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CHARACTERIZATION OF 2-DIMENSIONAL NORMAL MATHER-JACOBIAN LOG CANONICAL SINGULARITIES

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Abstract. In this paper we characterize 2-dimensional normal Mather-Jacobian log canonical singularities which are not complete intersections. We prove that a 2-dimensional normal singularity which is not a complete intersection is a Mather-Jacobian log canonical singularity if and only if it is a toric singularity with embedding dimension 4.

Introduction. In birational geometry, the notion of discrepancy plays an important role. By using this discrepancy, we can define canonical, log canonical, terminal and log terminal singularities. Those singularities are all normal $\mathbb{Q}$-Gorenstein singularities and these conditions are essential in birational geometry. But these conditions seem to be an unnecessary restriction for a singularity to be considered as a good singularity.

By using the jet schemes and the Nash blow-up of a variety, Ishii [10], de Fernex and Docampo [2] independently introduced the notion of Mather-Jacobian log discrepancy, which is a modification of the classical definition of discrepancy without the restriction of normal $\mathbb{Q}$-Gorenstein property. This discrepancy was introduced as an alternative discrepancy on which Inversion of Adjunction holds in full generality, and many good properties are drawn from this property. Henceforth, in this paper, we denote Mather-Jacobian by MJ for short.

According to this MJ-discrepancy, MJ-canonical and MJ-log canonical are defined in a similar way as the usual canonical and log canonical singularities (see [10] and [2]). These singularities have good properties: for example, stability under deformations, lower semi continuity of MJ-minimal log discrepancies and Shokurov's conjecture holds ([10], Corollary 3.15 and [2], Corollary 4.15). Then, it is natural to ask what kind of singularities are MJ-canonical or MJ-log canonical.

In [5], Ein and Ishii determined MJ-canonical singularities of dimension 2. They also determined complete intersection MJ-log canonical singularities of dimension 2 and showed that a non-complete intersection MJ-log canonical singularity of dimension 2 is embedded into an MJ-log canonical complete intersection surface.

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In this paper, we determine normal MJ-log canonical surface singularities which are not complete intersections. More precisely, we prove that a non-complete intersection normal MJ-log canonical surface singularity is exactly a rational triple point on a toric surface. By this, we complete the characterization of a normal MJ-log canonical surface singularity.

The paper is organized as follows:

In Section 2, we state some preliminaries about MJ-singularities and jet schemes for the convenience of the reader. We refer to [6] for further details. In Section 3, we state and prove the main result:

THEOREM 0.1 (Theorem 2.1). *Let (X, p) be a singularity on a normal surface X. We assume that (X, p) is not locally a complete intersection. Then the following are equivalent:*

(i) *(X, p) is MJ-log canonical singularity,*
(ii) *(X, p) is a toric singularity with multiplicity* 3,
(iii) *(X, p) is a toric singularity with embedding dimension* 4.

In Section 4, we give the appendix about toric varieties: give the defining ideal of a toric variety with embedding dimension 4 (This description of the ideal is used in Section 3 for the proof of the main theorem) and show that a toric surface singularity with embedding dimension 4 is MJ-log canonical. We also give an example for a toric singularity of dimension 3 with embedding dimension 6 which is not MJ-log canonical.

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1. Preliminaries on Mather-Jacobian minimal log discrepancy.

We start by recalling the definition and basic properties of Mather-Jacobian log discrepancy which is defined in [6]. We refer to [6] for further details.

Throughout this paper, by a variety we mean a reduced equi-dimensional separated scheme of finite type over $\mathbb{C}$.

Let X be a variety of dimension $\dim X = d$. The sheaf Ω_X^d is invertible over the smooth locus X_{reg} of X, hence the projection

$$\pi : P(\Omega_X^d) \longrightarrow X$$

is an isomorphism over X_{reg}. The Nash blow up $\widehat{X} \longrightarrow X$ is defined as the closure of $\pi^{-1}(X_{reg})$ in $P(\Omega_X^d)$.

DEFINITION 1.1. Let $\varphi : Y \longrightarrow X$ be a resolution of singularities of X that factors through the Nash blow-up of X. The image of the canonical homomorphism

$$\varphi^*(\Omega_X^d) \longrightarrow \Omega_Y^d$$

is an invertible sheaf of the form $Jac_f \Omega_Y^d$, where Jac_f is the relative Jacobian which is an invertible ideal on Y and defines an effective divisor supported on the exceptional locus of φ which is called the Mather discrepancy divisor and denoted by $\widehat{K}_{Y/X}$.

DEFINITION 1.2. Let $\varphi : Y \longrightarrow X$ be a log resolution of Jac_X, where Jac_X is the Jacobian ideal of a variety X. We denote by $J_{Y/X}$ the effective divisor on Y such that $Jac_X \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$.

Here, we note that every log resolution of Jac_X factors through the Nash blow-up, see for example, Remark 2.3, in [6].

DEFINITION 1.3. Let E be a prime divisor over X and $\varphi : Y \longrightarrow X$ be a log resolution of Jac_X, on which E appears. We define the Mather-Jacobian-log discrepancy at E as

$$a_{\mathrm{MJ}}(E; X) := \mathrm{ord}_E(\widehat{K}_{Y/X} - J_{Y/X}) + 1 .$$

Note that the Mather-Jacobian log discrepancy at a prime divisor E does not depend on the choice of φ. We denote $\mathrm{ord}_E \widehat{K}_{Y/X}$ by $\widehat{k}_E$.

DEFINITION 1.4. Let W be a closed subset of X such that it does not contain an irreducible component of X. The Mather-Jacobian minimal log discrepancy of X along W is defined as

$$\mathrm{mld}_{\mathrm{MJ}}(W; X) = \inf\{a_{\mathrm{MJ}}(E; X) \mid E \text{ prime divisor over } X \text{ with center in } W\}$$

if $\dim X \geq 2$, or $\dim X = 1$ and the infimum on the right hand side is non-negative; otherwise, we set $\mathrm{mld}_{\mathrm{MJ}}(W; X) = -\infty$.

