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CHARACTERISTIC CYCLES OF HIGHEST WEIGHT HARISH-CHANDRA MODULES AND THE WEYL GROUP ACTION ON THE CONORMAL VARIETY

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Abstract. We give an inductive algorithm that computes the action of simple reflections on a subset of basis-vectors of the Borel-Moore homology of the conormal variety associated to the symmetric pair $(\mathrm{Sp}(2n), \mathrm{GL}(n))$.

Introduction. The purpose of this paper is to establish an algorithm that computes the action of simple reflections on a subset of basis-vectors of the Borel-Moore homology of the conormal variety associated to the pair $(\mathrm{Sp}(2n), \mathrm{GL}(n))$.

Fix (G, K) a symmetric pair of complex Lie groups. Let $\mathfrak{g}$ denote the Lie algebra of G, and let $\mathcal{B}$ denote the variety of Borel subalgebras of $\mathfrak{g}$. Fix $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ a base point of $\mathcal{B}$. Write W for the Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$. The conormal variety, $T^*_K\mathcal{B}$, (or Generalized Steinberg variety) is the union the conormal bundles to K-orbits on $\mathcal{B}$. The topological construction of the Springer representation due to Kazhdan-Lusztig can be adapted to prove that the top Borel-Moore homology $H_{\mathrm{top}}(T^*_K\mathcal{B}, \mathbb{Z})$ is a representation of W. The fundamental classes of conormal bundles to K-orbits, $[T^*_{\mathcal{Q}}\mathcal{B}]$, are a basis of the space $H_{\mathrm{top}}(T^*_K\mathcal{B}, \mathbb{Z})$. The representation $(H_{\mathrm{top}}(T^*_K\mathcal{B}, \mathbb{Z}), W)$ plays a fundamental role in various areas within representation theory. Even when it has been the focus of intense study, there is not known formula for the action of a simple reflection on a basis-vector.

In this paper we consider the pair $(\mathrm{Sp}(2n), \mathrm{GL}(n))$ and we consider the set $T = \{\mathcal{Q}\}$ of K-orbits on $\mathcal{B}$ having the property that their closures are Schubert varieties. Such orbits are the support of the localization of highest weight Harish-Chandra modules. For a simple reflection we give an inductive algorithm to compute $s_\alpha \cdot T^*_{\mathcal{Q}}\mathcal{B}$ when $\mathcal{Q} \in T$. Our method is indirect. The Grothendieck group of Harish-Chandra modules with trivial infinitesimal character, $\mathcal{K}(\mathcal{M}_\rho(\mathfrak{g}, K))$ affords an action of W, via the coherent continuation representation. This action is effectively computable. The characteristic cycle of the localization of Harish-Chandra modules determines a map $CC : \mathcal{K}(\mathcal{M}_\rho(\mathfrak{g}, K)) \to H_{\mathrm{top}}(T^*_K\mathcal{B}, \mathbb{Z})$ which is known to be W-equivariant, see [13]. In [3], an algorithm is given that computes characteristic cycles of highest weight Harish-Chandra modules. We use the characteristic cycle computation to transfer information on coherent continuation to information on the W action on Borel-Moore homology. This is a constant theme in the present work. Our results are written using the combinatorial language of clans. The clans that occur determine, in an easy manner, K-

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orbits and Schubert cells. If prefered, the reader can view our arguments in the context of $(\mathfrak{g}, B)$-modules.

This paper is motivated by the work by W. McGovern in [10] and that by P. Trapa in [15]. Peter Trapa has repeatedly and explicitly called for transferring knowledge on the coherent continuation representation to information on $(H_{\mathrm{top}}(T_K^*\mathcal{B}, \mathbb{Z}), W)$.

The organization of this paper is as follows. The first section gives background material on the parametrization of K-orbits on $\mathcal{B}$ via the combinatorial notion of clans, reviews the notion of τ-invariant, and describes the action of operators $T_{\alpha,\beta}$ in clan notation. In Section 2 we review relevant results of [3] regarding the characteristic cycles of highest weight Harish-Chandra modules. Section 3 illustrates the main ideas of this paper in low rank examples. In Section 4 we derive very explicit and technical results on the coherent continuation representation. Section 5 contains the main theorems of the paper describing $s_\alpha \cdot T_{\mathcal{Q}}^*\mathcal{B}$ in an inductive manner.

1. Preliminaries. In this article we consider the pair of complex groups $(G, K) = (\mathrm{Sp}(2n), \mathrm{GL}(n))$ and the corresponding real form $G_{\mathbb{R}} = \mathrm{Sp}(2n, \mathbb{R})$. The realization of the complex group $G = \mathrm{Sp}(2n)$ that we use is

$$G = \left\{ g \in M_{n\times n}(\mathbb{C}) : g^t \begin{pmatrix} 0_n & S_n \\ -S_n & 0_n \end{pmatrix} g = \begin{pmatrix} 0_n & S_n \\ -S_n & 0_n \end{pmatrix} \right\}$$

where $S_n \in M_{n\times n}(\mathbb{C})$ satisfies $(S_n)_{ij} = \delta_{i+j,n+1}$.

If $I_n \in M_{2n\times 2n}(\mathbb{C})$ is the identity matrix and $I_{n,n} = \mathrm{diag}(I_n, -I_n)$, then the group K is the fixed point set of the involution $\theta = \mathrm{Ad}(I_{n,n})$. We denote by $\mathfrak{g}$ the Lie algebra of G and we let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexified Cartan decomposition. As K-representation, $\mathfrak{p}$ decomposes into the direct sum of two irreducible subrepresentations,

$$\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

with

$$\mathfrak{p}^+ = \left\{ \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} : S_n A^t S_n = A \right\}$$

and $\mathfrak{p}^-$ the transpose of $\mathfrak{p}^+$. Then, the diagonal matrices in $\mathfrak{g}$,

$$\mathfrak{h} = \{\mathrm{diag}(t_1, \ldots, t_n, -t_n, \ldots, -t_1) : t_i \in \mathbb{C}\}$$

form a Cartan subalgebra of both $\mathfrak{g}$ and $\mathfrak{k}$. We let $\varepsilon_i \in \mathfrak{h}^*$ be defined by

$$\varepsilon_i(\mathrm{diag}(t_1, \ldots, t_n, -t_n, \ldots, -t_1)) = t_i, \text{ for } 1 \leq i \leq n\,.$$

We once and for all fix the positive system of roots $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{h})$ determined by the set of simple roots

$$S := \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} : 1 \leq i \leq n-1\} \cup \{\alpha_n = 2\varepsilon_n\}\,. \tag{1.1}$$

The set of roots in $\mathfrak{p}^+$ is $\Delta(\mathfrak{p}^+) = \{\varepsilon_i + \varepsilon_j : 1 \leq i < j \leq n\}$ and the sets of positive compact roots in $\Delta_c^+ = \{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq n\}$. Our choice of Cartan subalgebra and positive system determines a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha\in\Delta^+} \mathfrak{g}^\alpha$. The connected subgroup of G with

Lie algebra $\mathfrak{b}$ is a Borel subgroup that we denote by B. We let $T^*\mathcal{B}$ stand for the cotangent bundle of the variety $\mathcal{B} \simeq G \cdot \mathfrak{b}$ of Borel subalgebras. We write $\mu : T^*\mathcal{B} \to \mathcal{N}^* \simeq \mathcal{N}$ the moment map with image in the nilpotent cone.

Our arguments are inductive in nature, reducing questions regarding the pair (G, K) to equivalent questions about smaller rank pairs. For each $1 \le i < n-1$, we consider

$$(G_i, K_i) \simeq (\mathrm{Sp}(2(n-2)), \mathrm{GL}(n-2)) . \tag{1.2}$$

The group G_i is embedded in G so that the Cartan subalgebra is $\mathfrak{h}_i = \{H \in \mathfrak{h} : \varepsilon_i(H) = \varepsilon_{i+1}(H) = 0\}$, $\Delta_i = \Delta(\mathfrak{g}_i, \mathfrak{h}_i) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) : \langle \alpha, \varepsilon_i \rangle = \langle \alpha, \varepsilon_{i+1} \rangle = 0\}$. The Lie algebra of G_i is

$$\mathfrak{g}_i = \mathfrak{h}_i \oplus \sum_{\alpha \in \Delta_i} \mathfrak{g}^\alpha .$$

We use the notation K_i for $K \cap G_i$ and B_i for $B \cap G_i$. It is useful to observe that roots in $S \cap \Delta_i$ are simple for $\Delta_i^+ = \Delta^+ \cap \Delta_i$. The pair

$$(G', K') \simeq (\mathrm{Sp}(2(n-1)), \mathrm{GL}(n-1)) , \tag{1.3}$$

where G' is embedded in G so that the Cartan subalgebra is $\mathfrak{h}' = \{H \in \mathfrak{h} : \varepsilon_1(H) = 0\}$ and

$$\mathfrak{g}' = \mathfrak{h}' \oplus \sum_{\substack{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}), \\ \langle \alpha, \varepsilon_1 \rangle = 0}} \mathfrak{g}^\alpha ,$$

is also relevant to our work. We write $K' = K \cap G'$ and $B' = B \cap G'$.

1.1. K-orbits in the flag variety of type C. The group K acts with finitely many orbits both on $\mathcal{B}$ and on $\mathcal{N} \cap \mathfrak{p}$. We consider nilpotent K-orbits that lie in $\mathfrak{p}^+$. These orbits have a particularly nice form. They are

$$\mathcal{O}_k^n = \left\{ \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} ; S_n X^t S_n = X, \ \mathrm{rank}(X) = k \right\}, k = 0, 1, \ldots, n .$$

Given $\mathcal{Q}$ a K-orbit in $\mathcal{B}$, we let $T^*_{\mathcal{Q}}\mathcal{B} \subset T^*\mathcal{B}$ denote the conormal bundle to $\mathcal{Q}$. The moment map image $\mu(T^*_{\mathcal{Q}}\mathcal{B})$ is a subvariety of the nilpotent cone. Observe that $T^*_{\mathcal{Q}}\mathcal{B}$ is invariant under the action of K, and since μ is proper and $T^*_{\mathcal{Q}}\mathcal{B}$ is irreducible, $\mu(T^*_{\mathcal{Q}}\mathcal{B})$ is an irreducible K-invariant subvariety of the nilpotent cone. Hence, $\mu(\overline{T^*_{\mathcal{Q}}\mathcal{B}})$ is the closure of a single nilpotent K-orbit. The K-orbits in $\mathcal{B}$ relevant to us are:

$$\mu^{-1}(\mathcal{O}_k^n) = \left\{ \mathcal{Q} \in K|\mathcal{B} : \mu(\overline{T^*_{\mathcal{Q}}\mathcal{B}}) = \overline{\mathcal{O}_k^n} \right\}, k = 0, 1, \ldots, n .$$

We parametrize K-orbits in $\cup_k \ \mu^{-1}(\mathcal{O}_k^n)$ in two different ways. We use clans, following the description in [21]. We identify a subset $\mathcal{W}$ of the Weyl group $W(\mathfrak{g}, \mathfrak{h})$ in bijection with $\cup_k \ \mu^{-1}(\mathcal{O}_k^n)$. Such bijection is explicitly described in [3, Lemma 17]. We summarize some relevant results.

For (G, K) the clans are $2n$-tuples $c = (c_1 \ \cdots c_{2n})$ satisfying the following.

(a) Each c_i is $+$, $-$ or a natural number.
(b) If $c_i \in \mathbb{N}$, then $c_i = c_j$ for exactly one $j \neq i$.

(c) The number of +'s that occur among the c_i's is the same as the number of −'s that occur.
(d) If $c_i = \pm$, then $c_{2n-i+1} = \mp$. If $c_i = c_j \in \mathbb{N}$, then $c_{2n-i+1} = c_{2n-j+1} \in \mathbb{N}$.

LEMMA 1.4 ([3, Corollary 14]). *Let $\mathcal{Q} \in \mu^{-1}(\mathcal{O}^n_k)$ for some $k = 0, \dots, n$. Let c be the clan that parametrizes $\mathcal{Q}$. Then, all entries c_i of c with $i \leq n$ is a + signs or a natural number. The clan $(+ \cdots + \mid - \cdots -)$ parametrizes $\mathcal{Q}_0 = K \cdot \mathfrak{b}$.*

REMARK 1.5. (a) The symmetric nature of the clans, described in part (d), allows us to identify a clan with its left half. This is the convention used in [3]. For example, the clan $(1\ +\ 2\ + \mid -\ 2\ -\ 1)$ will be written as $(1\ +\ 2\ +)$.
(b) If c parametrizes a K-orbit $K \cdot \mathfrak{b} \subset \mathcal{B}$, then c is of the form $(1\ c')$ or $(+\ c')$. The smaller clan (c') parametrizes a K'-orbit $K' \cdot (\mathfrak{b} \cap \mathfrak{g}')$, in the notation of (1.3).

Using clan notation we write:

$$\mu^{-1}(\mathcal{O}^n_k) \simeq \{c : \mu(T^*_c) = \mathcal{O}^n_k\} = \mathcal{C}_{\mathfrak{g}}(k)\,.$$

LEMMA 1.6 ([3, Lemma 26]). *Let $\mathcal{C}_{\mathfrak{g}}(k)$ be the set of all clans for (G, K) with $\mu(T^*_c) = \mathcal{O}^n_k$. Then,*

$$\mathcal{C}_{\mathfrak{g}}(n) = \{(+\ c') : c' \in \mathcal{C}_{\mathfrak{g}'}(n-1) \cup \mathcal{C}_{\mathfrak{g}'}(n-2)\}, \text{ and}$$
$$\mathcal{C}_{\mathfrak{g}}(k) = \{(+\ c') : c' \in \mathcal{C}_{\mathfrak{g}'}(k-2)\} \cup \{(1\ c') : c' \in \mathcal{C}_{\mathfrak{g}'}(k)\}, 2 \leq k \leq n-1,$$
$$\mathcal{C}_{\mathfrak{g}}(k) = \{(1\ c') : c' \in \mathcal{C}_{\mathfrak{g}'}(k)\}, k = 0, 1\,.$$

Next, we define $\mathcal{W} \subset W(\mathfrak{g}, \mathfrak{h})$. The closure of each orbit in $\cup_k\ \mu^{-1}(\mathcal{O}^n_k)$ is a Schubert variety, see [3]. The set $\mathcal{W}$ indexes the set of Schubert varieties that are K-orbit closures. In (1.1) we have fixed a positive system $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{h})$ which contains $\Delta(\mathfrak{p}^+)$. Write $\Delta^+_c \subset \Delta^+$ for the set of positive compact roots and let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha = (n, n-1, \dots, 1)\,.$$

DEFINITION 1.7.

$$\mathcal{W} = \{w \in W : -w\ \rho \text{ is } \Delta^+_c\text{-dominant}\}\,.$$

The Weyl group W consists of all permutations and sign changes of $\{\varepsilon_i\}$. This may be expressed in several ways. We will write $w = (w_1 w_2 \cdots w_n)$ where $w(\varepsilon_j)$ is ε_{w_j}, when $w_j > 0$ and $-\varepsilon_{w_j}$ when $w_j < 0$.

LEMMA 1.8 ([3, Lemma 17]). *The set $\cup_k\ \mathcal{C}_{\mathfrak{g}}(k)$ is in bijection with $\mathcal{W}$.*

Each clan $c \in \cup_k\ \mathcal{C}_{\mathfrak{g}}(k)$ gives a unique K-obit $\mathcal{Q}_c \in \mathcal{B}$. The closure $\overline{\mathcal{Q}_c}$ is a Schubert variety by $w_c \in \mathcal{W}$. The bijection assigns to c the Weyl group element w_c. Let

$$W' = \{w \in W : w(1) = 1\}\,.$$

LEMMA 1.9. *Given two clans, c_1, c_2, let $w_{c_1}, w_{c_2} \in W$ be the corresponding Weyl group elements under the bijection of Lemma* 1.8.

(1) *If* $c_1 = (1c_1')$ *and* $c_2 = (1\ c_2')$, *then* $w_{c_2}^{-1} w_{c_1} \in W'$.
(2) *If* $c_1 = (+\ c_1')$ *and* $c_2 = (+\ c_2')$, *then* $w_{c_2}^{-1} w_{c_1} \in W'$.

PROOF. The lemma follows from the easy algorithm in [3, Lemma 17]. □

1.2. Tau-invariant and the operators $T_{\alpha,\beta}$. The operators $T_{\alpha,\beta}$ of [17, Definition 3.4] will be repeatedly used throughout this paper. We give some necessary background. The τ-invariant of $w \in W$ is

$$\tau(w) = \{\alpha \in S : w\alpha < 0\}\,.$$

By Lemma 1.8, each $w_c \in \mathcal{W}$ corresponds to a clan c. We write

$$\tau(c) = \tau(w_c) = \tau_{weak}(\mathcal{Q}_c)\,.$$

LEMMA 1.10. *Let $c \in \cup_k\, \mathcal{C}_{\mathfrak{g}}(k)$. The simple root $\alpha_j = \varepsilon_j - \varepsilon_{j+1} \in \tau(c)$ if and only if the j and $j+1$ entries in c are $+\,+$, $k\ +$ or $k\ k+1$. The root $\alpha_n = 2\varepsilon_n \in \tau(c)$ when the last entry in c is an integer.*

PROOF. The lemma follows from Lemma 1.8 and the definition of τ-invariant. □

Given α, β consecutive simple roots in the Dynkin diagram, we say that w is in the domain of $T_{\alpha,\beta}$ when $\alpha \notin \tau(w)$ and $\beta \in \tau(w)$. When this is the case and α has the same length as β, then

$$T_{\alpha,\beta}(w) = \begin{cases} ws_\alpha, & \beta \notin \tau(ws_\alpha)\,, \\ ws_\beta, & \alpha \in \tau(ws_\beta)\,. \end{cases} \tag{1.11}$$

When α, β have different lengths

$$T_{\alpha,\beta}(w) = \left\{\widetilde{w} : \widetilde{w} \in \{ws_\alpha, ws_\beta\}, \alpha \in \tau(\widetilde{w}), \beta \notin \tau(\widetilde{w})\right\} \tag{1.12}$$

a set of either one or two elements.

