Tohoku Math. J.
71 (2019), 111–122

THE p-ADIC DUALITY FOR THE FINITE STAR-MULTIPLE POLYLOGARITHMS

SHIN-ICHIRO SEKI*

(Received May 23, 2016, revised August 30, 2016)

Abstract. We prove the $\boldsymbol{p}$-adic duality theorem for the finite star-multiple polylogarithms. That is a generalization of Hoffman's duality theorem for the finite multiple zeta-star values.

1. Introduction. We begin with the duality for the finite multiple zeta(-star) values (FMZ(S)Vs) in Subsection 1.1. Next, we explain the duality for the finite star-multiple polylogarithms (FSMPs) in Subsection 1.2. Our main results are Theorem 1.3, Theorem 1.5, and Theorem 3.4. The first two theorems are special cases of Theorem 3.4.

1.1. Duality for FMZVs. For any positive integer n and an index $\mathbf{k} = (k_1, \dots, k_m)$, we define the truncated multiple harmonic sums $\zeta_n(\mathbf{k})$ and $\zeta_n^\star(\mathbf{k})$ by

$$\zeta_n(\mathbf{k}) := \sum_{n \geq n_1 > \cdots > n_m \geq 1} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}},$$

$$\zeta_n^\star(\mathbf{k}) := \sum_{n \geq n_1 \geq \cdots \geq n_m \geq 1} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}},$$

respectively (we define $\zeta_n(\mathbf{k})$ as 0 for an empty summation). Then, the multiple zeta value (MZV) $\zeta(\mathbf{k})$ and the multiple zeta-star value (MZSV) $\zeta^\star(\mathbf{k})$ are defined by $\zeta(\mathbf{k}) := \lim_{n\to\infty} \zeta_n(\mathbf{k})$ and $\zeta^\star(\mathbf{k}) := \lim_{n\to\infty} \zeta_n^\star(\mathbf{k})$, respectively when $k_1 \geq 2$. The duality theorem for MZVs $\zeta(\mathbf{k}) = \zeta(\mathbf{k}')$ was conjectured firstly by Hoffman in [1] and proved by using the iterated integral (cf. [9]). Dualities for MZSVs are not found except a few cases (cf. [4]).

Recently, Kaneko and Zagier [5] introduced the *finite* multiple zeta values (FMZVs) and several people are studying relations among FMZVs. The FMZV $\zeta_{\mathcal{A}}(\mathbf{k})$ and the finite multiple zeta-star value (FMZSV) $\zeta_{\mathcal{A}}^\star(\mathbf{k})$ are defined by $\zeta_{\mathcal{A}}(\mathbf{k}) := (\zeta_{p-1}(\mathbf{k}) \bmod p)_p$ and $\zeta_{\mathcal{A}}^\star(\mathbf{k}) := (\zeta_{p-1}^\star(\mathbf{k}) \bmod p)_p$ respectively in the $\mathbb{Q}$-algebra $\mathcal{A} = \left(\prod_p \mathbb{Z}/p\mathbb{Z}\right)\Big/\left(\bigoplus_p \mathbb{Z}/p\mathbb{Z}\right)$, where p runs over all prime numbers. Around 2000, the duality theorem for FMZSVs was discovered and proved by Hoffman [2, Theorem 4.6]:

$$\zeta_{\mathcal{A}}^\star(\mathbf{k}) = -\zeta_{\mathcal{A}}^\star(\mathbf{k}^\vee), \tag{1}$$

where $\mathbf{k}^\vee$ is the Hoffman dual of the index $\mathbf{k}$ (See Definition 2.1). This is a counterpart to the duality theorem for MZVs. Comparing with the duality of MZVs, it is worth mentioning that

2010 *Mathematics Subject Classification.* Primary 11M32.

Key words and phrases. Finite multiple zeta values, finite multiple polylogarithms.

*Partly supported by the Grant-in-Aid for JSPS Fellows (JP16J01758), The Ministry of Education, Culture, Sports, Science and Technology, Japan.

such a simple duality is satisfied by FMZSVs rather than by non-star FMZVs. The duality (1) is one of basic relations among FMZ(S)Vs and some other proofs are given by Imatomi [3, Corollary 4.1] and Yamamoto [8, p. 3].

In order to rewrite the duality (1) to relations for non-star FMZVs, let us recall terminologies of Hoffman's algebra. Let $\mathfrak{H} := \mathbb{Q}\langle x, y\rangle$ be a non-commutative polynomial algebra in two variables and $\mathfrak{H}^1 := \mathbb{Q} + \mathfrak{H}y$ its subalgebra. Let $z_k := x^{k-1}y$ for a natural number k and $z_{\mathbf{k}} := z_{k_1} \cdots z_{k_m}$ for an index $\mathbf{k} = (k_1, \ldots, k_m)$. Then, $\mathfrak{H}^1$ is generated by z_k $(k = 1, 2, \ldots)$ as a non-commutative algebra. We define the $\mathbb{Q}$-linear map $Z_{\mathcal{A}} \colon \mathfrak{H}^1 \to \mathcal{A}$ characterized by $Z_{\mathcal{A}}(1) = 1$ and $Z_{\mathcal{A}}(z_{\mathbf{k}}) = \zeta_{\mathcal{A}}(\mathbf{k})$ for each index $\mathbf{k}$. We also define an algebra automorphism $\psi \colon \mathfrak{H}^1 \to \mathfrak{H}^1$ by $x \mapsto x+y$ and $y \mapsto -y$. In this setup, Hoffman proved that the duality theorem (1) is equivalent to the following relations for FMZVs:

THEOREM 1.1 ([2, Theorem 4.7]). *For any word $w \in \mathfrak{H}^1$, we have*

$$\psi(w) - w \in \ker(Z_{\mathcal{A}}).$$

Next, we recall the $\mathbb{Q}$-algebra $\widehat{\mathcal{A}}$ introduced by Rosen [6]. For any positive integer n, we define $\mathcal{A}_n$ to be the quotient ring $\left(\prod_p \mathbb{Z}/p^n\mathbb{Z}\right)\big/\left(\bigoplus_p \mathbb{Z}/p^n\mathbb{Z}\right)$. Then, the rings $\{\mathcal{A}_n\}$ becomes a projective system by natural projections and we define $\widehat{\mathcal{A}}$ to be the projective limit $\varprojlim_n \mathcal{A}_n$. We equip $\mathcal{A}_n$ with the discrete topology for each n and $\widehat{\mathcal{A}}$ with the projective limit topology. The $\mathbb{Q}$-algebra $\widehat{\mathcal{A}}$ is complete and not locally compact. There exist natural projections $\pi \colon \widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p \twoheadrightarrow \widehat{\mathcal{A}}$ and $\pi_n \colon \widehat{\mathcal{A}} \twoheadrightarrow \mathcal{A}_n$ for any n, where $\mathbb{Z}_p$ is the ring of p-adic integers. We redefine the FMZV $\zeta_{\widehat{\mathcal{A}}}(\mathbf{k})$ and the FMZSV $\zeta^{\star}_{\widehat{\mathcal{A}}}(\mathbf{k})$ to be $\pi((\zeta_{p-1}(\mathbf{k}))_p)$ and $\pi((\zeta^{\star}_{p-1}(\mathbf{k}))_p)$, respectively as elements of $\widehat{\mathcal{A}}$. Furthermore, $\zeta_{\mathcal{A}_n}(\mathbf{k}) := \pi_n(\zeta_{\widehat{\mathcal{A}}}(\mathbf{k}))$ and $\zeta^{\star}_{\mathcal{A}_n}(\mathbf{k}) := \pi_n(\zeta^{\star}_{\widehat{\mathcal{A}}}(\mathbf{k}))$ in $\mathcal{A}_n$. We define the element $\boldsymbol{p} := \pi((p)_p) \in \widehat{\mathcal{A}}$ and we also denote $\pi_n(\boldsymbol{p}) \in \mathcal{A}_n$ by $\boldsymbol{p}$ by abuse of notation. We can check that the topology of $\widehat{\mathcal{A}}$ is the $\boldsymbol{p}$-adic topology (see Subsection 2.2).

