r)$ is switchable to $V_1(x;s)$ at $v=\alpha \in (r, v_R)$ if there are constants $x_\alpha = x_\alpha(\alpha;r) > 0$, $y_\alpha = y_\alpha(\alpha;r) > 0$ and $s = s(\alpha, r) \in (\alpha, v_R)$ such that $V_1(-x_\alpha;s) = V_0(y_\alpha;r) = \alpha$ and $V_1'(-x_\alpha;s) = V_0'(y_\alpha;r)$. We claim that for each $r \in (v_L, v_R)$ there is a unique $\alpha_{0\to1} = \alpha_{0\to1}(r)$ such that $V_0(x;r)$ is switchable to $V_1(x;s)$ at $v = \alpha$ if and only if $r < \alpha < \alpha_{0\to1}(r)$. To prove this, we recall that

$$V_0'(x;r)^2 = G_0(r) - G_0(V_0(x;r)) \quad \text{and} \quad V_1'(x;s)^2 = G_1(s) - G_1(V_1(x;s)) .$$

Hence if there are $x_\alpha > 0$, $y_\alpha > 0$ and $s \in (\alpha, v_R)$ such that $V_0(y_\alpha;r) = V_1(-x_\alpha;s) = \alpha$, then $V_0'(y_\alpha;r) = V_1'(-x_\alpha;s)$ if and only if

$$G_0(r) - G_0(\alpha) = G_1(s) - G_1(\alpha) . \tag{49}$$

Now, put

$$\phi(v;r) = G_1(v) - G_0(v) + G_0(r) .$$

Then $\phi'(v;r) = g(h_1(v),v) - g(h_0(v),v) > 0$ and $\phi(v_L;r) < 0 < \phi(v_R;r)$. Hence, there exists a unique $\alpha_{0\to1}(r) \in (v_L, v_R)$ such that $\phi(\alpha_{0\to1}(r);r) = 0$. Note that $\phi(r;r) = G_1(r) < 0$, so that $r < \alpha_{0\to1}(r)$. Moreover, $\phi(v;r) < 0$ if and only if $v < \alpha_{0\to1}(r)$. Therefore, we find a unique $s = s(\alpha;r) < v_R$ such that $G_1(s) = \phi(\alpha;r)$, i.e., (49) is satisfied if and only if $r < \alpha < \alpha_{0\to1}(r)$. This verifies our assertion.

Step 2. We say that $V_1(x;s)$ is switchable to $V_0(x;r)$ at $v = \alpha \in (v_L, s)$ if there exist $x_\alpha = x_\alpha(\alpha;s) > 0$, $y_\alpha = y_\alpha(\alpha;s) > 0$ and $r = r(\alpha, s)$ such that $V_1(x_\alpha;s) = V_0(-y_\alpha;r) = \alpha$ and $V_1'(x_\alpha;s) = V_0'(-y_\alpha;r)$. One can prove as in Step 1 that for each $s \in (v_L, v_R)$ there is a unique $\alpha_{1\to0} = \alpha_{1\to0}(s)$ such that $V_1(x;s)$ is switchable to $V_0(x;r)$ at $v = \alpha$ if and only if $\alpha_{1\to0}(s) < \alpha < s$.

Step 3. We start with choosing an arbitrary r_1 such that $v_L < r_1 < v_R$. Then we choose an α_1 arbitrarily in the interval $r_1 < \alpha_1 < \alpha_{0\to1}(r_1)$. We now have a triplet $(y_{\alpha_1}, x_{\alpha_1}, s_1) = (y_{\alpha_1}(\alpha_1, r_1), x_{\alpha_1}(\alpha_1, r_1), s_1(\alpha_1, r_1))$ for which $V_0(y_{\alpha_1};r_1) = V_1(-x_{\alpha_1};s_1) = \alpha_1$ and $V_0'(y_{\alpha_1};r_1) = V_1'(-x_{\alpha_1};s_1)$ are satisfied. Here, $x_{\alpha_1} > 0$, $y_{\alpha_1} > 0$ and $\alpha_1 < s_1 < v_R$. Let $L_1 = x_{\alpha_1} + y_{\alpha_1}$ and define

$$W_1(x;r_1,\alpha_1) = \begin{cases} V_0(x;r_1) & \text{for } 0 \le x \le y_{\alpha_1} , \\ V_1(x-L_1;s_1) & \text{for } y_{\alpha_1} < x \le L_1 . \end{cases}$$

