ORTHOGONALITY OF DIVISORIAL ZARISKI DECOMPOSITIONS FOR CLASSES WITH VOLUME ZERO

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Abstract. We show that the orthogonality conjecture for divisorial Zariski decompositions on compact Kähler manifolds holds for pseudoeffective $(1, 1)$ classes with volume zero.

1. Introduction. The orthogonality conjecture of divisorial Zariski decompositions [4] states the following:

CONJECTURE 1.1. *Let (X^n, ω) be a compact Kähler manifold, and α a pseudoeffective $(1, 1)$ class. Then*

$$(1.1) \qquad \langle \alpha^{n-1} \rangle \cdot \alpha = \mathrm{Vol}(\alpha) \,.$$

Here $\mathrm{Vol}(\alpha)$ denotes the volume of the class α [1] and $\langle \cdot \rangle$ is the moving intersection product of classes as introduced by Boucksom [2, 4, 5, 6]. The name of this problem comes from the following observation. If we choose an approximate Zariski decomposition of the class α (see Section 2 for the precise details of the construction), given by suitable modifications $\mu_\delta : X_\delta \to X$, for all small $\delta > 0$, with $\mu_\delta^*(\alpha + \delta\omega) = \theta_\delta + [E_\delta]$ with θ_δ a semipositive class and E_δ an effective $\mathbb{R}$-divisor, then (1.1) is equivalent to

$$\lim_{\delta \downarrow 0} \int_{X_\delta} \theta_\delta^{n-1} \wedge [E_\delta] = 0 \,,$$

which explains the name. This conjecture was first raised by Nakamaye [14, p.566] when X is projective and $\alpha = c_1(L)$ for some line bundle L. This was solved by Boucksom-Demailly-Păun-Peternell [4] who more generally proved Conjecture 1.1 when X is projective and α belongs to the real Néron-Severi group, and posed it in the general case. It was observed in [4, 6] that Conjecture 1.1 is equivalent to several other powerful statements, including the fact that the dual cone of the pseudoeffective cone $\mathcal{E}$ is the movable cone $\mathcal{M}$, the weak transcendental Morse inequalities, and the C^1 differentiability of the volume function on the big cone. Very recently, Witt-Nyström [19] has proved Conjecture 1.1 when X is projective.

Our main result is the following:

THEOREM 1.2.   *Conjecture* 1.1 *holds if* $\mathrm{Vol}(\alpha) = 0$.

The strategy of proof is similar to the one in [4] with one crucial difference. Since the weak transcendental Morse inequality

$$(1.2) \qquad \mathrm{Vol}(\alpha - \beta) \geqslant \int_X \alpha^n - n \int_X \alpha^{n-1} \wedge \beta,$$

for the difference of two nef classes α, β remains conjectural, we employ instead the weaker version

$$(1.3) \qquad \mathrm{Vol}(\alpha - \beta) \geqslant \frac{\left(\int_X \alpha^n - n \int_X \alpha^{n-1} \wedge \beta \right)^n}{\left(\int_X \alpha^n \right)^{n-1}},$$

which was proved independently in [18, 17], using the mass concentration technique of Demailly-Păun [10] and its recent improvements by Chiose [7], Xiao [20] and Popovici [16]. Inequality (1.3) is too weak to prove Conjecture 1.1 in general, but the fact that the numerator on the right hand side has the correct form turns out to be enough to prove Theorem 1.2.

If we define the "difference function" $\mathcal{D} : \mathcal{E} \to \mathbb{R}$ on the pseudoeffective cone $\mathcal{E}$ by

$$(1.4) \qquad \mathcal{D}(\alpha) := \langle \alpha^{n-1} \rangle \cdot \alpha - \mathrm{Vol}(\alpha),$$

then Conjecture 1.1 simply states that $\mathcal{D}$ vanishes identically on $\mathcal{E}$. As a corollary of Theorem 1.2 we have:

COROLLARY 1.3.   *The function* $\mathcal{D} : \mathcal{E} \to \mathbb{R}$ *is nonnegative, continuous on* $\mathcal{E}$, *and vanishes on its boundary.*

Theorem 1.2 and Corollary 1.3 are proved in Section 2. In Section 3 we will make some further remarks on the function $\mathcal{D}$, and on the relation between Theorem 1.2 and the "cone duality" conjecture.