DEFINITION 1.5. We say that a point p of X is Mather-Jacobian log canonical (MJ-log canonical for short) singularity of X if the inequality $\mathrm{mld}_{\mathrm{MJ}}(p; X) \geq 0$ holds, i.e., for every exceptional prime divisor E over X with center $\{p\}$, we have inequality $a_{\mathrm{MJ}}(E; X) \geq 0$.

Here we note that the definition of MJ-log canonical singularity in [6] is different. But it is equivalent to the above definition by [5], Proposition 2.22, (i).

We will introduce the basic definition of jet schemes and the relation to Mather-Jacobian minimal log discrepancy. For the theory on jet schemes and arc space, see for example [7].

DEFINITION 1.6. Let X be a k-scheme, $K \supset k$ a field extension, and $m \in \mathbb{Z}_{\geq 0}$ a non-negative integer. A k-morphism $\mathrm{Spec}\, K[t]/(t^{m+1}) \longrightarrow X$ is called an m-jet of X and a k-morphism $\mathrm{Spec}\, K[[t]] \longrightarrow X$ is called an *arc* of X.

We denote the m-th jet scheme of X by X_m, and the space of arcs by X_∞. We have canonical morphisms $\psi_m : X_\infty \longrightarrow X_m$ and $\pi_m : X_m \longrightarrow X$. When we need to specify the scheme X, we denote them by $\psi_{X,m} : X_\infty \longrightarrow X_m$ and $\pi_{X,m} : X_m \longrightarrow X$.

The following proposition is given in Proposition 2.14, (2) in [5], which is used in the next section.

PROPOSITION 1.7. *Let $X \subset \mathbb{A}^N$ be a variety of dimension d. Then for a closed point $p \in X$,*

$$\mathrm{mld}_{\mathrm{MJ}}(p, X) = \mathrm{mld}(p, \mathbb{A}^N, I_X^{N-d}) = \inf_m\{(m+1)d - \dim\pi_m^{-1}(p)\}\,.$$

2. Normal MJ-log canonical surface singularity.

We are going to prove the following theorem:

THEOREM 2.1. *Let (X, p) be a singularity on a normal surface X. We assume that (X, p) is not locally a complete intersection. Then the following are equivalent:*

(i) *(X, p) is an MJ-log canonical singularity,*
(ii) *(X, p) is a toric singularity with multiplicity* 3,
(iii) *(X, p) is a toric singularity with embedding dimension* 4.

Note that the equivalence (ii)$\Leftrightarrow$(iii) follows immediately from Artin's formula $\mathrm{emb}(X, p) = \mathrm{mult}_p X + 1$ ([1]), because a toric singularity is rational and we can apply the formula.

The program of the rest of the proof is as follows:

First we prove that a 2-dimensional MJ-log canonical singularity which is not a complete intersection is a rational triple point (Proposition 2.2). Then by checking Tyurina's list of rational triple points, we pick up possible rational triple points for being MJ-log canonical and we prove that these singularities are toric triple points (Proposition 2.5). This completes the proof of (i)$\Rightarrow$(ii).

Then, in Proposition 2.6 we prove (iii)$\Rightarrow$(i) by determining the defining ideal of a toric variety with embedding dimension 4 and apply Theorem 5.6 in [5].

PROPOSITION 2.2. *Let (X, p) be a normal MJ-log canonical surface singularity. Then (X, p) is a log terminal singularity with multiplicity* 3 *or a complete intersection log canonical singularity.*

PROOF. By Theorem 3.19 in [5], X is log canonical in the sense of de Fernex and Hacon. De fernex and Hacon proved that for a normal surface, log canonical in their sense is equivalent to log canonical in the usual sense (Corollary 7.15 in [3]). Thus X is log canonical in the usual sense.

We assume that (X, p) is not a complete intersection singularity. Let r be a positive integer such that rK_X is Cartier and let $\mathfrak{d}_{r,X}$ be the lci-defect ideal of level r of X. Then $\mathrm{ord}_E(\mathfrak{d}_{r,X}) \geq 1$ for every prime divisor E over X whose center is p. By Proposition 3.4 in [2], for every prime divisor E over X whose center is x, we have

$$\widehat{k_E}(X) - \mathrm{ord}_E(Jac_X) = k_E(X) - \frac{1}{r}\mathrm{ord}_E(\mathfrak{d}_{r,X}).$$

Thus we have $k_E(X) > -1$ for every prime divisor E over X whose center is p. This implies that X is log terminal. Since log terminal singularities are rational singularities, p is a rational singularity.

By Proposition 1.7, we have $0 \le \mathrm{mld}_{\mathrm{MJ}}(p, X) \le 2(m+1) - \dim \pi_m^{-1}(p)$ for every $m \in \mathbb{N}$. By substituting $m = 1$, we obtain $\dim \pi_1^{-1}(p) \le 4$. This implies that $\mathrm{emb}(X, p) \le 4$. Since (X, x) is not a complete intersection singularity, $\mathrm{emb}(X, p) = 4$. As X is rational at p, we can apply Artin's formulain in [1] $\mathrm{mult}_p(X) = \mathrm{emb}(X, p) - 1 = 3$. □

The rational triple singularities are defined by 3 equations in $\mathbb{C}^4$ and classified into 9 types. The explicit equations were first obtained by Tyurina in [13].