Let $\widehat{\mathfrak{H}}^1$ be the completion of $\mathfrak{H}^1$. Namely, $\widehat{\mathfrak{H}}$ is defined as the non-commutative formal power series ring $\mathbb{Q}\langle\!\langle x, y\rangle\!\rangle$ and $\widehat{\mathfrak{H}}^1 := \mathbb{Q} + \widehat{\mathfrak{H}}y$. Then, the weighted finite multiple zeta function $Z_{\widehat{\mathcal{A}}} \colon \widehat{\mathfrak{H}}^1 \to \widehat{\mathcal{A}}$ is defined by

$$\sum_{\mathbf{k}} a_{\mathbf{k}} z_{\mathbf{k}} \mapsto \sum_{\mathbf{k}} a_{\mathbf{k}} \zeta_{\widehat{\mathcal{A}}}(\mathbf{k}) \boldsymbol{p}^{\mathrm{wt}(\mathbf{k})},$$

where $a_{\mathbf{k}} \in \mathbb{Q}$ and $\mathrm{wt}(\mathbf{k})$ is the weight of the index $\mathbf{k}$. The algebra automorphism ψ on $\mathfrak{H}^1$ is extended continuously to the map on $\widehat{\mathfrak{H}}^1$ and we define a continuous algebra automorphism $\Phi \colon \widehat{\mathfrak{H}}^1 \to \widehat{\mathfrak{H}}^1$ by

$$w \mapsto (1+y)\left(\frac{1}{1+y} * w\right).$$

Here, the harmonic product $* \colon \mathfrak{H}^1 \times \mathfrak{H}^1 \to \mathfrak{H}^1$ is defined $\mathbb{Q}$-bilinearly and inductively by

$$w * 1 = 1 * w = w, \quad z_k w_1 * z_l w_2 = z_k(w_1 * z_l w_2) + z_l(z_k w_1 * w_2) + z_{k+l}(w_1 * w_2)$$

for any positive integers k, l and words $w, w_1, w_2 \in \mathfrak{H}^1$ and $*$ is extended naturally to the product on $\widehat{\mathfrak{H}}^1$. Rosen generalized Theorem 1.1 as follows:

THEOREM 1.2 (Asymptotic duality theorem [6, Theorem 4.5]). *For any $w \in \widehat{\mathfrak{H}}^1$, we have*

$$\psi(w) - \Phi(w) \in \ker(Z_{\widehat{\mathcal{A}}}).$$

On the other hand, Zhao, Sakugawa, and the author proved the straightforward generalization of the duality (1) to an $\mathcal{A}_2$-relation ([10, Theorem 2.11] and [7, The equality (40)]). We can rewrite the relation as the following symmetric form:

$$\zeta^{\star}_{\mathcal{A}_2}(\mathbf{k}) + \zeta^{\star}_{\mathcal{A}_2}(1, \mathbf{k})\boldsymbol{p} = -\zeta^{\star}_{\mathcal{A}_2}(\mathbf{k}^{\vee}) - \zeta^{\star}_{\mathcal{A}_2}(1, \mathbf{k}^{\vee})\boldsymbol{p}.$$

In this paper, we give the following $\boldsymbol{p}$-adic version of the duality (1):

THEOREM 1.3 (The $\boldsymbol{p}$-adic duality theorem for FMZSVs). *Let $\mathbf{k}$ be an index. Then, we have*

$$\sum_{i=0}^{\infty} \zeta^{\star}_{\widehat{\mathcal{A}}}(\{1\}^i, \mathbf{k})\boldsymbol{p}^i = -\sum_{i=0}^{\infty} \zeta^{\star}_{\widehat{\mathcal{A}}}(\{1\}^i, \mathbf{k}^{\vee})\boldsymbol{p}^i$$

in the ring $\widehat{\mathcal{A}}$.

Here, the notation $(\{1\}^i, \mathbf{k})$ means $(\underbrace{1, \dots, 1}_{i}, k_1, \dots, k_m)$ for $\mathbf{k} = (k_1, \dots, k_m)$. We remark that if we take the $i = 0$ part of the equality in the above theorem, then we recover the equality (1).

1.2. Duality for FSMPs. In [7], Sakugawa and the author introduced the finite (star-) multiple polylogarithms (F(S)MPs) and proved the following dual functional equation of FSMPs which is a generalization of Hoffman's duality theorem (1):

THEOREM 1.4 (Sakugawa-Seki [7, Theorem 1.3]). *Let $\mathbf{k}$ be an index. Then, we have*

$$\widetilde{\pounds}^{\star}_{\mathcal{A},\mathbf{k}}(t) - \frac{1}{2}\zeta^{\star}_{\mathcal{A}}(\mathbf{k}) = \widetilde{\pounds}^{\star}_{\mathcal{A},\mathbf{k}^{\vee}}(1-t) - \frac{1}{2}\zeta^{\star}_{\mathcal{A}}(\mathbf{k}^{\vee})$$

in the ring $\mathcal{A}_{\mathbb{Z}[t]} = \left(\prod_p \mathbb{Z}/p\mathbb{Z}[t]\right) \Big/ \left(\bigoplus_p \mathbb{Z}/p\mathbb{Z}[t]\right)$.

See Definition 2.7 for the definition of FMPs.

In this paper, we prove the following $\boldsymbol{p}$-adic version of Theorem 1.4 which contains Theorem 1.3 as a special case:

THEOREM 1.5. *Let $\mathbf{k}$ be an index. Then, we have*

$$\sum_{i=0}^{\infty} \left(\widetilde{\pounds}^{\star}_{\widehat{\mathcal{A}},(\{1\}^i,\mathbf{k})}(t) - \frac{1}{2}\zeta^{\star}_{\widehat{\mathcal{A}}}(\{1\}^i, \mathbf{k})\right)\boldsymbol{p}^i = \sum_{i=0}^{\infty} \left(\widetilde{\pounds}^{\star}_{\widehat{\mathcal{A}},(\{1\}^i,\mathbf{k}^{\vee})}(1-t) - \frac{1}{2}\zeta^{\star}_{\widehat{\mathcal{A}}}(\{1\}^i, \mathbf{k}^{\vee})\right)\boldsymbol{p}^i \tag{2}$$

in the ring $\widehat{\mathcal{A}}_{\mathbb{Z}[t]} = \varprojlim_n \left(\prod_p \mathbb{Z}/p^n\mathbb{Z}[t]\right) \Big/ \left(\bigoplus_p \mathbb{Z}/p^n\mathbb{Z}[t]\right)$.