Finally a comment about notation. We will often use the same symbol (e.g. α) to denote a closed form or current and the cohomology class that it represents. This should not cause much confusion, since the meaning will be clear based on the context.

2.   The main theorem.   In this section we give the proof of Theorem 1.2 and Corollary 1.3.

PROOF OF THEOREM 1.2.   Let α be any pseudoeffective $(1, 1)$ class on X. By definition of moving intersection products, for any $1 \leqslant p \leqslant n$ the real (p, p) cohomology class $\langle \alpha^p \rangle$ is defined to be

$$\langle \alpha^p \rangle = \lim_{\delta \downarrow 0} \langle (\alpha + \delta \omega)^p \rangle,$$

where for $\delta > 0$ the class $\alpha + \delta \omega$ is big, and in this case we define

$$\langle (\alpha + \delta \omega)^p \rangle = [\langle T_{\min,\delta}^p \rangle],$$

where $T_{\min,\delta}$ is any positive current with minimal singularities in the class $\alpha + \delta \omega$, and $\langle T_{\min,\delta}^p \rangle$ denotes the non-pluripolar product [5].

The volume of α (see [1]) is in fact equal to the moving self-intersection product

$$\text{Vol}(\alpha) = \langle \alpha^n \rangle = \lim_{\delta \downarrow 0} \langle (\alpha + \delta \omega)^n \rangle \,.$$

We now review the well-known construction of approximate Zariski decompositions [2, 4, 5, 6], following roughly the argument in [5, Proposition 1.18]. Applying Demailly's regularization [9] to $T_{\min, \frac{\delta}{2}}$ we obtain a sequence of currents $T_{\delta, \varepsilon}, \varepsilon > 0$, in the big class $\alpha + \frac{\delta}{2}\omega$, with analytic singularities, with $T_{\delta, \varepsilon} \geqslant -\varepsilon \omega$, and with their potentials decreasing to that of $T_{\min, \frac{\delta}{2}}$ as $\varepsilon \to 0$. As long as $\varepsilon \leqslant \frac{\delta}{2}$, we have that $T_{\delta, \varepsilon} + \frac{\delta}{2}\omega$ and $T_{\min, \frac{\delta}{2}} + \frac{\delta}{2}\omega$ are closed positive currents in the class $\alpha + \delta\omega$ whose potentials are locally bounded away from the proper analytic subvariety $A := E_{nK}(\alpha + \frac{\delta}{2}\omega)$, and so by weak continuity of the Bedford-Taylor Monge-Ampère operator along decreasing sequences we have

$$\left(T_{\delta, \varepsilon} + \frac{\delta}{2}\omega \right)^p \to \left(T_{\min, \frac{\delta}{2}} + \frac{\delta}{2}\omega \right)^p \,,$$

weakly on $X \backslash A$ as $\varepsilon \to 0$, for all $1 \leqslant p \leqslant n$. It follows that

$$\int_{X \backslash A} \left(T_{\min, \frac{\delta}{2}} + \frac{\delta}{2}\omega \right)^p \wedge \omega^{n-p} \leqslant \liminf_{\varepsilon \to 0} \int_{X \backslash A} \left(T_{\delta, \varepsilon} + \frac{\delta}{2}\omega \right)^p \wedge \omega^{n-p}$$

$$\leqslant \int_{X \backslash A} (T_{\min, \delta})^p \wedge \omega^{n-p} \,,$$

where the last inequality follows from [5, Theorem 1.16], since all the currents involved have small unbounded locus. However, as $\delta \to 0$ both the left hand side and the right hand side converge to

$$\int_X \langle \alpha^p \rangle \wedge \omega^{n-p} \,,$$

and so we may choose a sequence $0 < \varepsilon(\delta) \ll \delta$ such that the currents $T_\delta := T_{\delta, \varepsilon(\delta)} - \frac{\delta}{2}\omega$ in the class α have analytic singularities, satisfy $T_\delta \geqslant -\delta\omega$, and are such that

$$\lim_{\delta \downarrow 0} \langle (T_\delta + \delta\omega)^p \rangle = \langle \alpha^p \rangle \,,$$

for all $1 \leqslant p \leqslant n$. Let $\mu_\delta : X_\delta \to X$ be a resolution of the singularities of $T_\delta + \delta\omega$, so that

$$\mu_\delta^*(T_\delta + \delta\omega) = \theta_\delta + [E_\delta] \,,$$

so that θ_δ is a smooth semipositive form, E_δ is an effective $\mathbb{R}$-divisor and $[E_\delta]$ denotes the current of integration. We refer to this construction as an approximate Zariski decomposition for the class α.

