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CLASSIFICATION OF BIHARMONIC C-PARALLEL LEGENDRIAN SUBMANIFOLDS IN 7-DIMENSIONAL SASAKIAN SPACE FORMS

TORU SASAHARA

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Abstract. In [5], D. Fetcu and C. Oniciuc presented the classification result for biharmonic C-parallel Legendrian submanifolds in 7-dimensional Sasakian space forms. However, it is incomplete. In this paper, all such submanifolds are explicitly determined.

1. Introduction. In [5, Theorem 5.1], Fetcu and Oniciuc presented the classification result for proper biharmonic C-parallel Legendrian submanifolds in 7-dimensional Sasakian space forms. The case (2) of the theorem is proved by applying Lemma 4.4 in [1]. However, the lemma is wrong, and hence Fetcu and Oniciuc's classification is incomplete. This paper corrects errors in [1], and moreover, completes the classification.

Our main result is the following, which determines explicitly all proper biharmonic C-parallel Legendrian submanifolds in 7-dimensional Sasakian space forms.

THEOREM 1.1. *Let $f : M^3 \to N^7(\varepsilon)$ be a 3-dimensional C-parallel Legendrian submanifold in a 7-dimensional Sasakian space form of constant φ-sectional curvature ε. Then M^3 is proper biharmonic if and only if either*:

(1) *M^3 is flat, $N^7(\varepsilon) = S^7(\varepsilon)$ with $\varepsilon > -1/3$, where $S^7(\varepsilon)$ is a unit sphere in $\mathbb{C}^4$ equipped with its canonical and deformed Sasakian structures, and $f(M^3)$ is an open part of*

$$f(u,v,w) = \left(\frac{\lambda}{\sqrt{\lambda^2+\alpha^{-1}}}\exp\left(i\left(\frac{1}{\alpha\lambda}u\right)\right),\right.$$
$$\frac{1}{\sqrt{\alpha(c-a)(2c-a)}}\exp(-i(\lambda u-(c-a)v))\,,$$
$$\frac{1}{\sqrt{\alpha\rho_1(\rho_1+\rho_2)}}\exp(-i(\lambda u+cv+\rho_1 w))\,, \tag{1.1}$$
$$\left.\frac{1}{\sqrt{\alpha\rho_2(\rho_1+\rho_2)}}\exp(-i(\lambda u+cv-\rho_2 w))\right),$$

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where $\alpha = 4/(\varepsilon+3)$, $\rho_{1,2} = (\sqrt{4c(2c-a)+d^2} \pm d)/2$ *and* λ, a, c, d *are real constants given by*

$$\begin{cases} (3\lambda^2-\alpha^{-1})(3\lambda^4-2(\varepsilon+1)\lambda^2+\alpha^{-2})+\lambda^4((a+c)^2+d^2)=0\,, \\ (a+c)(5\lambda^2+a^2+c^2-7\alpha^{-1}+4)+cd^2=0\,, \\ d(5\lambda^2+d^2+3c^2+ac-7\alpha^{-1}+4)=0\,, \\ \alpha^{-1}+\lambda^2+ac-c^2=0 \end{cases}$$

such that $-1/\sqrt{\alpha} < \lambda < 0$, $0 < a \le (\lambda^2-\alpha^{-1})/\lambda$, $a \ge d \ge 0$, $a > 2c$, $\lambda^2 \ne 1/(3\alpha)$; *or*

(2) M^3 *is non-flat,* $N^7(\varepsilon) = S^7(\varepsilon)$ *with* $\varepsilon \ge (-7+8\sqrt{3})/13$ *and* $f(M^3)$ *is an open part of*

$$f(x,\mathbf{y}) = \left(\sqrt{\frac{\mu^2}{\mu^2+1}}e^{-\frac{i}{\mu}x}, \sqrt{\frac{1}{\mu^2+1}}e^{i\mu x}\mathbf{y}\right), \tag{1.2}$$

where $\mathbf{y} = (y_1, y_2, y_3)$, $\|\mathbf{y}\| = 1$ *and*

$$\mu^2 = \begin{cases} 1 & (\varepsilon = 1)\,, \\ \dfrac{4\varepsilon+4 \pm \sqrt{13\varepsilon^2+14\varepsilon-11}}{3(3+\varepsilon)} & (\varepsilon \ne 1)\,. \end{cases} \tag{1.3}$$

REMARK 1.1. The flat case (1) of Theorem 1.1 has been proved by Fetcu and Oniciuc in [5, Theorem 5.1]. However, they did not give the explicit representation of non-flat biharmonic C-parallel Legendrian submanifolds in $S^7(\varepsilon)$.