PROPOSITION 2.3 ([13]). Let (X, p) be a rational triple point on a surface in $\mathbb{C}^4$. The defining ideal of X in $\mathbb{C}^4$ is one of the following (1) – (9) :

(1) $A_{k-1,l-1,m-1}$, $k \ge l \ge m \ge 1$

$$(x(y + w^k) - yw^m, yz - (y + w^k)w^l, xz - w^{l+m}).$$

(2) $B_{m,n}$, $m \ge 0$

$n = 2k - 1, k \ge 2$

$$(xz - y^{m+1}(y^k + yw), (y^k + yw)w - z^2, xw - y^{m+1}z),$$

$n = 2k, k \ge 2$

$$(x(z + y^k) - y^{m+2}w, yw^2 - z(z + y^k), xw - y^{m+1}z).$$

(3) $C_{m,n}$, $m \ge 3, n \ge 0$

$$(xz - y^{n+1}(y^2 + w^{m-1}), (y^2 + w^{m-1})w - z^2, xw - y^{n+1}z).$$

(4) D_n, $n \ge 0$

$$(x(y^2 + z) - y^{n+1}w^2, w^3 - z(y^2 + z), xw - y^{n+1}z).$$

(5) $E_{6,0}$

$$(x^2z - (y + w^2)y, yw - z^2, x^2w - (y + w^2)z).$$

(6) $E_{0,7}$

$$((x^2 + w^3)z - y^2, yw - z^2, (x^2 + w^3)w - yz).$$

(7) $E_{7,0}$

$$(w^2z - (y + x^2)y, yw - z^2, w^3 - (y + x^2)z).$$

(8) F_n, $n \ge 0$

$$(xz - y^{n+1}(y^3 + w^3), (y^3 + w^3)w - z^2, xw - y^{n+1}z).$$

(9) H_n

$n = 3k-1, k \geq 2$

$$((xw + x^k)z - y^2, yw - z^2, (xw + x^k)w - yz)\,,$$

$n = 3k, k \geq 2$

$$(xwz - (y + x^k)y, yw - z^2, xw^2 - (y + x^k)z)\,,$$

$n = 3k+1, k \geq 2$

$$(xw(z + x^k) - y^2, yw - (z + x^k)z, xw^2 - yz)\,.$$

LEMMA 2.4. *Let $X = \operatorname{Spec}\mathbb{C}[x_1, \dots, x_{2n}]/(f_1, \dots, f_r)$ be an n-dimensional variety with $\operatorname{emb}(X, p) = 2n$, where p is the origin. Let $I_Y = (\operatorname{in}(f_1), \dots, \operatorname{in}(f_r))$, where $\operatorname{in}(f_j)$ is the initial term of f_j, i.e. the sum of all monomials in f_j of smallest degree. Let $Y = \operatorname{Spec}\mathbb{C}[x_1, \dots, x_{2n}]/I_Y$ and $q \in Y$ be the origin. If X is MJ-log canonical at p, then $\operatorname{mld}(q, \mathbb{A}^{2n}, I_Y^n) \geq 0$.*

PROOF. In this proof we will use the formula in Proposition 1.7. So we need the description of the fiber of $\pi_{X,m} : X_m \longrightarrow X$. We use the notation in ([5], 2.11, Remark 2.12) which provides with basic facts on the fiber of $\pi_{X,m}$. We have $\pi_{X,m}^{-1}(p) = \operatorname{Spec}\mathbb{C}[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}]/(f_j^{(i)})_{1\leq j\leq r, 1\leq i\leq m}$, where $\mathbf{x}^{(i)}$ denotes the collection of variables $x_1^{(i)}, \dots, x_{2n}^{(i)}$ and $\Sigma_{i=1}^{\infty} f_j^{(i)} t^i$ is the Taylor expansion of $f_j(\Sigma_{i=1}^{\infty} \mathbf{x}^{(i)} t^i)$. By the assumption that the embedding dimension is $2n$, we have all monomials of f_j are of degree ≥ 2. Therefore we have $f_j^{(i)} \in \mathbb{C}[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(i-1)}]$. Let I_m be the defining ideal of $\pi_{X,m}^{-1}(p)$ in $\mathbb{C}[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}]$.

Let l be a positive integer. Let us denote $\overline{f}$ the function obtained by substituting the values $x_1^{(1)} = \cdots = x_{2n}^{(1)} = \cdots = x_1^{(l-1)} = \cdots = x_{2n}^{(l-1)} = 0$ into $f \in \mathbb{C}[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}]$.

CLAIM 1. For a monomial $h = x_{j_1} \cdots x_{j_d} \in \mathbb{C}[x_1, \dots, x_{2n}]$ of degree $d \geq 3$, it follows that $\overline{h^{(i)}} = 0$ $(1 \leq i \leq 3l-1)$.

In fact, $h^{(i)}$ is a sum of monomials $x_{j_1}^{(l_1)} \cdots x_{j_d}^{(l_d)}$ with $l_1 + \cdots + l_d = i \leq 3l-1$. For each monomial $x_{j_1}^{(l_1)} \cdots x_{j_d}^{(l_d)}$, there is k $(1 \leq k \leq d)$ such that $l_k < l$, because $\Sigma_{t=1}^{d} l_t \leq 3l-1$ and $d \geq 3$. This yields that $\overline{h^{(i)}} = 0$ for $1 \leq i \leq 3l-1$ as claimed.

CLAIM 2. For a polynomial $h \in \mathbb{C}[x_1, \dots, x_{2n}]$ whose monomials are all of degree ≥ 2, it follows that

$$\overline{h^{(i)}} = \overline{\operatorname{in}(h)^{(i)}} \quad (1 \leq i \leq 3l-1)\,.$$

In fact, when all monomials of h have degree ≥ 3, the both sides are zero by Claim 1. On the other hand, when there is a monomial of degree 2 in h, then all monomials of degree ≥ 3 vanish in $\overline{h^{(i)}}$ by Claim 1 and only the monomials of degree 2 survive in $\overline{h^{(i)}}$. Anyway the required equality holds.