Next we select α_2 arbitrarily in $s_1 < \alpha_2 < \alpha_{1\to0}(s_1)$ and obtain a triplet $(x_{\alpha_2}, y_{\alpha_2}, r_2)$ for which $V_1(x_{\alpha_2};s_1) = V_0(-y_{\alpha_2};r_2)$, $V_1'(x_{\alpha_2};s_1) = V_0'(-y_{\alpha_2};r_2)$ hold. Let $L_2 = L_1 + x_{\alpha_2} + y_{\alpha_2}$ and define $W_2 = W_2(x;r_1,\alpha_1,\alpha_2)$ by

$$W_2(x;r_1,\alpha_1,\alpha_2) = \begin{cases} V_1(x-L_1;s_1) & \text{for } L_1 \le x \le L_1 + x_{\alpha_2} , \\ V_0(x-L_2;r_2) & \text{for } L_1 + x_{\alpha_2} < x \le L_2 . \end{cases}$$

In the same way, we can define when j is even

$$W_j(x;r_1,\alpha_1,\ldots,\alpha_j) = \begin{cases} V_1(x-L_{j-1};s_{j-1}) & \text{for } L_{j-1} \le x \le L_{j-1} + x_{\alpha_j} , \\ V_0(x-L_j;r_j) & \text{for } L_{j-1} + x_{\alpha_j} < x \le L_j \end{cases}$$

and when j is odd

$$W_j(x; r_1, \alpha_1, \ldots, \alpha_j) = \begin{cases} V_0(x - L_{j-1}; r_j) & \text{for } L_{j-1} \le x \le L_{j-1} + y_{\alpha_j}, \\ V_1(x - L_j; s_j) & \text{for } L_{j-1} + y_{\alpha_j} < x \le L_j. \end{cases}$$

Here, we have defined $L_0 = 0$ and $L_j = L_{j-1} + x_{\alpha_j} + y_{\alpha_j}$. Therefore, for each positive integer k, we obtain a function on $[0, L_k]$ defined by

$$\widetilde{W}_{k,+}(x; r_1, \alpha_1, \ldots, \alpha_k) = W_j(x; r_1, \alpha_1, \ldots, \alpha_j) \quad \text{on } [L_{j-1}, L_j].$$

Finally we define $v_{k,+}(x; r_1, \alpha_1, \ldots, \alpha_k)$ by

$$v_{k,+}(x; r_1, \alpha_1, \ldots, \alpha_k) = \widetilde{W}_{k,+}\left(\frac{Lk}{l}x; r_1, \alpha_1, \ldots, \alpha_k\right).$$

Put

$$u_{k,+}(x; r_1, \alpha_1, \ldots, \alpha_k) = \begin{cases} h_0(v_{k,+}(x; r_1, \alpha_1, \ldots, \alpha_k)) & \text{on } [L_{j-1}, L_j] \text{ if } j \text{ is odd}, \\ h_1(v_{k,+}(x; r_1, \alpha_1, \ldots, \alpha_k)) & \text{on } [L_{j-1}, L_j] \text{ if } j \text{ is even}. \end{cases}$$

Moreover, let

$$D(r_1, \alpha_1, \ldots, \alpha_k) = l^2/(Lk^2).$$

We therefore have a discontinuous steady state $(u_{k,+}(x; r_1, \alpha_1, \ldots, \alpha_k), v_{k,+}(x; r_1, \alpha_1, \ldots, \alpha_k))$ of (1.2) for $D = D(r_1, \alpha_1, \ldots, \alpha_k)$. If $k \ge 2$ and $\alpha_i \ne \alpha_j$ for some $i \ne j$, then this solution is called an asymmetric steady state of mode k since $v'_{k,+}(x; r_1, \alpha_1, \ldots, \alpha_k)$ has exactly $k - 1$ simple zeros in the open interval $0 < x < l$.

Similarly we can construct a steady state of mode k starting from $V_1(x; s_1)$ instead of $V_0(x; r_1)$. We omit the detail.

6.3. Stability of steady states with jump discontinuity. In this subsection, we study the stability of steady states with jump discontinuity. In Subsections 6.1 and 6.2, we constructed various discontinuous steady states of (1), in which $u(x)$ has finitely many jump discontinuities and $v(x)$ has jump discontinuities in the second order derivative. First we state the definition of (ε_0, E)-stability defined in [32]. Let $H^1(I) = \left\{u \in L^2(I) \mid u' \in L^2(I)\right\}$ and $\|u\|_{H^1(I)} = \|u\|_{L^2(I)} + \|u'\|_{L^2(I)}$.