REMARK 1.2. The immersion (1.1) can be rewritten as

$$f(u,v,w) = (z_1(u), z_2(u)\mathbf{y}(v,w))\,,$$

where $(z_1(u), z_2(u))$ is a Legendre curve with constant curvature $(\lambda^2-\alpha^{-1})/\lambda$ in $S^3(\varepsilon)$ given by

$$(z_1(u), z_2(u)) = \left(\frac{\lambda}{\sqrt{\lambda^2+\alpha^{-1}}}e^{i\frac{1}{\alpha\lambda}u}, \frac{1}{\sqrt{\alpha\lambda^2+1}}e^{-i\lambda u}\right)$$

and $\mathbf{y}(u,v)$ is a Legendrian surface in $S^5(\varepsilon)$ given by

$$\mathbf{y}(v,w) = \left(\frac{\sqrt{\alpha\lambda^2+1}}{\sqrt{\alpha(c-a)(2c-a)}}e^{i(c-a)v}, \frac{\sqrt{\alpha\lambda^2+1}}{\sqrt{\alpha\rho_1(\rho_1+\rho_2)}}e^{-i(cv+\rho_1 w)}, \frac{\sqrt{\alpha\lambda^2+1}}{\sqrt{\alpha\rho_2(\rho_1+\rho_2)}}e^{-i(cv-\rho_2 w)}\right).$$

REMARK 1.3. (i) For each fixed x, (1.2) has constant Gauss curvature $(\mu^2+1)/\alpha$ with respect to the induced metric from $S^7(\varepsilon)$. We can check that the surface is an integral C-parallel surface in $S^7(\varepsilon)$.

(ii) The curve

$$z(x) := \left(\sqrt{\frac{\mu^2}{\mu^2+1}}e^{-\frac{i}{\mu}x}, \sqrt{\frac{1}{\mu^2+1}}e^{i\mu x}\right)$$

given in (1.2) is a Legendre curve with constant curvature $(\mu^2 - 1)/(\mu\sqrt{\alpha})$ in $S^3(\varepsilon)$.

REMARK 1.4. (i) In [5, Theorem 5.1], it is stated that when $\varepsilon = 5/9$, M^3 is locally isometric to a product $\gamma \times \bar{M}^2$, where γ is a curve of constant curvature $1/\sqrt{2}$ in $S^7(5/9)$ and $\bar{M}^2$ is a surface of constant Gauss curvature $4/3$. However, $1/\sqrt{2}$ should be replaced by $2/3$ because γ coincides with $z(x)$ in Remark 1.3.

(ii) The function λ in the case (2) of [5, Theorem 5.1] and the function μ in (1.3) are related by the equation $\mu^2 = \alpha\lambda^2$. Hence, in view of Remark 1.3, the case $\varepsilon = 1$ and the case $\mu^2 = (4\varepsilon + 4 + \sqrt{13\varepsilon^2 + 14\varepsilon - 11})/(3(3+\varepsilon))$ with $\varepsilon > 1$ in (2) of Theorem 1.1 are missing from [5, Theorem 5.1].

Applying Theorem 1.1, we have the following result which corrects [5, Corollary 5.2].

COROLLARY 1.1. *Let $f : M^3 \to S^7(1)$ be a C-parallel Legendrian submanifold. Then M^3 is proper biharmonic if and only if either*:

(1) *M^3 is flat, and $f(M^3)$ is an open part of*

$$
\begin{aligned}
f(u,v,w) = \Bigg(&-\frac{1}{\sqrt{6}}\exp(-i\sqrt{5}u),\\
&\frac{1}{\sqrt{6}}\exp\left(i\left(\frac{1}{\sqrt{5}}u - \frac{4\sqrt{3}}{\sqrt{10}}v\right)\right),\\
&\frac{1}{\sqrt{6}}\exp\left(i\left(\frac{1}{\sqrt{5}}u + \frac{\sqrt{3}}{\sqrt{10}}v - \frac{3\sqrt{2}}{2}w\right)\right),\\
&\frac{1}{\sqrt{2}}\exp\left(i\left(\frac{1}{\sqrt{5}}u + \frac{\sqrt{3}}{\sqrt{10}}v + \frac{\sqrt{2}}{2}w\right)\right)\Bigg); \text{ or}
\end{aligned}
$$

(2) *M^3 is non-flat, and $f(M^3)$ is an open part of*

$$
f(x,\mathbf{y}) = \frac{1}{\sqrt{2}}(e^{ix}, e^{-ix}\mathbf{y}), \tag{1.4}
$$

where $\mathbf{y} = (y_1, y_2, y_3)$ and $\|\mathbf{y}\| = 1$.

REMARK 1.5. The flat case (1) of Corollary 1.1 has been proved in [5, Corollary 5.2]. However, the non-flat submanifold (1.4) is missing from [5, Corollary 5.2].

REMARK 1.6. The author classified proper biharmonic Legendrian surfaces in 5-dimensional Sasakian space forms (see [10] and [12]). Those surfaces are flat and C-parallel.

In the last section, by the same argument as in the proof of Theorem 1.1, we determine explicitly all proper biharmonic parallel Lagrangian submanifolds in 3-dimensional complex projective space.