We remark that if we take the $i = 0$ part of the equality (2), then we recover Theorem 1.4. More generally, Sakugawa and the author proved the multi-variable and $\mathcal{A}_2$-version of

Theorem 1.4 ([7, Theorem 3.12]) and the main result of this paper is the $\boldsymbol{p}$-adic dual functional equation for the multi-variable FSMPs (= Theorem 3.4) which contains [7, Theorem 3.12] and Theorem 1.5 as special cases.

This paper is organized as follows. In Section 2, we prepare some notation and define the finite multiple polylogarithms. In Section 3, we prove the $\boldsymbol{p}$-adic reversal theorem for FMPs and state the $\boldsymbol{p}$-adic duality theorem for FSMPs. In Section 4, we complete the proof of main results.

Acknowledgement. The author would like to express his sincere gratitude to his advisor Professor Tadashi Ochiai for carefully reading the manuscript and helpful comments. The author also thanks Dr. Kenji Sakugawa for useful discussion and helpful advice. In addition, the author would like to thank the anonymous referee for pointing out several errors and useful suggestions. The proof of Proposition 4.2 was greatly shortened by his/her idea of using Lemma 4.1, though the author's original proof was based on more complicated calculations.

2. Notation and Definitions. For a tuple of indeterminates $\mathbf{t} = (t_1, \ldots, t_m)$, we define $\mathbf{t}_1$, $1 - \mathbf{t}$, $\mathbf{t}^{-1}$, and $\overline{\mathbf{t}}$ to be $(t_1, \ldots, t_{m-1}, 1)$, $(1 - t_1, \ldots, 1 - t_m)$, $(t_1^{-1}, \ldots, t_m^{-1})$, and $(t_m, \ldots, t_1)$ respectively. We use the notation $R[\mathbf{t}]$ as a polynomial ring $R[t_1, \ldots, t_m]$ for a ring R.

2.1. Indices. We call a tuple of positive integers $\mathbf{k} = (k_1, \ldots, k_m)$ an index and we define the weight $\mathrm{wt}(\mathbf{k})$ (resp. depth $\mathrm{dep}(\mathbf{k})$) of $\mathbf{k}$ to be $k_1 + \cdots + k_m$ (resp. m).

Let $\mathbf{k} = (k_1, k_2, \ldots, k_m)$, $\mathbf{k}' = (k'_1, \ldots, k'_m)$, and $\mathbf{l} = (l_1, \ldots, l_n)$ be indices. Then, we define the reverse index $\overline{\mathbf{k}}$ of $\mathbf{k}$, the summation $\mathbf{k} \oplus \mathbf{k}'$, and the concatenation index $(\mathbf{k}, \mathbf{l})$ by

$$\begin{aligned} \overline{\mathbf{k}} &:= (k_m, \ldots, k_1)\,, \\ \mathbf{k} \oplus \mathbf{k}' &:= (k_1 + k'_1, \ldots, k_m + k'_m)\,, \\ (\mathbf{k}, \mathbf{l}) &:= (k_1, \ldots, k_m, l_1, \ldots, l_n)\,, \end{aligned}$$

respectively. We use the same notation for tuples of non-negative integers or indeterminates.

Let W be the free monoid generated by the set $\{0, 1\}$. We set $W_1 := W1$. Then, there exists a bijection from the set of all indices I to W_1 induced by the correspondence

$$\mathbf{k} = (k_1, \ldots, k_m) \mapsto \underbrace{0 \cdots 0}_{k_1 - 1} 1 \underbrace{0 \cdots 0}_{k_2 - 1} 1 \cdots 1 \underbrace{0 \cdots 0}_{k_m - 1} 1$$

and we denote the bijection as w. Let $\tau \colon W \to W$ be a monoid homomorphism defined by $\tau(0) = 1$ and $\tau(1) = 0$.

DEFINITION 2.1 (cf. [2, Section 3]). For an index $\mathbf{k}$, we define the Hoffman dual $\mathbf{k}^\vee$ of $\mathbf{k}$ by the relation $w(\mathbf{k}^\vee) = \tau(w(\mathbf{k})1^{-1})1$.

For any index $\mathbf{k}$, $\mathrm{wt}(\mathbf{k}) = \mathrm{wt}(\mathbf{k}^\vee)$ and $\mathrm{dep}(\mathbf{k}) + \mathrm{dep}(\mathbf{k}^\vee) = \mathrm{wt}(\mathbf{k}) + 1$ hold.

2.2. The adelic ring. In order to define the finite multiple polylogarithms, we introduce some adelic rings in a general setting.

DEFINITION 2.2. Let R be a commutative ring and Σ an infinite family of ideals of R. Then, we define the ring $\mathcal{A}^{\Sigma}_{n,R}$ for each positive integer n by

$$\mathcal{A}^{\Sigma}_{n,R} := \left(\prod_{I\in\Sigma} R/I^n\right)\Big/\left(\bigoplus_{I\in\Sigma} R/I^n\right).$$

Then, $\{\mathcal{A}^{\Sigma}_{n,R}\}$ becomes a projective system by natural projections and we define the ring $\widehat{\mathcal{A}^{\Sigma}_{R}}$ by

$$\widehat{\mathcal{A}^{\Sigma}_{R}} := \varprojlim_{n} \mathcal{A}^{\Sigma}_{n,R}.$$

We put the discrete topology on $\mathcal{A}^{\Sigma}_{n,R}$ for each n and we define the topology of $\widehat{\mathcal{A}^{\Sigma}_{R}}$ to be the projective limit topology.

LEMMA 2.3. *We use the same notation as in Definition* 2.2 *and we define the I-adic completion $\widehat{R}_I$ of R to be $\varprojlim_n R/I^nR$. Then, there exists the following natural surjective ring homomorphism*:

$$\pi\colon \prod_{I\in\Sigma} \widehat{R}_I \longrightarrow \widehat{\mathcal{A}^{\Sigma}_{R}}.$$

PROOF. For a short exact sequence of projective systems of rings

$$0 \longrightarrow \left\{\bigoplus_{I\in\Sigma} R/I^n\right\} \longrightarrow \left\{\prod_{I\in\Sigma} R/I^n\right\} \longrightarrow \left\{\mathcal{A}^{\Sigma}_{n,R}\right\} \longrightarrow 0,$$

the system $\left\{\bigoplus_{I\in\Sigma} R/I^n\right\}$ satisfies the Mittag-Leffler condition. Therefore, there exists a natural surjection

$$\prod_{I\in\Sigma} \widehat{R}_I \simeq \varprojlim_{n} \prod_{I\in\Sigma} R/I^n \longrightarrow \widehat{\mathcal{A}^{\Sigma}_{R}}.$$

□

REMARK 2.4. We assume that some topology of R/I^n is defined for any $I \in \Sigma$. If we put the product topology on $\prod_{I\in\Sigma} R/I^n$ and the quotient topology on $\mathcal{A}^{\Sigma}_{n,R}$ by $\prod_{I\in\Sigma} R/I^n \twoheadrightarrow \mathcal{A}^{\Sigma}_{n,R}$, then the topology is indiscrete. However, we consider the discrete topology of $\mathcal{A}^{\Sigma}_{n,R}$ in this paper.

