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LARGE DEVIATIONS FOR CONTINUOUS ADDITIVE FUNCTIONALS OF SYMMETRIC MARKOV PROCESSES

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Abstract. Let X be a locally compact separable metric space and m a positive Radon measure on X with full topological support. Let $\mathbf{M} = (P_x, X_t)$ be an m-symmetric Markov process on X. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be the Dirichlet form on $L^2(X; m)$ generated by $\mathbf{M}$. Let μ be a positive Radon measure in the *Green-tight Kato class* and A_t^μ the positive continuous additive functional in the Revuz correspondence to μ. Under certain conditions, we establish the large deviation principle for positive continuous additive functionals A_t^μ of symmetric Markov processes.

Introduction. Let X be a locally compact separable metric space and m a positive Radon measure on X with full topological support. Let $\mathbf{M} = (P_x, X_t)$ be an irreducible, conservative, m-symmetric Markov process on X with the doubly Feller property. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be the Dirichlet form on $L^2(X; m)$ generated by $\mathbf{M}$. We assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is regular and transient. Let μ be a positive Radon measure in the *Green-tight Kato class* (in notation $\mu \in \mathcal{K}_\infty$) and A_t^μ the positive continuous additive functional in the Revuz correspondence to μ.

We define

$$\gamma(\theta) := \inf\left\{\mathcal{E}(u,u) : u \in \mathcal{D}(\mathcal{E}),\ \theta \int_X u^2 d\mu = 1\right\},\quad \theta \in \mathbb{R}^1 . \tag{1}$$

Let θ_0 be a unique value such that $\gamma(\theta_0) = 1$. We define the functions $C(\theta)$ and $\widetilde{C}(\theta)$ by

$$C(\theta) = -\inf\left\{\mathcal{E}(u,u) - \theta \int_X u^2 d\mu : u \in C_0(X) \cap \mathcal{D}(\mathcal{E}),\ \int_X u^2 dm = 1\right\},$$

and

$$\widetilde{C}(\theta) = \begin{cases} C(\theta), & \theta \geq \theta_0 \\ 0, & \theta < \theta_0 . \end{cases}$$

Here $C_0(X)$ is the space of continuous functions on X with compact support.
Let $I(\lambda)$ (resp. $\widetilde{I}(\lambda)$) be the Legendre transform of $C(\theta)$ (resp. $\widetilde{C}(\theta)$):

$$I(\lambda) = \sup_{\theta\in\mathbb{R}^1}\{\lambda\theta - C(\theta)\} \quad \left(\text{resp. } \widetilde{I}(\lambda) = \sup_{\theta\in\mathbb{R}^1}\{\lambda\theta - \widetilde{C}(\theta)\}\right),\quad \lambda \in \mathbb{R}^1 .$$

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In [7], [9], a large deviation principle was proved for additive functionals of Brownian motion corresponding to Kato measures. In [12], it was extended to the case of symmetric α-stable process. A main objective of this paper is to extend these results in [7], [9] and [12] to more general symmetric Markov processes:

THEOREM 0.1. *Suppose* **M** *satisfies* **(I)**, **(DF)**, **(C)** *and* **(LU)** *below. Let* $\mu \in \mathcal{K}_\infty$. *Then*

(i) *For any open set* $G \subset \mathbb{R}^1$,

$$\liminf_{t\to\infty} \frac{1}{t} \log P_x\left(\frac{A_t^\mu}{t} \in G\right) \geq -\inf_{\lambda\in G} I(\lambda)\,.$$

(ii) *For any closed set* $K \subset \mathbb{R}^1$,

$$\limsup_{t\to\infty} \frac{1}{t} \log P_x\left(\frac{A_t^\mu}{t} \in K\right) \leq -\inf_{\lambda\in K} \widetilde{I}(\lambda)\,.$$

We can show that I equals $\widetilde{I}$ on $[C'(\theta_0+),\infty)$, where $C'(\theta_0+) = \lim_{\varepsilon\to 0} C'(\theta_0+\varepsilon)$ for $\varepsilon > 0$. As a corollary of Theorem 0.1, for $A \subset [C'(\theta_0+),\infty)$ with $\inf_{\lambda\in A^\circ} I(\lambda) = \inf_{\lambda\in\bar{A}} I(\lambda)$,

$$\lim_{t\to\infty} \frac{1}{t} \log P_x\left(\frac{A_t^\mu}{t} \in A\right) = -\inf_{\lambda\in A} I(\lambda)\,.$$

In particular, if $C = \widetilde{C}$, that is, $C(\theta) = 0$ for $\theta \leq \theta_0$, then the large deviation principle for A_t^μ/t holds.