DEFINITION 6.7. *For positive constants ε_0 and E, the steady state $(\tilde{u}, \tilde{v})$ of* (1) *is said to be (ε_0, E)-stable if, for some $I_0 \subset I$ with* meas $(I \setminus I_0) < \varepsilon^4$ *and for some $\varepsilon \in (0, \varepsilon_0)$, the initial functions (u_0, v_0) satisfy*

$$\|u_0 - \tilde{u}\|^2_{L^\infty(I_0)} + \|v_0 - \tilde{v}\|^2_{H^1(I)} < \varepsilon^2,$$

then for all $t > 0$

$$\|u(t, \cdot) - \tilde{u}\|^2_{L^\infty(I_0)} + \|v(t, \cdot) - \tilde{v}\|^2_{H^1(I)} < E\varepsilon^2.$$

THEOREM 6.8. *Let $(\tilde{u}, \tilde{v})$ be a steady state of* (1) *with finitely many discontinuities constructed in Subsections* 6.1 *and* 6.2. *If $v_L < \min_{0 \le x \le l} \tilde{v}(x) < \max_{0 \le x \le l} \tilde{v}(x) < v_R$, then $(\tilde{u}, \tilde{v})$ is (ε_0, E)-stable for a certain pair (ε_0, E) with $0 < \varepsilon_0,\ E < \infty$.*

PROOF. We apply Theorem 2.3 in [10]. First, we note that Assumption 2.1 in [10] is satisfied in our case by virtue of Theorem 3.2. Let $J(x) = (a_{ij}(x))_{1\le i,j\le 2}$ denote the Jacobian matrix at the steady state:

$$J(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} = \begin{pmatrix} f'_\beta(\tilde{u}(x)) & -1 \\ \sigma & -\gamma \end{pmatrix}. \tag{50}$$

We have to check the three conditions: (i) $a_{11}(x) \le -c_1 < 0$, (ii) $a_{22}(x) \le -c_1 < 0$ and (iii) $\det J(x) > 0$ for all $0 \le x \le l$, where c_1 is a positive constant. Conditions (ii) and (iii) are automatically satisfied since $a_{22}(x) = -\gamma$, $\det J(x) = -\gamma f'_\beta(\tilde{u}(x)) + \sigma$ and $f'_\beta(\tilde{u}(x)) < 0$ for all $x \in [0,l]$. By the assumption $v_L < \min \tilde{v}(x) < \max \tilde{v}(x) < v_R$, condition (i) is fulfilled because of $a_{11}(x) = f'_\beta(\tilde{u}(x)) < 0$.

Therefore, all the conditions of Theorem 2.3 in [10] are satisfied and we conclude that $(\tilde{u}, \tilde{v})$ is (ε_0, E)-stable. □

7. Concluding remarks. We discuss the relationship between steady states of the reaction-diffusion system (DFHN) where both species diffuse and those of our system (1). For this purpose we set $\varepsilon = \varepsilon_1$ and $D = 1/\varepsilon_2$ in (DFHN).

Let us recall some known results on the existence and stability of steady states of (DFHN). For proofs see, e.g., [20, 22, 23]. Assume that $g(u,v) = 0$ intersects $f(u,v) = 0$ at exactly one point on the branch $\mathscr{B}_2$ (excluding the end points (u_L, v_L) and (u_R, v_R)). Let α^* be a unique value in the interval $v_L < \alpha < v_R$ satisfying

$$\int_{h_0(\alpha^*)}^{h_1(\alpha^*)} f(s, \alpha^*)\, ds = 0\,. \tag{51}$$

Let $(u_{1,\pm}(x;\alpha^*, D), v_{1,\pm}(x;\alpha^*, D))$ with $D > D_c(\alpha^*)$ be the monotone steady states of (1) given by Corollary 6.2. For each positive integer n, let $(U^n_\pm(x;\alpha^*, D), V^n_\pm(x;\alpha^*, D))$ denote the symmetric steady state of (1) with n jump discontiuities in u obtained from $(u_{1,\pm}, v_{1,\pm})$ by applying the "folding-up method" in Subsection 5.3.

Then for any fixed $D_0 > D_c(\alpha^*)$ and positive integer n, there exists an $\varepsilon_0 > 0$ such that (DFHN) has an (ε, D)-family of steady states $(u^n_\pm(x;\varepsilon, D), v^n_\pm(x;\varepsilon, D))$ for $(\varepsilon, D) \in \{(\varepsilon, D) \mid 0 < \varepsilon < \varepsilon_n,\ D \ge D_0\}$ which satisfy

$$\lim_{\varepsilon\downarrow 0} u^n_\pm(x;\varepsilon, D) = U^n_\pm(x;\alpha^*, D) \quad \text{locally uniformly on } [0,l] \setminus \{x_{*,j} \mid j = 1, 2, \dots, n\}$$