LEMMA 2.5. *We use the same notation as in Definition* 2.2 *and Definition* 2.3. *We assume that $I\widehat{R}_I$ is a principal ideal of $\widehat{R}_I$ for any $I \in \Sigma$. Furthermore, we define an ideal $\boldsymbol{I}$ of $\widehat{\mathcal{A}^{\Sigma}_{R}}$ to be $\pi((I\widehat{R}_I)_{I\in\Sigma})$. Let π_n be the natural projection $\pi_n\colon \widehat{\mathcal{A}^{\Sigma}_{R}} \twoheadrightarrow \mathcal{A}^{\Sigma}_{n,R}$ for any positive integer n. Then, we have* $\ker(\pi_n) = \boldsymbol{I}^n$. *In particular, the topology of $\widehat{\mathcal{A}^{\Sigma}_{R}}$ coincides with the $\boldsymbol{I}$-adic topology and $\widehat{\mathcal{A}^{\Sigma}_{R}}$ is complete with respect to the $\boldsymbol{I}$-adic topology.*

PROOF. Let n be a positive integer. Take any element x of $\ker(\pi_n)$. Then, there exists an element $\{x_I\}_{I\in\Sigma}$ of $\prod_{I\in\Sigma}\widehat{R_I}$ such that $x=\pi((x_I)_{I\in\Sigma})$ by Lemma 2.3. By the commutative diagram

$$\begin{array}{ccc} \prod_{I\in\Sigma}\widehat{R_I} & \xrightarrow{\ \pi\ } & \widehat{\mathscr{A}}_R^{\Sigma} \\ {\scriptstyle(\bmod I^n)_{I\in\Sigma}}\downarrow & & \downarrow{\scriptstyle\pi_n} \\ \prod_{I\in\Sigma} R/I^n & \xrightarrow{\ \rho_n\ } & \mathscr{A}_{n,R}^{\Sigma} \end{array}$$

we have

$$\pi_n(x)=\pi_n\circ\pi((x_I)_{I\in\Sigma})=\rho_n((x_I \bmod I^n)_{I\in\Sigma})=0\,.$$

Here, ρ_n is the canonical projection. Therefore, there exists a subset Σ' of Σ such that $\Sigma\setminus\Sigma'$ is finite and $x_I\in I^n\widehat{R_I}$ for every $I\in\Sigma'$. We can take a generator a_I of $I\widehat{R_I}$ for any $I\in\Sigma$ by the assumption. Then, there exists an element $\{y_I\}_{I\in\Sigma'}$ of $\prod_{I\in\Sigma'}\widehat{R_I}$ such that $x_I=a_I^n y_I$ holds for any $I\in\Sigma'$. We define y_I to be zero for $I\in\Sigma\setminus\Sigma'$. Then, we have

$$x=\pi((x_I)_{I\in\Sigma})=\pi((a_I^n y_I)_{I\in\Sigma})=(\pi((a_I)_{I\in\Sigma}))^n\cdot\pi((y_I)_{I\in\Sigma})\in\boldsymbol{I}^n$$

and we obtain the inclusion $\ker(\pi_n)\subset\boldsymbol{I}^n$. The opposite inclusion is trivial and the last assertion follows from the fact that $\{\ker(\pi_n)\}$ is a neighborhood basis of zero. $\square$

In the rest of this paper, we only use the case $\Sigma=\{pR\mid p \text{ is a prime number}\}$ and we omit the notation Σ. We will define the finite multiple polylogarithms as elements of the $\mathbb{Q}$-algebra $\widehat{\mathscr{A}}_{\mathbb{Z}[\mathbf{t}]}$ in the next subsection. Let $\pi\colon\prod_p\widehat{\mathbb{Z}[\mathbf{t}]}_p\twoheadrightarrow\widehat{\mathscr{A}}_{\mathbb{Z}[\mathbf{t}]}$ be the natural surjection obtained by Lemma 2.3 where $\widehat{\mathbb{Z}[\mathbf{t}]}_p=\varprojlim_n\mathbb{Z}[\mathbf{t}]/p^n\mathbb{Z}[\mathbf{t}]$ is the p-adic completion of $\mathbb{Z}[\mathbf{t}]$. Let $\pi_n\colon\widehat{\mathscr{A}}_{\mathbb{Z}[\mathbf{t}]}\twoheadrightarrow\mathscr{A}_{n,\mathbb{Z}[\mathbf{t}]}$ be the natural projection for each n. The topology of $\widehat{\mathscr{A}}_{\mathbb{Z}[\mathbf{t}]}$ coincides with the $\boldsymbol{p}$-adic topology and $\widehat{\mathscr{A}}_{\mathbb{Z}[\mathbf{t}]}$ is complete with respect to the topology by Lemma 2.5. Since an equality $\pi((\sum_{i=0}^{\infty}a_i^{(p)}p^i)_p)=\sum_{i=0}^{\infty}(a_i^{(p)})_p\boldsymbol{p}^i$ holds, in order to obtain a $\boldsymbol{p}$-adic relation, it is sufficient to show the p-adic relations given by taking the p-components for all but finitely many prime numbers p. Here, $a_i^{(p)}\in\mathbb{Z}_{(p)}[\mathbf{t}]$ and $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at p, and note that the opposite assertion does not hold in general.