As discussed above, we have

$$\text{Vol}(\alpha) = \lim_{\delta \downarrow 0} \int_X \langle (T_\delta + \delta\omega)^n \rangle = \lim_{\delta \downarrow 0} \int_{X_\delta} \langle (\mu_\delta^*(T_\delta + \delta\omega))^n \rangle = \lim_{\delta \downarrow 0} \int_{X_\delta} \theta_\delta^n \,,$$

$$\langle \alpha^{n-1} \rangle \cdot \alpha = \lim_{\delta \downarrow 0} \int_X \langle (T_\delta + \delta\omega)^{n-1} \rangle \wedge \alpha = \lim_{\delta \downarrow 0} \int_X \langle (T_\delta + \delta\omega)^{n-1} \rangle \wedge (\alpha + \delta\omega)$$

$$= \lim_{\delta \downarrow 0} \int_{X_\delta} \langle (\mu_\delta^*(T_\delta + \delta\omega))^{n-1} \rangle \wedge \mu_\delta^*(\alpha + \delta\omega)$$

$$= \lim_{\delta \downarrow 0} \int_{X_\delta} (\theta_\delta^n + \theta_\delta^{n-1} \wedge [E_\delta]) ,$$

so that the orthogonality relation (1.1) is in general equivalent to the statement that

$$(2.1) \qquad\qquad \lim_{\delta \downarrow 0} \int_{X_\delta} \theta_\delta^{n-1} \wedge [E_\delta] = 0 .$$

We now follow [4], and fix a constant C_0 such that $C_0\omega \pm (\alpha + \delta\omega)$ is nef. We write

$$E_\delta = \mu_\delta^*(\alpha + \delta\omega + C_0\omega) - (\theta_\delta + C_0\mu_\delta^*\omega) ,$$

as the difference of two nef classes. For $t \in [0, 1]$ write

$$\theta_\delta + tE_\delta = A - B ,$$

where

$$A = \theta_\delta + t\mu_\delta^*(\alpha + \delta\omega + C_0\omega) ,$$

$$B = t(\theta_\delta + C_0\mu_\delta^*\omega) ,$$

and A, B are nef. Then

$$\mathrm{Vol}(\alpha + \delta\omega) = \mathrm{Vol}(\theta_\delta + E_\delta) \geqslant \mathrm{Vol}(\theta_\delta + tE_\delta) = \mathrm{Vol}(A - B) .$$

We use [18, Theorem 1.1] (also independently obtained in [17, Theorem 3.5]) and obtain

$$(2.2) \qquad\qquad \mathrm{Vol}(A - B) \geqslant \frac{\left(\int_{X_\delta} A^n - n \int_{X_\delta} A^{n-1} \wedge B \right)^n}{\left(\int_{X_\delta} A^n \right)^{n-1}} .$$

We follow the same argument as in [4, p.218] (see also [12, (11.15)]) and estimate

$$\int_{X_\delta} A^n - n \int_{X_\delta} A^{n-1} \wedge B \geqslant \int_{X_\delta} \theta_\delta^n + nt \int_{X_\delta} \theta_\delta^{n-1} \wedge [E_\delta] - 5n^2 t^2 C_0^n \int_X \omega^n ,$$

as long as $t \leqslant \frac{1}{10n}$. We choose

$$t = \frac{\int_{X_\delta} \theta_\delta^{n-1} \wedge [E_\delta]}{10nC_0^n \int_X \omega^n} ,$$

which is easily seen to be less than $\frac{1}{10n}$, and so we obtain

$$\int_{X_\delta} A^n - n \int_{X_\delta} A^{n-1} \wedge B \geqslant \int_{X_\delta} \theta_\delta^n + \frac{1}{20} \frac{\left(\int_{X_\delta} \theta_\delta^{n-1} \wedge [E_\delta] \right)^2}{C_0^n \int_X \omega^n} ,$$

and plugging this into (2.2) we obtain

$$\mathrm{Vol}(\alpha + \delta\omega) \geqslant \frac{\left(\int_{X_\delta} \theta_\delta^n + \frac{1}{20} \frac{\left(\int_{X_\delta} \theta_\delta^{n-1} \wedge [E_\delta]\right)^2}{C_0^n \int_X \omega^n}\right)^n}{\left(\int_{X_\delta} A^n\right)^{n-1}}.$$