Let $T_{j\ j+1}$ stand for $T_{\alpha_j,\alpha_{j+1}}$. In clan notation the formulae (1.11) reads as follows, see [3]. Write the $j, j+1, j+2$ entries of the clans. We have in the case of equal lengths:

$$\begin{aligned} T_{j\ j+1}(\cdots +\ k\ + \cdots) &= (\cdots +\ +k\ \cdots) \\ T_{j\ j+1}(\cdots +\ k-1\ k \cdots) &= (\cdots k-1\ +\ k\ \cdots) \\ T_{j\ j+1}(\cdots\ +\ +k \cdots) &= (\cdots\ +\ k\ +\ \cdots) \\ T_{j\ j+1}(\cdots\ k-1\ +\ k \cdots) &= (\cdots\ +\ k-1\ k\ \cdots)\,. \end{aligned} \tag{1.13}$$

When the roots α_j, α_{j+1} have different lengths, we have

$$\begin{aligned} &T_{n\ n-1}(\cdots\ +\ +) = (\cdots\ +\ k) \\ &T_{n\ n-1}(\cdots\ \ k\ +) = (\cdots\ k-1\ +\ k) \\ &T_{n-1\ n}(\cdots\ +\ k) = \{(\cdots\ +\ k\ +), (\cdots\ +\ +)\}\,. \end{aligned} \tag{1.14}$$

LEMMA 1.15. *If* $c = (c_1 \cdots c_{j-1} c_j \cdots c_{i-1} + 1\ c_{i+2} \cdots c_n)$ *then, there exists a sequence of operators* $T_{i,i-1}, T_{i+1,i+2}, \ldots, T_{2,1}$ *so that*

$$T_{2,1} \cdots T_{i,i-1} c = (+1\ c_1 \cdots c_{j-1} c_j \cdots c_{i-1} c_{i+2} \cdots c_n)\,.$$

PROOF. The lemma follows from the explicit formulas for $T_{i,i-1} \cdot c$. A similar computation is included in the proof of [3, Lemma 51]. □

For a simple root α we let $\mathcal{F}_\alpha$ denote the variety of all parabolic subalgebras of $\mathfrak{g}$ conjugate to $\mathfrak{h} \oplus \mathfrak{g}^{-\alpha} \oplus \sum_{\beta \in \Delta^+} \mathfrak{g}^\beta$. We let $\pi_\alpha : \mathcal{B} \to \mathcal{F}_\alpha$ be the natural projection. It is known that for $\alpha \notin \tau_{weak}(\mathcal{Q}_c)$, $\pi_\alpha^{-1}\pi_\alpha(\mathcal{Q}_c)$ contains a dense K-orbit of dimension $\dim(\mathcal{Q}_c) + 1$. We denote this orbit by

$$s_\alpha \circ \mathcal{Q}_c \simeq s_\alpha \circ c\,.$$

It is useful to write $s_\alpha \circ c$ explicitly in clan notation. Let $\alpha = \varepsilon_j - \varepsilon_{j+1}$ and write the $j-1, j, j+1$ entries of c. Then

$$s_\alpha \circ c = s_\alpha \circ (\cdots c_{j-1} + \ell \cdots) = (\cdots c_{j-1}\ \ell + \cdots)\,. \tag{1.16}$$

When $\alpha = 2\varepsilon_n$

$$s_\alpha \circ c = s_\alpha \circ (\cdots c_{n-1}\ +) = (\cdots c_{n-1}\ \ell)\,. \tag{1.17}$$

LEMMA 1.18. *If* $c \in \mathcal{C}_\mathfrak{g}(k)$ *and* $\alpha \notin \tau(c)$, *then* $s_\alpha \circ c \in \mathcal{C}_\mathfrak{g}(k) \cup \mathcal{C}_\mathfrak{g}(k-1)$.

PROOF. The lemma follows from (1.16), (1.17), and the algorithm that computes moment map images in clan notation, see [21] and [14]. □

1.3. Weyl group action on the conormal variety. We let $T^*_K\mathcal{B}$ denote the conormal variety to the K-action on $\mathcal{B}$, that is, the union of conormal bundles to K-orbits in $\mathcal{B}$. The variety $T^*_K\mathcal{B}$ is pure of dimension $\dim(\mathcal{B})$ and its irreducible components are the closures $\overline{T^*_\mathcal{Q}\mathcal{B}}$ of conormal bundles. The set $\left\{[\overline{T^*_\mathcal{Q}\mathcal{B}}]\right\}$ of fundamental classes of conormal bundle closures is a basis for the top Borel-Moore homology $H_{\mathrm{top}}(T^*_K\mathcal{B}, \mathbb{Z})$. The latter affords a W-module structure. See for example, [7], [8], [4], [12], [11]. Although the W action on Borel-Moore homology has been the focus of intense studies, there is no known method to explicitly determine the action of a simple reflection on the basis $\left\{[\overline{T^*_\mathcal{Q}\mathcal{B}}]\right\}$. It is our goal to give such formulae when $\mathcal{Q} \in \cup_k\ \mu^{-1}(\mathcal{O}^n_k)$. In this section we review some basic known facts. In order to avoid introducing more notation, we assume $\mathcal{Q} = \mathcal{Q}_c \in \mu^{-1}(\mathcal{O}^n_k)$ for $\mathcal{O}^n_k \subset \mathcal{N} \cap \mathfrak{p}^+$. We keep the notation $\tau_{weak}(\mathcal{Q}) = \tau(c)$.

THEOREM 1.19 ([11], [4], [12], [3]). *Let* $\mathcal{Q}$ *be a* K*-orbit in* $\mathcal{B}$ *with* $\mu(T^*_\mathcal{Q}\mathcal{B}) = \mathcal{O}^n_k$.

(1) *If* $\alpha \in \tau_{weak}(\mathcal{Q})$, *then*

$$s_\alpha \cdot T^*_\mathcal{Q}\mathcal{B} = -T^*_\mathcal{Q}\mathcal{B}\,.$$

(2) *If $\alpha \notin \tau_{weak}(\mathcal{Q})$, then there exist positive integers $m^{\alpha}(\mathcal{Q}, \mathcal{Q}_j)$ so that*

$$s_\alpha \cdot T^*_{\mathcal{Q}}\mathcal{B} = T^*_{\mathcal{Q}}\mathcal{B} + T^*_{s_\alpha \circ \mathcal{Q}}\mathcal{B} + \sum_{\substack{\mathcal{Q}_j \subset \overline{\mathcal{Q}},\ \alpha \in \tau_{weak}(\mathcal{Q}_j) \\ \mu(T^*_{\mathcal{Q}_j}) = \mu(T^*_{\mathcal{Q}})}} m^{\alpha}(\mathcal{Q}, \mathcal{Q}_j)\, T^*_{\mathcal{Q}_j}\mathcal{B}\,. \tag{1.20}$$

REMARK 1.21. A convolution construction defines a $H_{\mathrm{top}}(T^*_{\mathrm{diag}\,(G)}\mathcal{B} \times \mathcal{B}, \mathbb{Z})$ module structure on $H_{\mathrm{top}}(T^*_K\mathcal{B}, \mathbb{Z})$, which by a theorem of Kazhdan-Lusztig, is the group algebra $\mathbb{Z}[W]$. It follows from this construction that

$$\sum_{\mathcal{Q}_j :\, \mu(T^*_{\mathcal{Q}_j}) \subset \overline{\mathcal{O}^n_k}} \mathbb{Z}\,[T^*_{\mathcal{Q}_j}\mathcal{B}] \tag{1.22}$$

is W invariant. Since our nilpotent orbit $\mathcal{O}^n_k \subset \mathfrak{p}^+$, each orbit $\mathcal{Q}_j$ in (1.22) has $\mu(T^*_{\mathcal{Q}_j}) \subset \mathfrak{p}^+$. By [3, Proposition 16], the combined conditions $\mathcal{Q}_j \subset \overline{\mathcal{Q}}$ and $\mu(T^*_{\mathcal{Q}_j}) \subset \mathfrak{p}^+$, imply that $\mu(T^*_{\mathcal{Q}}\mathcal{B}) \subset \overline{\mu(T^*_{\mathcal{Q}_j}\mathcal{B})}$. Thus, the equality of moment map images in formula (1.20) follows from the inclusion $\mu(T^*_{\mathcal{Q}}\mathcal{B}) \subset \overline{\mu(T^*_{\mathcal{Q}_j}\mathcal{B})}$ and (1.22).

2. Characteristic cycle.

2.1. Invariants of Harish-Chandra modules. Let $\mathcal{M}_\rho(\mathfrak{g}, K)$ denote the category of finitely generated $(\mathfrak{g}, K)$-modules of infinitesimal character ρ. This category is equivalent to the category $\mathcal{M}_c(\mathcal{D}, K)$ of coherent K-equivariant $\mathcal{D}$-modules on the flag variety $\mathcal{B}$. Localization implements the equivalence of categories, see [1]. If $X \in \mathcal{M}_\rho(\mathfrak{g}, K)$ we write $\mathfrak{X}$ for its localization. The irreducible modules $\mathfrak{X}$ are parametrized by pairs $(\mathcal{Q}, \chi)$, where $\mathcal{Q}$ is a K-orbit on $\mathcal{B}$ and χ is a K-equivariant local system; $\mathrm{supp}(\mathfrak{X}) = \overline{\mathcal{Q}}$.

A fundamental invariant of X is the characteristic cycle of $\mathfrak{X}$. That is,

$$CC(\mathfrak{X}) = [\overline{T^*_{\mathcal{Q}}\mathcal{B}}] + \sum_j m_{\mathcal{Q}_j}\ [\overline{T^*_{\mathcal{Q}_j}\mathcal{B}}],$$

viewed as an element of the top degree Borel-Moore homology of the conormal variety. If $m_{\mathcal{Q}_j} \neq 0$, then $\mathcal{Q}_j$ is in the singular locus of $\mathrm{supp}(\mathfrak{X})$.

We refer to the support of $\mathfrak{X}$ as the support of X. A Harish-Chandra module written as $X(\mathcal{Q})$ is assumed to have $\mathrm{supp}(\mathfrak{X}) = \overline{\mathcal{Q}}$.

The moment map image of a conormal bundle closure $\overline{T^*_{\mathcal{Q}_j}\mathcal{B}}$ is the closure of a single nilpotent K-orbit. The characteristic cycle and associated variety are related through the moment map $\mu : T^*\mathcal{B} \to \mathcal{N}$. We have,

$$\mathrm{AV}(X) = \mu(\overline{T^*_{\mathcal{Q}}\mathcal{B}}) \underset{m_{\mathcal{Q}_j} \neq 0}{\cup} \mu(\overline{T^*_{\mathcal{Q}_j}\mathcal{B}})\,.$$

The *leading term cycle* is defined to be

$$\mathrm{LTC}(X) = \sum m_{\mathcal{Q}}\ [\overline{T^*_{\mathcal{Q}}\mathcal{B}}],$$

summing over the K-orbits with $\dim(\mu(\overline{T^*_{\mathcal{Q}}\mathcal{B}})) = \dim\,(\mathrm{AV}(X))$.

An important observation is that the family of highest weight Harish-Chandra modules may be considered as sitting in $\mathcal{M}_\rho(\mathfrak{g}, K)$ or in $\mathcal{M}(\mathfrak{g}, B)$, the category of finitely generated $(\mathfrak{g}, B)$-modules with trivial infinitesimal character. The characteristic cycle of a $\mathcal{D}$-module is defined independent of which category we are in.

2.2. Invariants of highest weight Harish-Chandra modules. We give a short summary of results in [3] relevant to our work. We keep the notation introduced in Section 1.

LEMMA 2.1. *A $(\mathfrak{sp}(2n), GL(n))$ module with trivial infinitesimal character is a highest weight Harish-Chandra modules if and only if its associated variety is a subvariety of $\mathfrak{p}^+$.*

PROOF. See [3, Appendix B], for example. □

NOTATION 2.2. *We use the shorthand notation T^*_c for $\overline{T^*_{\mathcal{Q}_c}\mathcal{B}}$, when c is the clan parametrizing the K-orbit $\mathcal{Q}_c$. We write the highest weight module $X(\mathcal{Q}_c)$ as $X(c)$.*

THEOREM 2.3 ([3]). *Let $\mathcal{Q}'$ for the K'-orbit determined by a clan (c') and assume that*

$$CC(X(c')) = \sum_{c'_j} m_{c'_j}\ [T^*_{c'_j}]\,.$$

(1) *If $c = (+\ c')$, then*

$$CC(X(c)) = \sum_{c'_j} m_{c'_j}\ [T^*_{(+\ c'_j)}]\,.$$

Moreover, when $(+\ c') \in \mathcal{C}_{\mathfrak{g}}(n) \cup \bigcup_{k\ even} \mathcal{C}_{\mathfrak{g}}(k)$, we have

$$LTC(X(c)) = CC(X(c))\,.$$

(2) *If $c = (1\ c')$ and $(1\ c') \in \mathcal{C}_{\mathfrak{g}}(k)$ with $k < n-1$, then*

$$CC(X(c)) = \sum_{c'_j} m_{c'_j}\ [T^*_{(1\ c'_j)}]\,.$$

Moreover, $LTC(X(c)) = CC(X(c))$ if and only if $(1\ c') \in \mathcal{C}_{\mathfrak{g}}(k)$ with k even.

(3) *If $c = (1\ c')$ and $(1\ c') \in \mathcal{C}_{\mathfrak{g}}(n-1)$, then*

$$CC(X(c)) = \sum_{c'_j} m_{c'_j}\ [T^*_{(1\ c'_j)}]\ +\ \sum_{c'_j} m_{c'_j}\ [T^*_{(+\ c'_j)}],\ \text{if } n \text{ is even.}$$

When n is odd, we have

$$CC(X(c)) = \sum_{c'_j} m_{c'_j}\ [T^*_{(1\ c'_j)}]\,.$$

COROLLARY 2.4. *Let $\mathcal{Q}'$ be the K'-orbit determined by the clan (c') and assume that there exist integers $n_\ell = \pm$ so that*

$$T^*_{c'} = \sum_{\ell} n_\ell\ CC(X(c_\ell))$$

for some highest weight modules Harish-Chandra modules $X(c_\ell)$. Then,

$$T^*_{(+\ c')} = \sum_\ell n_\ell\ CC(X(+\ c_\ell)),$$
$$T^*_{(1\ c')} = \sum_\ell n_\ell\ CC(X((1\ c_\ell)),\ \textit{when } k < n-1 \textit{ and } (1\ c') \in \mathcal{C}_{\mathfrak{g}}(k),$$
$$T^*_{(1\ c')} = \sum_\ell n_\ell\ CC(X((1\ c_\ell)),\ \textit{when } n \textit{ is odd } (1\ c') \in \mathcal{C}_{\mathfrak{g}}(n-1),$$
$$T^*_{(1\ c')} = \sum_\ell n_\ell\ [CC(X((1\ c_\ell)) - CC(X(+\ c_\ell))]c'),$$
$$\textit{when } n \textit{ is even and } (1\ c') \in \mathcal{C}_{\mathfrak{g}}(n-1)\,.$$

3. Low rank examples. In this section we illustrate our method for computing the action of simple reflections on conormal bundles $\{T^*_c\}$ in low rank cases. We use ATLAS software to compute the coherent continuation action of simple reflections on irreducible highest weight $(\mathfrak{g}, K)$ modules. We keep the notation introduced in prior sections. We write s_i for the simple reflection through the simple root α_i.

3.1. Type C_2. By [3], we have:

$$T^*_{(+\ +)} = CC\,(X(+\ +)) \qquad T^*_{(+\ 1)} = CC\,(X(+\ 1))$$
$$T^*_{(1\ 2)} = CC\,(X(1\ 2)) \qquad T^*_{(1\ +)} = CC\,(X(1\ +)) - CC\,(X(+\ +))\ .$$

Observe that for appropriate integers $n_\ell = \pm 1$, each conormal bundle T^*_c is given in the form $T^*_c = \sum n_\ell\ CC\,(X(c_\ell))$, where each $X(c_\ell)$ is an irreducible highest weight $(\mathfrak{g}, K)$-module. Since the characteristic cylce functor is W-equivariant we conclude that

$$s_\alpha \cdot T^*_c = \sum n_\ell\ CC\,(s_\alpha \cdot X(c_\ell))\ . \tag{3.1}$$

We compute:

c	$s_1 \cdot X(c)$	$s_2 \cdot X(c)$
$(++)$	$-\ X(+\ +)$	$X(+\ +) +\ X(+\ 1)$
$(1\ +)$	$-\ X(1\ +)$	$X(1\ +) +\ X(+\ 1) + X(1\ 2)$
$(+\ 1)$	$X(+\ 1) + X(+\ +) + X(1\ +)$	$-\ X(+\ 1)$
$(1\ 2)$	$-\ X(1\ 2)$	$-X(1\ 2)$

The modules occurring in $s_\alpha \cdot X(c)$ are highest weight $(\mathfrak{g}, K)$-modules. Hence, their chracteristic cycles are given by the algorithm in [3]. This observation and (3.1) give:

$$s_2 \cdot T^*_{(+\ +)} = T^*_{(+\ +)} + T^*_{(+\ 1)}\,,$$
$$s_2 \cdot T^*_{(1\ +)} = T^*_{(1\ +)} + T^*_{(1\ 2)}\,,$$
$$s_1 \cdot T^*_{(+\ 1)} = T^*_{(+\ 1)} + 2\,T^*_{(+\ +)} + T^*_{(1\ +)}\,.$$

3.2. Type C_3. When AV$X(c) = \mathcal{O}_3$, the computations in [3] yield:

$$T^*_{(+\,+\,+)} = CC\,(X(+\;+\;+))\,, \qquad T^*_{(+\,+\,1)} = CC\,(X(+\;+\;1))$$
$$T^*_{(+\,1\,+)} = CC\,(X(+\,1\,+)) - CC\,(X(+\;+\;+))\;.$$

Coherent continuation gives:

c	$s_1 \cdot X(c)$	$s_2 \cdot X(c)$
$(+\,+\,+)$	$-\,X(c)$	$-\,X(c)$
$(+\,1\,+)$	$X(c) + X(1\;+\,+) + X(+\,+\;1)$	$-\,X(c)$
$(+\,+\;1)$	$-\,X(c)$	$X(c) + X(+\,1\,+) + X(+\,+\,+)$

c	$s_3 \cdot X(c)$
$(+\,+\,+)$	$X(c) + X(+\,+\;1)$
$(+\,1\,+)$	$X(c) + X(+\,+\;1) + X(+\,1\,2)$
$(+\,+\;1)$	$-\,X(c)$

Combining the given information, as we did in the type C_2 case, we obtain:

$$s_1 \cdot T^*_{(+\,1\,+)} = T^*_{(+\,1\,+)} + 2\,T^*_{(+\,+\,+)} + T^*_{(+\,+\,1)} + T^*_{(1\,+\,+)}\,,$$
$$s_3 \cdot T^*_{(+\,1\,+)} = T^*_{(+\,1\,+)} + T^*_{(+\,1\,2)}\,,$$
$$s_2 \cdot T^*_{(+\,+\,1)} = T^*_{(+\,+\,1)} + 2\,T^*_{(+\,+\,+)} + T^*_{(+\,1\,+)}\,,$$
$$s_3 \cdot T^*_{(+\,+\,+)} = T^*_{(+\,+\,+)} + T^*_{(+\,+\,1)}\,.$$

Similarly, when AV$(X(c)) = \mathcal{O}_2$, we compute:

$$s_1 \cdot T^*_{(+\,1\,2)} = T^*_{(+\,1\,2)} + T^*_{(1\,+\,2)}\,,$$
$$s_2 \cdot T^*_{(1\,+\,2)} = T^*_{(1\,+\,2)} + 2\,T^*_{(1\,+\,+)} + T^*_{(+\,1\,2)} + T^*_{(1\,2\,+)}\,,$$
$$s_3 \cdot T^*_{(1\,+\,+)} = T^*_{(1\,+\,+)} + T^*_{(1\,+\,2)}\,,$$
$$s_3 \cdot T^*_{(1\,2\,+)} = T^*_{(1\,2\,+)} + T^*_{(1\,2\,3)}\,.$$

3.3. Type C_4. There are sixteen highest weight Harish-Chandra modules for Sp$(8,\mathbb{R})$. We only write the action of simple reflections on $T^*_{(1\,+\,2\,+)}$ and $T^*_{(+\,1\,2\,+)}$ due to space considerations.