2.3. The finite multiple polylogarithms.

DEFINITION 2.6. Let n be a positive integer, $\mathbf{k}=(k_1,\ldots,k_m)$ an index, and $\mathbf{t}=(t_1,\ldots,t_m)$ a tuple of indeterminates. Then, we define the four kinds of the truncated multiple polylogarithms which are elements of $\mathbb{Q}[\mathbf{t}]$ as follows:

$$£^{*}_{n,\mathbf{k}}(\mathbf{t}):=\sum_{n\ge n_1>\cdots>n_m\ge1}\frac{t_1^{n_1}\cdots t_m^{n_m}}{n_1^{k_1}\cdots n_m^{k_m}},$$

$$£^{*,\star}_{n,\mathbf{k}}(\mathbf{t}):=\sum_{n\ge n_1\ge\cdots\ge n_m\ge1}\frac{t_1^{n_1}\cdots t_m^{n_m}}{n_1^{k_1}\cdots n_m^{k_m}},$$

$$\pounds^{ш}_{n,\mathbf{k}}(\mathbf{t}) := \sum_{n\geq n_1>\cdots>n_m\geq 1} \frac{t_1^{n_1-n_2}\cdots t_{m-1}^{n_{m-1}-n_m}t_m^{n_m}}{n_1^{k_1}\cdots n_m^{k_m}},$$

$$\pounds^{ш,\star}_{n,\mathbf{k}}(\mathbf{t}) := \sum_{n\geq n_1\geq\cdots\geq n_m\geq 1} \frac{t_1^{n_1-n_2}\cdots t_{m-1}^{n_{m-1}-n_m}t_m^{n_m}}{n_1^{k_1}\cdots n_m^{k_m}}.$$

DEFINITION 2.7. Let $\mathbf{k}=(k_1,\dots,k_m)$ be an index and $\mathbf{t}=(t_1,\dots,t_m)$ a tuple of indeterminates. Then, we define the four kinds of the finite multiple polylogarithms which are elements of $\widehat{\mathcal{A}}_{\mathbb{Z}[\mathbf{t}]}$ as follows:

$$\pounds^{*}_{\widehat{\mathcal{A}},\mathbf{k}}(\mathbf{t}) := \pi((\pounds^{*}_{p-1,\mathbf{k}}(\mathbf{t}))_p) \ \ (\textit{the finite harmonic multiple polylogarithm}),$$
$$\pounds^{*,\star}_{\widehat{\mathcal{A}},\mathbf{k}}(\mathbf{t}) := \pi((\pounds^{*,\star}_{p-1,\mathbf{k}}(\mathbf{t}))_p) \ \ (\textit{the finite harmonic star-multiple polylogarithm}),$$
$$\pounds^{ш}_{\widehat{\mathcal{A}},\mathbf{k}}(\mathbf{t}) := \pi((\pounds^{ш}_{p-1,\mathbf{k}}(\mathbf{t}))_p) \ \ (\textit{the finite shuffle multiple polylogarithm}),$$
$$\pounds^{ш,\star}_{\widehat{\mathcal{A}},\mathbf{k}}(\mathbf{t}) := \pi((\pounds^{ш,\star}_{p-1,\mathbf{k}}(\mathbf{t}))_p) \ \ (\textit{the finite shuffle star-multiple polylogarithm}).$$

This definition is well-defined since $\pounds^{\circ,\bullet}_{p-1,\mathbf{k}}(\mathbf{t})$ is an element of $\mathbb{Z}_{(p)}[\mathbf{t}]$ for each prime number p, $\circ\in\{*,ш\}$, and $\bullet\in\{\emptyset,\star\}$. We also define the finite multiple polylogarithm $\pounds^{\circ,\bullet}_{\mathcal{A}_n,\mathbf{k}}(\mathbf{t})$ as elements of $\mathcal{A}_{n,\mathbb{Z}[\mathbf{t}]}$ by

$$\pounds^{\circ,\bullet}_{\mathcal{A}_n,\mathbf{k}}(\mathbf{t}) := \pi_n(\pounds^{\circ,\bullet}_{\widehat{\mathcal{A}},\mathbf{k}}(\mathbf{t}))$$

for each positive integer n, $\circ\in\{*,ш\}$, and $\bullet\in\{\emptyset,\star\}$. We define 1-variable finite (star-) multiple polylogarithms as follows:

$$\pounds^{\bullet}_{\widehat{\mathcal{A}},\mathbf{k}}(t) := \pounds^{*,\bullet}_{\widehat{\mathcal{A}},\mathbf{k}}(t,\{1\}^{m-1}) = \pounds^{ш,\bullet}_{\widehat{\mathcal{A}},\mathbf{k}}(\{t\}^m)\in\widehat{\mathcal{A}}_{\mathbb{Z}[t]},$$
$$\widetilde{\pounds}^{\bullet}_{\widehat{\mathcal{A}},\mathbf{k}}(t) := \pounds^{*,\bullet}_{\widehat{\mathcal{A}},\mathbf{k}}(\{1\}^{m-1},t) = \pounds^{ш,\bullet}_{\widehat{\mathcal{A}},\mathbf{k}}(\{1\}^{m-1},t)\in\widehat{\mathcal{A}}_{\mathbb{Z}[t]},$$

where t is an indeterminate and $\bullet\in\{\emptyset,\star\}$.

3. The $\boldsymbol{p}$-adic reversal theorem and the $\boldsymbol{p}$-adic duality theorem.

3.1. The $\boldsymbol{p}$-adic reversal theorem for FMPs. The reversal relation for FMZVs ([2, Theorem 4.5]) has been extended to several general cases. For example, Rosen proved the $\boldsymbol{p}$-adic reversal relation for FMZVs in $\widehat{\mathcal{A}}$ ([6, Theorem 4.1]) and Sakugawa and the author proved the reversal relation for FMPs in $\mathcal{A}_{2,\mathbb{Z}[\mathbf{t}]}$ ([7, Proposition 3.11]). Here, we prove the $\boldsymbol{p}$-adic reversal relation for FMPs in $\widehat{\mathcal{A}}_{\mathbb{Z}[\mathbf{t}]}$.

THEOREM 3.1. *Let $\mathbf{k}=(k_1,\dots,k_m)$ be an index, $\mathbf{t}=(t_1,\dots,t_m)$ a tuple of indeterminates, and $\bullet\in\{\emptyset,\star\}$. Then, we have the following $\boldsymbol{p}$-adic relation in $\widehat{\mathcal{A}}_{\mathbb{Z}[\mathbf{t}]}$:*

$$\pounds^{*,\bullet}_{\widehat{\mathcal{A}},\overline{\mathbf{k}}}(\mathbf{t}) = (-1)^{\mathrm{wt}(\mathbf{k})}(t_1\cdots t_m)^{\boldsymbol{p}}\sum_{i=0}^{\infty}\sum_{\substack{\mathbf{l}=(l_1,\dots,l_m)\in\mathbb{Z}^m_{\geq 0}\\ \mathrm{wt}(\mathbf{l})=i}}\left[\prod_{j=1}^{m}\binom{k_j+l_j-1}{l_j}\right]\pounds^{*,\bullet}_{\widehat{\mathcal{A}},\mathbf{k}\oplus\mathbf{l}}(\overline{\mathbf{t}^{-1}})\boldsymbol{p}^i,$$

where $(t_1\cdots t_m)^{\boldsymbol{p}}=((t_1\cdots t_m)^p)_p\in\widehat{\mathcal{A}}_{\mathbb{Z}[\mathbf{t}]}$ and $\pounds^{*,\bullet}_{\widehat{\mathcal{A}},\mathbf{k}\oplus\mathbf{l}}(\overline{\mathbf{t}^{-1}})$ is an element of $\widehat{\mathcal{A}}_{\mathbb{Z}[\mathbf{t}^{-1}]}$.