where $x_{*,j}$ is the j-th discontinuity point of $U^n_\pm(x;\alpha^*, D)$, and

$$\lim_{\varepsilon\downarrow 0} v^n_\pm(x,\varepsilon, D) = V^*_\pm(x;\alpha^*, D)$$

uniformly on $[0,l]$. We call this solution $(u^n_\pm(x;\varepsilon, D), v^n_\pm(x;\varepsilon, D))$ the *normal n-layered solution* of (DFHN). (See [20] or [23, Theorem 3.3 and Corollary 3.9].) Moreover, there exists $\varepsilon_n(D) > 0$ for $D > D_0$ such that $\varepsilon_n(D) \to 0$ as $n \to \infty$ and the normal n-layered solution is asymptotically stable if $0 < \varepsilon < \varepsilon_n(D)$ (see [23, Theorem 3.27]). This result seem to continue to hold true for our situation that $g(u,v) = 0$ meets $f(u,v) = 0$ at three points on the branch

$\mathcal{B}_2$. Therefore, the special family of steady states $(U^n_\pm(x;\alpha^*, D), V^n_\pm(x;\alpha^*, D))$ is captured as the limit of *stable* steady-states of (DFHN) as $\varepsilon \downarrow 0$.

It is also to be noted that the standard bifurcation analysis for (DFHN) yields the following: Let (u_j, v_j), $j = 1, 2, 3$, be three equilibra with $u_L < u_1 < u_3 < u_2 < u_R$ on the branch $\mathcal{B}_2$. Then these give rise to constant steady-states of (DFHN). For $\nu = 1, 2$ and a positive integer j, let $\varepsilon^{[\nu]}_j$ and $D^{[\nu]}_j(\varepsilon)$ be defined by

$$\varepsilon^{[\nu]}_j = \sqrt{\frac{f'_\beta(u_\nu)}{\ell_j}}, \quad D^{[\nu]}_j(\varepsilon) = \frac{\sigma - \gamma f'_\beta(u_\nu) + \gamma \ell_j \varepsilon^2}{\ell_j(f'_\beta(u_\nu) - \ell_j\varepsilon^2)}, \quad \text{where } \ell_j = (\pi j/l)^2 .$$

Then, for each pair of distinct positive integers (j, k) there is a unique $\varepsilon^{[\nu]}_{j,k}$ such that $D^{[\nu]}_j(\varepsilon^{[\nu]}_{j,k}) = D^{[\nu]}_k(\varepsilon^{[\nu]}_{j,k})$. For $j = 0$, $k = 1$, we define $\varepsilon^{[\nu]}_{0,1} := \varepsilon^{[\nu]}_1$. For each $j \geq 1$ and $\varepsilon^{[\nu]}_{j,j+1} < \varepsilon < \varepsilon^{[\nu]}_{j-1,j}$, the constant steady state (u_ν, v_ν) is stable $0 < D < D^{[\nu]}_j(\varepsilon)$, but it is unstable if $D > D^{[\nu]}_j(\varepsilon)$, namely the DDI occurs for (u_ν, v_ν) at $D = D^{[\nu]}_j(\varepsilon)$. Moreover, the linearized operator

$$\begin{pmatrix} \varepsilon^2 d^2/dx^2 + f'_\beta(u_\nu) & -1 \\ \sigma & Dd^2/dx^2 - \gamma \end{pmatrix}$$

has 0 as a simple eigenvalue if $D = D^{[\nu]}_j(\varepsilon)$ and $\varepsilon \neq \varepsilon^{[\nu]}_{j,k}$ for $k \neq j$, hence $\left((u_\nu, v_\nu), D^{[\nu]}_j(\varepsilon)\right)$ is a bifurcation point when D is regarded as the bifurcation parameter. (See [22, pp. 561–562].) We observe that

$$D^{[\nu]}_j(\varepsilon) \to D^{[\nu]}_j \quad \text{as } \varepsilon \downarrow 0,$$

in which $D^{[\nu]}_j = \det J/(f'_\beta(u_\nu)\ell_j)$ (see Lemma 4.2). Therefore, the bifurcation points $\left((u_\nu, v_\nu), D^{[\nu]}_j\right)$ for (1) are regarded as the limit of those $\left((u_\nu, v_\nu), D^{[\nu]}_j(\varepsilon)\right)$ for (DFHN) as $\varepsilon \downarrow 0$. Also, it is not difficult to check that, as $\varepsilon \downarrow 0$, the bifurcating solutions of (DFHN) converge to the bifurcating solutions of (1) given by Theorem 4.6. It is very interesting to ask whether there exist steady states of (DFHN) for $\varepsilon > 0$ sufficiently small in the neighbourhood of a solution on the branch $\mathscr{C}^{(3)}_k$ in the case $G_2(v_3) < \min\{G_2(v_L), G_2(v_R)\}$ (see §§5.1–5.3).