By Corollary 2.4, we have

$$T^*_{(1\,+\,2\,+)} = [CC\,(X(1\;+\;2\,+))\;-CC\,(X(+\;+\;1\,+))\,]$$
$$-\,[CC\,(X(1\;+\;+\,+))\;-CC\,(X(+\;+\;+\,+))\,]\,,$$
$$T^*_{(+\,1\,2\,+)} = CC\,(X(+\,1\,2\,+))\;-CC\,(X(+\,1\;+\;+))\;.$$

Computing as in prior examples we obtain:

$$s_1 \cdot T^*_{(+\,1\,2\,+)} = T^*_{(+\,1\,2\,+)} + T^*_{(1\,+\,2\,+)}\,,$$
$$s_4 \cdot T^*_{(+\,1\,2\,+)} = T^*_{(+\,1\,2\,+)} + T^*_{(+\,1\,2\,3)}\,.$$

REMARK 3.2. It is relevant to observe that the formula for $s_4 \cdot T^*_{(+\,1\,2\,+)}$ and that for $s_3 \cdot T^*_{(1\,2\,+)}$ differ by a + at the begging of each clan.

$$s_2 \cdot T^*_{(1\,+\,2\,+)} = T^*_{(1\,+\,2\,+)} + 2\,T^*_{(1\,+\,+\,+)} + T^*_{(1\,2\,+\,+)} + T^*_{(+\,1\,2\,+)} + T^*_{(1\,+\,+\,2)},$$
$$s_4 \cdot T^*_{(1\,+\,2\,+)} = T^*_{(1\,+\,2\,+)} + T^*_{(1\,+\,2\,3)}.$$

REMARK 3.3. The following observations will be relevant to us.

(a) If $T^*_{(1\ c)}$ occurs in $s_4 \cdot T^*_{(1\,+\,2\,+)}$, then T^*_c occurs in $s_3 \cdot T^*_{(+\,1\,+)}$.
(b) If T^*_c occurs in $s_3 \cdot T^*_{(+\,1\,+)}$, then $T^*_{(1\ c)}$ occurs in $s_4 \cdot T^*_{(1\,+\,2\,+)}$.
(c) The conormal bundle contributes to $T^*_{(1\ c)}$ occurs in $s_2 \cdot T^*_{(1\,+\,2\,+)}$ if and only if T^*_c occurs in $s_1 \cdot T^*_{(+\,1\,+)}$.

3.4. One example in type C_6. The examples considered so far suggest an inductive algorithm to compute the action of simple reflections on the conormal variety. For example, one can check that:

$$\begin{aligned} s_2 \cdot {} & T^*_{(1\,+\,2\,+)} \\ & = T^*_{(1\,+\,2\,+)} + 2\,T^*_{(1\,+\,+\,+)} + T^*_{(1\,2\,+\,+)} + T^*_{(+\,1\,2\,+)} + T^*_{(1\,+\,+\,2)}. \\ s_3 \cdot {} & T^*_{(+\,1\,+\,2\,+)} \\ & = T^*_{(+\,1\,+\,2\,+)} + 2\,T^*_{(+\,1\,+\,+\,+)} + T^*_{(+\,1\,2\,+\,+)} + T^*_{(+\,+\,1\,2\,+)} + T^*_{(+\,1\,+\,+\,2)}. \end{aligned} \tag{3.4}$$

On the other hand, the formula

$$\begin{aligned} s_4 \cdot T^*_{(1\,+\,2\,+\,3\,+)} = {} & T^*_{(1\,+\,2\,+\,3\,+)} + 2\,T^*_{(1\,+\,2\,+\,+\,+)} + T^*_{(1\,+\,2\,3\,+\,+)} \\ & + T^*_{(1\,+\,+\,2\,3\,+)} + T^*_{(1\,+\,2\,+\,+\,3)} + T^*_{(+\,+\,1\,2\,3\,+)}, \end{aligned} \tag{3.5}$$

shows that the inductive argument needed is more subtle than the one used in [3]. Inducing data from the pair $(G', K') \simeq \mathrm{Sp}(2(n-1)), \mathrm{GL}(n-1))$ to (G, K) provides useful but partial information.

Observe that $T^*_{(+\,+\,1\,2\,3\,+)}$ occurs in $s_4 \cdot T^*_{(1\,+\,2\,+\,3\,+)}$ and $T^*_{(+\,1\,2\,+)}$ occurs in $s_{\alpha_3} \cdot T^*_{(1\,+2\,+)}$. The clan (+ 1 2 +) is obtained from (+ + 1 2 3 +) by deleting the symbols +1 in the second and third entries.

4. Coherent continuation. Our study of the W-module structure on $H_{\mathrm{top}}(T^*_K\mathcal{B}, \mathbb{Z})$ is indirect. The Weyl group W acts on the Grothendieck group $\mathcal{K}(\mathcal{M}_\rho(\mathfrak{g}, K))$ of Harish-Chandra modules of infinitesimal character ρ via coherent continuation. The characteristic cycle of the localization of Harrish-Chandra modules induces a W-equivariant map between $\mathcal{K}(\mathcal{M}_\rho(\mathfrak{g}, K))$ and $H_{\mathrm{top}}(T^*_K\mathcal{B}, \mathbb{Z})$, see [13]. Concrete knowledge of the characteristic cycles of highest weight Harish-Chandra modules, available in [3], allow us to transfer information on the coherent continuation representation to information about the W-action on the Borel-Moore homology of the conormal variety.

4.1. A brief survey on the coherent continuation representation. We include a brief summary of results on the coherent continuation representation. The reader might want to consult [18, Chapter 7, 8], [16], [19], [20]. In order to avoid introducing further notation we write relevant results in the context of highest weight $(\mathrm{Sp}(2n), \mathrm{GL}(n))$ modules.

We use the Beilinson-Bernstein classification of irreducible $(\mathfrak{g}, K)$-modules of infinitesimal character ρ. Highest weight $(\mathfrak{g}, K)$ modules are parametrized by pairs $(\mathcal{Q}_c, \mathbf{1})$ consisting of a K-orbit in $\mathcal{B}$, as described in Subsection 1.1, and the trivial local system over $\mathcal{Q}_c$. We keep the notation of prior sections and we write $X(\mathcal{Q}_c) = X(c)$ for the highest weight Harish-Chandra module attached to the pair $(\mathcal{Q}_c, \mathbf{1})$. In particular, the localization of $X(\mathcal{Q}_c)$ has support $\overline{\mathcal{Q}_c}$. When consulting the reference the reader might want to be aware of various classifications of Harish-Chandra modules. For example, in [20], irreducible $(\mathfrak{g}, K)$-modules are parametrized by regular characters. The correspondence of parameters, $(\mathcal{Q}_c, \mathbf{1}) \leftrightarrow \gamma_{\mathbf{c}}$, is explained in [19, Proposition 2.7]. The modules $X(\mathcal{Q}_c)$, we are interested in, lie in both categories $\mathcal{M}_\rho(\mathfrak{g}, K)$ and $\mathcal{M}(\mathfrak{g}, B)$. As an element of $\mathcal{M}(\mathfrak{g}, B)$, $X(\mathcal{Q}_c)$ is the irreducible highest weight module $L(w_c)$. The localization of $L(w_c)$ has support equal to the Schubert variety $\overline{B_{w_c}} = \overline{\mathcal{Q}_c}$. Hence, $s_\alpha \cdot X(\mathcal{Q}_c)$ can be viewed in the context of $(\mathfrak{g}, B)$-modules.

We refer the reader to [18, 7.3.8] for the notion of τ-invariant of an irreducible Harish-Chandra module. It follows from the remarks surrounding [18, 7.3.8] that for $X(c) = X(\mathcal{Q}_c)$, a highest weight $(\mathfrak{g}, K)$-module, $\tau(c) = \tau(X(Q_c)) = \tau_{weak}(\mathcal{Q}_c)$.

THEOREM 4.1 ([18, Corollary 7.3.18], [16, Theorem 3.10], [19, Theorem 5.10], [20, Lemma 14.7]). *Let $X(c)$ be an irreducible highest weight $(\mathfrak{g}, K)$-module. If $\alpha \in \Delta^+$ is simple and $\alpha \in \tau(X(c))$, then*

$$s_\alpha \cdot X(c) = -\, X(c)\,.$$

If $\alpha \in \Delta^+$ is simple and $\alpha \notin \tau(X(c))$, then there exist positive integers $\mu^\alpha(c, c_i)$ so that

$$s_\alpha \cdot X(c) = X(c) + X(s_\alpha \circ c) + \sum_i \mu^\alpha(c, c_i)\, X(c_i)\,. \tag{4.2}$$

For each clan c_i with $\mu^\alpha(c, c_i) \neq 0$ the following holds:

(1) $\mathrm{AV}(X(c_i)) \subset \mathrm{AV}(X(c))$;
(2) $\dim(\mathcal{Q}_{c_i}) \leq \dim(\mathcal{Q}_c) - 1$, and $\mathcal{Q}_{c_i} \subset \overline{\mathcal{Q}}$;
(3) $\alpha \in \tau(X(c_i))$;
(4) if γ is a simple root perpendicular to α and $\gamma \in \tau(X(c))$, then $\gamma \in \tau(X(c_i))$.

Moreover, if β is a simple root with $\langle \alpha, \beta \rangle$ of type A_2, then there is exactly one c_j with $\mu^\alpha(c, c_j) \neq 0$ and $\beta \notin \tau(X(c_j))$. In this case $\mu^\alpha(c, c_j) = 1$.

REMARK 4.3. (1) By [18, Chapter7], $s_\alpha \cdot X = X + U_\alpha$. In [19, Theorem 5.10], the author proves that U_α is completely reducible. Information on the constituents of U_α can be found in [18, Theorem 8.5.18] and [18, 8.3.2].

(2) The statements in Theorem 4.1 regarding τ-invariant follow from [16, Theorem 3.10].
(3) The inclusion of associated varieties follows, for example, from [20, Lemma 4.7]. Note that if $X(c)$ is a highest weight Harish-Chandra modules, then $\mathrm{AV}(X(c)) \subset \mathfrak{p}^+$. Thus, each module Y that contributes to $s_\alpha \cdot X(c)$ has $\mathrm{AV}(Y) \subset \mathfrak{p}^+$. It follows from Lemma 2.1, that Y is a highest weight module.

(4) When $X(c_i)$ contributes to $s_\alpha \cdot X(c)$, the corresponding Kazhdan-Lusztig-Vogan polynomial is of maximal possible degree and $\mu^\alpha(c, c_i)$ is the coefficient of the largest power of q, see [20, Lemma 14.7]. The coefficients $\mu^\alpha(c, c_i)$ encode important geometric information on the singularities of $\overline{\mathcal{Q}_c} = \overline{B_{w_c}}$, see [6, Theorem 4.3] and [9, Theorem 1.2].

4.2. The theory of coherent continuation will be used in §5 to help describe an inductive algorithm that computes $s_\alpha \cdot T_c^*$. Our inductive procedure is compatible with the induction used in [2] in the study of coherent continuation. In this section we relate coefficients $\mu^\alpha(c, d)$ to appropriate coefficients $\mu^\alpha(c', d')$ occurring in coherent continuation formulae for smaller pairs. We also study the behavior of $\mu^\alpha(c, d)$ when operators $T_{i,i-1}$ are applied to the clans c, d.

We continue with our notation (G, K) for $(\mathrm{Sp}(2n), \mathrm{GL}(n))$, $\mathfrak{b}$ for the Borel subalgebra of $\mathfrak{g}$ as in §1, Δ^+ for the corresponding positive system, S for the set of simple roots, etc. We use the notation (G', K') for $(\mathrm{Sp}(2(n-1)), \mathrm{GL}(n-1))$, $\mathfrak{b}' = \mathfrak{b} \cap \mathfrak{g}'$, Δ'^+ for the corresponding positive system, etc. For $\alpha \in S \cap \Delta'^+ \subset S$, we write $\mu^\alpha(\mathrm{clan}_1, \mathrm{clan}_2)$ for the multiplicity of $X(\mathrm{clan}_2)$ in $s_\alpha \cdot X(\mathrm{clan}_1)$.

LEMMA 4.4. *Let $X(\mathcal{Q}_{c_1})$, $X(\mathcal{Q}_{c_2})$ be highest weight $(\mathfrak{g}, K)$ modules attached to K-orbits with parametrizing clans c_1, c_2, respectively.*

(1) *If $c_1 = (1\ c_1')$ and $c_2 = (1\ c_2')$ and $\alpha \in S \cap \Delta'^+ \subset \Delta^+$ satisfies $\alpha \notin \tau(c_1')$ and $\alpha \in \tau(c_2')$, then $\alpha \notin \tau(c_1)$ and $\alpha \in \tau(c_2)$ and*
$$\mu^\alpha((1\ c_1'), (1\ c_2')) = \mu^\alpha(c_1', c_2') .$$

(2) *If $c_1 = (+\ c_1')$ and $c_2 = (+\ c_2')$ and $\alpha \in S \cap \Delta'^+ \subset \Delta^+$ satisfies $\alpha \notin \tau(c_1')$ and $\alpha \in \tau(c_2')$, then $\alpha \notin \tau(c_1)$ and $\alpha \in \tau(c_2)$ and*
$$\mu^\alpha((+\ c_1'), (+\ c_2')) = \mu^\alpha(c_1', c_2') .$$

PROOF. By Lemma 1.8 and Lemma 1.9, associated to the clans c_1, c_2 are Weyl group elements w_{c_1}, w_{c_2} so that $X(\mathcal{Q}_{c_1}) = L(w_{c_1})$ and $X(\mathcal{Q}_{c_2}) = L(w_{c_2})$ and $w_{c_2}^{-1} w_{c_1} \in W'$. The lemma follows from [2, Lemma 3.10]. □

LEMMA 4.5. *Let α_i, α_{i+1} be consecutive short simple roots in Δ^+. Assume c, d are clans with $\alpha_i \notin \tau(c) \cup \tau(d)$ and $\alpha_{i+1} \in \tau(c) \cap \tau(d)$. Furthermore, assume β is a simple root perpendicular to α_i, α_{i+1} and so that $\beta \notin \tau(c)$ but $\beta \in \tau(d)$. Then,*
$$\mu^\beta(c, d) = \mu^\beta(T_{i,i+1} \cdot c, T_{i,i+1} \cdot d) .$$

PROOF. This lemma is a special case of Theorem 4.2 in [5], written in clan notation. □

LEMMA 4.6. *Let α be a simple root perpendicular to the long simple root. Let c be a clan of the form $c = (1\ c'+)$. Assume $\alpha \notin \tau(c)$ and let $X(c)$ be the highest weight $(\mathfrak{g}, K)$ module with support $\overline{\mathcal{Q}_c}$. If $X(+ \cdots +)$ occurs as a summand in $s_\alpha \cdot X(1\ c'+)$, then $X(+ \cdots\ 1)$ occurs as a summand in $s_\alpha \cdot X(T_{n,n-1} \cdot (1\ c'+))$.*

PROOF. Let α_n denote the long simple root. Since $\alpha \in \{\alpha_n\}^{\perp}$,

$$s_{\alpha_n} s_\alpha \cdot X(1\ c'+) = s_\alpha s_{\alpha_n} \cdot X(1\ c'+)\,.$$

We argue that $X(+\cdots+\ \ 1)$ contributes to $s_{\alpha_n} s_\alpha \cdot X(1\ c'+)$. Hence, $X(+\cdots+\ \ 1)$ occurs as a summand in $s_\alpha s_{\alpha_n} \cdot X(1\ c'+)$. Comparing the τ-invariant of the constituents of $s_\alpha s_{\alpha_n} \cdot X(1\ c'+)$, we conclude that $X(+\cdots+\ \ 1)$ occurs in $s_\alpha \cdot X(T_{n,n-1} \cdot (1\ c'+))$.

Write

$$s_\alpha \cdot X(1\ c'+) = X(1\ c'+) + \mu^\alpha\left((1\ c'+),(+\cdots+)\right)\ X(+\cdots+) + Z_{\alpha_n} + Z_{\not\alpha_n}\,, \tag{4.7}$$

where Z_{α_n} ($Z_{\not\alpha_n}$) is a sum of irreducible modules that have α and α_n in their τ-invariant (sum of irreducible modules that have α but not α_n in their τ-invariant, resp.).