PROOF. Let p be a prime number. Since a p-adically convergent identity

$$\frac{1}{(p-n)^k} = (-1)^k \sum_{l=0}^{\infty} \binom{k+l-1}{l} \frac{p^l}{n^{k+l}}$$

holds for a positive integer $n < p$, by the substitutions $n_i \mapsto p - n_{m+1-i}$, we have

$$\begin{aligned}
\text{\pounds}^*_{p-1,\overline{\mathbf{k}}}(\mathbf{t}) &= \sum_{p-1\geq n_1>\cdots>n_m\geq 1} \frac{t_1^{n_1}\cdots t_m^{n_m}}{n_1^{k_m}\cdots n_m^{k_1}} \\
&= \sum_{p-1\geq p-n_m>\cdots>p-n_1\geq 1} \frac{t_1^{p-n_m}\cdots t_m^{p-n_1}}{(p-n_m)^{k_m}\cdots(p-n_1)^{k_1}} \\
&= (-1)^{\mathrm{wt}(\mathbf{k})}(t_1\cdots t_m)^p \\
&\quad\times \sum_{p-1\geq n_1>\cdots>n_m\geq 1} \sum_{(l_1,\dots,l_m)\in\mathbb{Z}^m_{\geq 0}} \left[\prod_{j=1}^{m}\binom{k_j+l_j-1}{l_j}\right] \frac{t_m^{-n_1}\cdots t_1^{-n_m}}{n_1^{k_1+l_1}\cdots n_m^{k_m+l_m}} p^{l_1+\cdots+l_m} \\
&= (-1)^{\mathrm{wt}(\mathbf{k})}(t_1\cdots t_m)^p \sum_{i=0}^{\infty} \sum_{\substack{\mathbf{l}=(l_1,\dots,l_m)\in\mathbb{Z}^m_{\geq 0}\\ \mathrm{wt}(\mathbf{l})=i}} \left[\prod_{j=1}^{m}\binom{k_j+l_j-1}{l_j}\right] \text{\pounds}^*_{p-1,\mathbf{k}\oplus\mathbf{l}}(\overline{\mathbf{t}^{-1}}) p^i
\end{aligned}$$

in the ring $\widehat{\mathbb{Z}[\mathbf{t}]}_p$. Therefore, we have the conclusion for non-star case. The star case is similar. $\square$

3.2. The *p*-adic duality theorem for FSMPs. In this subsection, we state the main results. Let $\mathbf{k}$ be an index and $\mathbf{t}$ a tuple of $\mathrm{dep}(\mathbf{k})$ indeterminates. We define a $\boldsymbol{p}$-adically convergent series $\mathcal{L}^{\star}_{\widehat{\mathcal{A}},\mathbf{k}}(\mathbf{t})$ with FSSMPs-coefficients by

$$\mathcal{L}^{\star}_{\widehat{\mathcal{A}},\mathbf{k}}(\mathbf{t}) := \sum_{i=0}^{\infty} \left(\text{\pounds}^{ш,\star}_{\widehat{\mathcal{A}},(\{1\}^i,\mathbf{k})}(\{1\}^i,\mathbf{t}) - \frac{1}{2}\text{\pounds}^{ш,\star}_{\widehat{\mathcal{A}},(\{1\}^i,\mathbf{k})}(\{1\}^i,\mathbf{t}_1) \right) \boldsymbol{p}^i . \tag{3}$$

THEOREM 3.2. *Let w be a positive integer and $\mathbf{t}$ a tuple of w indeterminates. Then, we have*

$$\mathcal{L}^{\star}_{\widehat{\mathcal{A}},\{1\}^w}(\mathbf{t}) = \mathcal{L}^{\star}_{\widehat{\mathcal{A}},\{1\}^w}(1-\mathbf{t})$$

in the ring $\widehat{\mathcal{A}}_{\mathbb{Z}[\mathbf{t}]}$.

We will give a proof of Theorem 3.2 in the next section. Since finite multiple polylogarithms in (3) are of shuffle type, the case $\mathbf{k} = \{1\}^w$ (= Theorem 3.2) is essential. In fact, the following lemma holds:

LEMMA 3.3. *Let $\mathbf{k} = (k_1,\dots,k_m)$ be an index, $w = \mathrm{wt}(\mathbf{k})$, $\mathbf{t} = (t_1,\dots,t_m)$ a tuple of indeterminates, and $\bullet \in \{\emptyset,\star\}$. Then, we have*

$$\text{\pounds}^{ш,\bullet}_{\widehat{\mathcal{A}},\mathbf{k}}(\mathbf{t}) = \text{\pounds}^{ш,\bullet}_{\widehat{\mathcal{A}},\{1\}^w}(\{0\}^{k_1-1},t_1,\dots,\{0\}^{k_m-1},t_m)\,.$$

PROOF. We can easily check it by the definition of the finite shuffle multiple polylogarithms. $\square$

The main result of this paper is as follows:

THEOREM 3.4. *Let r be a positive integer, $\mathbf{k}_1, \ldots, \mathbf{k}_r$ indices, and $\mathbf{t} = (t_1, \ldots, t_r)$ a tuple of indeterminates. We define an index $\mathbf{k}$ to be $(\mathbf{k}_1, \ldots, \mathbf{k}_r)$ and $\mathbf{k}'$ to be $(\mathbf{k}_1^\vee, \ldots, \mathbf{k}_r^\vee)$. Furthermore, we define l_i and l_i' by $l_i := \mathrm{dep}(\mathbf{k}_i)$ and $l_i' := \mathrm{dep}(\mathbf{k}_i^\vee)$ respectively for $i = 1, \ldots, r$. Then, we have*

$$\mathcal{L}^\star_{\widehat{\mathcal{A}},\mathbf{k}}(\{1\}^{l_1-1}, t_1, \ldots, \{1\}^{l_r-1}, t_r) = \mathcal{L}^\star_{\widehat{\mathcal{A}},\mathbf{k}'}(\{1\}^{l_1'-1}, 1-t_1, \ldots, \{1\}^{l_r'-1}, 1-t_r)$$

in the ring $\widehat{\mathcal{A}}_{\mathbb{Z}[\mathbf{t}]}$.

PROOF. We denote $\mathbf{k}_i$ and $\mathbf{k}_i^\vee$ as $(k_1^{(i)}, \ldots, k_{l_i}^{(i)})$ and $({k'}_1^{(i)}, \ldots, {k'}_{l_i'}^{(i)})$ respectively for $i = 1, \ldots, r$. Let $w := \mathrm{wt}(\mathbf{k})$. Then, by Lemma 3.3, Theorem 3.2, and Definition 2.1, we have

$$\begin{aligned}
&\mathcal{L}^\star_{\widehat{\mathcal{A}},\mathbf{k}}(\{1\}^{l_1-1}, t_1, \ldots, \{1\}^{l_r-1}, t_r)\\
&= \mathcal{L}^\star_{\widehat{\mathcal{A}},\{1\}^w}(\ldots, \{0\}^{k_1^{(i)}-1}, 1, \ldots, \{0\}^{k_{l_i-1}^{(i)}-1}, 1, \{0\}^{k_{l_i}^{(i)}-1}, t_i, \ldots)\\
&= \mathcal{L}^\star_{\widehat{\mathcal{A}},\{1\}^w}(\ldots, \{1\}^{k_1^{(i)}-1}, 0, \ldots, \{1\}^{k_{l_i-1}^{(i)}-1}, 0, \{1\}^{k_{l_i}^{(i)}-1}, 1-t_i, \ldots)\\
&= \mathcal{L}^\star_{\widehat{\mathcal{A}},\{1\}^w}(\ldots, \{0\}^{{k'}_1^{(i)}-1}, 1, \ldots, \{0\}^{{k'}_{l_i'-1}^{(i)}-1}, 1, \{0\}^{{k'}_{l_i'}^{(i)}-1}, 1-t_i, \ldots)\\
&= \mathcal{L}^\star_{\widehat{\mathcal{A}},\mathbf{k}'}(\{1\}^{l_1'-1}, 1-t_1, \ldots, \{1\}^{l_r'-1}, 1-t_r)\,.
\end{aligned}$$