2. Preliminaries.

2.1. Sasakian space forms. A $(2n+1)$-dimensional manifold N^{2n+1} is called an *almost contact manifold* if it admits a unit vector field ξ, a one-form η and a $(1,1)$-tensor field φ

satisfying

$$\eta(\xi) = 1\,, \quad \varphi^2 = -I + \eta \otimes \xi\,.$$

Every almost contact manifold admits a Riemannian metric g satisfying

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)\,.$$

The quadruplet (φ, ξ, η, g) is called an *almost contact metric structure*. An almost contact metric structure is said to be *normal* if the tensor field S defined by

$$S(X, Y) = \varphi^2[X, Y] + [\varphi X\,, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X\,, \varphi Y] + 2d\eta(X, Y)\xi$$

vanishes identically. A normal almost contact structure is said to be *Sasakian* if it satisfies

$$d\eta(X, Y) := (1/2)\,(X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])) = g(X, \varphi Y)\,.$$

The tangent plane in T_pN^{2n+1} which is invariant under φ is called a *φ-section*. The sectional curvature of φ-section is called the *φ-sectional curvature*. Complete and connected Sasakian manifolds of constant φ-sectional curvature are called *Sasakian space forms*. Denote Sasakian space forms of constant φ-sectional curvature ε by $N^{2n+1}(\varepsilon)$.

Let $S^{2n+1} \subset \mathbb{C}^{n+1}$ be the unit hypersphere centered at the origin. Denote by z the position vector field of S^{2n+1} in $\mathbb{C}^{n+1}$ and by g_0 the induced metric. Let $\xi_0 = -Jz$, where J is the usual complex structure of $\mathbb{C}^{n+1}$ which is defined by $JX = iX$ for $X \in T\mathbb{C}^{n+1}$. Let η_0 be a 1-form defined by $\eta_0(X) = g_0(\xi_0, X)$ and φ_0 be the tensor field defined by $\varphi_0 = s \circ J$, where $s : T_z\mathbb{C}^{n+1} \to T_zS^{2n+1}$ denotes the orthogonal projection. Then, $(S^{2n+1}, \varphi_0, \xi_0, \eta_0, g_0)$ is a Sasakian space form of constant φ-sectional curvature 1. If we put

$$\eta = \alpha\eta_0\,, \quad \xi = \alpha^{-1}\xi_0\,, \quad \varphi = \varphi_0\,, \quad g = \alpha g_0 + \alpha(\alpha - 1)\eta_0 \otimes \eta_0$$

for a positive constant α, then $(S^{2n+1}, \varphi, \xi, \eta, g)$ is a Sasakian space form of constant ϕ sectional curvature $\varepsilon = (4/\alpha) - 3 > -3$. We denote it by $S^{2n+1}(\varepsilon)$. Tanno [13] showed that a simply connected Sasakian space form $N^{2n+1}(\varepsilon)$ with $\varepsilon > -3$ is isomorphic to $S^{2n+1}(\varepsilon)$; i.e., there exists a C^∞-diffeomorphism which maps the structure tensors into the corresponding structure tensors.

2.2. Legendrian submanifolds in Sasakian space forms. Let M^m be an m-dimensional submanifold M in a Sasakian space form $N^{2n+1}(\varepsilon)$. If η restricted to M^m vanishes, then M^m is called an *integral submanifold*, in particular if $m = n$, it is called a *Legendrian submanifold*. In particular a Legendrian submanifold in a 3-dimensional Sasakian space form is called a *Legendre curve*. One can see that a curve $z(s)$ in $S^3(\varepsilon) \subset \mathbb{C}^2$ is a Legendre curve if and only if it satisfies $\mathrm{Re}(z'(s), iz(s)) = 0$ identically in $\mathbb{C}^2$, where $(\cdot, \cdot)$ is the standard Hermitian inner product on $\mathbb{C}^2$.

We denote the second fundamental form, the shape operator and the normal connection of a submanifold by h, A and D, respectively. The mean curvature vector field H is defined by $H = (1/m)\mathrm{Tr}\ h$. If it vanishes identically, then M^m is called a *minimal submanifold*. In particular, if $h \equiv 0$, then M^m is called a *totally geodesic submanifold*. A Legendrian submanifold in a Sasakian manifold is parallel, i.e., satisfies $\bar{\nabla} h = 0$ if and only if it is totally geodesic. Here, $\bar{\nabla} h$ is defined by

$$(\bar{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)\,.$$

A Legendrian submanifold is called *C-parallel* if $\bar{\nabla} h$ is parallel to ξ.

For a Legendrian submanifold M in a Sasakian space form, we have (cf. [2])

$$A_\xi = 0\,,\quad \varphi h(X, Y) = -A_{\varphi Y} X\,,\quad \langle h(X, Y)\,, \varphi Z\rangle = \langle h(X, Z)\,, \varphi Y\rangle \tag{2.1}$$

for any vector fields X, Y and Z tangent to M, where $\langle\cdot,\cdot\rangle$ is the inner product. We denote by K_{ij} the sectional curvature determined by an orthonormal pair $\{X_i, X_j\}$. Then from the equation of Gauss we have

$$K_{ij} = (\varepsilon + 3)/4 + \left\langle h(X_i, X_i), h(X_j, X_j)\right\rangle - \|h(X_i, X_j)\|^2\,. \tag{2.2}$$

The following Legendrian submanifolds can be regarded as the simplest Legendrian submanifolds next to totally geodesic ones in Sasakian space forms.