By Lemma 1.10, $\alpha_{n-1} \in \tau\left((1\ c'+)\right)$ and $\alpha_{n-1} \notin \tau\left((+\cdots+1)\right)$. If $\alpha \in \{\alpha_{n-1}\}^{\perp}$, then Theorem 4.1 guarantees that each irreducible summand of Z_n has α_{n-1} in its τ-invariant. Hence, $X(+\cdots+\ \ 1)$ can not be one such summand. If $\alpha = \alpha_{n-2}$, then $X\left(T_{n-2,n-1} \cdot (1\ c'+)\right)$ is the only summand in $s_\alpha \cdot X(1\ c'+)$ with α_{n-2} and not α_{n-1} in its τ-invariant. As $(+\cdots+1)$ starts with $+$, $T_{n-2,n-1} \cdot (1\ c'+) \neq (+\cdots+1)$. Thus, $X(+\cdots+1)$ does not occur in Z_n. It follows that $X(+\cdots 1)$ is not cancelled out in

$$\begin{aligned} s_{\alpha_n} \cdot \left(s_\alpha \cdot X(1\ c'+)\right) = {} & s_{\alpha_n} \cdot X(1\ c'+) + s_{\alpha_n} \cdot Z_{\not\alpha_n} \\ & + \mu^\alpha\left((1\ c'+),(+\cdots+)\right)\ [X(+\cdots+) + X(+\cdots 1)] - Z_{\alpha_n}\,. \end{aligned} \tag{4.8}$$

Hence, $X(+\cdots+1)$ contributes to $s_{\alpha_n} s_\alpha \cdot X(1\ c'+) = s_\alpha s_{\alpha_n} \cdot X(1\ c'+)$.

On the other hand,

$$s_{\alpha_n} \cdot X(1\ c'+) = X(1\ c'+) + X(T_{n,n-1} \cdot (1\ c'+)) + Y_\alpha + Y_{\not\alpha}\,,$$

where Y_α and $Y_{\not\alpha}$ are sums of irreducible modules. The irreducible modules occurring in Y_α have α_{n-1}, α_n, and α in their τ-invariant. The irreducible modules contributing to $Y_{\not\alpha}$ have α_{n-1}, α_n, but not α in their τ-invariant. By Lemma 1.10, the last two symbols of the clans that parametrize constituents of $Y_{\not\alpha}$ are natural numbers. Then,

$$s_\alpha \cdot \left(s_{\alpha_n} \cdot X(1\ c'+)\right) = s_\alpha \cdot X(1\ c'+) + s_\alpha \cdot X(T_{n,n-1} \cdot (1\ c'+)) - Y_\alpha + s_\alpha \cdot Y_{\not\alpha}\,. \tag{4.9}$$

We know that $X(+\cdots+1)$ does not occur in $s_\alpha \cdot X(1\ c'+)$. In order to complete the proof we need to show that $X(+\cdots+1)$ does not contribute to $s_\alpha \cdot Y_{\not\alpha}$. If $\alpha \in \{\alpha_{n-1}\}^{\perp}$, each constituent of $s_\alpha \cdot Y_{\not\alpha}$ has α_{n-1} in its τ-invariant. Hence, $X(+\cdots+1)$ is not one of them. Assume $\alpha = \alpha_{n-2}$. Since $X(+\cdots+1)$ has α_{n-2} and not α_{n-1} in its τ-invariant, for $X(+\cdots+1)$ to occur in $s_\alpha \cdot Y_{\not\alpha}$ we would need to have $X(c)$, summand of $Y_{\not\alpha}$, with

$$X(+\cdots+1) = X\left(T_{n-2,n-1}\ c\right)\,. \tag{4.10}$$

Our observation on the shape of such a clan c indicates that equation (4.10) can not hold. □

4.3. We continue our study of the coherent continuation representation.

PROPOSITION 4.11. *Let $X(c)$ be the highest weight $(\mathfrak{g}, K)$ module with support $\overline{\mathcal{Q}_c}$. Assume c is of the form $c = (1\ c'+)$. If α is a simple root perpendicular to the long simple root with $\alpha \notin \tau(c)$, then $X(+\cdots+)$ does not contribute to $s_\alpha \cdot X(1\ c'+)$.*

PROOF. We use induction on n to obtain a contradiction. The cases $n \leq 3$ have been established in §3. Let $n > 3$ and assume that the $(\mathrm{Sp}(2n), \mathrm{GL}(n))$ module $X(+\cdots+)$ occurs in $s_\alpha \cdot X(1\ c'+)$. By Lemma 4.6, if

$$T_{n,n-1}(1\ c'+) = \begin{cases} T_{n,n-1}(1\ c''++) & = (1\ c''+k), \quad \text{or} \\ T_{n,n-1}(1\ c''\ k\ +) & = (1\ c''+k), \end{cases}$$

then $\mu^\alpha\ (T_{n,n-1}(1\ c'+), (+\cdots+\ 1)) \neq 0$. Both $(+\cdots+\ 1)$ and $(1\ c''+k)$ are in the domain of the operator $T_{n-1,n-2}$. Hence, Lemma 4.5 gives

$$\mu^\alpha\ (T_{n,n-1}(1\ c'+), (+\cdots+\ 1)) = \mu^\alpha\ (T_{n-1,n-2}(1\ c''+k), T_{n-1,n-2}(+\cdots+\ 1)) \neq 0\,.$$

By choosing an appropriate sequence of operators $T_{i,i-1}$, as in Lemma 1.15, and applying Lemma 4.5 we conclude that

$$\mu^\alpha\ ((+\ k\ 1\ c''), (+1+\cdots+)) \neq 0\,.$$

Now, for clans corresponding to a smaller pair $(\mathrm{Sp}(2(n-2)), \mathrm{GL}(n-2))$, Lemma 4.4 gives,

$$\mu^\alpha\ ((1\ c''), (+\cdots+)) \neq 0\,. \tag{4.12}$$

Note that $\tau\,((1\ c'')) \subset \tau\,((+\cdots+))$, implies that the clan $(1\ c'')$ ends on a + sign. Hence, (4.12) contradicts our induction hypothesis. □

PROPOSITION 4.13. *Let $c = (1\ c')$ be a clan consisting of n symbols. Assume α is a simple root not in $\tau(c)$. If $\mu^\alpha\ ((1\ c'), (+\ d')) \neq 0$, then either*

(1) *$c = (1 + c'')$, $\alpha = \varepsilon_2 - \varepsilon_3$ and $(+\ d') = T_{2,1}\ c$, or*
(2) *$c = (1 + \cdots + c_j = 2\ c'')$, d is of the form $(+\cdots+d_j = 1\ d'')$, and $\alpha \neq \varepsilon_{j-1} - \varepsilon_j$.*

Moreover, there is at most one clan $d = (+\cdots+d_j = 1\ d'')$ with

$$\mu^\alpha((1+\cdots+c_j = 2\ c''), (+\cdots+d_j = 1\ d'')) \neq 0\,.$$

We prove an auxiliary lemma.

LEMMA 4.14. *Let $c = (c_1\ c_2 \cdots c_k \cdots c_n)$ be a clan consisting of n symbols. Let α be a simple root not in $\tau(c)$.*

(1) *Assume $k > 1$, $c_1 \cdots c_k \in \mathbb{N}$ and $c_{k+1} = +$. If $X(d)$ contributes to $s_\alpha \cdot X(c)$, then d is of the form $d = (1\ d')$.*
(2) *Assume for $k \geq 2$ $(c_1\ c_2\ \cdots c_{k+1}) = (1\ +\cdots+2)$. If $\alpha = \varepsilon_k - \varepsilon_{k+1}$ and $X(d)$ occurs in $s_\alpha \cdot X(c)$, then either $k = 2$ and $d = T_{2,1}\ c$ or $k \neq 2$ and d is of the form $(1\ d')$.*
(3) *Assume $\alpha \neq \varepsilon_2 - \varepsilon_3$ and $c_1 \in \mathbb{N}$. If c has all simple roots but α in its τ-invariant and $X(d)$ contributes to $s_\alpha \cdot X(c)$, then d is of the form $d = (1\ d')$.*

PROOF. We prove (1) by induction on n. The cases $n \leq 3$ have been established in §3. Let $n > 3$ and assume that there exists a clan of the form $d = (+\ d')$ for which $\mu^\alpha(c,d) \neq 0$. Then, by Theorem 4.1, $\alpha \in \tau(d)$. Moreover, each simple root $\gamma \in \{\alpha\}^\perp \cap \tau(c)$ is in $\tau(d)$. The description of the τ-invariant of clans in $\cup_k\ C_{\mathfrak{g}}(k)$ given in Lemma 1.10, implies that for some $\ell \geq k+1$ d is of the form $(+ \cdots + d_\ell = 1\ d'')$.

We claim that the clans c, d are in the domain of the operator $T_{\ell-1,\ell-2}$. Observe that α is not adjacent to $\alpha_{\ell-1} = \varepsilon_{\ell-1} - \varepsilon_\ell$. Indeed, if α were adjacent to $\alpha_{\ell-1}$, by Lemma 1.10, we would have $\alpha \notin \tau(c), \alpha_{\ell-1} \in \tau(c), \alpha \in \tau(d), \alpha_{\ell-1} \notin \tau(d)$. In such a case, Theorem 4.1 would force $d = T_{\alpha,\alpha_{\ell-1}} c$. That is not possible as the first entry in c is an integer and the first entry in d is a sign. Since α not adjacent to $\alpha_{\ell-1}$ and $\alpha_{\ell-1} \notin \tau(d)$, we conclude that $\alpha_{\ell-1} \notin \tau(c)$. The root $\alpha_{\ell-2} \in \tau(c) \cap \tau(d)$ according to description of the τ-invariant of clans in Lemma 1.10. Thus, both c and d are in the domain of $T_{\ell-1,\ell-2}$. Lemma 4.5 gives

$$\mu^\alpha(c,d) = \mu^\alpha(T_{\ell-1,\ell-2} \cdot c, T_{\ell-1,\ell-2} \cdot d) \neq 0\,.$$

Choosing an appropriate sequence of operators $T_{i,i-1}$, as in Lemma 1.15, Lemma 4.5 yields

$$\mu^\alpha((+1\ c_1 \cdots \not{c}_{\ell-1}\ \not{c}_\ell \cdots c_n), (+1 + \cdots \not{+} \not{1}\ d_{\ell+1} \cdots d_n)) \neq 0\,. \tag{4.15}$$

Applying Lemma 4.4 to (4.15), we obtain clans associated to a smaller rank pair $(\mathrm{Sp}(2(n-2)), \mathrm{GL}(n-2))$, for which

$$\mu^\alpha((c_1 \cdots \not{c}_{\ell-1}\ \not{c}_\ell \cdots c_n), (+ \cdots \not{+} \not{1}\ d_{\ell+1} \cdots d_n)) \neq 0\,,$$

a contradiction to our inductive hypothesis.

The proof of (2) is similar. For (3), observe that each clan d with $\mu^\alpha(c,d) \neq 0$ satisties (3) and (4) of Theorem 4.1. Assume there is a clan d that starts with a $+$ sign satisfying $\mu^\alpha(c,d) \neq 0$. Since the clan c starts with an integer $d \notin T_{\alpha,\beta} c$ for any root β adjacent to α. Hence, $\tau(d)$ contains α and the roots adjacent to α. On the other hand, (4) of Theorem 4.1 and our assumptions on $\tau(c)$, imply that $\{\alpha\}^\perp \cap S \subset \tau(d)$. Thus $\tau(d) = S$. By Lemma 1.10, there is no clan d with its first entry equal to $+$ and $\tau(d) = S$. □

PROOF OF PROPOSITION 4.13. We use induction on n. The cases $n \leq 3$ have been established in §3. Let $n > 3$ and assume that for a clan $d = (+\ d')$, $\mu^\alpha(c,d) \neq 0$. Note that Proposition 4.11 shows $d \neq (+ \cdots +)$. Moreover, Lemma 4.14 implies that c is of the form $(1 + \cdots c_j = 2\ c'')$ for some $j \geq 3$. When $j = 3$, $\alpha = \varepsilon_2 - \varepsilon_3$ and $(+\ d') = T_{2,1}\ c$, Theorem 4.1 gives $\mu^\alpha(c,d) \neq 0$. This is case (1) in the statement of the proposition.

If $j > 3$, by part (2) in Lemma 4.14, $\alpha \neq \varepsilon_{j-1} - \varepsilon_j$. We have two possibilities, either $\alpha_{j-1} \in \tau(d)$ or $\alpha_{j-1} \notin \tau(d)$. If $\alpha_{j-1} \in \tau(d)$, then for some $\ell > j$, $d = (+ \cdots + c_{\ell+1} = 1\ d'')$. Arguing as in the proof of Lemma 4.14, we conclude that both clans c and d are in the domain of the operator $T_{\ell,\ell-1}$. Thus,

$$\mu^\alpha(c,d) = \mu^\alpha(T_{\ell,\ell-1} \cdot c, T_{\ell,\ell-1} \cdot d) \neq 0\,.$$

Choosing an appropriate sequence of operators $T_{i,i-1}$, as in Lemma 1.15, Lemma 4.5 and Lemma 4.4 produce a pair a clans with $n-2$ symbols with

$$\mu^{\alpha}((1\ +\cdots+c_j=2\cdots \not{c}_{\ell-1}\ \not{c}_{\ell}\cdots c_n),(+\cdots \not+\not1\ d_{\ell+1}\cdots d_n))\neq 0\,.$$

This is a contradiction to our inductive hypothesis. Hence, $\alpha_{j-1}\notin\tau(d)$ and d is of the form prescribed by the proposition.

A similar inductive argument gives the uniqueness statement in the proposition. □

PROPOSITION 4.16. *Let c be a clan of the form $(+\ 1\ c')$. Assume $T=\{\alpha:\alpha$ is a short simple root $\alpha\neq\alpha_1$ and $\alpha\notin\tau(c)\}\neq\emptyset$. Let j be the least integer so that $\alpha_j=\varepsilon_j-\varepsilon_{j+1}\in T$. If $\mu^{\alpha_1}((+\ 1\ c'),(+\ +\ d_3\ d_4\cdots d_n))\neq 0$, then for all $i\leq j$ $d_i=+$ and $d_{j+1}=1$.*

PROOF. It is clear from Lemma 1.10 that $\alpha_j\neq\alpha_2$. Assume there is a clan $d=(+\ +\ \ d')$ with $\alpha_j\in\tau(d)$ and $\mu^{\alpha_1}(c,d)\neq 0$. Since $\alpha_j\notin\tau(c)$, by Theorem 4.1, $\mu^{\alpha_j}((+\ 1\ c'),(+\ +\ \ d'))\neq 0$. By Lemma 4.4, we conclude that for smaller clans

$$\mu^{\alpha_j}((1\ c'),(+\ d'))=\mu^{\alpha_j}(c,d)\neq 0\,. \tag{4.17}$$

Now, (4.17) contradicts Lemma 4.14. Hence, $\alpha_j\notin\tau(d)$ and the proposition follows form (3) and (4) of Theorem 4.1. □

PROPOSITION 4.18. *Let c be a clan consisting of n symbols of the form $c=(+\ 1\ c')$. Then $\mu^{\alpha_1}((+\ 1\ c'),(+\cdots+))\neq 0$ if and only if either*

$$\begin{cases} c=(+\ 1), d=(++) & n=2 \textit{ or}\\ c=(+\ 1\ 2+\cdots+) & n \textit{ is even.}\end{cases} \tag{4.19}$$

PROOF. We first argue that $\mu^{\alpha_1}((+\ 1\ c'),(+\cdots+))\neq 0$ implies that c satisfies (4.19). It follows from Proposition 4.16 that the tau-invariant $\tau((+\ 1\ c'))=S-\{\alpha_1,\alpha_n\}$. Hence, the clan c is of the form $c=(+\ 1\ 2\cdots k+\cdots+)$. On the other hand, if $\mu^{\alpha_1}((+\ 1\ c'),(+\cdots+))\neq 0$ then $X(+\cdots+)$ contributes to $s_{\alpha_1}\cdot X(+1\ c')$. By Theorem 4.1, $\overline{O^n_n}=\mathrm{AV}(X(+\cdots+))\subset\mathrm{AV}(X(+\ 1\ c'))$. Hence, we must have $AV(X((+\ 1\ c'))=\overline{O^n_n}$. The computations of associated varieties in [3, Proposition 36] force us to have

$$\mu(T^*_{(+1\ 2\cdots k+\cdots+)})=\begin{cases} O^n_n \text{ or } O^n_{n-1} & \text{if n is even, or}\\ O^n_n & \text{if n is odd.}\end{cases} \tag{4.20}$$

Using the algorithm that computes moment map images, [21], we conclude that (4.20) can only hold if

$$c=(+\ 1\ 2\cdots k+\cdots+)=\begin{cases}(+\ 1+\cdots+) & \text{or}\\ (+\ 1\ 2+\cdots+) & \text{with n even.}\end{cases}$$

It remains to show that $\mu^{\alpha_1}((+\ 1+\cdots+),(+\cdots+))\neq 0$ if and only if $n=2$. The examples in Section 3 give $\mu^{\alpha_1}((+\ 1),(++))\neq 0$ and $\mu^{\alpha_1}((+\ 1+),(+\ +\ +))=0$. We proceed by induction on n to arrive to a contradiction. Assume that clans with k symbols so that $3\leq k<n$

have $\mu^{\alpha_1}((+\ 1+\cdots+),(+\cdots+))=0$. Let $c=(+\ 1+\cdots+)$ consists of n symbols and have $\mu^{\alpha_1}(c,(+\cdots+))\neq 0$. By Lemma 4.6,

$$\mu^{\alpha_1}((+\ 1+\cdots 2),(+\cdots 1))\neq 0\,. \tag{4.21}$$

Applying an appropriate sequence of operators $T_{i,i-1}$ to the clans in displayed equation (4.21) and arguing as in the proof of Proposition 4.13, we conclude that

$$\mu^{\alpha_1}((\not{+}\not{1}+2+\cdots+),(\not{+}\not{1}+\cdots+))\neq 0,$$

a contradiction to our inductive hypothesis.

We conclude that $\mu^{\alpha_1}((+1+\cdots+),(+\cdots+))$ vanishes when for $n>2$. In order to complete the proof of this proposition we need to show, for n even, that $\mu^{\alpha_1}((+1\ 2+\cdots+),(+\cdots+))\neq 0$. The argument is identical to that used in the proof of Proposition 4.11. □

4.4.

NOTATION 4.22. A sequence of consecutive integers (or a sequence of consecutive + signs) in a clan is called a block. If the i-th block in a clan consists of + signs, we denote by r_i the number of symbols in the i-th block. If the i-th block of a clan consists of integers, we write t_i for the size of the i-th block. We refer to the sequence $[r_1,t_1,\ldots,r_s,t_s]$ as the size-type of the clan.