Now we return to the proof of the lemma. Note that all monomials of f_j are of degree ≥ 2. Therefore, by the claims, we obtain

$$\{\overline{f_1^{(2l)}}, \dots, \overline{f_r^{(2l)}}, \dots, \overline{f_1^{(3l-1)}}, \dots, \overline{f_r^{(3l-1)}}\}$$

$$= \{\overline{\mathrm{in}(f_1)^{(2l)}}, \dots, \overline{\mathrm{in}(f_r)^{(2l)}}, \dots, \overline{\mathrm{in}(f_1)^{(3l-1)}}, \dots, \overline{\mathrm{in}(f_r)^{(3l-1)}}\}\,.$$

We replace $\mathbf{x}^{(m)}$ by $\mathbf{x}^{(m-(l-1))}$ for every $m \geq l$. Then $\overline{\mathrm{in}(f_j)^{(m)}}$ becomes $\mathrm{in}(f_j)^{(m-2(l-1))}$ for f_j with $\mathrm{ord}(f_j) = 2$. Indeed, this follows from the fact that $\mathrm{in}(f_j)(\Sigma_{i=1}^{\infty}\mathbf{x}^{(i)}t^i) = \Sigma_{i=1}^{\infty}\mathrm{in}(f_j)^{(i)}t^i$ and $\mathrm{in}(f_j)(\Sigma_{i=l}^{\infty}\mathbf{x}^{(i)}t^i) = \Sigma_{i=2l}^{\infty}\overline{\mathrm{in}(f_j)^{(i)}}t^i$ for f_j with $\mathrm{ord}(f_j) = 2$. Thus we get

$$\begin{aligned}&\mathrm{ht}(\mathrm{in}(f_1)^{(2)}, \dots, \mathrm{in}(f_r)^{(2)}, \dots, \mathrm{in}(f_1)^{(l+1)}, \dots, \mathrm{in}(f_r)^{(l+1)})\\ &\quad\geq \mathrm{ht}(\overline{f_1^{(2l)}}, \dots, \overline{f_r^{(2l)}}, \dots, \overline{f_1^{(3l-1)}}, \dots, \overline{f_r^{(3l-1)}})\,.\end{aligned}$$

Next we will prove that

$$\mathrm{ht}(\mathrm{in}(f_1)^{(2)}, \dots, \mathrm{in}(f_r)^{(2)}, \dots, \mathrm{in}(f_1)^{(l+1)}, \dots, \mathrm{in}(f_r)^{(l+1)}) \geq nl\,.$$

Since X is MJ-log canonical at p, $\mathrm{mld}_{\mathrm{MJ}}(p, X) \geq 0$. By Proposition 1.7, we have $n(m + 1) - \dim\pi_{X,m}^{-1}(p) \geq 0$ for any $m \in \mathbb{N}$. This implies that $\mathrm{ht}(I_m) \geq n(m - 1)$. When we consider the case $m = 3l - 1$, we have $\mathrm{ht}I_{3l-1} \geq n(3l - 2)$. On the other hand it follows that

$$I_{3l-1} \subset (\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(l-1)}, \overline{f_1^{(2l)}}, \dots, \overline{f_r^{(2l)}}, \dots, \overline{f_1^{(3l-1)}}, \dots, \overline{f_r^{(3l-1)}})\,,$$

because every monomial in $f^{(i)}$ $(1 \leq i \leq 2l - 1)$ has a factor in $\mathbf{x}^{(j)}$, $j \leq l - 1$. Thus we have

$$\mathrm{ht}(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(l-1)}, \overline{f_1^{(2l)}}, \dots, \overline{f_r^{(2l)}}, \dots, \overline{f_1^{(3l-1)}}, \dots, \overline{f_r^{(3l-1)}}) \geq n(3l - 2)\,.$$

This implies that

$$\mathrm{ht}(\overline{f_1^{(2l)}}, \dots, \overline{f_r^{(2l)}}, \dots, \overline{f_1^{(3l-1)}}, \dots, \overline{f_r^{(3l-1)}}) \geq nl\,,$$

since $\mathrm{ht}(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(l-1)}) = 2n(l-1)$ and $\overline{f_j^{(i)}} \in \mathbb{C}[\mathbf{x}^{(l)}, \dots, \mathbf{x}^{(m)}]$ for $1 \leq j \leq r, 2l \leq i \leq 3l-1, m \geq l$. Thus we have

$$\mathrm{ht}(\mathrm{in}(f_1)^{(2)}, \dots, \mathrm{in}(f_r)^{(2)}, \dots, \mathrm{in}(f_1)^{(l+1)}, \dots, \mathrm{in}(f_r)^{(l+1)}) \geq nl\,,$$

as required.

Note that all monomials of f_j are of degree ≥ 2. Therefor we have

$$\pi_{Y,m}^{-1}(q) = \mathrm{Spec}\,\mathbb{C}[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}]/(\mathrm{in}(f_j)^{(i)})_{1\leq j\leq r, 2\leq i\leq m}\,.$$

By Proposition 1.7, we have

$$\begin{aligned}\mathrm{mld}(q, \mathbb{A}^{2n}, I_Y^n) &= \inf_m\{n(m + 1) - \dim\pi_{Y,m}^{-1}(q)\}\\ &= \inf_m\{n(m + 1) - 2nm + \mathrm{ht}(\mathrm{in}(f_1)^{(2)}, \dots, \mathrm{in}(f_r)^{(2)}, \dots, \mathrm{in}(f_1)^{(m)}, \dots, \mathrm{in}(f_r)^{(m)})\}\\ &\geq \inf_m\{n(m + 1) - 2nm + n(m - 1)\} \geq 0\,.\end{aligned}$$

□

Note that here Y is not necessarily the tangent cone of X at p.

PROPOSITION 2.5. *Let (X, p) be a normal MJ-log canonical surface singularity. If p is a log terminal singularity with multiplicity* 3*, then it is a toric singularity.*

PROOF. Since p is a log terminal singularity with multiplicity 3, p is a rational triple point. We may assume that the defining ideals I_X of X in $\mathbb{C}^4$ is one of the ideals (1) – (9). Let $I_X = (f_1, f_2, f_3)$ and $I_Y = (\mathrm{in}(f_1), \mathrm{in}(f_2), \mathrm{in}(f_3))$. Let $Y = \mathrm{Spec}\mathbb{C}[x, y, z, w]/I_Y$ and $q \in Y$ be the origin.