DEFINITION 2.1. An n-dimensional Legendrian submanifold M^n in a Sasakian space form is called *H-umbilical* if every point has a neighborhood V on which there exists an orthonormal frame field $\{e_1, \dots, e_n\}$ such that the second fundamental form takes the following form:

$$\begin{aligned} &h(e_1, e_1) = \lambda\varphi e_1\,, \quad h(e_2, e_2) = \cdots = h(e_n, e_n) = \mu\varphi e_1\,,\\ &h(e_1, e_j) = \mu\varphi e_j\,, \quad h(e_j, e_k) = 0\,, \quad j \neq k\,, \quad j, k = 2, \dots, n\,, \end{aligned}$$

where λ and μ are some functions on V.

REMARK 2.1. If in Definition 2.1 we assume that the mean curvature vector field is nowhere vanishing, then $e_1 = -\varphi H/\|H\|^2$ holds and hence it is a globally defined differentiable vector field, and λ is also a globally defined differentiable function. Moreover, at each point p of M^n, the shape operator A_{JH} has only one eigenvalue $\mu(p)$ on $D(p) = \{X \in T_p M^n | \langle X, JH\rangle = 0\}$. Since $\mu = (n\|H\| - \lambda)/(n - 1)$ holds, it is also a globally defined differentiable function.

2.3. Biharmonic submanifolds. Let $f : M^n \to N$ be a smooth map between two Riemannian manifolds. The *tension field* $\tau(f)$ of f is a section of the vector bundle f^*TN defined by

$$\tau(f) := \sum_{i=1}^{n} \left\{\nabla^f_{e_i} df(e_i) - df(\nabla_{e_i} e_i)\right\}\,,$$

where ∇^f, ∇ and $\{e_i\}$ denote the induced connection, the connection of M^n and a local orthonormal basis of M^n, respectively.

A smooth map f is called a *harmonic map* if it is a critical point of the energy functional

$$E(f) = \int_\Omega \|df\|^2 dv$$

over every compact domain Ω of M^n, where dv is the volume form of M^n. A smooth map f is harmonic if and only if $\tau(f) = 0$ at each point on M^n (cf. [4]).

The bienergy functional $E_2(f)$ of f over compact domain $\Omega \subset M^n$ is defined by

$$E_2(f) = \int_\Omega \|\tau(f)\|^2 dv\,.$$

Thus E_2 provides a measure for the extent to which f fails to be harmonic. If f is a critical point of E_2 over every compact domain Ω, then f is called a *biharmonic map*. In [6], Jiang proved that f is biharmonic if and only if its bitension field defined by

$$\tau_2(f) := \sum_{i=1}^{n} \left\{ (\nabla^f_{e_i}\nabla^f_{e_i} - \nabla^f_{\nabla_{e_i}e_i})\tau(f) + R^N(\tau(f), df(e_i))df(e_i) \right\}$$

vanishes identically, where R^N is the curvature tensor of N.

A submanifold is called a *biharmonic submanifold* if the isometric immersion that defines the submanifold is biharmonic map. Minimal submanifolds are biharmonic. A biharmonic submanifold is said to be a *proper* biharmonic submanifold if it is non-minimal.

Loubeau and Montaldo introduced a class which includes biharmonic submanifolds as follows.

DEFINITION 2.2 ([9]). An isometric immersion $f : M \to N$ is called *biminimal* if it is a critical point of the bienergy functional E_2 with respect to all *normal variation* with compact support. Here, a normal variation means a variation f_t through $f = f_0$ such that the variational vector field $V = df_t/dt|_{t=0}$ is normal to $f(M)$. In this case, M or $f(M)$ is called a *biminimal submanifold* in N.

An isometric immersion f is biminimal if and only if the normal part of $\tau_2(f)$ vanishes identically. Clearly, biharmonic submanifolds are biminimal. Biminimal H-umbilical Legendrian submanifolds in Sasakian space forms have been classified by the author as follows.