LEMMA 4.23. *Let $A\in\mathbb{N}$ and let $c=(A+\ c')$ be a clan consisting of n symbols and of size-type $[1,r,t,\ldots]$. Let $d=(+\ d')$ be a clan consisting of n symbols and of size-type $[r+1,s,\ldots]$. Assume α is a simple root with $\alpha\notin\tau(c')$. If $\mu^{\alpha}(c,d)\neq 0$, then either*

(1) *$r=s=t$; or*
(2) *$r=t<s$; or*
(3) *$s=t<r$.*

PROOF. First observe that $\alpha\neq\varepsilon_{r+1}-\varepsilon_{r+2}$. Indeed, this is the content Lemma 4.14, (2). We consider thirteen possibilities, i.e.,

Case 1: $r<s<t$, Case 2: $r<t<s$, Case 3: $s<r<t$, Case 4: $s<t<r$

Case 5: $t<r<s$, Case 6: $t<s<r$, Case 7: $s<r=t$, Case 8: $s=t<r$

Case 9: $t=r<s$, Case 10: $r<s=t$, Case 11: $s=r<t$, Case 12: $t=r=s$

Case 13: $t<r=s$.

The goal is to show that all cases other than Cases 8, 9, and 12 contradict our results in Subsection 4.3. We argue that Cases 1, 3, 6 can not hold. Other possibilities are excluded by similar arguments.

Assume $\mu^{\alpha}(c,d)\neq 0$ and $r<s<t$. By Lemma 4.5, we have

$$\mu^{\alpha}(T_{2,1}\cdots T_{r+1,r}\ c,T_{2,1}\cdots T_{r+1,r}\ d)=\mu^{\alpha}(c,d)\neq 0\,.$$

That is,

$$\mu^{\alpha}\left((+1\ A+\cdots \not{+}\not{1}\ 2\cdots t+\cdots),(+1+\cdots \not{+}\not{1}\ 2\cdots s+\cdots)\right)\neq 0\,.$$

Moreover, Lemma 4.4 gives clans with $n-2$ symbols with

$$\mu^{\alpha}\left((A+\cdots \not{+}\not{1}\ 2\cdots t+\cdots),(+\cdots \not{+}\not{1}\ 2\cdots s+\cdots)\right)\neq 0\,.$$

Repeating the argument, we get

$$\mu^{\alpha}\,(A\ 1\ \cdots(t-r)+\cdots),(+1\ 2\cdots(s-r)+\cdots))\neq 0\,. \tag{4.24}$$

Note that the block $A\ 1\cdots(t-r)$ consists of at least two integers. Also note that $\alpha_1=\varepsilon_1-\varepsilon_2$ is perpendicular to α , $\alpha_1\in\tau\,((A\ 1\ \cdots(t-r)+\cdots))$ and $\alpha_1\notin\tau\,((+\ 1\ \cdots(s-r)+\cdots))$. Thus, equation (4.24) contradicts both (4) of Theorem 4.1 and Lemma 4.14. We conclude that Case 1 can not hold.

Assume $\mu^{\alpha}\,(c,d)\neq 0$ and $s<r<t$. After applying an appropriate sequence of operators $T_{i,i-1}$ to both c and d, Lemma 4.4 yield

$$\begin{aligned}0\ &\neq \mu^{\alpha}\,(c,d)\\ &=\mu^{\alpha}\,((A+\cdots+c_{r-s+2}=1\ 2\cdots(t-s)+\cdots),(+\cdots+d_{\ell}=1\cdots))\ ,\end{aligned} \tag{4.25}$$

where $\ell\geq r-s+3$. Since $c_{r-s+2}=1$, equation (4.25) contradicts Proposition 4.13. Hence, Case 3 can not hold.

Next, we consider the case $\mu^{\alpha}\,(c,d)\neq 0$ and $t<s<r$. Once again, after applying an appropriate sequence of operators $T_{i,i-1}$ to both c and d, Lemma 4.4 yield

$$\mu^{\alpha}(\hat{c},\hat{d}\,)=\mu^{\alpha}\,((A+\cdots+c_{r-t+2}=+\cdots),(+\cdots+d_{r=t+2}=1\cdots))\neq 0\,. \tag{4.26}$$

Note that $\alpha\neq\varepsilon_{r-t+2}-\varepsilon_{r-t+3}$. Otherwise, equation (4.26) would imply that $\hat{d}=T_{r-t+2,r-t+1}\ \hat{c}$. This is not possible as the clan $\hat{d}$ starts with + while $\hat{c}$ starts with an integer. Then, $\varepsilon_{r-t+2}-\varepsilon_{r-t+3}\in\tau(\hat{c})$ but not in $\tau(\hat{d})$. This is a contradiction to (4) of Theorem 4.1. We conclude that Case 6 can not hold. □

PROPOSITION 4.27. *Let $A\in\mathbb{N}$ and let $c=(A+\ c')$ be a clan consisting of n symbols and of size-type $[1,r_1,t_1,r_2,t_2,\ldots]$. Let $d=(+\ d')$ be a clan consisting of n symbols and of size-type $[r_1+1,s_1,\ldots]$. Assume α is a simple root with $\alpha\notin\tau(c)$. If $\mu^{\alpha}(c,d)\neq 0$, then either $r_1=t_1$ or there exists and index j so that $t_j=\sum_1^{j-1}(r_k-t_k)+r_j$.*

PROOF. By Lemma 4.23, either $r_1=t_1$ or $r_1>t_1=s_1$.

Assume $r_1>t_1=s_1$. We claim that clan d has its entry $d_{r_1+r_2+t_1+2}\in\mathbb{N}$. In particular, $\alpha\neq\varepsilon_{1+r_1+t_1+r_2}-\varepsilon_{2+r_1+t_1+r_2}$. We prove the claim arguing by contradiction. Assume $d_{r_1+r_2+t_1+2}=+$. Let $\hat{c}$ be the clan obtained from c by deleting its third block and t_1 signs from its second block. Write $\hat{d}$ for the clan obtained from d by deleting its second block and t_1 signs from its first block. Lemma 4.5 and Lemma 4.4 give

$$\begin{aligned}\mu^{\alpha}(c,d)&=\mu^{\alpha}(\hat{c},\hat{d})\\ &=\mu^{\alpha}((A+\cdots c_{1+r_1-t_1+r_2}=1\cdots),(+\cdots c_{1+r_1-t_1+r_2}=+\cdots))\neq 0,\end{aligned}$$

contradicting part (2) of Lemma 4.14. Hence, the size-type of c is $[1, r_1, t_1, r_2, t_2, \dots]$ and the size-type of d is $[r_1 + 1, t_1, r_2, s_2, \dots]$. Moreover,

$$\begin{aligned} &\mu^{\alpha}(c, d) \\ &= \mu^{\alpha}((A + \cdots c_{1+r_1-t_1+r_2} = 1 \cdots t_2 + \cdots), (+ \cdots c_{1+r_1-t_1+r_2} = 1 \cdots s_2 + \cdots)) . \end{aligned} \tag{4.28}$$

By Lemma 4.23, for (4.28) to hold either $t_2 = (r_1 - t_1) + r_2$, or $(r_1 - t_1) + r_2 > t_2 = s_2$. If $t_2 = (r_1 - t_1) + r_2$, the proposition holds. If $(r_1 - t_1) + r_2 > t_2 = s_2$, we repeat the above argument to conclude that either $t_3 = \sum_1^2 (r_k - t_k) + r_3$ or $\sum_1^2 (r_k - t_k) + r_3 > t_3 = s_3$. Using the description of τ-invariants in Lemma 1.10, we gather that if $\alpha \neq \alpha_n$, then for some index j we must have $t_j = \sum_1^{j-1} (r_k - t_k) + r_j$.

In order to complete the proof of the proposition we assume that $\alpha = \alpha_n$ is the only simple root with $\alpha \notin \tau(c)$ but $\alpha \in \tau(d)$. Assume that the size-type of c is $[1, r_1, t_1, r_2, \dots, t_{\ell-1}, r_\ell]$ and the size-type of d is $[1 + r_1, t_1, r_2, \dots, t_{\ell-1} + r_\ell]$. If $t_{\ell-1} < \sum_1^{\ell-2} (r_k - t_k) + r_{\ell-1}$, then $\mu^{\alpha}(c, d) = \mu^{\alpha_n}((A + \cdots + \cdots +), (+ \cdots + 1 \cdots r_\ell))$. This is a contradiction to part (4) of Theorem 4.1. □

COROLLARY 4.29. *Let $c = (1 + \ c')$ be a clan consisting of n symbols. Assume that $\mu(T^*_{c'}) = O^{n-2}_{n-2}$. If α is a simple root with $\alpha \notin \tau(c)$, then $\mu^{\alpha}((1 + c'), (+ + \ d')) = 0$ for all clans $(+ + \ d')$.*

PROOF. We use the algorithm that computes moment map images in [21] to conclude that if c' has $\mu(T^*_{c'}) = O^{n-2}_{n-2}$ then c' is of the form $(+ \cdots + c'_{\ell+1} = 1 \cdots t_1 + \cdots)$ with $\ell > t_1$. Write the type-size of c as $[1, r_1 = \ell + 1, t_1, r_2, \dots]$. Assume that for some clan $(+ + \ d')$, $\mu^{\alpha}((1 + c'), (+ + \ d')) \neq 0$. As $\ell + 1 > t_1$, Proposition 4.27 implies that for some index j, $t_j = \sum_1^{j-1} (r_k - t_k) = r_j$ and $r_k \geq t_k$ for all $k = 1, \dots, j-1$. It is easy to see, using the algorithm in [21], that no clan c' of size-type $[\ell, t_1, r_2, t_2 \cdots \sum_1^{j-1} (r_k - t_k) + r_j \cdots]$ has $\mu(T^*_{c'}) = O^{n-2}_{n-2}$. □

COROLLARY 4.30. *Let $c = (+ \ 1 + \ c')$ be a clan consisting of n symbols. Assume that $\mu(T^*_{c'}) = O^{n-3}_{n-3}$. If α is a simple root with $\alpha \notin \tau(c)$, then $\mu^{\alpha_1}((+1 + \ c'), (+ + + \ d')) = 0$ for all clans $(+ + + \ d')$.*

PROOF. The are two cases. Either α_1 is the only simple root not in the τ-invariant of c, or there exists a simple root $\alpha_j \neq \alpha_1$ not in $\tau(c)$. Assume $S - \tau(c) = \{\alpha_1\}$ and further assume that $\mu^{\alpha_1}((+1 + c'), (+ + + \ d')) \neq 0$. By Theorem 4.1, $(+ + \ d') = (+ \cdots +)$. This is a contradiction to Proposition 4.18. Hence, we must have $\alpha_j \neq \alpha_1$ in $S - \tau(c)$. As $\mu(T^*_{c'}) = O^{n-3}_{n-3}$, the clan c' starts with a + sign and $\alpha_j \neq \varepsilon_3 - \varepsilon_4$. The examples §3 show that the Corollary holds for $n = 4$. We proceed by induction on n to arrive to a contradiction. Let $n > 4$. Assume that $\mu^{\alpha_1}((+1 + c'), (+ + + \ d')) \neq 0$. If $\alpha_j \notin \tau((+ + d'))$, using an appropriate sequence of operators $T_{i,i-1}$, Lemma 4.4 produces smaller clans with $n - 2$ symbols $\hat{c} = (+1 + \cdots \not+ \cancel{c_{j+1} = 1} \cdots)$ and $\hat{d} = (+ + + \cdots \not+ \cancel{c_{j+1} = 1} \cdots)$ with $\mu^{\alpha_1}(\hat{c}, \hat{d}) \neq 0$. This is a contradiction to our induction hypothesis. Thus, we have $\alpha_j \neq \alpha_1$, $\alpha_j \notin \tau(c)$, and $\alpha_j \in \tau(d)$. If $\mu^{\alpha_1}((+1 + c'), (+ + + \ d')) \neq 0$, then $\mu^{\alpha_j}((+1 + c'), (+ + + \ d')) \neq 0$. By Lemma 4.4, $\mu^{\alpha_j}((1 + c'), (+ + + \ d')) \neq 0$. This is a contradiction to Corollary 4.29. □

COROLLARY 4.31. *If* $\mu^{\alpha}\,((1+c'),(++\ d'))\neq 0$, *then* $\mu(T^*_{d'})\subsetneq\mu(T^*_{c'})$.
If $\mu^{\alpha_1}\,((+1+c'),(+++\,d'))\neq 0$, *then* $\mu(T^*_{d'})\subsetneq\mu(T^*_{c'})$.

PROOF. We prove the first statement of the Corollary. The proof of the second statement is similar. Write the size type of $(1+\ c')$ as $[1,r_1,t_1,r_2,\dots,r_j,t_j,\dots]$. Similarly, let $[r_1+1,s_1,\dots]$ be the size type of $(++\ d')$. If $r_1=t_1$ and $s_1>t_1$, then we must have $s_1=t_1+1$. Indeed, by Lemma 4.5 and Lemma 4.4, $\mu^{\alpha}\,((1+c'),(++\ d'))=\mu^{\alpha}\,(\hat{c},\hat{d}\,)\neq 0$, where $\hat{c}$ is the clan obtained from $(1+\ c')$ by deleting the second and third block and $\hat{d}$ is the clan obtained from $(++\ d')$ by deleting the first r_1 numerical entries and r_1 signs at the beginning of the clan. The clan $\hat{c}$ is of the form $(1+\cdots)$ and $\hat{d}$ is of the form $(+1\cdots)$. Hence, $\hat{c}$ and $\hat{d}$ are related by an operator of the form T_{α_2,α_1}. The claim $s_1=t_1+1$ follows. The algorithm that computes moment map images proves the Corollary in this case.

When $r_1\neq t_1$, Proposition 4.27 and Proposition 4.11 guarantee that for some index j, $s_j>t_j=\sum_1^{j-1}(r_k-t_k)+r_j$. We argue that $s_j=t_j+1$. Indeed, $\mu^{\alpha}\,((1+c'),(++\ d'))=\mu^{\alpha}\,(\widetilde{c},\widetilde{d})\neq 0$, where $\widetilde{c}$ is obtained from $(1+c')$ by deleting the second through the j-th blocks and $\widetilde{d}$ is obtained from $(++\ d')$ by deleting the second through the $\sum_1^{j}(r_i+t_i)$-th entries. Note that $\widetilde{c}$ is of the form $(1+\cdots)$ and $\widetilde{d}$ is of the form $(+\ 1\cdots)$. Since $\mu^{\alpha}\,(\widetilde{c},\widetilde{d})\neq 0$, we must have $\widetilde{c}=T_{\varepsilon_1-\varepsilon_2,\varepsilon_2-\varepsilon_3}\cdot\widetilde{d}$. In particular, $s_j=t_j+1$. Moreover, the size-type of clan $(1+c')$ is $[1,\ r_1,\ t_1,\dots,t_j,\ r_{j+1},\dots,r_k,\ t_k,\dots]$ and the type-size of $(++d')$ is $[1+r_1,\ t_1,\dots,t_j+1,\ r_{j+1}-1,\dots,r_k,\ t_k,\dots]$. A careful implementation of the algorithm that computes moment map images settles the corollary. □

5. The conormal variety. In this section we compute the action of simple reflections on conormal bundles $T^*_{\mathcal{Q}}$ when $\overline{\mathcal{Q}}$ is the support of a highest weight $(\mathfrak{g},K)$ module.

5.1. We summarize known results on the action of the Weyl group W on the conormal variety $T^*_K\mathfrak{B}$. We begin by proving a refined version of Theorem 1.19.

PROPOSITION 5.1. *Let* $\mathcal{Q}$ *be a* K*-orbit in* $\mathcal{B}$. *Assume that* $\overline{\mathcal{Q}}$ *is the support of a highest weight* $(\mathfrak{g},K)$*-module,* $X(\mathcal{Q})$. *If* $\alpha\notin\tau_{weak}(\mathcal{Q})$, *then*

$$s_\alpha\cdot T^*_{\mathcal{Q}}=T^*_{\mathcal{Q}}+\ T^*_{s_\alpha\circ\mathcal{Q}}+\sum_{\substack{\mathcal{Q}_j\subset\overline{\mathcal{Q}},\ \alpha\in\tau_{weak}(\mathcal{Q}_j)\\ \mu(T^*_{\mathcal{Q}_j})=\mu(T^*_{\mathcal{Q}})}} m^{\alpha}(\mathcal{Q},\mathcal{Q}_j)\,T^*_{\mathcal{Q}_j}\,.$$

Moreover, if $\gamma\in\tau_{weak}(\mathcal{Q})$ *is perpendicular to* α, *then* $\gamma\in\tau_{weak}(\mathcal{Q}_j)$ *for each* $\mathcal{Q}_j$ *with* $m^{\alpha}(\mathcal{Q},\mathcal{Q}_j)\neq 0$.

PROOF. The first statement is Theorem 1.19. We need to show that for each j, $\tau_{weak}(\mathcal{Q})\cap\{\alpha\}^{\perp}\subset\tau_{weak}(\mathcal{Q}_j)$. By [3], we have

$$CC(X(\mathcal{Q}))=T^*_{\mathcal{Q}}+\sum_{\substack{\mathcal{Q}_\ell\subset\overline{\mathcal{Q}},\\ \tau_{weak}(\mathcal{Q})\subset\tau_{weak}(\mathcal{Q}_\ell)}} T^*_{\mathcal{Q}_\ell}\,.$$

W-equivariance of the characteristic cycle functor implies

$$s_\alpha\cdot T^*_{\mathcal{Q}}=CC(s_\alpha\cdot X(\mathcal{Q}))-\sum s_\alpha\cdot T^*_{\mathcal{Q}_\ell}\,,$$

where

$$s_\alpha \cdot T^*_{\mathcal{Q}_\ell} = \begin{cases} -\, T^*_{\mathcal{Q}_\ell} & \text{if } \alpha \in \tau_{weak}(\mathcal{Q}_\ell) \\ \sum_t n_{\ell,t}\, T^*_{\mathcal{Q}_{\ell_t}} & \text{if } \alpha \notin \tau_{weak}(\mathcal{Q}_\ell)\,, \end{cases}$$

for some positive integers $n_{\ell,t}$.

Thus,

$$s_\alpha \cdot T^*_{\mathcal{Q}} = CC(s_\alpha X(\mathcal{Q})) + \sum_{\mathcal{Q}_\ell : \alpha \in \tau_{weak}(\mathcal{Q}_\ell)} T^*_{\mathcal{Q}_\ell} - \sum_{(\ell,t):\alpha \notin \tau_{weak}(\mathcal{Q}_\ell)} n_{\ell,t}\, T^*_{\mathcal{Q}_{\ell_t}}\,.$$

Theorem 1.19, asserts that $s_\alpha \cdot T_{\mathcal{Q}}$ is a positive integer combination of conormal bundles. Hence, $\sum_{(\ell,t):\alpha\notin\tau_{weak}(\mathcal{Q}_\ell)} n_{\ell,t}\, T^*_{\mathcal{Q}_{\ell_t}}$ is cancelled out by $CC(s_\alpha \cdot X(\mathcal{Q}))$.