$\square$

By considering the case $r = 1$, we obtain Theorem 1.5. Theorem 1.3 is obtained by the substitution $t = 1$ in the equality (2).

4. Proof of Theorem 3.2.

LEMMA 4.1. *Let p be a prime number and n a positive integer satisfying $n < p$. Then, we have the following p-adic expansion*:

$$(-1)^n\binom{p-1}{n} = (-1)^{p-1}\left(1 - \frac{p}{n}\right)\sum_{i=0}^{\infty}\sum_{p-1\ge m_1\ge\cdots\ge m_i\ge n}\frac{p^i}{m_1\cdots m_i}.$$

PROOF. We can calculate as follows:

$$\begin{aligned}(-1)^n\binom{p-1}{n} &= (-1)^n\binom{p-1}{p-1-n} = (-1)^n\frac{p-n}{n}\binom{p-1}{p-n}\\ &= (-1)^{p-1}\left(1-\frac{p}{n}\right)\prod_{m=n}^{p-1}\left(1-\frac{p}{m}\right)^{-1}\\ &= (-1)^{p-1}\left(1-\frac{p}{n}\right)\prod_{m=n}^{p-1}\left(1+\frac{p}{m}+\frac{p^2}{m^2}+\cdots\right)\\ &= (-1)^{p-1}\left(1-\frac{p}{n}\right)\sum_{i=0}^{\infty}\sum_{p-1\geq m_1\geq\cdots\geq m_i\geq n}\frac{p^i}{m_1\cdots m_i}.\end{aligned}$$

This completes the proof of the lemma. □

PROPOSITION 4.2. *Let p be an odd prime number and $\mathbf{t}=(t_1,\dots,t_w)$ a tuple of indeterminates. Then, we have the following p-adic expansion*:

$$\begin{aligned}&\sum_{p-1\geq n_1\geq\cdots\geq n_w\geq 1}(-1)^{n_1}\binom{p-1}{n_1}\frac{t_1^{n_1-n_2}\cdots t_{w-1}^{n_{w-1}-n_w}t_w^{n_w}}{n_1\cdots n_w}\\ &\quad= £^{ш,\star}_{p-1,\{1\}^w}(\mathbf{t})+\sum_{i=1}^{\infty}\left(£^{ш,\star}_{p-1,\{1\}^{w+i}}(\{1\}^i,\mathbf{t})-£^{ш,\star}_{p-1,(\{1\}^{i-1},2,\{1\}^{w-1})}(\{1\}^{i-1},\mathbf{t})\right)p^i.\end{aligned}$$

PROOF. By Lemma 4.1, we have

$$\begin{aligned}&\sum_{p-1\geq n_1\geq\cdots\geq n_w\geq 1}(-1)^{n_1}\binom{p-1}{n_1}\frac{t_1^{n_1-n_2}\cdots t_{w-1}^{n_{w-1}-n_w}t_w^{n_w}}{n_1\cdots n_w}\\ &= \sum_{p-1\geq n_1\geq\cdots\geq n_w\geq 1}\frac{t_1^{n_1-n_2}\cdots t_{w-1}^{n_{w-1}-n_w}t_w^{n_w}}{n_1\cdots n_w}\left(1-\frac{p}{n_1}\right)\sum_{i=0}^{\infty}\sum_{p-1\geq m_1\geq\cdots\geq m_i\geq n_1}\frac{p^i}{m_1\cdots m_i}\\ &= \sum_{i=0}^{\infty}\sum_{p-1\geq m_1\geq\cdots\geq m_i\geq n_1\geq\cdots n_w\geq 1}\left(1-\frac{p}{n_1}\right)\frac{t_1^{n_1-n_2}\cdots t_{w-1}^{n_{w-1}-n_w}t_w^{n_w}}{m_1\cdots m_i n_1\cdots n_w}p^i\\ &= £^{ш,\star}_{p-1,\{1\}^w}(\mathbf{t})+\sum_{i=1}^{\infty}\left(£^{ш,\star}_{p-1,\{1\}^{w+i}}(\{1\}^i,\mathbf{t})-£^{ш,\star}_{p-1,(\{1\}^{i-1},2,\{1\}^{w-1})}(\{1\}^{i-1},\mathbf{t})\right)p^i.\end{aligned}$$

This completes the proof of the proposition. □

PROPOSITION 4.3. *Let N and w be positive integers. Then, the following polynomial identity holds in* $\mathbb{Q}[t_1,\dots,t_w]$:

$$\begin{aligned}&\sum_{N\geq n_1\geq\cdots\geq n_w\geq 1}(-1)^{n_1}\binom{N}{n_1}\frac{t_1^{n_1-n_2}\cdots t_{w-1}^{n_{w-1}-n_w}(t_w^{n_w}-\frac{1}{2})}{n_1\cdots n_w}\\ &\qquad= \sum_{N\geq n_1\geq\cdots\geq n_w\geq 1}\frac{(1-t_1)^{n_1-n_2}\cdots(1-t_{w-1})^{n_{w-1}-n_w}\{(1-t_w)^{n_w}-\frac{1}{2}\}}{n_1\cdots n_w}.\end{aligned}$$

PROOF. By [7, Theorem 2.5], we have

$$\begin{aligned}\sum_{N\geq n_1\geq\cdots\geq n_w\geq 1}&(-1)^{n_1}\binom{N}{n_1}\frac{t_1^{n_1-n_2}\cdots t_{w-1}^{n_{w-1}-n_w}t_w^{n_w}}{n_1\cdots n_w}\\&=\sum_{N\geq n_1\geq\cdots\geq n_w\geq 1}\frac{(1-t_1)^{n_1-n_2}\cdots(1-t_{w-1})^{n_{w-1}-n_w}\{(1-t_w)^{n_w}-1\}}{n_1\cdots n_w},\end{aligned}$$

and by the substitution $t_w = 1$, we have

$$\begin{aligned}\sum_{N\geq n_1\geq\cdots\geq n_w\geq 1}&(-1)^{n_1}\binom{N}{n_1}\frac{t_1^{n_1-n_2}\cdots t_{w-1}^{n_{w-1}-n_w}}{n_1\cdots n_w}\\&=-\sum_{N\geq n_1\geq\cdots\geq n_w\geq 1}\frac{(1-t_1)^{n_1-n_2}\cdots(1-t_{w-1})^{n_{w-1}-n_w}}{n_1\cdots n_w}.\end{aligned}$$