THEOREM 2.3 ([12]). *Let $f : M^n \to N^{2n+1}(\varepsilon)$ be a non-minimal biminimal H-umbilical Legendrian submanifold, where $n \geq 3$. Then $N^{2n+1}(\varepsilon) = S^{2n+1}(\varepsilon)$ with*

$$\varepsilon \geq \frac{-3n^2 - 2n + 5 + 32\sqrt{n}}{n^2 + 6n + 25} \quad (> -3)$$

and $f(M^n)$ is an open part of

$$f(x, \mathbf{y}) = \left(\sqrt{\frac{\mu^2}{\mu^2+1}}\, e^{-\frac{i}{\mu}x}, \sqrt{\frac{1}{\mu^2+1}}\, e^{i\mu x}\mathbf{y} \right),$$

where $\mathbf{y} = (y_1, \ldots, y_n)$, $\|\mathbf{y}\| = 1$ and

$$\mu^2 = \begin{cases} 1 & (\varepsilon = 1)\,, \\ \dfrac{(n+5)\varepsilon + 3n - 1 \pm \sqrt{P(n,\varepsilon)}}{2(3+\varepsilon)n} & (\varepsilon \neq 1)\,, \end{cases}$$

where $P(n, \varepsilon) := (n^2 + 6n + 25)\varepsilon^2 + (6n^2 + 4n - 10)\varepsilon + 9n^2 - 42n + 1$.

REMARK 2.2. Submanifolds given in Theorem 2.3 are in fact proper biharmonic.

3. C-parallel Legendrian submanifolds.

3.1. A special orthonormal basis. We recall a special local orthonormal basis which is used in [1] (see also [5]). Let M be a non-minimal Legendrian submanifold of $N^7(\varepsilon)$. Let p be an arbitrary point of M, and denote by U_pM the unit sphere in T_pM. We consider the function $f_p : U_pM \to \mathbb{R}$ given by

$$f_p(u) = \langle h(u,u)\,, \varphi u\rangle\ .$$

A function f_p attains a critical value at X if and only if $\langle h(X,X), \varphi Y\rangle = 0$ for all $Y \in U_pM$ with $\langle X, Y\rangle = 0$, i.e., X is an eigenvector of $A_{\varphi X}$.

We take X_1 as a vector at which f_p attains its maximum. Then there exists a local orthonormal basis $\{X_1, X_2, X_3\}$ of T_pM such that the shape operators take the following forms (cf. [1]):

$$A_{\varphi X_1} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},\ A_{\varphi X_2} = \begin{pmatrix} 0 & \lambda_2 & 0 \\ \lambda_2 & a & b \\ 0 & b & c \end{pmatrix},\ A_{\varphi X_3} = \begin{pmatrix} 0 & 0 & \lambda_3 \\ 0 & b & c \\ \lambda_3 & c & d \end{pmatrix}, \tag{3.1}$$

where

$$\lambda_1 > 0\,,\ \ \lambda_1 \geq 2\lambda_2\,,\ \ \lambda_1 \geq 2\lambda_3\,,\ \ a \geq 0,\ \ a^2 \geq d^2\,, \tag{3.2}$$

and moreover, if $\lambda_2 = \lambda_3$, then $b = 0$ and $a \geq 2c$.

LEMMA 3.1. *The vector $X_1 \in T_pM$ can be differentiably extended to a vector field $X_1(x)$ on a neighborhood V of p such that at every point x of V, f_x attains a critical value at $X_1(x)$, that is, $X_1(x)$ is an eigenvector of $A_{\varphi X_1(x)}$.*

PROOF. Let $E_1(x)$, $E_2(x)$, $E_3(x)$ be an arbitrary local differentiable orthonormal frame field on a neighborhood V of p, such that $E_i(p) = X_i$. The purpose is to find a local differentiable vector field $X_1(x) = \sum y^i(x)E_i(x)$ such that $(y^1)^2 + (y^2)^2 + (y^3)^2 = 1$ and at every point x of V, f_x attains a critical value at $X_1(x)$. As in the proof of Theorem A in [7], we apply Lagrange's multiplier method.

Consider the following function:

$$F(x, y^1, y^2, y^3, \lambda) := \sum_{i,j,k} h_{ijk} y^i y^j y^k - \lambda\{(y^1)^2 + (y^2)^2 + (y^3)^2 - 1\}\,,$$

where $h_{ijk} := \langle h(E_i(x), E_j(x)), \varphi E_k(x)\rangle$. We shall show that there exist differentiable functions y^1, y^2, y^3 defined a neighborhood of p satisfying the following system of equations:

$$\begin{cases} \dfrac{\partial F}{\partial y^i} = 3\displaystyle\sum_{j,k} h_{ijk}(x) y^j y^k - 2y^i\lambda = 0, \ \ i \in \{1,2,3\}\,, \\ \dfrac{\partial F}{\partial \lambda} = (y^1)^2 + (y^2)^2 + (y^3)^2 - 1 = 0\,. \end{cases} \tag{3.3}$$

Define functions G_i by

$$\begin{cases} G_i(x, y^1, y^2, y^3, \lambda) = 3\sum_{j,k} h_{ijk}(x) y^j y^k - 2y^i\lambda \ \text{ for } \ i = 1, 2, 3\,. \\ G_4(x, y^1, y^2, y^3) = (y^1)^2 + (y^2)^2 + (y^3)^2 - 1\,. \end{cases}$$

Since $X_1(p) = X_1 = E_1(p)$, we have $(y^1, y^2, y^3) = (1, 0, 0)$ at p. It follows from (3.1) and (3.3) that $2\lambda(p) = 3\lambda_1$. We set $y^4 = \lambda$. A straightforward computation yields

$$\det\left(\frac{\partial G_\alpha}{\partial y^\beta}\right)(p) = 36(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)\,. \tag{3.4}$$

By (3.2), we have $\lambda_2 \neq \lambda_1 \neq \lambda_3$. Hence the RHS of (3.4) is not zero. The implicit function theorem shows that there exist local differentiable functions $y^1(x)$, $y^2(x)$, $y^3(x)$, $\lambda(x)$ on a neighborhood of p satisfying (3.3). The proof is finished. □

REMARK 3.1. In [1] and [5], the differentiablity of $X_1(x)$ is not proved.