Let $\gamma \in \tau_{weak}(\mathcal{Q}) \cap \{\alpha\}^\perp$. The inclusion $\tau_{weak}(\mathcal{Q}) \subset \tau_{weak}(\mathcal{Q}_\ell)$ implies that $\gamma \in \tau_{weak}(\mathcal{Q}_\ell)$ for each ℓ. On the other hand, since in our context $\tau(X(\mathcal{Q})) = \tau_{weak}(\mathcal{Q})$, Theorem 4.1 gives

$$s_\alpha \cdot X(\mathcal{Q}) = X(\mathcal{Q}) + X(s_\alpha \circ \mathcal{Q}) + \sum_{\substack{\mathcal{Q}_i \subset \overline{\mathcal{Q}} \\ \alpha,\gamma \in \tau(X(\mathcal{Q}_i))}} \mu^\alpha(\mathcal{Q}, \mathcal{Q}_i)\, X(\mathcal{Q}_i)\,. \tag{5.2}$$

It follows form identity (5.2) and [3] that the conormal bundles contributing to the characteristic cycles of the modules on the right hand side of (5.2) contain γ in their weak τ-invariant. The proposition follows. □

THEOREM 5.3. *Let $X(\mathcal{Q}')$ be a highest weight (G', K') module. Write c' for the clan that parametrizes $\mathcal{Q}'$. Let $\alpha \in \Delta(\mathfrak{g}', \mathfrak{h}')$ be simple with $\alpha \notin \tau(c')$. Assume*

$$s_\alpha \cdot T^*_{c'} = \sum_i m^\alpha(c', c_i)\, T^*_{c_i}\,.$$

Then $\alpha \notin \tau((+\, c'))$ and

$$s_\alpha \cdot T^*_{+\,c'} = \sum_i m^\alpha(c', c_i)\, T^*_{+\,c_i}\,.$$

PROOF. By Corollary 2.4, there exist integers n_ℓ so that

$$T^*_{c'} = \sum_\ell n_\ell\, CC(X(c'_\ell))$$
$$T^*_{+\,c'} = \sum_\ell n_\ell\, CC(X(+\, c'_\ell))\,,$$

where $X(c'_\ell)$ and $X(+\, c'_\ell)$ are highest weight Harish-Chandra modules. The W-equivariance of the characteristic cycle functor yields;

$$s_\alpha \cdot T^*_{c'} = \sum_\ell n_\ell\, CC(s_\alpha \cdot X(c'_\ell)) = \sum_i m^\alpha(c', c_i)\, T^*_{c_i} \tag{5.4}$$
$$s_\alpha \cdot T^*_{+\,c'} = \sum_\ell n_\ell\, CC(s_\alpha \cdot X(+\, c'_\ell))\,.$$

When $\alpha \notin \tau(c'_\ell)$, by Theorem 4.1 and Lemma 4.4, we have

$$s_\alpha \cdot X(c'_\ell) = X(c'_\ell) + \sum_t \mu^\alpha(c'_\ell, c'_t)\, X(c'_t)\,,$$

$$s_\alpha \cdot X(+\, c'_\ell) = X(+\, c'_\ell) + \sum_t \mu^\alpha(c'_\ell, c'_t)\, X(+\, c'_t)\,. \tag{5.5}$$

Thus,

$$\begin{aligned} s_\alpha \cdot T^*_{c'} &= \sum_i m^\alpha(c', c_i)\, T^*_{c_i} \\ &= -\sum_{\substack{\ell \\ \alpha \in \tau(c'_\ell)}} n_\ell\, CC(X(c'_\ell)) + \sum_{\substack{(\ell,t) \\ \alpha \notin \tau(c'_\ell)}} n_\ell\, \mu^\alpha(c'_\ell, c'_t)\, CC\,(X(c'_t))\ . \end{aligned}$$

$$\begin{aligned} &s_\alpha \cdot T^*_{+\, c'} \\ &\quad = -\sum_{\substack{\ell \\ \alpha \in \tau(c'_\ell)}} n_\ell\, CC(X(+\, c'_\ell)) + \sum_{\substack{(\ell,t) \\ \alpha \notin \tau(c'_\ell)}} n_\ell\, \mu^\alpha(c'_\ell, c'_t)\, CC\,(X(+\, c'_t))\ . \end{aligned}$$

Since for each conormal bundle T^*_d that contributes to $CC(X(c'_t))$, $T^*_{(+\, d)}$ contributes to $CC\,(X(+\, c'_t))$, the theorem follows from (5.4), (5.5), and part (1) of Theorem 2.3. □

THEOREM 5.6. *Let $X(\mathfrak{Q}')$ be a highest weight (G', K') module. Write c' for the clan that parametrizes $\mathfrak{Q}'$. Let $\alpha \in \Delta(\mathfrak{g}', \mathfrak{h}')$ be simple with $\alpha \notin \tau(c')$. Assume*

$$s_\alpha \cdot T^*_{c'} = \sum_i m^\alpha(c', c_i)\, T^*_{c_i}\,.$$

Then $\alpha \notin \tau((1\ c'))$ and each conormal bundle $T^*_{1\ c_i}$ occurs in $s_\alpha \cdot T^*_{1\ c'}$. Moreover, the coefficient $m^\alpha((1\ c'), (1\ c_i))$ of $T^*_{1\ c_i}$ in $s_\alpha \cdot T^*_{1\ c'}$ equals $m^\alpha(c', c_i)$.

PROOF. The argument is similar to the one used in the proof of Theorem 5.3. By Corollary 2.4, there exist integers n_ℓ so that

$$T^*_{c'} = \sum_\ell n_\ell\, CC(X(c'_\ell)) \tag{5.7}$$

$$T^*_{1\ c'} = \begin{cases} \sum_\ell n_\ell\, CC(X(1\ c'_\ell)) & \text{if } (1\ c') \in C_{\mathfrak{g}}(k), k < n-1\,, \\ \sum_\ell n_\ell\, CC(X(1\ c'_\ell)) & \text{if } (1\ c') \in C_{\mathfrak{g}}(n-1),\ n \text{ odd}, \\ \sum_\ell n_\ell\, [CC(X(1\ c'_\ell)) - CC(X(+\, c'_\ell))] & \text{if } (1\ c') \in C_{\mathfrak{g}}(n-1),\ n \text{ even}; \end{cases}$$

where for each ℓ, $X(+\, c'_\ell)$ and $X(1\ c'_\ell)$ are highest weight $(\mathfrak{g}, K)$ modules. Since the characteristic cycle functor is W-equivariant, we have

$$s_\alpha \cdot T^*_{c'} = \sum_\ell n_\ell\, CC(s_\alpha \cdot X(c'_\ell)) = \sum_i m(c', c_i)\, T^*_{c_i} \tag{5.8}$$

$$s_\alpha \cdot T^*_{1\ c'} = \begin{cases} \sum_\ell n_\ell\, CC(s_\alpha \cdot X(1\ c'_\ell)) \text{ or }, \\ \sum_\ell n_\ell\, [CC(s_\alpha \cdot X(1\ c'_\ell)) - CC(s_\alpha \cdot X(+\, c'_\ell))]\,. \end{cases}$$

When $\alpha \notin \tau(c'_\ell)$, the coherent continuation formulae in Theorem 4.1, Lemma 4.4 and Proposition 4.13 imply

$$s_\alpha \cdot X(c'_\ell) = X(c'_\ell) + \sum_{t_\ell} \mu^\alpha(c'_\ell, c_{t_\ell})\, X(c_{t_\ell}),$$

$$s_\alpha \cdot X(1\ c'_\ell) = X(1\ c'_\ell) + \sum_{t_\ell} \mu^\alpha((\not{1}\ c'_\ell), (\not{1}\ c_{t_\ell}))\, X(1\ c_{t_\ell}) + \mu^\alpha((1\ c'_\ell), (+\ c_{r_\ell}))\, X(+\ c_{r_\ell}),$$

$$s_\alpha \cdot X(+\ c'_\ell) = X(+\ c'_\ell) + \sum_{t_\ell} \mu^\alpha(c'_\ell, c_{t_\ell})\, X(+\ c_{t_\ell}).$$

In particular, $X(1\ c_{t_\ell})$ contributes to $s_\alpha \cdot X(1\ c'_\ell)$ if and only if $X(c_{t_\ell})$ occurs in $s_\alpha \cdot X(c'_\ell)$ and $\mu^\alpha((1\ c'_\ell), (1\ c_{t_\ell})) = \mu^\alpha(c'_\ell, c_{t_\ell}) = \mu^\alpha((+\ c'_\ell), (+\ c_{t_\ell}))$.

On the other hand, it is important to observe that the conormal bundles contributing to $CC\,(X(+\ c_{r_\ell}))$ or $CC\,(X(+\ c_{t_\ell}))$ are parametrized by clans that start with + sign. Moreover, by Theorem 2.3, a conormal bundle $T^*_{(1\ d)}$ contributes to $CC\,(X(1\ c_{t_\ell}))$ if and only if T^*_d contributes to $CC\,(X(c_{t_\ell}))$ and their multiplicity are both equal to one. The theorem follows. □

REMARK 5.9. Theorem 5.6 does not provide a formula for $s_\alpha \cdot T^*_{1\ c'}$. Our example in §3.4 shows that conormal bundles of the form $T^*_{+\ +\ d'}$ might contribute to $s_\alpha \cdot T^*_{1\ c'}$. Such occurrence is not accounted for in Theorem 5.6.

5.2. In this section we determine the action of simple reflections on conormal bundles of the form $T^*_{1\ c'}$. Some of the proofs require the study of various subcases. In occasions, and due to space considerations, we have included complete arguments for some subcases and gave enough information for the reader to produce the argument that settle the left easier cases.

THEOREM 5.10. *Let $c = (1+\ c')$ be a clan consisting of n symbols and so that $\mu(T^*_{c'}) = O^{n-2}_{n-2}$. Let $\alpha \in \Delta(\mathfrak{g}', k')$ be a simple root so that $\alpha \notin \tau\,(c')$. If*

$$s_\alpha \cdot T^*_{(+\ c')} = \sum_i m^\alpha((+\ c'), c_i)\, T^*_{c_i},$$

then

$$s_\alpha \cdot T^*_{(1\ +\ c')} = \sum_i m^\alpha((+\ c'), c_i)\, T^*_{(1\ c_i)}.$$

PROOF. Write

$$T^*_{(+\ c')} = \sum_\ell n_\ell\, CC\,(X(+\ d_\ell)) \text{ with } n_\ell \in \mathbb{Z},$$

as prescribed by Corollary 2.4. By Theorem 2.3, $CC\,(X(c')) = LTC\,(X(c'))$. It follows that, for each ℓ, $\mu(T^*_{d_\ell}) = O^{n-2}_{n-2}$. Thus, each clan d_ℓ has a + as its first symbol and $\mu(T^*_{(+\ d_\ell)}) = O^{n-1}_{n-1}$. Corollary 2.4 gives

$$T^*_{(1+\ c')} = \begin{cases} \sum_\ell n_\ell\, CC\,(X(1\ +\ d_\ell)) & \text{when } n \text{ is odd} \\ \sum_\ell n_\ell\, [CC\,(X(1\ +\ d_\ell)) - CC\,(X(+\ +\ d_\ell))] & \text{when } n \text{ is even.} \end{cases}$$

As the characteristic cycle functor is W equivariant, in order to compute $s_\alpha \cdot T^*_{(1+\ c')}$ it is necessary to understand the coherent continuation action of s_α on $X(1+d_\ell)$ and $X(++\ d_\ell)$. On the one hand we know, by Corollary 4.29, that every irreducible module occurring in $s_\alpha \cdot X(1+d_\ell)$ is parametrized by a clan that starts with 1. On the other hand, the shape of the clans d_ℓ and (2) of Theorem 4.1 imply that the irreducible summands of $s_\alpha \cdot X(1+\ d_\ell)$ are parametrized by clans of the form $(1 + \cdots)$. Moreover, by Lemma 4.4, $\mu^\alpha\ ((1+\ d_\ell), (1+\ d)) \neq 0$ if and only if $\mu^\alpha\ ((+\ d_\ell), (+\ d)) \neq 0$. When this is the case, we have

$$\begin{aligned}(5.11)\qquad \mu^\alpha\ ((1\ +\ d_\ell), (1+\ d)) &= \mu^\alpha\ ((+\ d_\ell), (+d)) = \mu^\alpha\ (d_\ell, d)\\ &= \mu^\alpha\ ((++\ d_\ell), (++\ d))\ .\end{aligned}$$

In particular, we know that $X(1+\ d)$ contributes to $s_\alpha \cdot X(1+\ d_\ell)$ if and only if $X(d)$ contributes to $s_\alpha \cdot X(d_\ell)$, if and only if $X(++\ d)$ contributes to $s_\alpha \cdot X(++\ d_\ell)$, if and only if $X(+\ d)$ contributes to $s_\alpha \cdot X(+\ d_\ell)$.

In view of Theorem 2.3, in order to compare the characteristic cycles of the relevant modules we must keep track of $\mu(T^*_{(+\ d)})$. By (1) of Theorem 4.1, we have

$$\mu(T^*_{(+\ d)}) \subseteq \mathrm{AV}\,(X(+\ d)) \subseteq \mathrm{AV}\,(X(+\ d_\ell)) = \overline{O^{n-1}_{n-1}}\ .$$

When $d = s_\alpha \circ d_\ell$, by Lemma 1.18, $\mu(T^*_d) \in O^{n-2}_{n-2} \cup O^{n-2}_{n-3}$. Hence , in this case we have $\mu(T^*_{(+\ d)}) = O^{n-1}_{n-1}$. When $d \neq s_\alpha \circ d_\ell$, the orbit closure inclusion $\mathcal{Q}_d \subset \overline{\mathcal{Q}_{d_\ell}}$ implies $O^{n-2}_{n-2} \subset \mu(T^*_d)$, see [3, Prop 38]. Hence, for each such clan, $O^{n-1}_{n-1} = \mu(T^*_{(+\ d)})$.

The theorem follows form (5.11) and the computation of characteristic cycles in Theorem 2.3. □

Proposition 5.12. *Let $c = (1\ 2 \cdots r +\ c')$ be a clan consisting of n symbols where $r \geq 2$. Assume $\alpha \in \Delta(\mathfrak{g}', \mathfrak{h}')$ is simple and $\alpha \notin \tau\ ((2 \cdots r +\ c'))$. If*

$$s_\alpha \cdot T^*_{(2\cdots r+\ c')} = \sum_i m^\alpha((2\cdots r +\ c'), c_i)\ T^*_{c_i}\,,$$

then

$$s_\alpha \cdot T^*_{(1\ 2\cdots r+\ c')} = \sum_i m^\alpha((2\cdots r +\ c'), c_i)\ T^*_{(1\ c_i)}\,.$$

By Theorem 5.6, each conormal bundle $T^*_{(1\ c_i)}$ occurs in $s_\alpha \cdot T^*_{(1\ 2\cdots r+\ c')}$ with multiplicity $m^\alpha((2\cdots r +\ c'), c_i)$. The content of Proposition 5.12 is that no other conormal bundle contributes to $s_\alpha \cdot T^*_{(1\ 2\cdots r+\ c')}$.

Proof. We prove the proposition when $r = 2$, n is odd and $(+\ c') \in \mathcal{C}_{\mathfrak{sp}(2(n-2))}(n-2)$. Other cases are easier to handle by similar techniques.

Assume $r = 2$, n is odd and $\mu(T^*_{(+\ c')}) = O^{n-2}_{n-2}$. We start the proof with an argument similar to the one used in the proof of Theorem 5.10. In particular, we observe that under our assumptions, Theorem 2.3, Corollary 2.4 and W-equivariance of CC give,

$$s_\alpha \cdot T^*_{2\ +\ c'} = \sum_\ell n_\ell\ [CC(s_\alpha \cdot X(2\ +c'_\ell)) - CC(s_\alpha \cdot X(+\ +\ c'_\ell))] \tag{5.13}$$
$$s_\alpha \cdot T^*_{1\ 2+\ c'} = \sum_\ell n_\ell\ [CC(s_\alpha \cdot X(1\ 2+c'_\ell)) - CC(s_\alpha \cdot X(1\ \ ++c'_\ell))],$$

where for each $\ell, n_\ell \in \mathbb{Z}$ and $(+\ c'_\ell) \in C_{\mathfrak{sp}(2(n-2))}(n-2)$. Observe that $(2+\ c'_\ell) \in C_{\mathfrak{sp}(2(n-1))}(n-2)$.

The formulas for coherent continuation in Theorem 4.1, Lemma 4.4 and Lemma 4.14 (a) imply that

$$s_\alpha \cdot X(1\ 2+c'_\ell) = X(1\ 2+c'_\ell) + \sum_j \mu^\alpha((2\ +c'_\ell), d_{\ell_j})\ X(1\ d_{\ell_j}), \tag{5.14}$$

when $\alpha \notin \tau((2\ +\ c'_\ell))$. In particular, $X(1\ d_{\ell_j})$ contributes to $s_\alpha \cdot X(1\ 2+c'_\ell)$ if and only if $X(d_{\ell_j})$ contributes to $s_\alpha \cdot X(2+c'_\ell)$. Hence, $\mu(T^*_{d_{\ell_j}}) \subseteq \mathrm{AV}(X(2+c'_\ell)) = \overline{\mathcal{O}^{n-1}_{n-1}}$.

Similarly, Theorem 4.1, Lemma 4.4 and Corollary 4.29 yield

$$s_\alpha \cdot X(1\ \ ++c'_\ell) = X(1\ \ ++c'_\ell) + \sum_j \mu^\alpha((+\ +c'_\ell), r_{\ell_j})\ X(1\ r_{\ell_j}), \tag{5.15}$$

with $\mu(T^*_{r_{\ell_j}}) \subseteq \mathcal{O}^{n-1}_{n-1}$. Note that every irreducible module in the right hand side of equation (5.15) is parametrized by a clan that starts with 1. This statement is proved in Corollary 4.29, as $\mu(T^*_{(+\ c'_\ell)}) = \mathcal{O}^{n-2}_{n-2}$.