By combining these two identities, we obtain the desired identity. $\square$

In order to prove Theorem 3.2, it is sufficient to show the following theorem:

THEOREM 4.4. *Let n and w be positive integers and $\mathbf{t}$ a tuple of w indeterminates. We define $\mathcal{L}^{\star}_{\mathcal{A}_n,\{1\}^w}(\mathbf{t})$ to be*

$$\mathcal{L}^{\star}_{\mathcal{A}_n,\{1\}^w}(\mathbf{t}) := \sum_{i=0}^{n-1}\left(£^{\text{ш},\star}_{\mathcal{A}_n,\{1\}^{w+i}}(\{1\}^i,\mathbf{t})-\frac{1}{2}£^{\text{ш},\star}_{\mathcal{A}_n,\{1\}^{w+i}}(\{1\}^i,\mathbf{t}_1)\right)\boldsymbol{p}^i.$$

Then, we have

$$\mathcal{L}^{\star}_{\mathcal{A}_n,\{1\}^w}(\mathbf{t}) = \mathcal{L}^{\star}_{\mathcal{A}_n,\{1\}^w}(1-\mathbf{t}) \tag{4}$$

in $\mathcal{A}_{n,\mathbb{Z}[\mathbf{t}]}$.

PROOF. We prove the equality (4) by induction on n. By combining Proposition 4.2 with Proposition 4.3, we have

$$\begin{aligned}£^{\text{ш},\star}_{\widehat{\mathcal{A}},\{1\}^w}(\mathbf{t})-\frac{1}{2}£^{\text{ш},\star}_{\widehat{\mathcal{A}},\{1\}^w}(\mathbf{t}_1)+\sum_{i=1}^{\infty}&\left\{\left(£^{\text{ш},\star}_{\widehat{\mathcal{A}},\{1\}^{w+i}}(\{1\}^i,\mathbf{t})-\frac{1}{2}£^{\text{ш},\star}_{\widehat{\mathcal{A}},\{1\}^{w+i}}(\{1\}^i,\mathbf{t}_1)\right)\right.\\&\left.-\left(£^{\text{ш},\star}_{\widehat{\mathcal{A}},(\{1\}^{i-1},2,\{1\}^{w-1})}(\{1\}^{i-1},\mathbf{t})-\frac{1}{2}£^{\text{ш},\star}_{\widehat{\mathcal{A}},(\{1\}^{i-1},2,\{1\}^{w-1})}(\{1\}^{i-1},\mathbf{t}_1)\right)\right\}\boldsymbol{p}^i\\=£^{\text{ш},\star}_{\widehat{\mathcal{A}},\{1\}^w}(1-\mathbf{t})-\frac{1}{2}£^{\text{ш},\star}_{\widehat{\mathcal{A}},\{1\}^w}((1-\mathbf{t})_1)&.\end{aligned}\tag{5}$$

We see that the equality (4) for $n = 1$ holds by the projection $\pi_1 : \widehat{\mathcal{A}}_{\mathbb{Z}[\mathbf{t}]} \twoheadrightarrow \mathcal{A}_{\mathbb{Z}[\mathbf{t}]}$. We assume that the equation (4) for $n-1$ holds for any tuple of indeterminates. By the equality (5) and the projection $\pi_n\colon \widehat{\mathcal{A}}_{\mathbb{Z}[\mathbf{t}]} \twoheadrightarrow \mathcal{A}_{n,\mathbb{Z}[\mathbf{t}]}$, we have

$$\begin{aligned}\mathcal{L}^{\star}_{\mathcal{A}_n,\{1\}^w}(\mathbf{t}) &= \pounds^{\text{ш},\star}_{\mathcal{A}_n,\{1\}^w}(1-\mathbf{t}) - \frac{1}{2}\pounds^{\text{ш},\star}_{\mathcal{A}_n,\{1\}^w}((1-\mathbf{t})_1) \\ &\quad + \sum_{i=1}^{n-1}\left(\pounds^{\text{ш},\star}_{\mathcal{A}_n,\{1\}^{w+i}}(\{1\}^{i-1}, t_0, \mathbf{t}) - \frac{1}{2}\pounds^{\text{ш},\star}_{\mathcal{A}_n,\{1\}^{w+i}}(\{1\}^{i-1}, t_0, \mathbf{t}_1)\right)\boldsymbol{p}^i\Bigg|_{t_0=0}.\end{aligned} \tag{6}$$

On the other hand, by the induction hypothesis, we have

$$\mathcal{L}^{\star}_{\mathcal{A}_{n-1},\{1\}^{w+1}}(t_0, \mathbf{t}) = \mathcal{L}^{\star}_{\mathcal{A}_{n-1},\{1\}^{w+1}}(1-t_0, 1-\mathbf{t}).$$

Therefore, the right-hand side of (6) coincides with $\mathcal{L}^{\star}_{\mathcal{A}_n,\{1\}^w}(1-\mathbf{t})$ and the equality (4) for n holds. Here, note that there exists the canonical isomorphism $\mathcal{A}_{n-1,\mathbb{Z}[\mathbf{t}]} \simeq \boldsymbol{p}\mathcal{A}_{n,\mathbb{Z}[\mathbf{t}]}$. □

References

[1] M. Hoffman, Multiple harmonic series, Pacific J. Math. 152 (1992), no. 2, 275–290.

[2] M. Hoffman, Quasi-symmetric functions and mod p multiple harmonic sums, Kyushu J. Math. 69 (2015), 345–366.

[3] K. Imatomi, Multi-poly-Bernoulli-star numbers and finite multiple zeta-star values, Integers 14 (2014), Paper No. A51, 10 pp.

[4] M. Kaneko and Y. Ohno, On a kind of duality of multiple zeta-star values, Int. J. Number Theory 6 (2010), no. 8, 1927–1932.

[5] M. Kaneko and D. Zagier, Finite multiple zeta values, in preparation.

[6] J. Rosen, Asymptotic relation for truncated multiple zeta values, J. Lond. Math. Soc. (2) 91, 554–572.

[7] K. Sakugawa and S. Seki, On functional equations of finite multiple polylogarithms, J. Algebra 469 (2017), 323–357.

[8] S. Yamamoto, Multiple zeta-star values and multiple integrals, RIMS Kôkyûroku Bessatsu, B68 (2017), 3–14.

[9] D. Zagier, Values of zeta functions and their applications, in ECM volume, Progr. Math. 120 (1994), 497–512.

[10] J. Zhao, Wolstenholme type theorem for multiple harmonic sums, Int. J. Number Theory 4 (2008), no. 1, 73–106.

Mathematical Institute
Tohoku University
6–3, Aoba, Aramaki, Aoba-ku
Sendai 980–8578
Japan

E-mail address: shinichiro.seki.b3@tohoku.ac.jp