(1) Let X be $A_{k-1,l-1,m-1}$, $k \geq l \geq m \geq 1$, the defining ideal of X is

$$I_X = (x(y + w^k) - yw^m, yz - (y + w^k)w^l, xz - w^{l+m}) .$$

We assume $m \geq 2$. Then we have $I_Y = (xy, yz, xz)$. Let

$$(\pi_3^Y)^{-1}(q) = \mathbb{C}[x^{(1)}, y^{(1)}, z^{(1)}, w^{(1)}, \dots, x^{(3)}, y^{(3)}, z^{(3)}, w^{(3)}]/I_3 .$$

Since $I_3 \subset (x^{(1)}, y^{(1)}, z^{(1)})$, by the form of the generators of I_Y, we have $\dim(\pi_{Y,3})^{-1}(q) \geq 9$. By Proposition 1.7,

$$\begin{aligned}\mathrm{mld}(q, \mathbb{A}^4, I_Y^2) &= \inf_m\{2(m + 1) - \dim\pi_{Y,m}^{-1}(q)\} \\ &\leq 8 - \dim(\pi_{Y,3})^{-1}(q) \leq -1 .\end{aligned}$$

This implies that $\mathrm{mld}(q, \mathbb{A}^4, I_Y^2) = -\infty$. It follows by Proposition 2.4 that X is not MJ-log canonical at p. Therefore, for X to be MJ-log canonical at p, m must be 1.

(2) Let X be $B_{m,n}$, $m \geq 0$.
If $n = 2k - 1, k \geq 2$, then the defining ideal of X is

$$I_X = (xz - y^{m+1}(y^k + yw), (y^k + yw)w - z^2, xw - y^{m+1}z) .$$

If $n = 2k, k \geq 2$, then the defining ideal of X is

$$I_X = (x(z + y^k) - y^{m+2}w, yw^2 - z(z + y^k), xw - y^{m+1}z) .$$

We assume that $m = 0$. Then we have $I_Y = (xz, z^2, xw - yz)$. Let $(\pi_{Y,4})^{-1}(q) = \mathbb{C}[x^{(1)}, y^{(1)}, z^{(1)}, w^{(1)}, \dots, x^{(4)}, y^{(4)}, z^{(4)}, w^{(4)}]/I_4$. Since $I_4 \subset (x^{(1)}, x^{(2)}, z^{(1)}, z^{(2)}, x^{(3)}w^{(1)} - y^{(1)}z^{(3)})$, we have $\dim(\pi_{Y,4})^{-1}(q) \geq 11$. By Proposition 1.7,

$$\begin{aligned}\mathrm{mld}(q, \mathbb{A}^4, I_Y^2) &= \inf_m\{2(m + 1) - \dim\pi_{Y,m}^{-1}(q)\} \\ &\leq 10 - \dim(\pi_{Y,4})^{-1}(q) \leq -1 .\end{aligned}$$

This implies that $\mathrm{mld}(q, \mathbb{A}^4, I_Y^2) = -\infty$. It follows by Proposition 2.4 that X is not MJ-log canonical at p. We assume that $m \geq 1$. Then we have $I_Y = (xz, z^2, xw)$. Let $(\pi_{Y,3})^{-1}(q) = \mathbb{C}[x^{(1)}, y^{(1)}, z^{(1)}, w^{(1)}, \dots, x^{(3)}, y^{(3)}, z^{(3)}, w^{(3)}]/I_3$. Since $I_3 \subset (x^{(1)}, z^{(1)}, w^{(1)})$, we have $\dim(\pi_{Y,3})^{-1}(q) \geq 9$. By Proposition 1.7,

$$\begin{aligned}\mathrm{mld}(q, \mathbb{A}^4, I_Y^2) &= \inf_m\{2(m + 1) - \dim\pi_{Y,m}^{-1}(q)\} \\ &\leq 8 - \dim(\pi_{Y,3})^{-1}(q) \leq -1 .\end{aligned}$$

This implies that $\mathrm{mld}(q, \mathbb{A}^4, I_Y^2) = -\infty$. It follows by Proposition 2.4 that X is not MJ-log canonical at p.

(3) Let X be $C_{m,n}$, $m \geq 3, n \geq 0$, then the defining ideal of X is

$$I_X = (xz - y^{n+1}(y^2 + w^{m-1}), (y^2 + w^{m-1})w - z^2, xw - y^{n+1}z) .$$

We assume that $n = 0$, then we have $I_Y = (xz, z^2, xw - yz)$. In (2) we proved $\mathrm{mld}(q, \mathbb{A}^4, I_Y^2) = -\infty$. It follows by Proposition 2.4 that X is not MJ-log canonical at p. Next we assume that $n \geq 1$, then we have $I_Y = (xz, z^2, xw)$. In (2) we proved $\mathrm{mld}(q, \mathbb{A}^4, I_Y^2) = -\infty$. It follows by Proposition 2.4 that X is not MJ-log canonical at p.

(4) Let X be D_n, $n \geq 0$, then the defining ideal of X is

$$I_X = (x(y^2 + z) - y^{n+1}w^2, w^3 - z(y^2 + z), xw - y^{n+1}z) .$$

We assume that $n = 0$, then we have $I_Y = (xz, z^2, xw - yz)$. In (2) we proved $\mathrm{mld}(q, \mathbb{A}^4, I_Y^2) = -\infty$. It follows by Proposition 2.4 that X is not MJ-log canonical at p. We assume that $n \geq 1$. Then we have $I_Y = (xz, z^2, xw)$. In (2) we proved $\mathrm{mld}(q, \mathbb{A}^4, I_Y^2) = -\infty$. It follows by Proposition 2.4 that X is not MJ-log canonical at p.