A. Appendix: Equilibria of the kinetic system. In this appendix we consider the number of intersection points of the curve $C = \{(u, f_\beta(u)) \mid -\infty < u < +\infty\}$ and the straight line $\ell = \{(u, mu - r) \mid -\infty < u < +\infty\}$, where m is a positive number and r is a real number.

Put

$$\Phi_m(u) = mu - f_\beta(u) \left(= u^3 - (1+\beta)u^2 + (\beta + m)u\right) . \tag{52}$$

Clearly, the number of intersection points are given by the number of real (distinct) roots of the cubic equation $mu - r - f_\beta(u) = 0$, i.e.,

$$\Phi_m(u) = r . \tag{53}$$

Since $\Phi'_m(u) = 3u^2 - 2(1+\beta)u + (\beta + m)$ and $(1+\beta)^2 - 3(\beta + m) = 1 - \beta + \beta^2 - 3m$, we see that

(a) if $m \geq (1-\beta+\beta^2)/3$ then $\Phi'_m \geq 0$ for all u;
(b) if $0 < m < (1-\beta+\beta^2)/3$ then $\Phi_m(u)$ attains one strict local maximum at $u = u_-(m)$ and one strict local minimum at $u = u_+(m)$, where

$$u_\pm(m) = \frac{1}{3}\left(1+\beta \pm \sqrt{1-\beta+\beta^2-3m}\right).$$

Therefore we can classify the number of intersection points immediately. However, we would like to know also the location of intersection points. For this purpose the following observations are useful.

(c) $u'_-(m) > 0 > u'_+(m)$ for $0 < m < (1-\beta+\beta^2)/3$ and $u_L = (1+\beta-\sqrt{1-\beta+\beta^2})/3 = u_-(0) < u_-(m) < u_-((1-\beta+\beta^2)/3) = (1+\beta)/3 = u_+((1-\beta+\beta^2)/3) < u_+(m) < u_+(0) = (1+\beta+\sqrt{1-\beta+\beta^2})/3 = u_R$ for $0 < m < (1-\beta+\beta^2)/3$, since $\Phi_0(u) = -f_\beta(u)$;
(d) the straight line $v = m(u-u_*) + f_\beta(u_*)$ passing through a point $(u_*, f_\beta(u_*))$ on C meets the v-axis at $(0, -\Phi_m(u_*))$;
(e) if $0 < m < (1-\beta+\beta^2)/3$, then there are exactly two straight lines which are tangent to the curve C, and they are given by $v = m(u-u_\pm(m)) + f_\beta(u_\pm(m))$;
(f) $\Phi_m(u_R) - \Phi_m(u_L) = \frac{2}{3}\sqrt{1-\beta+\beta^2}\left\{m - \frac{2}{9}(1-\beta+\beta^2)\right\}$.

PROPOSITION A.1. *If $m \geq (1-\beta+\beta^2)/3$, then C and ℓ intersect at precisely one point (u_1, v_1) with*

(i) $u_R < u_1$ *if* $r > \Phi_m(u_R)$;
(ii) $u_1 = u_R$ *if* $r = \Phi_m(u_R)$;
(iii) $u_L < u_1 < u_R$ *if* $\Phi_m(u_L) < r < \Phi_m(u_R)$;
(iv) $u_1 = u_L$ *if* $r = \Phi_m(u_L)$;
(v) $u_1 < u_L$ *if* $r < \Phi_m(u_L)$.

PROOF. Since $v = \Phi_m(u)$ is strictly monotone increasing if $m > 1-\beta+\beta^2$, the assertion follows immediately. □

The case $0 < m < (1-\beta+\beta^2)/3$ splits into three: (i) $(1-\beta+\beta^2)/4 < m < (1-\beta+\beta^2)/3$, (ii) $m = (1-\beta+\beta^2)/4$ and (iii) $0 < m < (1-\beta+\beta^2)/4$. This is due to the following

LEMMA A.2. *If $m_\star = (1-\beta+\beta^2)/4$ then the lines $v = m_\star(u-u_L) + f_\beta(u_L)$ and $v = m_\star(u-u_R) + f_\beta(u_R)$ are tangent to the curve C. If $m \neq m_\star$, then the lines $v = m(u-u_L) + f_\beta(u_L)$ and $v = m(u-u_R) + f_\beta(u_R)$ are never tangent to C.*