We also have

$$(2.3) \qquad \int_{X_\delta} A^n = \int_{X_\delta} (\theta_\delta + t\mu_\delta^*(\alpha + \delta\omega + C_0\omega))^n \leqslant \int_{X_\delta} \theta_\delta^n + Ct \leqslant C,$$

and so

$$\int_{X_\delta} \theta_\delta^n + \frac{1}{20} \frac{\left(\int_{X_\delta} \theta_\delta^{n-1} \wedge [E_\delta]\right)^2}{C_0^n \int_X \omega^n} \leqslant C\mathrm{Vol}(\alpha + \delta\omega)^{\frac{1}{n}} \to C\mathrm{Vol}(\alpha)^{\frac{1}{n}} = 0,$$

as $\delta \to 0$, and we conclude that

$$\int_{X_\delta} \theta_\delta^{n-1} \wedge [E_\delta] \to 0,$$

which proves (2.1). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

REMARK 2.1.   Some of the estimates in this proof, such as for example (2.3), are far from being sharp. It is easy to make them sharp, but this does not appear to give any useful improvement.

PROOF OF COROLLARY 1.3.   Using the notation as in the proof of Theorem 1.2, we have that

$$\mathcal{D}(\alpha) = \lim_{\delta\downarrow 0} \int_{X_\delta} \theta_\delta^{n-1} \wedge [E_\delta] \geqslant 0.$$

It follows easily from the definitions that moving intersection products are upper-semicontinuous on the pseudoeffective cone and continuous in its interior (the big cone), while it was proved in [1] that the volume function is continuous on the whole pseudoeffective cone. It follows that $\mathcal{D}$ is continuous on the big cone and upper-semicontinuous on the pseudoeffective cone. By Theorem 1.2, $\mathcal{D}$ vanishes on its boundary, and hence it is continuous on all of $\mathcal{E}$. $\square$

3. Further Remarks.   In this section we collect some further remarks on the function $\mathcal{D}$ and on the cone duality conjecture of [4].

The function $\mathcal{D}$ defined in (1.4) clearly vanishes on the nef cone, where moving intersection products equal usual intersection products (see e.g. [5]). More generally we have the following result, which is a simple consequence of the author's work with Collins [8] and was also observed by Deng [11]:

PROPOSITION 3.1.   *Let α be a pseudoeffective $(1, 1)$ class, and write $\alpha = P + N$ for its divisorial Zariski decomposition. If the class P is nef then $\mathcal{D}(\alpha) = 0$.*

Recall here that the positive part is given by $P = \langle\alpha\rangle$, which is in general only nef in codimension 1 [3, 15].

PROOF.   First, we show that

$$\langle \alpha^{n-1} \rangle \cdot N = 0 \,.$$

Since moving products are unchanged if we replace a class by its positive part (see [4, 6]), this is equivalent to showing that

$$\langle P^{n-1} \rangle \cdot N = 0 \,,$$

and since by assumption P is nef, this is also equivalent to showing that

$$P^{n-1} \cdot N = 0 \,.$$

But by definition the irreducible components of N are contained in $E_{nK}(\alpha) = E_{nK}(P)$ (see e.g. [13, Claim 4.7]), and so are also irreducible components of $E_{nK}(P)$. By the main theorem of [8] we have therefore $P^{n-1} \cdot N = 0$. On the other hand since P is nef we also have

$$\langle \alpha^{n-1} \rangle \cdot \alpha = \langle \alpha^{n-1} \rangle \cdot P = \langle P^{n-1} \rangle \cdot P = P^n = \mathrm{Vol}(P) = \mathrm{Vol}(\alpha) \,,$$

as claimed.   □

REMARK 3.2.   In fact, it is not hard to see (cf. [4]) that in general (1.1) is equivalent to the following two relations both holding

(3.1) $$\langle \alpha^{n-1} \rangle \cdot P = \mathrm{Vol}(\alpha) \,,$$

(3.2) $$\langle \alpha^{n-1} \rangle \cdot N = 0 \,.$$

It was proved in [4] that Conjecture 1.1 implies (and is in fact equivalent to) the "cone duality" conjecture, which states that the dual cone of the pseudoeffective cone $\mathcal{E}$ of a compact Kähler manifold equals the movable cone $\mathcal{M} \subset H^{n-1,n-1}(X, \mathbb{R})$. It is easy to see that $\mathcal{E} \subset \mathcal{M}^\vee$, so the point is to prove the reverse inclusion. The proof given there starts by assuming that there is a class $\alpha \in \partial \mathcal{E} \cap (\mathcal{M}^\vee)^\circ$, and derives a contradiction, assuming that X is projective and that the class belongs to the real Néron-Severi group.