(5) Let X be $E_{6,0}$, then the defining ideal of X is

$$I_X = (x^2z - (y + w^2)y, yw - z^2, x^2w - (y + w^2)z) .$$

Then we have $I_Y = (y^2, yw - z^2, yz)$.
Let $(\pi_{Y,3})^{-1}(q) = \mathbb{C}[x^{(1)}, y^{(1)}, z^{(1)}, w^{(1)}, \ldots, x^{(3)}, y^{(3)}, z^{(3)}, w^{(3)}]/I_3$. Since $I_3 \subset (y^{(1)}, z^{(1)}, w^{(1)})$, we have $\dim(\pi_{Y,3})^{-1}(q) \geq 9$. By Proposition 1.7,

$$\begin{aligned}\mathrm{mld}(q, \mathbb{A}^4, I_Y^2) &= \inf_m\{2(m + 1) - \dim\pi_{Y,m}^{-1}(q)\}\\ &\leq 8 - \dim(\pi_{Y,3})^{-1}(q) \leq -1 .\end{aligned}$$

This implies that $\mathrm{mld}(q, \mathbb{A}^4, I_Y^2) = -\infty$. It follows by Proposition 2.4 that X is not MJ-log canonical at p.

(6) Let X be $E_{0,7}$, then the defining ideal of X is

$$I_X = ((x^2 + w^3)z - y^2, yw - z^2, (x^2 + w^3)w - yz) .$$

Then we have $I_Y = (y^2, yw - z^2, yz)$.
In (5) we proved $\mathrm{mld}(q, \mathbb{A}^4, I_Y^2) = -\infty$. It follows by Proposition 2.4 that X is not MJ-log canonical at p.

(7) Let X be $E_{7,0}$, then the defining ideal of X is

$$I_X = (w^2z - (y + x^2)y, yw - z^2, w^3 - (y + x^2)z) .$$

Then we have $I_Y = (y^2, yw - z^2, yz)$.
In (5) we proved $\mathrm{mld}(q, \mathbb{A}^4, I_Y^2) = -\infty$. It follows by Proposition 2.4 that X is not MJ-log canonical at p.

(8) Let X be F_n, $n \geq 0$, then the defining ideal of X is

$$I_X = (xz - y^{n+1}(y^3 + w^3), (y^3 + w^3)w - z^2, xw - y^{n+1}z)\,.$$

We assume that $n = 0$, then we have $I_Y = (xz, z^2, xw - yz)$.
In (2) we proved $\mathrm{mld}(q, \mathbb{A}^4, I_Y^2) = -\infty$. It follows by Proposition 2.4 that X is not MJ-log canonical at p. We assume that $n \geq 1$. Then we have $I_Y = (xz, z^2, xw)$. In (2) we proved $\mathrm{mld}(q, \mathbb{A}^4, I_Y^2) = -\infty$. It follows by Proposition 2.4 that X is not MJ-log canonical at p.

(9) Let X be H_n.
If $n = 3k - 1, k \geq 2$, then the defining ideal of X is

$$I_X = ((xw + x^k)z - y^2, yw - z^2, (xw + x^k)w - yz)\,.$$

If $n = 3k, k \geq 2$, then the defining ideal of X is

$$I_X = (xwz - (y + x^k)y, yw - z^2, xw^2 - (y + x^k)z)\,.$$

If $n = 3k + 1, k \geq 2$, then the defining ideal of X is

$$I_X = (xw(z + x^k) - y^2, yw - (z + x^k)z, xw^2 - yz)\,.$$

Then, in any case, we have $I_Y = (y^2, yw - z^2, yz)$, which is the same ideal as in (5).
In (5) we proved $\mathrm{mld}(q, \mathbb{A}^4, I_Y^2) = -\infty$. It follows by Proposition 2.4 that X is not MJ-log canonical at p.

By these discussions, we have that (X, p) is $A_{k-1,l-1,0}$ for some $k \geq l \geq 1$. Tyurina proved in [13] that the exceptional curve on the minimal resolution of this singularity is a chain of $\mathbb{P}^1$'s. It is well known that such a singularity is a toric singularity (see, for example [9], Theorem 7.4.17) .

□

PROPOSITION 2.6. *Let (X, p) be a toric singularity of embedding dimension* 4*, then X is MJ-log canonical at p.*

PROOF. According to the Theorem 5.6, in [5], we have the folllowing :

(i) In case (X, p) is locally a complete intersection:
X is MJ-log canonical at p if and only if $\widehat{O_{X,p}} \simeq k[[x_1, x_2, x_3, x_4]]/(f, g)$, where f, g satisfy the conditions that $\mathrm{mult}_p f = \mathrm{mult}_p g = 2$ and $V(\mathrm{in}(f), \mathrm{in}(g)) \subset \mathbb{P}^3$ is a reduced curve with at worst ordinary double points. Here $\mathrm{in}(f)$ is the initial form of f according to the degree.

(ii) In case (X, p) is not locally a complete intersection: X is MJ-log canonical at p if and only if X is a subscheme of a 2-dimensional locally complete intersection scheme M which is MJ-log canonical at p.

We will see in Proposition 3.2 in the next section, that a toric variety X with embedding dimension 4 at p is described as

$$X = \mathrm{Spec}\mathbb{C}[x, y, z, w]/(xz - y^{n+2}, yw - z^{m+2}, xw - y^{n+1}z^{m+1})\ (m \geq 0, n \geq 0)\,.$$

Let

$$W = \mathrm{Spec}\mathbb{C}[x, y, z, w]/(xz - y^{n+2}, yw - z^{m+2})\,.$$

Then W is a complete intersection surface containing X. By (ii), it is enough to show that W is MJ-log canonical at p. Since $V(\mathrm{in}(xz - y^{n+2}), \mathrm{in}(yw - z^{m+2})) \subset \mathbb{P}^3$ is a reduced curve with at worst ordinary nodes, W is MJ-log canonical at p by (i). □

This completes the proof of Theorem 2.1.

3. Appendix on toric varieties. In this section, we give a description of a toric surface singularity, with embedding dimension 4, and we also give a direct proof of the implication (iii)⇒(i) in Theorem 2.1.

The proof of (iii)⇒(i) in the previous section used the result of [5] which is based on the classification of space curves. So we think that it makes sense for us to give a direct proof based on toric discussions.