PROOF. If the line $v = m(u-u_L) + f_\beta(u_L)$ is tangent to C at $(u_0, f_\beta(u_0))$ then $m = f'_\beta(u_0)$, so that $\Phi'_m(u_0) = 0$. This means that $u_0 = u_\pm(m)$. Hence, $f_\beta(u_0) = f'_\beta(u_0)(u_0 - u_L) + f_\beta(u_L)$. Note that $f_\beta(b) - f_\beta(a) = (b-a)\left(f'_\beta(a) + \frac{1}{2}f''_\beta(a)(b-a) + (b-a)^2\right)$ and $f'_\beta(u_L) = 0$. Therefore, $f_\beta(u_0) - f_\beta(u_L) = (u_0-u_L)^2\left(\frac{1}{2}f''_\beta(u_L) + (u_0-u_L)\right)$. On the other hand, $f'_\beta(u_0) = f'_\beta(u_0) - f'_\beta(u_L) = f''_\beta(u_L)(u_0 - u_L) + 3(u_0 - u_L)^2$ because of $f'''_\beta(u) = 6$; hence, our equation reduces to

$$f''_\beta(u_L) + 3(u_0 - u_L) = \frac{1}{2} f''_\beta(u_L) + u_0 - u_L \,.$$

Therefore, $u_0 = u_L + \frac{1}{4} f''_\beta(u_L) = u_L + (1+\beta)/2 - 3u_L/2 = (1+\beta-u_L)/2$, i.e., $2u_0 = 1 + \beta - u_L = \frac{1}{3}(2(1+\beta) + \sqrt{1-\beta+\beta^2})$. This shows that $u_0 = u_+(m)$ is the only choice and we get $2\sqrt{1-\beta+\beta^2-3m} = \sqrt{1-\beta+\beta^2}$, yielding $m = (1-\beta+\beta^2)/4$.

In the same way, corresponding to u_R, we see that $u_0 = u_-(m)$ and $m = (1-\beta+\beta^2)/4$. □

We now state the classification table.

PROPOSITION A.3. *If $(1-\beta+\beta^2)/4 < m < (1-\beta+\beta^2)/3$, then C and ℓ intersect*

(i) *at precisely one point (u_1, v_1) with $u_R < u_1$ if $r > \Phi_m(u_R)$;*
(ii) *at precisely one point (u_1, v_1) with $u_1 = u_R$ if $r = \Phi_m(u_R)$;*
(iii) *at precisely one point (u_1, v_1) with $u_L < u_1 < u_R$ if $\Phi_m(u_-(m)) < r < \Phi_m(u_R)$;*
(iv) *at precisely two points (u_1, v_1) and (u_2, v_2) with $u_L < u_1 = u_-(m) < u_2 < u_R$ if $r = \Phi_m(u_-(m))$;*
(v) *at precisely three points (u_1, v_1), (u_2, v_2) and (u_3, v_3) with $u_L < u_1 < u_3 < u_2 < u_R$ if $\Phi_m(u_+(m)) < r < \Phi_m(u_-(m))$;*
(vi) *at precisely two points (u_1, v_1) and (u_2, v_2) with $u_L < u_1 < u_2 = u_+ < u_R$ if $r = \Phi_m(u_+(m))$;*
(vii) *at precisely one point (u_1, v_1) with $u_L < u_1 < u_R$ if $\Phi_m(u_L) < r < \Phi_m(u_+(m))$;*
(viii) *at precisely one point (u_1, v_1) with $u_1 = u_L$ if $r = \Phi_m(u_L)$;*
(ix) *at precisely one point (u_1, v_1) with $u_1 < u_L$ if $r < \Phi_m(u_L(m))$.*

PROOF. By (c) above, if $m_\star < m < (1-\beta+\beta^2)/3$ then $u_-(m_\star) < u_-(m) < u_+(m) < u_+(m_\star)$. Hence the straight line $v = m(u - u_-(m)) + f_\beta(u_-(m))$ ($v = m(u - u_+(m)) + f_\beta(u_+(m))$, respectively) tangent to C with gradient m must intersect C at a point $(u_\sharp, f_\beta(u_\sharp))$ with $u_L < u_\sharp < u_-(m)$ ($u_+(m) < u_\sharp < u_R(m)$, respectively).