If the eigenvalues of $A_{\varphi X_1(x)}$ have constant multiplicities on a neighborhood V of p, we can extend X_2 and X_3 differentiably to vector fields $X_2(x)$ and $X_3(x)$ on V. We work on the open dense set of M defined by this property.

3.2. Correction to a paper by Biakoussis, Blair and Koufogiorgos. Let M be a C-parallel Legendrian submanifold of $N^7(\varepsilon)$. The condition that M is C-parallel is equivalent to $\nabla\varphi h = 0$, where ∇ is the Levi-Civita connection of M. Hence we have

$$R \cdot \varphi h = 0\,, \tag{3.5}$$

where R is the curvature tensor of M.

By using (2.2), (3.1) and (3.5), Biakoussis et al. obtained a system of algebraic equations with respect to λ_1, λ_2, λ_3, a, b, c, d, K_{12}, K_{13} and K_{23} (see [1, pages 211–212]).

However, the equation (3.19)-(iv) in [1], i.e., $c(a-2c)(\lambda_2-\lambda_3) = 0$ is incorrect. It should be replaced by

$$b(a - 2c)(\lambda_2 - \lambda_3) = 0\,,$$

which is obtained by $\langle (R(X_1, X_2) \cdot \varphi h)(X_2, X_2), X_3\rangle = 0$.

In [1, Lemma 4.4], it is stated that if $\lambda_1 = 2\lambda_3 \neq 2\lambda_2$, then $\varepsilon = -3$. However, the proof is based on the the wrong equation (3.19)-(iv) (see page 214, line 11), and hence the statement is also wrong. The following is a counterexample to [1, Lemma 4.4]: The submanifold (1.4) is a H-umbilical Legendrian submanifold such that, with respect to some orthonormal local frame field e_1, e_2, e_3 with $e_1 = \partial/\partial x$, the second fundamental form h satisfies

$$h(e_1, e_1) = 0\,, \quad h(e_2, e_2) = h(e_3, e_3) = \varphi e_1\,,$$
$$h(e_1, e_2) = \varphi e_2\,, \quad h(e_1, e_3) = \varphi e_3\,, \ \ h(e_2, e_3) = 0\,.$$

We put $X_1 = (e_1 - \sqrt{2}e_2)/\sqrt{3}$, $X_2 = (\sqrt{2}e_1 + e_2)/\sqrt{3}$ and $X_3 = e_3$. Then the shape operators take the forms (3.1) with $\lambda_1 = 2/\sqrt{3}$, $\lambda_2 = -1/\sqrt{3}$, $\lambda_3 = 1/\sqrt{3}$, $a = c = \sqrt{2}/\sqrt{3}$ and $b = d = 0$.

On the other hand, following the wrong statement of [1, Lemma 4.4], the non-flat case (2) of [5, Theorem 5.1] is investigated. Therefore, the classification presented in the theorem is incomplete.

3.3. Biharmonic C-parallel Legendrian submanifolds. We shall prove Theorem 1.1. First, we recall the following.

PROPOSITION 3.1 ([5]). *A C-parallel Legendrian submanifolds in a 7-dimensional Sasakian space form $N^7(\varepsilon)$ is proper biharmonic if and only if $\varepsilon > -1/3$ and*

$$\operatorname{Tr} h(\cdot, A_H \cdot) = (3\varepsilon + 1)/2 . \tag{3.6}$$

By applying the proof of [1, Lemmas 4.2–4.6] and Proposition 3.1, we obtain the following.

PROPOSITION 3.2. *Let M^3 be a proper biharmonic C-parallel Legendrian submanifold in $N^7(\varepsilon)$. If M is non-flat, then it is H-umbilical.*

PROOF. By [1, Lemma 4.2], the case $\lambda_1 \neq 2\lambda_2 \neq 2\lambda_3 \neq \lambda_1$ cannot hold. According to the proof of [1, Lemma 4.6], the case $\lambda_1 = 2\lambda_2 = 2\lambda_3$ cannot hold for $\varepsilon > -3$. Hence, by Proposition 3.1 the proof is divided into the following three cases.