Let $\sigma \subset \mathbb{R}^n$ be a strongly convex rational polyhedral cone of maximal dimension. $\mathcal{H}$ is called the Hilbert basis of $\sigma^\vee \cap \mathbb{Z}^n$ if $\mathcal{H}$ is the minimal generating set of $\sigma^\vee \cap \mathbb{Z}^n$ with respect to inclusion.

LEMMA 3.1. *Let σ be a 2-dimensional strongly convex cone in standard form*

$$\sigma = Cone(e_2, de_1 - ke_2)$$

where $e_1 = (1, 0)$, $e_2 = (0, 1)$, $d > 0$, $0 \leq k < d$, and $gcd(d, k) = 1$.
Then the algebra $A_\sigma = \mathbb{C}[S_\sigma]$ has Hilbert basis $X^{s_i}Y^{t_i}$ for $i = 1, \dots, e$ where e is the embedding dimension and the exponents are defined as:
Let $d/(d - k) = [[b_2, \dots, b_{e-1}]]$ be the Hirzebrunch-Jung continued fraction expansion with $b_i \geq 2$, then

$$\begin{aligned} s_1 &= d, & t_1 &= 0\,, \\ s_2 &= d - k, & t_2 &= 1\,, \\ s_{i+1} &= b_i s_i - s_{i-1}, & t_{i+1} &= b_i t_i - t_{i-1}, \quad i = 2, \dots, e-1\,. \end{aligned}$$

PROOF. See Proposition 2.8, in [14], or Section 2.6 in [8]. □

PROPOSITION 3.2. *Let X_σ be the toric variety defined by the cone σ. Then X_σ has embedding dimension 4 at 0 if and only if the Hirzebruch-Jung continued fraction expansion of $d/(d - k) = [[2 + n, 2 + m]]$ where $m, n \geq 0$. In that case, we have*

$$\begin{aligned} A_\sigma &= \mathbb{C}[X^{(n+2)(m+2)-1}, X^{m+2}Y, XY^{n+2}, Y^{(n+2)(m+2)-1}] \\ &= \mathbb{C}[x, y, z, w]/(xz - y^{n+2}, yw - z^{m+2}, xw - y^{n+1}z^{m+1})\,. \end{aligned}$$

PROOF. Apply the previous lemma when $e = 4$ and let $b_2 = n + 2$, $b_3 = m + 2$, $m, n \geq 0$. □

In the following, we will give a direct proof of (iii)⇒(i) in Theorem 3.1.

LEMMA 3.3. *Let $d/k = [[a_1, \ldots, a_s]]$ be the Hirzebruch-Jung continued fraction of d/k. And define $k[U_i, V_i]$, $i = 0, \ldots, s$ as follows*

$$\begin{aligned} &U_0 = X^d, \quad V_0 = Y/X^k \\ &U_i = V_{i-1}^{-1}, \quad V_i = U_{i-1}V_{i-1}^{a_i}, \quad i = 1, \ldots, s. \end{aligned}$$

Let $A_i := \mathbb{C}[U_i, V_i]$, and $Y_i = \mathrm{Spec}\,\mathbb{C}[U_i, V_i]$ then the ring homorphisms $A_\sigma \to A_i$ define a resolution of singularities $Y = Y_1 \cup \cdots \cup Y_s \to X$ of X.

PROOF. See Section 3 in [12]. □

PROPOSITION 3.4. *Let $d/(d-k) = [[2+n, 2+m]]$, then U_i, V_i is defined as:*

$$\begin{aligned} &U_0 = X^{(n+2)(m+2)-1}, && V_0 = Y/X^{(n+1)(m+2)-1} \\ &U_i = X^{(n+2-i)(m+2)-1}/Y^i, && V_i = Y^{i+1}/X^{(n+1-i)(m+2)-1}, \quad i = 1, \ldots, n \\ &U_{n+1} = X^{m+1}/Y^{(n+2)-1}, && V_{n+1} = Y^{2(n+2)-1}/X^m \\ &U_{n+1+i} = X^{(m+1)-i}/Y^{(i+1)(n+2)-1}, && V_{n+1+i} = Y^{(i+2)(n+2)-1}/X^{m-i}, \quad i = 1, \ldots, m. \end{aligned}$$

PROOF. If $d/(d-k) = [[2+n, 2+m]]$, then by induction on n and m, we have $d/k = [[2, 2, \ldots, 2, 3, 2, \ldots, 2]]$ where the first group of twos has n elements, and the second group has m elements. Then by applying the previous lemma we have the formula for U_i, V_i as above. □

LEMMA 3.5. *The ring homomorphisms $A_\sigma \to A_i$ are defined as the canonical inclusions:*

$$\mathbb{C}[U_i^{i+1}V_i^i, U_iV_i, U_i^{n+1-i}V_i^{n+2-i}, U_i^{(n+1-i)(m+2)-1}V_i^{(n+2-i)(m+2)-1}] \to \mathbb{C}[U_i, V_i]$$

for $i = 0, \ldots, n$.

$$\begin{aligned} &\mathbb{C}[U_{n+1+i}^{(i+2)(n+2)-1}V_{n+1+i}^{(i+1)(n+2)-1}, U_{n+1+i}^{i+2}V_{n+1+i}^{i+1}, U_{n+1+i}V_{n+1+i}, U_{n+1+i}^{m-i}V_{n+1+i}^{m+1-i}] \\ &\qquad \to \mathbb{C}[U_{n+1+i}, V_{n+1+i}] \end{aligned}$$

for $i = 0, \ldots, m$.