Keeping this remark in mind, we can easily verify the assertions (i)–(ix). □

PROPOSITION A.4. *Let $m_\star = (1-\beta+\beta^2)/4$. If $m = m_\star$, then C and ℓ intersect*

(i) *at precisely one point (u_1, v_1) with $u_R < u_1$ if $r > \Phi_{m_\star}(u_-(m_\star))$;*
(ii) *at precisely two points (u_1, v_1) and (u_2, v_2) with $u_L < u_1 < u_2 = u_R$ if $r = \Phi_{m_\star}(u_R)$;*
(iii) *at precisely three points (u_1, v_1), (u_2, v_2) and (u_3, v_3) with $u_L < u_1 < u_3 < u_2 < u_R$ if $\Phi_{m_\star}(u_-(m_\star)) < r < \Phi_{m_\star}(u_+(m_\star))$;*
(iv) *at precisely two points (u_1, v_1) and (u_2, v_2) with $u_1 = u_L < u_2 < u_R$ if $r = \Phi_{m_\star}(u_L)$;*
(v) *at precisely one point (u_1, v_1) with $u_1 < u_L$ if $r < \Phi_{m_\star}(u_L)$.*

PROOF. This proposition is proved in the same way as in Proposition A.3. □

We notice that, by (c) above, if $0 < m < m_\star$ then $u_-(m) < u_-(m_\star) < u_+(m_\star) < u_+(m)$. Hence the straight line $v = m(u - u_-(m)) + f_\beta(u_-(m))$ ($v = m(u - u_+(m)) + f_\beta(u_+(m))$, respectively) tangent to C with gradient m must intersect C at a point $(u_\flat, f_\beta(u_\flat))$ with $u_R < u_\flat$ ($u_\flat < u_L$, respectively). Moreover, by (f) above, $\Phi_m(u_L) < \Phi_m(u_R)$ if $m > 2(1-\beta+\beta^2)/9$ and $\Phi_m(u_L) > \Phi_m(u_R)$ if $m < 2(1-\beta+\beta^2)/9$.

These observations are sufficient to complete the classification in Propositions A.5–A.7 below:

PROPOSITION A.5. *If $2(1-\beta+\beta^2)/9 < m < (1-\beta+\beta^2)/4$, then C and ℓ intersect*

(i) *at precisely one point (u_1, v_1) with $u_R < u_1$ if $r > \Phi_m(u_-(m))$;*
(ii) *at precisely two points (u_1, v_1) and (u_2, v_2) with $u_L < u_1 = u_-(m) < u_R < u_2$ if $r = \Phi_m(u_-m)$;*
(iii) *at precisely three points (u_1, v_1), (u_2, v_2) and (u_3, v_3) with $u_L < u_1 < u_-(m) < u_3 < u_R < u_2$ if $\Phi_m(u_R) < r < \Phi_m(u_-(m))$;*
(iv) *at precisely three points (u_1, v_1), (u_2, v_2) and (u_3, v_3) with $u_L < u_1 < u_3 < u_2 = u_R$ if $r = \Phi_m(u_R)$;*
(v) *at precisely three points (u_1, v_1), (u_2, v_2) and (u_3, v_3) with $u_L < u_1 < u_3 < u_2 < u_R$ if $\Phi_m(u_L) < r < \Phi_m(u_R)$;*
(vi) *at precisely three points (u_1, v_1), (u_2, v_2) and (u_3, v_3) with $u_1 = u_L < u_3 < u_2 < u_R$ if $r = \Phi_m(u_L)$;*
(vii) *at precisely three point (u_1, v_1), (u_2, v_2) and (u_3, v_3) with $u_1 < u_L < u_3 < u_2 < u_R$ if $\Phi_m(u_-(m)) < r < \Phi_m(u_L)$;*
(viii) *at precisely two points (u_1, v_1) and (u_2, v_2) with $u_1 < u_L < u_2 = u_+(m) < u_R$ if $u = \Phi_m(u_+(m))$;*
(ix) *at precisely one point (u_1, v_1) with $u_1 < u_L$ if $r < \Phi_m(u_+(m))$.*

PROPOSITION A.6. *If $m = 2(1-\beta+\beta^2)/9$, then C and ℓ intersect*

(i) *at precisely one point (u_1, v_1) with $u_R < u_1$ if $r > \Phi_m(u_-(m))$;*
(ii) *at precisely two points (u_1, v_1) and (u_2, v_2) with $u_L < u_1 = u_-(m) < u_R < u_2$ if $r = \Phi_m(u_-(m))$;*
(iii) *at precisely three points (u_1, v_1), (u_2, v_2) and (u_3, v_3) with $u_L < u_1 < u_3 < u_R < u_2$ if $\Phi_m(u_L) = \Phi_m(u_R) < r < \Phi_m(u_-(m))$;*
(iv) *at precisely three points (u_1, v_1), (u_2, v_2) and (u_3, v_3) with $u_1 = u_L < u_3 < u_2 = u_R$ if $r = \Phi_m(u_L) = \Phi_m(u_R)$;*
(v) *at precisely three points (u_1, v_1), (u_2, v_2) and (u_3, v_3) with $u_1 < u_L < u_3 < u_2 < u_R$ if $\Phi_m(u_+(m)) < r < \Phi_m(u_L)$;*
(vi) *at precisely two points (u_1, v_1) and (u_2, v_2) with $u_1 < u_L < u_2 = u_+(m) < u_R$ if $u = \Phi_m(u_+(m))$;*
(ix) *at precisely one point (u_1, v_1) with $u_1 < u_L$ if $r < \Phi_m(u_+(m))$.*