Combining (5.13), (5.14) and (5.15) we obtain,

(5.16)
$$\begin{aligned} s_\alpha \cdot T^*_{(2\ +\ c')} &= \sum_i m^\alpha((2+\ c'), c_i)\ T^*_{c_i} \\ &= -\sum_{\substack{\ell \\ \alpha\in\tau((+\ c'_\ell))}} n_\ell CC(X(2\ +c'_\ell)) + \sum_{\substack{(\ell,j) \\ \alpha\notin\tau((+\ c'_\ell))}} n_\ell\ \mu^\alpha((2\ +c'_\ell), d_{\ell_j}) CC(X(d_{\ell_j})) \\ &\quad + \sum_{\substack{\ell \\ \alpha\in\tau((+\ c'_\ell))}} n_\ell\ CC(X(+\ +c'_\ell)) - \sum_{\substack{(\ell,j) \\ \alpha\notin\tau((+\ c'_\ell))}} n_\ell\ \mu^\alpha((+\ +c'_\ell), r_{\ell_j}) CC(X(r_{\ell_j}))\,. \end{aligned}$$

Similarly,

(5.17)
$$\begin{aligned} s_\alpha &\cdot T^*_{1\ 2+\ c'} \\ &= -\sum_{\substack{\ell \\ \alpha\in\tau((+\ c'_\ell))}} n_\ell CC(X(1\ 2\ +c'_\ell)) + \sum_{\substack{(\ell,j) \\ \alpha\notin\tau((+\ c'_\ell))}} n_\ell \mu^\alpha((2\ +c'_\ell), d_{\ell_j}) CC(X(1\ d_{\ell_j})) \\ &\quad + \sum_{\substack{\ell \\ \alpha\in\tau((+\ c'_\ell))}} n_\ell CC(X(1\ +\ +c'_\ell)) - \sum_{\substack{(\ell,j) \\ \alpha\notin\tau((+\ c'_\ell))}} n_\ell \mu^\alpha((+\ +c'_\ell), r_{\ell_j}) CC(X(1\ r_{\ell_j}))\,. \end{aligned}$$

It is important to observe that

$$\mu(T^*_{(++\ c'_\ell)}) = \mathcal{O}^{n-1}_{n-1} \text{ where } n-1 \text{ is even}$$

$$\mu(T^*_{d_{\ell_j}}), \mu(T^*_{r_{\ell_j}}) = \begin{cases} \mathcal{O}^{n-1}_k & \text{for some } k < n-1, \text{or} \\ \mathcal{O}^{n-1}_{n-1} & \text{where } n-1 \text{ is even,} \end{cases}$$

as by Theorem 2.3 (b) and (c) T^*_d contributes to $CC(X(1\ d_{\ell_j}))$ $(CC(X(1\ r_{\ell_j})))$ if and only if d is of the form $(1\ d')$ and $T^*_{d'}$ occurs in $CC(X(d_{\ell_j}))$ $(CC(X(r_{\ell_j})))$, resp. Now, the proposition follows from (5.16), (5.17) and Theorem 2.3. □

In the following theorem we view $G' \simeq \mathrm{Sp}(2(n-1))$ and $G_i \simeq \mathrm{Sp}(2(n-2))$ as subgroups of $G = \mathrm{Sp}(2n)$ as in Section 1. The group $\hat{G}_i \simeq \mathrm{Sp}(2(n-3))$ is embedded into G so that the Cartan subalgebra is $\hat{h} = \{H \in \mathfrak{h} : \varepsilon_1(H) = \varepsilon_i(H) = \varepsilon_{i+1}(H) = 0\}$, and $\Delta(\hat{\mathfrak{g}_i}, \hat{\mathfrak{h}_i}) = \{\alpha \in \Delta : \langle \alpha, \varepsilon_1 \rangle = \langle \alpha, \varepsilon_i \rangle = \langle \alpha, \varepsilon_{i+1} \rangle = 0\}$.

It is useful to observe that:

Simple roots in $\Delta(\mathfrak{g}, \mathfrak{h}) = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, 2\varepsilon_n\}$,

Simple roots in $\Delta(\mathfrak{g}', \mathfrak{h}') = \{\varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \ldots, 2\varepsilon_n\}$,

Simple roots in $\Delta(\mathfrak{g}_i, \mathfrak{h}_i) = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{i-2} - \varepsilon_{i-1}, \varepsilon_{i-1} - \varepsilon_{i+2}, \varepsilon_{i+2} - \varepsilon_{i+3}, \ldots, 2\varepsilon_n\}$,

Simple roots in $\Delta(\hat{\mathfrak{g}_i}, \hat{\mathfrak{h}_i}) = \{\varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{i-2} - \varepsilon_{i-1}, \varepsilon_{i-1} - \varepsilon_{i+2}, \varepsilon_{i+2} - \varepsilon_{i+3}, \ldots, 2\varepsilon_n\}$.

In particular, each root $\alpha = \varepsilon_j - \varepsilon_{j+1}$ with $j \geq i+2$ is simple for $\Delta^+(\mathfrak{g}, \mathfrak{h})$, $\Delta^+(\mathfrak{g}', \mathfrak{h}')$, $\Delta^+(\mathfrak{g}_i, \mathfrak{h}_i)$ and $\Delta^+(\hat{\mathfrak{g}_i}, \hat{\mathfrak{h}_i})$.

THEOREM 5.18. *Let $c = (1 +\ d)$ be a clan consisting of n symbols. Assume $\mu(T^*_d) = \mathcal{O}^{n-2}_k$ with $k \leq (n-3)$. Write $c = (1+\ d) = (1 + (+ \cdots + c_{i+1} = 2\ c'))$ where $i \geq 2$. Let $\alpha = \varepsilon_j - \varepsilon_{j+1}$ for some $j \geq i+2$ with $\alpha \notin \tau(c')$. Then,*

$$\alpha \notin \tau(c) \cup \tau\left((+\ d)\right) \cup \tau\left((1 + (+ \cdots \not{+}\, \not{c}_{i+1}\ c'))\right) .$$

Moreover, the following statements hold.

(1) *A conormal bundle of the form $T^*_{(1\ \omega)}$ contributes to $s_\alpha \cdot T^*_{(1+\ d)}$ if and only if T^*_ω contributes to $s_\alpha \cdot T^*_{(+\ d)}$. When this is the cases, we have*

$$m^\alpha\left((1+\ d), (1\ \omega)\right) = m^\alpha\left((+\ d), (\omega)\right) .$$

(2) *A conormal bundle of the form $T^*_{(+\ \omega)}$ contributes to $s_\alpha \cdot T^*_{(1+\ d)}$ if and only if $(+\ \omega) = (+ \cdots + w_{i+1}\ =\ 1\ \omega')$ and the conormal bundle $T^*_{(+ \cdots \not{+}\, \not{w}_{i+1}\ \omega')}$ contributes to $s_\alpha \cdot T^*_{\left(1+(+\cdots\not{+}\, \not{w}_{i+1}\ c')\right)}$. When this is the cases*

$$m^\alpha\left((1+d), (+\ \omega)\right) = m^\alpha\left((1 + + \cdots \not{+}\, \not{c}_{i+1}\ c'), (+ \cdots \not{+}\, \not{w}_{i+1}\ \omega')\right) .$$

Statement (1) of Theorem 5.18 is Theorem 5.6. When proving statement (2), in order to simplify the exposition, we will assume that $i = 2$. An identical argument settles the theorem when $i > 2$. Hence, we take $c = (1 +\ 2\ c')$ and consider two cases.

CASE I. $\mu(T^*_{c'}) = \mathcal{O}^{n-3}_k$ with $k < (n-3)$.

CASE II. $\mu(T^*_{c'}) = \mathcal{O}^{n-3}_k$ with $k = (n-3)$.
The remainder of this section is devoted to the proof of this theorem.

PROPOSITION 5.19. *Let $c = (1 +\ 2\ c')$ be a clan consisting of n symbols. Assume that $\mu(T^*_{c'}) = \mathcal{O}^{n-3}_k$ with $k < (n-3)$. Let $\alpha = \varepsilon_j - \varepsilon_{j+1}$ for some $j \geq 4$ with $\alpha \notin \tau(c')$. Then, $\alpha \notin \tau((1 +\ 2\ c')) \cup \tau((1 \not+ \not2\ c'))$. Moreover, a conormal bundle of the form $T^*_{(+\ \omega)}$ contributes to $s_\alpha \cdot T^*_{(1+\ 2\ c')}$ if and only if $(+\ \omega) = (+ +\ 1\ \omega')$ and $T^*_{(+\not+\not1\ \omega')}$ contributes to $s_\alpha \cdot T^*_{(1\not+\not2\ c')}$. When this is the cases*

$$m^\alpha\left((1 + 2\ c'), (+ +\ 1\ \omega)\right) = m^\alpha\left((1\not+\not2\ c'), (+\not+\not1\ \omega')\right).$$

PROOF. The description of characteristic cycles of highest weight $(\mathfrak{g}.K)$ modules given in Theorem 2.3 and its Corollary 2.4 allows us to find integers n_ℓ such that

$$T^*_{c'} = \sum_\ell n_\ell\ CC(X_{d_\ell}), \text{ where } \mu(T^*_{d_\ell}) = \begin{cases} \mathcal{O}^{n-3}_k & \text{when } k \text{ is even,} \\ \mathcal{O}^{n-3}_k, \text{ or } \mathcal{O}^{n-3}_{k+1} & \text{when } k \text{ is odd.} \end{cases}$$

It is then easy to show that Theorem 2.3 yield,

$$T^*_{(1\ c')} = \sum_\ell n_\ell\ CC\left(X(1\ d_\ell)\right)$$
$$T^*_{(1+\ 2\ c')} = \sum_\ell n_\ell\ CC\left(X(1 +\ 2\ d_\ell)\right),$$

where

$$\mu(T^*_{(1\ d_\ell)}) = \begin{cases} \mathcal{O}^{n-2}_k & \text{when } k \text{ is even,} \\ \mathcal{O}^{n-2}_k, \text{ or } \mathcal{O}^{n-2}_{k+1} & \text{when } k \text{ is odd;} \end{cases} \tag{5.20}$$

$$\mu(T^*_{(1+\ 2\ d_\ell)}) = \begin{cases} \mathcal{O}^{n}_{k+2} & \text{when } k \text{ is even,} \\ \mathcal{O}^{n}_{k+2}, \text{ or } \mathcal{O}^{n}_{k+3} & \text{when } k \text{ is odd.} \end{cases}$$

Thus,

$$s_\alpha \cdot T^*_{(1+\ 2\ c')} = \sum_\ell n_\ell\ CC\left(s_\alpha \cdot X(1 +\ 2\ d_\ell)\right).$$

When $\alpha \notin \tau(d_\ell)$, by Theorem 4.1 and Lemma 4.4 , $CC(s_\alpha \cdot X_{(1+\ 2\ d_\ell)})$ equals

$$CC(X_{(1+\ 2\ d_\ell)}) + \sum_j \mu^\alpha\left((+\ 2\ d_\ell), (+\ d_{\ell_j})\right)\ CC(X(1 +\ d_{\ell_j}))$$
$$+ \mu^\alpha\left((1\not+\not2\ d_\ell), (+\not+\not1\ c_{\ell_r})\right)\ CC(X(+ +\ 1\ c_{\ell_r})).$$

In particular, $X(1 +\ d_{\ell_j})$ contributes to $s_\alpha \cdot X(1 +\ 2\ d_\ell)$ if and only if $X(+\ d_{\ell_j})$ contributes to $s_\alpha \cdot X(+\ 2\ d_\ell)$. By (1) of Theorem 4.1,

$$\mu(T^*_{(+\ d_{\ell_j})}) \subseteq \mathrm{AV}(X(+\ 2\ d_{\ell_j})) \subseteq \mathrm{AV}(X(+\ 2\ d_\ell)) = \begin{cases} \overline{\mathcal{O}^{n-1}_{k+2}} & \text{when } k \text{ is even,} \\ \overline{\mathcal{O}^{n-1}_{k+3}} & \text{when } k \text{ is odd.} \end{cases} \tag{5.21}$$

Inclusion (5.21) and Theorem 2.3 imply that each conormal bundle that occurs in $CC(X(1+\ d_{\ell_j}))$ is parametrized by a clan of the form $(1\cdots)$. Similarly, the module $X(++\ 1\ c_{\ell_r})$ occurs in $s_\alpha \cdot X(1+\ 2\ d_\ell)$ if and only if $X(+\ c_{\ell_r})$ occurs in $s_\alpha \cdot X(1\ d_\ell)$. Hence,

$$\mu(T^*_{(+\ c_{\ell_r})}) \subseteq \operatorname{AV}\left(X(+\ c_{\ell_r})\right) \subseteq \operatorname{AV}\left(X(1\ d_\ell)\right) = \begin{cases} \overline{O^{n-2}_{k}} & \text{when } k \text{ is even,} \\ \overline{O^{n-2}_{k+1}} & \text{when } k \text{ is odd.} \end{cases} \tag{5.22}$$

Since $k+1 < n-2$, inclusion (5.22) and Theorem 2.3 imply that each conormal bundle that occurs in $CC\left((X(++\ 1\ c_{\ell_r})\right)$ is parametrized by a clan of the form $(++\ 1\ \omega')$. This settles the first statement of the proposition.

It is important to note that $T^*_{(++\,1\,\omega')}$ contributes to $s_\alpha \cdot T^*_{(1+\,2\,c')}$ if and only if it occurs in

$$\sum_{\substack{\ell_r \\ \alpha\notin\tau(d_\ell)}} n_\ell\ \mu^\alpha\left((1\not+\not2\ d_\ell), (+\not+\not1\ c_{\ell_r})\right)\ CC(X(++\ 1\ c_{\ell_r}))\,. \tag{5.23}$$

In order to prove the second statement of the proposition we need to compare $s_\alpha \cdot T^*_{(1+\,2\,c')}$ to $s_\alpha \cdot T^*_{(1\ c')} = \sum_\ell n_\ell\ CC\left(s_\alpha \cdot X(1\ d_\ell)\right)$. When $\alpha \notin \tau(d_\ell)$, by Lemma 4.5 and Lemma 4.4, we have

$$\begin{aligned} CC\left(s_\alpha \cdot X(1\ d_\ell)\right) = X(1\ d_\ell) + &\sum_i \mu^\alpha\left((1\ d_\ell), (1\ r_i)\right)\ CC\left(X(1\ r_i)\right) \\ &+ \mu^\alpha\left((1\not+\not2\ d_\ell), (+\not+\not1\ c_{\ell_r})\right)\ CC(X(+\not+\not1\ c_{\ell_r}))\,. \end{aligned}$$

We observe that $\mu(T^*_{(1\ r_i)}) \subset \operatorname{AV}\left(X(1\ d_\ell)\right)$. It follows from (5.20) and Theorem 2.3 that each conormal bundle that contributes to $CC\left(X(1\ r_i)\right)$ is parametrized by a clan of the form $(1\cdots)$. Moreover, a conormal bundle of the form $T^*_{(+\cdots)}$ occurs in $s_\alpha \cdot T^*_{(1\ c')}$ if and only if it contributes to

$$\sum_{\substack{\ell_r \\ \alpha\notin\tau(d_\ell)}} n_\ell\ \mu^\alpha\left((1\not+\not2\ d_\ell), (+\not+\not1\ c_{\ell_r})\right)\ CC(X(+\ c_{\ell_r}))\,. \tag{5.24}$$

By Theorem 2.3 and (5.22), $T^*_{(++\,1\,\omega')}$ contributes to $CC\left(X(++\ 1\ c_{\ell_r})\right)$ if and only if $T^*_{(+\,\omega')}$ contributes to $CC\left(X(+\ c_{\ell_r})\right)$. Now the proposition follows from (5.23) and (5.24). □

Proof of Theorem 5.18. Let $c = (1+\ 2\ c')$ and assume that $\mu(T^*_{c'}) = O^{n-3}_{n-3}$. When n is odd, the argument used in the proof of Proposition 5.19 settles the theorem. Assume n is even. Write

$$T^*_{c'} = \sum_\ell n_\ell\ CC(X_{d_\ell}), \text{ where } \mu(T^*_{d_\ell}) = O^{n-3}_{n-3}\,.$$

By Corollary 2.4 we have

$$T^*_{(1\ c')} = \sum_\ell n_\ell\ [CC\,(X(1\ d_\ell)) - CC\,(X(+\ d_\ell))],$$

$$T^*_{(1+\ 2\ c')} = \sum_\ell n_\ell\ [CC\,(X(1+\ 2\ d_\ell)) - CC\,(X(+\,+\ 1\ d_\ell))]$$
$$- \sum_\ell n_\ell\ [CC\,(X(1+\ +\ d_\ell)) - CC\,(X(+\,+\ +\ d_\ell))]\,.$$

If $\alpha \notin \tau(d_\ell)$, then Theorem 4.1 and Corollary 4.29 imply that

$$s_\alpha \cdot X(1\ +\ +\ d_\ell) = X(1+\ +\ d_\ell) + \sum_i \mu^\alpha\ ((1\ +\ +\ d_\ell), (1+\ d_{\ell_i}))\ \ X(1+\ d_{\ell_i})\,.$$

On the other hand, by Lemma 4.5 and Lemma 4.4, we have

$$\mu^\alpha\ ((1\ +\ +\ d_\ell), (1+\ d_{\ell_i})) = \mu^\alpha\ ((+\ +\ +\ d_\ell), (+\,+\ d_{\ell_i}))\ .$$

Thus,

$$s_\alpha \cdot X(+\,+\ +\ d_\ell) = X(+\,+\ +\ d_\ell) + \sum_i \mu^\alpha\ ((1\ +\ +\ d_\ell), (1\ +\ d_{\ell_i}))\ \ X(+\,+\ d_{\ell_i})\,.$$

It follows that

$$\begin{aligned}(5.25)\quad & CC\,(s_\alpha \cdot X(1+\ +\ d_\ell) - s_\alpha \cdot X(+\,+\ +\ d_\ell)) \\ &= CC\,(X(1+\,+\ d_\ell) - X(+\,+\ +\ d_\ell)) \\ &+ \sum_{\substack{i \\ (1+\ d_{\ell_i}) \neq s_\alpha \circ (1++d_\ell)}} \mu^\alpha\ ((1\ +\ +\ d_\ell), (1+\ d_{\ell_i}))\,[CC\,(X(1\ +\ d_{\ell_i}) - X(+\,+\ d_{\ell_i}))] \\ &+ [CC\,(X(s_\alpha \circ (1\ +\,+\ d_\ell)) - X(s_\alpha \circ (+\,+\,+\ d_\ell)))]\,.\end{aligned}$$

We claim that all the conormal bundles that contribute to (5.25) are parametrized by clans of the form $(1 \cdots)$. Indeed, part (1) of Theorem 4.1 gives the inclusion $\mathrm{AV}(X(1\ +\ d_{\ell_j})) \subseteq \mathrm{AV}\,(X(1\ +\ +\ d_\ell)) = \overline{\mathcal{O}^n_n}$. On the other hand, [3, Proposition 16] gives $\mu(T^*_{(1+\ +\ d_\ell)}) \subset \mu(T^*_{(1+\ d_{\ell_i})})$; as $\mathcal{Q}_{(1+\ d_{\ell_i})} \subset \overline{\mathcal{Q}_{(1\ ++\ d_\ell)}}$. Hence, $\mathcal{O}^n_{n-1} \subseteq \mu(T^*_{(1+\ d_{\ell_i})}) \subseteq \overline{\mathcal{O}^n_n}$. The description of the set $C_{\mathfrak{g}}(n)$ in Lemma 1.6 allow us to conclude that $\mu(T^*_{(1+\ d_{\ell_i})}) = \mathcal{O}^n_{n-1}$ and $\mu(T^*_{(+\ d_{\ell_i})}) = \mathcal{O}^{n-1}_{n-1}$. Part (3) of Theorem 2.3, then implies that each conormal bundle contributing to the first two summands of equation (5.25) are parametrized by clans of the form $(1 \cdots)$.