In general we have the following:

PROPOSITION 3.3.   *Let $\alpha \in \partial \mathcal{E} \cap (\mathcal{M}^\vee)^\circ$. Then we must have $\langle \alpha^{n-1} \rangle = 0$.*

PROOF.   Assume for a contradiction that $\langle \alpha^{n-1} \rangle \neq 0$ in $H^{n-1,n-1}(X, \mathbb{R})$. Since this class is represented by a closed nonnegative $(n-1, n-1)$ current, it follows that

$$\int_X \langle \alpha^{n-1} \rangle \wedge \omega > 0 \,,$$

where ω is a fixed Kähler metric. We choose an approximate Zariski decomposition T_δ with resolutions $\mu_\delta : X_\delta \to X$ as before, so that we have also

$$0 < 2\eta \leqslant \int_X \langle \alpha^{n-1} \rangle \wedge \omega = \lim_{\delta \downarrow 0} \int_{X_\delta} \theta_\delta^{n-1} \wedge \mu_\delta^* \omega = \lim_{\delta \downarrow 0} \int_X (\mu_\delta)_* (\theta_\delta^{n-1}) \wedge \omega \,,$$

for some fixed $\eta > 0$. Up to modifying the classes θ_δ (and $[E_\delta]$) by subtracting a small multiple of $\mathrm{Exc}(\mu_\delta)$, we may also assume that the classes θ_δ are Kähler on X_δ, so that the pushforwards $(\mu_\delta)_* (\theta_\delta^{n-1})$ are movable classes on X.

Let $\varepsilon > 0$ be such that $\alpha - \varepsilon\omega \in \mathcal{M}^\vee$. Integrating $\alpha - \varepsilon\omega$ against the class $(\mu_\delta)_*(\theta_\delta^{n-1}) \in \mathcal{M}$ we obtain

$$\int_X \alpha \wedge (\mu_\delta)_*(\theta_\delta^{n-1}) \geqslant \varepsilon \int_X \omega \wedge (\mu_\delta)_*(\theta_\delta^{n-1}) \geqslant \varepsilon\eta > 0\,,$$

for all $\delta > 0$ small. But we also have

$$\int_X \alpha \wedge (\mu_\delta)_*(\theta_\delta^{n-1}) = \int_{X_\delta} \mu_\delta^*\alpha \wedge \theta_\delta^{n-1} \leqslant \int_{X_\delta} \mu_\delta^*(\alpha + \delta\omega) \wedge \theta_\delta^{n-1} = \int_{X_\delta}(\theta_\delta^n + \theta_\delta^{n-1} \wedge [E_\delta])\,,$$

and putting these together we have

$$\int_{X_\delta}(\theta_\delta^n + \theta_\delta^{n-1} \wedge [E_\delta]) \geqslant \varepsilon\eta\,,$$

for all $\delta > 0$ small. Since $\text{Vol}(\alpha) = 0$, this implies

$$\int_{X_\delta} \theta_\delta^{n-1} \wedge [E_\delta] \geqslant \frac{\varepsilon\eta}{2}\,,$$

for all $\delta > 0$ small, which is a contradiction to Theorem 1.2. Therefore we must have that $\langle \alpha^{n-1} \rangle = 0$. $\qquad\square$

REMARK 3.4. Even though the class $\alpha \in \partial\mathcal{E} \cap (\mathcal{M}^\vee)^\circ$ does satisfy the orthogonality conjecture (say as in (2.1)) by Theorem 1.2, because it has volume zero, this is not enough to derive a contradiction in general. Indeed, to make the argument in [4] go through, one would need the following quantitative version of orthogonality

$$\left(\int_{X_\delta} \theta_\delta^{n-1} \wedge [E_\delta]\right)^2 \leqslant C\left(\text{Vol}(\alpha + \delta\omega) - \int_{X_\delta} \theta_\delta^n\right)\,,$$

(cf. [4, Theorem 4.1]) which does not follow from the arguments of Theorem 1.2.

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