PROPOSITION A.7. *If $0 < m < 2(1-\beta+\beta^2)/9$, then C and ℓ intersect*

(i) *at precisely one point (u_1, v_1) with $u_R < u_1$ if $r > \Phi_m(u_-(m))$;*
(ii) *at precisely two points (u_1, v_1) and (u_2, v_2) with $u_L < u_1 = u_-(m) < u_R < u_2$ if $r = \Phi_m(u_-(m))$;*

(iii) *at precisely three points* (u_1, v_1), (u_2, v_2) *and* (u_3, v_3) *with* $u_L < u_1 < u_3 < u_R < u_2$ *if* $\Phi_m(u_L) < r < \Phi_m(u_-(m))$ *;*

(iv) *at precisely three points* (u_1, v_1), (u_2, v_2) *and* (u_3, v_3) *with* $u = u_L < u_3 < u_R < u_2$ *if* $r = \Phi_m(u_L)$ *;*

(v) *at precisely three points* (u_1, v_1), (u_2, v_2) *and* (u_3, v_3) *with* $u_1 < u_L < u_3 < u_R < u_2$ *if* $\Phi_m(u_R) < r < \Phi_m(u_L)$ *;*

(vi) *at precisely three points* (u_1, v_1), (u_2, v_2) *and* (u_3, v_3) *with* $u_1 < u_L < u_3 < u_2 = u_R$ *if* $r = \Phi_m(u_R)$ *;*

(vii) *at precisely three point* (u_1, v_1), (u_2, v_2) *and* (u_3, v_3) *with* $u_1 < u_L < u_3 < u_2 < u_R$ *if* $\Phi_m(u_+(m)) < r < \Phi_m(u_R)$ *;*

(viii) *at precisely two points* (u_1, v_1) *and* (u_2, v_2) *with* $u_1 < u_L < u_2 = u_+(m) < u_R$ *if* $u = \Phi_m(u_+(m))$ *;*

(ix) *at precisely one point* (u_1, v_1) *with* $u_1 < u_L$ *if* $r < \Phi_m(u_+(m))$.

Notice that $\Phi_m(u_+(m)) \leq 0$ for $m \leq (1-\beta)^2/4$. Therefore, if we restrict the range of r to $0 < r < +\infty$, some of the cases in Propositions A.5–A.7 should be ignored. By an elementary computation we see that $2(1-\beta+\beta^2)/9 > (1-\beta)^2/4$ if $\beta > 5-\sqrt{24}$ and $2(1-\beta+\beta^2)/9 < (1-\beta)^2/4$ if $0 < \beta < 5-\sqrt{24}$. Therefore, for instance, if $\beta > 5-\sqrt{24}$, then new classification reads as follows:

PROPOSITION A.8. *Assume that* $5-\sqrt{24} < \beta < 1$. (a) *If* $(1-\beta)^2/4 < m < 2(1-\beta+\beta^2)/9$, *then the classification is the same as Proposition* A.7. (b) *If* $0 < m \leq (1-\beta)^2/4$, *then* C *and* ℓ *intersect*

(i) *at precisely one point* (u_1, v_1) *with* $u_R < u_1$ *if* $r > \Phi_m(u_-(m))$ *;*

(ii) *at precisely two points* (u_1, v_1) *and* (u_2, v_2) *with* $u_L < u_1 < u_R < u_2$ *if* $r = \Phi_m(u_-(m))$ *;*

(iii) *at precisely three points* (u_1, v_1), (u_2, v_2) *and* (u_3, v_3) *with* $u_L < u_1 < u_3 < u_R < u_2$ *if* $\Phi_m(u_L) < r < \Phi_m(u_-(m))$ *;*

(iv) *at precisely three points* (u_1, v_1), (u_2, v_2) *and* (u_3, v_3) *with* $u_1 = u_L < u_3 < u_R < u_2$ *if* $r = \Phi_m(u_L)$ *;*

(v) *at precisely three points* (u_1, v_1), (u_2, v_2) *and* (u_3, v_3) *with* $0 < u_1 < u_L < u_3 < u_2 < u_R$ *if* $0 < r < \Phi_m(u_L)$.

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