The clan $s_\alpha \circ (1\ +\,+\ d_\ell) = (1\ +\,+\ s_\alpha \circ d_\ell)$. By Lemma 1.18, $\mu(T^*_{s_\alpha \circ d_\ell}) = \mathcal{O}^{n-3}_{n-3} \cup \mathcal{O}^{n-3}_{n-4}$. In either case, by Theorem 2.3, the third summand in equation (5.25) is a combination of conormal bundles parametrized by clans of the form $(1 \cdots)$.

Similarly, when $\alpha \notin \tau(d_\ell)$, we have

$$s_\alpha \cdot X(1+\ 2\ d_\ell) = X(1+\ 2\ d_\ell) + \sum_j \mu^\alpha((1\ +\ 2\ d_\ell), (1+\ d_{\ell_j}))\ X(1+\ d_{\ell_j})$$
$$+ \mu^\alpha((1\not+\ \not2\ d_\ell), (+\not+\not1\ c_{\ell_t}))\ X(+\,+\ 1\ c_{\ell_t})\,.$$

Observe that $\mu^\alpha((1+\ 2\ d_\ell),(1+\ d_{\ell_j}))\neq 0$ if and only if

$$\begin{aligned}\mu^\alpha((1\ +\ 2\ d_\ell),(1+\ d_{\ell_j})) &= \mu^\alpha((+\ 2\ d_\ell),(+\ d_{\ell_j})) = \mu^\alpha((2\ d_\ell),d_{\ell_j}) \\ &= \mu^\alpha((+\!+\ 1\ d_\ell),(+\!+\ d_{\ell_j}))\neq 0\,.\end{aligned}$$

Thus,

$$s_\alpha\cdot X(+\!+\ 1\ d_\ell) = X(+\!+\ 1\ d_\ell) + \sum_j \mu^\alpha((+\ 1\ d_\ell),(+\ d_{\ell_j}))\ X(+\!+\ d_{\ell_j})\,.$$

Combining the above information we conclude that

$$\begin{aligned}&CC\,(s_\alpha\cdot X(1+\ 2\ d_\ell) - s_\alpha\cdot X(+\!+\ 1\ d_\ell)) = CC\,(X(1+2\ \ d_\ell) - X(+\!+\ 1\ d_\ell)) \\ &+ \sum_{\substack{j\\(1+\ d_{\ell_j})\neq s_\alpha\circ c}} \mu^\alpha((1\ +\ 2\ d_\ell),(1+\ d_{\ell_j}))[CC(X(1\ +\ d_{\ell_j}) - X(+\!+\ d_{\ell_j}))] \\ &+ [CC\,(X(s_\alpha\circ(1\ +2\ d_\ell) - X(s_\alpha\circ(+\!+1\ d_\ell))] \\ &+ \sum_{\ell_t}\mu^\alpha((1\not+\not2\ d_\ell),(+\not+\not1\ c_{\ell_t}))\ CC\,(X(+\!+\ 1\ c_{\ell_t}))\ .\end{aligned} \tag{5.26}$$

Once again, the first two summands in equation (5.26) are linear combinations of conormal bundles parametrized by clans of the form $(1\cdots)$. So is the third term when $\mu(T^*_{s_\alpha\circ d_\ell}) = O^{n-3}_{n-3}$.

We conclude that $T^*_{(+\!+\ 1\ \omega')}$ contributes to $s_\alpha\cdot T^*_{(1+\ 2\ c')}$ if and only if it occurs in

$$\begin{aligned}&\sum_{\substack{\ell_t\\ \alpha\notin\tau(d_\ell)}} n_\ell\ \mu^\alpha((1\not+\not2\ d_\ell),(+\not+\not1\ c_{\ell_t}))\ CC\,(X(+\!+\ 1\ c_{\ell_t})) \\ &- \sum_{\substack{\ell\\ \mu(T^*_{s_\alpha\circ d_\ell})=O^{n-3}_{n-4}}} n_\ell\ CC\,(X(s_\alpha\circ(+\!+1\ d_\ell))\ .\end{aligned} \tag{5.27}$$

In order to prove the theorem we need to show that a conormal bundle $T^*_{(+\!+\ 1\ \omega')}$ contributes to $s_\alpha\cdot T^*_{(1+\ 2\ c')}$ if and only if $T^*_{(+\ \omega')}$ contributes to $s_\alpha\cdot T^*_{(1\ c')}$. We compute $s_\alpha\cdot T^*_{(1\ c')} = \sum_\ell n_\ell\ [CC(s_\alpha\cdot X_{(1\ d_\ell)}) - CC(s_\alpha\cdot X_{(+\ d_\ell)})]$. When $\alpha\notin\tau(c')$ we have,

$$\begin{aligned}&CC\,(s_\alpha\cdot X(1\ d_\ell) - s_\alpha\cdot X(+\ d_\ell)) = [CC(X(1\ d_\ell) - X(+\ d_\ell))] \\ &+ \sum_{\substack{j\\(1\ s_{\ell_j})\neq s_\alpha\circ(1\ d_\ell)}} \mu^\alpha((1\ d_\ell),(1\ s_{\ell_j}))[CC(X(1\ s_{\ell_j}) - X(+\ s_{\ell_j}))] \\ &+ [CC\,(X(s_\alpha\circ(1\ \ d_\ell) - X(s_\alpha\circ(+\ d_\ell))] \\ &+ \sum_{\ell_t}\mu^\alpha((1\not+\not2\ d_\ell),(+\not+\not1\ c_{\ell_t}))\ CC(X(+\ c_{\ell_t}))\,.\end{aligned} \tag{5.28}$$

As before, the first two summands of equation (5.28) are linear combinations of conormal bundles parametrized by clans of the form $(1\cdots)$. So is the third term when $\mu(T^*_{s_\alpha\circ d_\ell}) =$

O^{n-3}_{n-3}. Now, it is important to note that a conormal bundle of the form $T^*_{(+\cdots)}$ contributes to $s_\alpha \cdot T^*_{(1\ c')}$ if and only if it occurs in

$$\sum_{\substack{\ell_t \\ \alpha\notin\tau(d_\ell)}} n_\ell\ \mu^\alpha\big((1 \not{\ } \not{2}\ d_\ell), (+ \not{\ } \not{1}\ c_{\ell_t})\big)\ CC\big(X(+\ c_{\ell_t})\big)$$
$$- \sum_{\substack{\ell \\ \mu(T^*_{s_\alpha\circ d_\ell}) = O^{n-3}_{n-4}}} n_\ell\ CC\big(X(s_\alpha \circ (+\ d_\ell))\big)\,. \tag{5.29}$$

We compare the characteristic cycles of the modules that occur in (5.27) and (5.29) to complete the argument. This is done by using Theorem 2.3. □

PROPOSITION 5.30. *Let $c = (1\ c')$ be a clan consisting of n symbols. Assume c is of the form $c = (1 + (+\cdots + c_{i+1} = 2\ c'))$. Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. If T^*_d occurs in $s_\alpha \cdot T^*_c$, then either*

(1) *$d = (1\ d')$, or*
(2) *$i = 2$ and T^*_d contributes to $CC\ (X(T_{2,1} \cdot c))$.*

Moreover, the multiplicity $m^{\alpha_i}\ (c, (1\ d')) = m^{\alpha_{i-1}}(d', c')$.

PROOF. The statement about multiplicities is Theorem 5.6. The argument needed to prove the proposition is identical to the one used in the proof of Theorem 5.10. The necessary bookkeeping of the coherent continuation action is included in Lemma 4.14, Proposition 4.11, Corollary 4.29 and Corollary 4.31. □

5.3. We compute the action of the reflection s_{α_1} on conormal bundles of the form $T^*_{(+\ 1\cdots)}$.

PROPOSITION 5.31. *Assume $j \geq 1$ and let $c = (+1\ 2\cdots j\ \ +\cdots+)$. Then,*

$$s_{\alpha_1} \cdot T^*_{(+\ 1\ +\cdots+)} = T^*_{(+\ 1\ +\cdots+)} + T^*_{(1\ +\ +\cdots+)} + T^*_{(+\ +\ 1\ +\cdots+)} + 2\ T^*_{(+\ +\ +\cdots+)}$$
$$s_{\alpha_1} \cdot T^*_{(+\ 1\ 2\cdots j\ +\cdots+)} = T^*_{(+\ 1\ 2\cdots j\ +\cdots+)} + T^*_{(1\ +\ 2\cdots j\ +\cdots+)}\,.$$

PROOF. Write

$$T^*_{(+\ 1\ +\ +\cdots+)} = \begin{cases} CC\ (X(+\ 1\ \ +\ \ +\cdots+)) & \text{if } n-2 \text{ is even} \\ CC\ (X(+\ 1\ \ +\ \ \ +\cdots+)) - T^*_{(+\ +\ +\cdots+)} & \text{if } n-2 \text{ is odd.} \end{cases}$$

When $j > 1$, write

$$T^*_{(+\ 1\ 2\cdots j\ +\cdots+)} = \begin{cases} CC\ (X(+\ 1\ 2\cdots j\ \ +\cdots+)) \\ \text{if } n-(j+1) \text{ is even} \\ CC\ (X(+\ 1\ 2\cdots j\ \ +\cdots+)) - T^*_{(+\ 1\ 2\cdots j-1\ +\cdots+)} \\ \text{if } n-(j+1) \text{ is odd.} \end{cases}$$

The coherent continuation action of s_{α_1} on the relevant modules is computed by using Theorem 4.1, Proposition 4.11, Corollary 4.30, and Proposition 4.18. We obtain

$$
\begin{aligned}
& s_{\alpha_1} \cdot T^*_{(+\ 1\ 2 \cdots j\ + \cdots +)} \\
(5.32) \qquad & = \begin{cases}
CC\,(X(+\ 1\ 2 \cdots j\ \ + \cdots +) + X(1\ \ +\ \ 2 \cdots j\ \ + \cdots +)) \\
\qquad\qquad\qquad\qquad \text{if } n-(j+1) \text{ is even}\,, \\
CC\,(X(+\ 1\ 2 \cdots j\ \ + \cdots +) + X(1\ \ +\ \ 2 \cdots j\ \ + \cdots +)) \\
-s_{\alpha_1} \cdot T^*_{(+\ 1\ 2 \cdots j-1\ + \cdots +)} \qquad \text{if } n-(j+1) \text{ is odd, } j \neq 2\,, \\
CC\,(X(+\ 1\ 2\ \ + \cdots +) + X(1\ \ +\ \ 2\ \ + \cdots +) + X(+ \cdots +)) \\
-s_{\alpha_1} \cdot T^*_{(+\ 1\ +\ + \cdots +)} \qquad \text{if } n-3 \text{ is odd, } j = 2.
\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
& s_{\alpha_1} \cdot T^*_{(+\ 1\ + \cdots +)} \\
& = \begin{cases}
CC\,(X(+\ 1\ \ + \cdots +) + X(1\ \ + \cdots +) + X(+\ +\ \ 1 + \cdots +)) \\
\text{if } n-2 \text{ is even, } n \neq 2\,, \\
CC\,(X(+\ 1) + X(1\ +) + X(+\ +)) \\
\text{if } n = 2\,, \\
CC\,(X(+\ 1\ \ + \cdots +) + X(1\ \ + \cdots +) + X(+\ \ +\ \ 1 \cdots +)) + T^*_{(+\ +\ + \cdots +)} \\
\text{if } n-2 \text{ is odd, } n \neq 2.
\end{cases}
\end{aligned}
$$

We apply Theorem 2.3 to complete the computation of $s_{\alpha_1} \cdot T^*_{(+\ 1\ + \cdots +)}$. The general case follows from (5.32) by induction on j. □

THEOREM 5.33. *Let $c = (+\ 1\ \ +\ \ c')$ be a clan consisting of n symbols. If $\mu(T^*_{c'}) = \mathcal{O}^{n-3}_{n-3}$, then*

$$s_{\alpha_1} \cdot T^*_{(+\ 1\ +\ c')} = T^*_{(+\ 1\ +\ c')} + \ T^*_{(1\ +\ +c')} + \ T^*_{(+\ +\ 1\ c')} + \ 2\, T^*_{(+\ +\ +c')}\,.$$

PROOF. The computation is identical to the computation of $s_{\alpha_1} \cdot T^*_{(+\ 1\ + \cdots +)}$ in Proposition 5.31. □

THEOREM 5.34. *Let $c = (+\ 1\ \ +\ \ c')$ be a clan consisting of n symbols. Write $c = (+\ 1\ \ + (+ \cdots + c_i = 2\ c''))$ with $i \geq 4$. Assume $\mu(T^*_{c'}) = \mathcal{O}^{n-3}_k$ with $k < n-3$. Then,*

$$
\begin{aligned}
& s_{\alpha_1} \cdot T^*_{(+\ 1\ +\ c')} = T^*_{(+\ 1\ +\ c')} + \ T^*_{(1\ +\ +\ c')} + \ T^*_{(+\ +\ 1\ c')} + \\
& \sum_\ell m^{\varepsilon_1 - \varepsilon_2}((+\ 1\ \ + (+ \cdots \not+ \not c_i = 2\ c''))\,, (+\ \ +\ \ \cdots \not+\ \not d_i = 1\ d'_\ell))\ T^*_{(+\ + \cdots +\ d_i = 1\ d'_\ell)}\,.
\end{aligned}
$$

PROOF. First, we write $T^*_{(+\ 1\ +\ c')}$ in the form

$$T^*_{(+\ 1\ +\ c')} = \sum_\ell n_\ell\ CC\,(X(d_\ell))\,, \ \text{with } n_\ell \in \mathbb{Z}\,.$$

Second, we use W-equivariance of the characteristic cycle functor to write

$$s_{\alpha_1} \cdot T^*_{(+\ 1\ +\ c')} = \sum_{\ell} n_\ell \; CC\left(s_{\alpha_1} \cdot X(d_\ell)\right) \; . \tag{5.35}$$

The explicit computation of the right hand side of (5.35) depends on the parity of n and on $\mu(T^*_{c'})$. It is necessary to consider various cases. It is not difficult to verify, using Proposition 4.16 and Theorem 2.3, that $s_{\alpha_1} \cdot T^*_{(+\ 1\ +\ c')}$ is of the following form:

$$\sum_{\ell} n_\ell \; \mu^{\varepsilon_1-\varepsilon_2}\left((+\ 1\ \ +(+\cdots \not{+} \not{c_i}\ c'')), (+\ \cdots \not{+}\ \not{d_i}\ d'_\ell)\right) CC(X(+\cdots +\ \ d_i = 1\ d'_\ell))$$
$$+\, T^*_{(+\ 1\ +\ c')} + T^*_{(1\ ++\ c')} + T^*_{(++\ 1\ c')} + 2\, T^*_{(+++\ c')}\,,$$

when n is odd and $\mu(T^*_{c'}) = \mathcal{O}^{n-3}_{n-4}$;

$$\sum_{\ell} n_\ell \; \mu^{\varepsilon_1-\varepsilon_2}\left((+\ 1\ \ +(+\cdots \not{+} \not{c_i}\ c'')), (+\ \cdots \not{+}\ \not{d_i}\ d'_\ell)\right) CC(X(+\cdots +\ \ d_i = 1\ d'_\ell))$$
$$+\, T^*_{(+\ 1\ +\ c')} + T^*_{(1\ ++\ c')} + T^*_{(++\ 1\ c')} + T^*_{(+++\ c')}\,,$$

when n is even and $\mu(T^*_{c'}) = \mathcal{O}^{n-3}_{n-4} \cup \mathcal{O}^{n-3}_{n-5}$; and

$$\sum_{\ell} n_\ell \; \mu^{\varepsilon_1-\varepsilon_2}\left((+\ 1\ \ +(+\cdots \not{+} \not{c_i}\ c'')), (+\ \cdots \not{+}\ \not{d_i}\ d'_\ell)\right) CC(X(+\cdots +\ \ d_i = 1\ d'_\ell))$$
$$+\, T^*_{(+\ 1\ +\ c')} + T^*_{(1\ ++\ c')} + T^*_{(++\ 1\ c')}\,,$$

either when $\mu(T^*_{c'}) = \mathcal{O}^{n-3}_{j}$ with $j < n-5$ or when n is odd and $\mu(T^*_{c'}) = \mathcal{O}^{n-3}_{n-5}$.

By Corollary 4.31, $\mu(T^*_{d'_\ell}) \subsetneq \mu(T^*_{c''})$. It follows that a conormal bundle $T^*_{(+\cdots+1\ r)}$ contributes to $CC(X(+\cdots +\ \ 1\ d'_\ell))$ if and only if $CC(X(+\cdots \not{+} \not{1}\ d'_\ell))$ has $T^*_{(+\cdots \not{+} \not{1} r)}$ as a summand. When this is the case, the multiplicities agree.

The action of the single reflection $s_{\varepsilon_1-\varepsilon_2}$ on $T^*_{(+\ 1\ +(+\cdots \not{+} \not{c_i}\ c''))}$ is analogous. The theorem follows from a careful comparison of the resulting formulae. $\square$

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