CASE (i). $\lambda_1 = 2\lambda_2 \neq 2\lambda_3$. In the proof of [1, Lemma 4.3], we have

$$\lambda_1 = 2\lambda_2 = -\lambda_3 = \sqrt{2(\varepsilon+3)}/4 , \quad a = c = d = 0 , \quad b = \pm\sqrt{6(\varepsilon+3)}/8 . \tag{3.7}$$

We choose a local orthonormal frame field $\{e_1, e_2, e_3\}$ as follows:

$$e_1 = (X_1 \pm \sqrt{3}X_3)/2 , \quad e_2 = X_2 , \quad e_3 = (\mp\sqrt{3}X_1 + X_3)/2 ,$$

where the $\pm$ signs are determined by the sign of b. Then, by a straightforward computation using (3.7), we obtain

$$\begin{aligned} &h(e_1, e_1) = -(\sqrt{2(\varepsilon+3)}/4)\varphi e_1 , \quad h(e_2, e_2) = h(e_3, e_3) = (\sqrt{2(\varepsilon+3)}/4)\varphi e_1 , \\ &h(e_2, e_3) = 0 , \quad h(e_1, e_i) = (\sqrt{2(\varepsilon+3)}/4)\varphi e_i , \quad i \in \{2, 3\} , \end{aligned} \tag{3.8}$$

which implies that M is H-umbilical. Moreover, from (3.6) and (3.8) we have $\varepsilon = 5/9$ (see the subcase (a) of (2) in [5, Theorem 5.1]).

CASE (ii). $\lambda_1 = 2\lambda_3 \neq 2\lambda_2$. Following the proof of [1, Lemma 4.4] (page 214, lines 7–10), we have

$$K_{12} = 0 , \tag{3.9}$$

$$b = d = 0 , \quad c \neq 0 . \tag{3.10}$$

Moreover, in [1, (3.16)-(iv), (3.21)]) the following equations have been obtained:

$$c(K_{23} + \lambda_3(\lambda_2 - \lambda_3)) = 0 , \tag{3.11}$$

$$(\lambda_2 - \lambda_3)(K_{23} - b^2 - c^2) = 0\,. \tag{3.12}$$

From (3.10), (3.11), (3.12) and $\lambda_2 \neq \lambda_3$, we have

$$K_{23} + \lambda_3(\lambda_2 - \lambda_3) = 0\,, \quad K_{23} = c^2\,. \tag{3.13}$$

We note that (3.10) and (3.13) show $\lambda_3 \neq 0$. It follows from (2.2), (3.1), (3.9) and (3.13) that

$$\begin{cases} \lambda_2^2 = 4c^2 - 2ac - \beta\,, \\ \lambda_3^2 = 3c^2 - ac - \beta\,, \\ \lambda_2\lambda_3 = 2c^2 - ac - \beta\,, \end{cases} \tag{3.14}$$

where $\beta = (\varepsilon + 3)/4$.

We choose a local orthonormal frame field $\{e_1, e_2, e_3\}$ as follows:

$$e_1 = (\lambda_3 X_1 + cX_2)/\sqrt{\lambda_3^2 + c^2}\,, \quad e_2 = (-cX_1 + \lambda_3 X_2)/\sqrt{\lambda_3^2 + c^2}, \quad e_3 = X_3\,.$$

We set

$$k(a, c) := 8c^4 - 6ac^3 + (a^2 - 3\beta)c^2 + a\beta c\,.$$

Then, by a straightforward computation using (3.14), we obtain

$$\begin{aligned} &\langle h(e_1, e_1), \varphi e_2\rangle = \frac{ck(a, c)}{\lambda_3(\lambda_3^2 + c^2)^{3/2}}\,, \quad \langle h(e_1, e_1)\,, \varphi e_3\rangle = 0\,, \\ &\langle h(e_2, e_2) - h(e_3, e_3)\,, \varphi e_1\rangle = -\frac{k(a, c)}{(\lambda_3^2 + c^2)^{3/2}}\,, \quad h(e_2, e_3) = 0\,, \\ &\langle h(e_2, e_2), \varphi e_2\rangle = -\frac{\lambda_3 k(a, c)}{c(\lambda_3^2 + c^2)^{3/2}}\,, \quad \langle h(e_3, e_3), \varphi e_3\rangle = 0\,. \end{aligned} \tag{3.15}$$

On the other hand, substituting (3.14) into the identity $\lambda_2^2\lambda_3^3 - (\lambda_2\lambda_3)^2 = 0$ gives

$$k(a, c) = 0\,.$$

Hence, it follows from (2.1) and (3.15) that M is H-umbilical.

CASE (iii). $\lambda_1 \neq 2\lambda_2 = 2\lambda_3$. By rotating the vector fields X_2 and X_3, if necessary, we may assume that $b = 0$. In [1, Lemma 4.5], it is proved that if M is non-flat, then $a \neq 2c$ and $a = c = d = 0$. Thus, M is H-umbilical. □

PROOF OF THEOREM 1.1. The flat case (1) has been proved in (1) of [5, Theorem 5.1]. Applying Proposition 3.2 and Theorem 2.3 for $n = 3$, we can prove the non-flat case (2). □

REMARK 3.2. In [5], the case (ii) of Proposition 3.2 was not investigated.