From this lemma, we have that the ring homomorphisms $A_\sigma \to A_i$ is one of the following forms:

(i) $\mathbb{C}[U, UV, U^\alpha V^\beta, U^\gamma V^\delta] \to \mathbb{C}[U, V]$;
(ii) $\mathbb{C}[U^\alpha V^\beta, UV, U^\gamma V^\delta, U^\zeta V^\eta] \to \mathbb{C}[U, V]$;
(iii) $\mathbb{C}[U^\alpha V^\beta, U^\gamma V^\delta, UV, U^\zeta V^\eta] \to \mathbb{C}[U, V]$;
(iv) $\mathbb{C}[U^\alpha V^\beta, U^\gamma V^\delta, UV, V] \to \mathbb{C}[U, V]$.

LEMMA 3.6. *For every prime divisor E over X, we have the Mather-Jacobian log discrepancy at E:*

$$a_{\mathrm{MJ}}(E; X) := \mathrm{ord}_E(\widehat{K}_{Y/X} - J_{Y/X}) + 1 \geq 0.$$

PROOF. We have

$$\begin{aligned} X &= \operatorname{Spec} \mathbb{C}[X^{(n+2)(m+2)-1}, X^{m+2}Y, XY^{n+2}, Y^{(n+2)(m+2)-1}] \\ &= \operatorname{Spec} \mathbb{C}[x, y, z, w]/(xz - y^{n+2}, yw - z^{m+2}, xw - y^{n+1}z^{m+1}) . \end{aligned}$$

Then the Jacobian ideal of X is:

$$Jac_X = (x^2, xy, xz, yz, yw, zw, w^2) .$$

In the case (i), we have $\mathcal{O}_Y(-\widehat{K}_{Y/X}) = U$ and $\mathcal{O}_Y(-J_{Y/X}) = U^2$.
In the case (iv), we have $\mathcal{O}_Y(-\widehat{K}_{Y/X}) = V$ and $\mathcal{O}_Y(-J_{Y/X}) = V^2$.
Then $a_{\mathrm{MJ}}(E; X) \geq 0$ for all prime divisor E over X.

In the case (ii) and (iii), we have

$$\begin{aligned} \operatorname{ord}_U(\widehat{K}_{Y/X}) &= \min\{\alpha, \gamma, \zeta\} \\ \operatorname{ord}_U(J_{Y/X}) &= \min\{2\alpha, \alpha + 1, \gamma + 1, \zeta + 1, 2\zeta\} \end{aligned}$$

therefore $a_{\mathrm{MJ}}(U, X) = 0$.
Similarly, we have

$$\begin{aligned} \operatorname{ord}_V(\widehat{K}_{Y/X}) &= \min\{\beta, \delta, \eta\} \\ \operatorname{ord}_V(J_{Y/X}) &= \min\{2\beta, \beta + 1, \delta + 1, \eta + 1, 2\eta\} \end{aligned}$$

therefore $a_{\mathrm{MJ}}(V, X) = 0$.

So in all cases, we all have $a_{\mathrm{MJ}}(E, X) \geq 0$. □

This lemma complete the direct proof of (iii)⇒(i) in Theorem 3.1.

Now we have two proofs that all 2-dimensional toric singularity with embedding dimension 4 is MJ-log canonical. Then it is natural to ask whether all d-dimensional toric singularities with embedding dimension $2d$ are MJ-log canonical. (Here we note that the embedding dimension of a d-dimensional MJ-log canonical singularity is less than or equal to $2d$.)

The following is a counter example to this question.

EXAMPLE 3.7. Let $\sigma = Cone((0, 0, 1), (2, 4, 1), (4, 6, 1), (4, 0, 1))$, then we have the Hilbert basis of the dual cone $\sigma^\vee$ is

$$Hilb(\sigma) = \{(0, 1, 0), (2, -1, 0), (1, -1, 2), (-1, 0, 4), (0, 0, 1), (1, 0, 0)\} .$$

Hence the embedding dimension of X_σ is 6.

On the other hand, X_σ is described as

$$X_\sigma = \operatorname{Spec} \mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]/I$$

where

$$I = (x_1x_2 - x_6^2, x_5^2x_6 - x_1x_3, x_5^4 - x_4x_6, x_2x_5^2 - x_3x_6, x_3x_5^2 - x_2x_4, x_1x_3^2 - x_2x_4x_6) .$$

And by Proposition 1.7,

$$\mathrm{mld}_{\mathrm{MJ}}(X, x) = \inf\{3(m + 1) - \dim \pi_m^{-1}(x)\} \leq 12 - \dim \pi_3^{-1}(x)$$

where

$$\pi_3^{-1}(x) = k[x^{(1)}, x^{(2)}, x^{(3)}]/J_3$$

and J_3 is generated by

$$
\begin{aligned}
&x_1^{(1)}x_2^{(1)} - (x_6^{(1)})^2, \qquad x_1^{(1)}x_2^{(2)} + x_1^{(2)}x_2^{(1)} - 2x_6^{(1)}x_6^{(2)},\\
&x_1^{(1)}x_3^{(1)}, \qquad (x_5^{(1)})^2x_6^{(1)} - x_1^{(2)}x_3^{(1)} - x_1^{(1)}x_3^{(2)},\\
&x_4^{(1)}x_6^{(1)}, \qquad x_2^{(1)}(x_5^{(1)})^2 - x_3^{(1)}x_6^{(2)} - x_3^{(2)}x_6^{(1)},\\
&x_3^{(1)}x_6^{(1)}, \qquad x_3^{(1)}(x_5^{(1)})^2 - x_2^{(1)}x_4^{(2)} - x_2^{(2)}x_4^{(1)},\\
&x_2^{(1)}x_4^{(1)}, \qquad x_1^{(1)}(x_3^{(1)})^2 - x_2^{(1)}x_4^{(1)}x_6^{(1)},\\
&x_4^{(2)}x_6^{(1)} + x_4^{(1)}x_6^{(2)}.
\end{aligned}
$$

We can see that $J_3 \subset (x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)})$, then we have $\mathrm{ht}(J_3) \leq 5$. Hence $\dim \pi_3^{-1}(x) \geq 13$, thus $\mathrm{mld}_{\mathrm{MJ}}(X, 0) < 0$, i.e., X is not MJ-log canonical singularity at 0.

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