4. Biharmonic parallel Lagrangian submanifolds. Let $\mathbb{C}P^n(4)$ denote the complex projective space of complex dimension n and constant holomorphic sectional curvature 4. We denote by J the almost complex structure of $\mathbb{C}P^n(4)$. An n-dimensional submanifold M^n of $\mathbb{C}P^n(4)$ is said to be *Lagrangian* if J interchanges the tangent and the normal spaces at each point.

In [5, Theorem 6.3], Fetcu and Oniciuc presented the classification result of proper biharmonic parallel Lagrangian submanifolds in $\mathbb{C}P^3(4)$. However, the theorem is proved by applying the wrong statement of [1, Lemma 4.4], and hence the classification is incomplete. This section completes it. First, we recall the following.

PROPOSITION 4.1 ([5]). *Let $L : M^3 \to \mathbb{C}P^3(4)$ be a proper biharmonic parallel Lagrangian immersion. Then L is locally given by $\pi \circ f$, where $\pi : S^{2n+1}(1) \to \mathbb{C}P^n(4)$ is the Hopf fibration and $f : M^3 \to S^7(1)$ is a non-minimal C-parallel Legendrian immersion satisfying*

$$\operatorname{Tr} h(\cdot, A_H \cdot) = 6H\,.$$

The following theorem determines explicitly all proper biharmonic parallel Lagrangian submanifolds in $\mathbb{C}P^3(4)$.

THEOREM 4.1. *Let $L : M^3 \to \mathbb{C}P^3(4)$ a proper biharmonic parallel Lagrangian submanifold. Then L is locally congruent to $\pi \circ f$, where $f : M^3 \to S^7(1)$ is one of the following*:

(1) *M^3 is flat and*

$$\begin{aligned} f(u,v,w) = \Bigg(&\frac{\lambda}{\sqrt{\lambda^2+1}} \exp\left(i\left(\frac{1}{\lambda}u\right)\right), \frac{1}{\sqrt{(c-a)(2c-a)}} \exp(-i(\lambda u - (c-a)v))\,, \\ &\frac{1}{\sqrt{\rho_1(\rho_1+\rho_2)}} \exp(-i(\lambda u + cv + \rho_1 w)), \frac{1}{\sqrt{\rho_2(\rho_1+\rho_2)}} \exp(-i(\lambda u + cv - \rho_2 w)) \Bigg), \end{aligned} \tag{4.1}$$

where $\rho_{1,2} = (\sqrt{4c(2c-a)+d^2} \pm d)/2$ and the 4-tuple (λ, a, c, d) is given by one of the following:

$$\left(-\sqrt{\frac{4-\sqrt{13}}{3}}, \sqrt{\frac{7-\sqrt{13}}{6}}, -\sqrt{\frac{7-\sqrt{13}}{6}}, 0\right),$$

$$\left(-\sqrt{\frac{1}{5+2\sqrt{3}}}, \sqrt{\frac{45+21\sqrt{3}}{13}}, -\sqrt{\frac{6}{21+11\sqrt{3}}}, 0\right),$$

$$\left(-\sqrt{\frac{1}{6+\sqrt{13}}}, \sqrt{\frac{523+139\sqrt{13}}{138}}, -\sqrt{\frac{79-17\sqrt{13}}{138}}, \sqrt{\frac{14+2\sqrt{13}}{3}}\right);$$

(2) *M^3 is non-flat and*

$$f(x,\mathbf{y}) = \left(\sqrt{\frac{\mu^2}{\mu^2+1}}e^{-\frac{i}{\mu}x}, \sqrt{\frac{1}{\mu^2+1}}e^{i\mu x}\mathbf{y}\right), \tag{4.2}$$

where $\mathbf{y} = (y_1, y_2, y_3)$, $\|\mathbf{y}\| = 1$ *and* $\mu^2 = (4 \pm \sqrt{13})/3$.

PROOF. The flat case (1) has been proved in [5, Corollary 6.4]. Applying Propositions 3.2 and 4.1 and modifying the second equation of [12, (5.33)] to $\lambda^2 + 2\mu^2 = 6$, we can prove the non-flat case (2). □

REMARK 4.1. Fetcu and Oniciuc [5] did not give the explicit representation of non-flat proper biharmonic parallel Lagrangian submanifolds in $\mathbb{C}P^3(4)$.

REMARK 4.2. The immersion (4.1) can be rewritten as the one with $\alpha = 1$ in Remark 1.2 (cf. [3], [8]).

REMARK 4.3. The immersion (4.2) has the same properties as in Remark 1.3, where $\alpha = 1$. From this, we see that (4.2) with $\mu^2 = (4 + \sqrt{13})/3$ is missing from [5, Theorem 6.3].

REMARK 4.4. The author classified proper biharmonic Lagrangian surfaces of constant mean curvature in $\mathbb{C}P^2(4)$ (see [11]). Those surfaces are flat and parallel.

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