In [9], [12], they showed that C equals $\widetilde{C}$ for the Brownian motion or α-stable process. In general, C does not equals $\widetilde{C}$ when $C(0) < 0$ ([10, Theorem 3.1 (ii)]).

In the proof of the large deviation principle for the positive continuous additive functional A_t^μ in the Revuz correspondence with μ, we use the Gärtner-Ellis Theorem. The function $\widetilde{C}(\theta)$ is regarded as the logarithmic moment generating function of A_t^μ. In the Gärtner-Ellis theorem, the differentiability of logarithmic moment generating functions is a sufficient condition for obtaining the lower bound. Needless to say, it is impossible to show the differentiability for continuous additive functionals of general symmetric Markov processes. Indeed, if $\theta_0 > 0$ and $C(0) < 0$, then the right derivative of $\widetilde{C}$ at $\theta = \theta_0$ is positive because it is equal to $C'(\theta_0)$ and $\widetilde{C}(\theta)$ is convex, but the left derivative is 0. Therefore, the logarithmic moment generating function $\widetilde{C}(\theta)$ is not differentiable at θ_0.

We prove first the lower bound for the absorbing symmetric Markov process $\mathbf{M}^G$ on a relatively compact open set $G \subset X$. For $\theta \in \mathbb{R}^1$, let

$$C^G(\theta) = -\inf\left\{\mathcal{E}^{\theta\mu,G}(u,u) : u \in \mathcal{D}(\mathcal{E}^G),\ \int_G u^2 dm = 1\right\}\,,$$

where $\mathcal{D}(\mathcal{E}^G) = \{u \in \mathcal{D}(\mathcal{E}) : u = 0 \text{ q.e. on } X\setminus G\}$. Here $\mathcal{E}^{\theta\mu,G}$ is the Schrödinger form on G defined in (7). Combining the local ultra-contractivity with the analytic perturbation theory, we can obtain that $C^G(\theta)$ is an analytic function in θ. Applying the Gärtner-Ellis theorem, we can show that the lower bound for absorbing symmetric Markov process $\mathbf{M}^G$. Then by approximating of X by G_n, where $\{G_n\}$ is an increasing sequence of relatively compact open sets with $\bigcup_{n=1}^{\infty} G_n = X$, we obtain the lower bound for the Markov process $\mathbf{M}$ on the whole space X.

On the other hand, to show the upper bound, we use two facts, L^p-independence of spectral bounds of Keynman-Kac semigroups and gaugeability for Schrödinger type operator. We show by the L^p-indepencence that for $\theta \geq \theta_0$ the logarithmic moment generating function of A^{μ} exists and equals $\widetilde{C}$, and by the gaugeability that for $\theta \leq \theta_0$ it equals 0. Hence, applying Gärtner-Ellis theorem, we have the upper bound.

Finally, we treat the 1-dimensional Brownian motion (P_x^w, B_t) with a positive drift k as an example. At this time, (P_x^w, B_t) satisfies the assumptions in Theorem 0.1 . We can choose the Dirac measure δ_0 at 0 as a positive Radon measure in the Green-tight Kato class. Then the local time l_t of the Brownian motion (P_x^w, B_t) at the origin is the continuous additive functional in the Revuz correspondence to δ_0. Let $\mathcal{L} = \frac{1}{2}\frac{d^2}{dx^2} + k\frac{d}{dx}$ be the infinitesimal generator of (P_x^w, B_t). Then $\mathcal{L}^{\delta_0} := \mathcal{L} + \delta_0$ is a self-adjoint operator on $L^2(\mathbb{R}, e^{2kx}dx)$. Since $C(\theta)$ is equal to the bottom of spectrum of $\mathcal{L}^{\delta_0}$, $C(\theta)$ is negative on $\theta < k$. Therefore we can see that $C(\theta) \neq \widetilde{C}(\theta)$ on $\theta < k$, and hence $I(\lambda) \neq \widetilde{I}(\lambda)$ on $0 \leq \lambda < k$. In particular, for $A \subset [k, \infty)$ with $\inf_{\lambda \in A^\circ} I(\lambda) = \inf_{\lambda \in \bar{A}} I(\lambda)$, we have

$$\lim_{t\to\infty} \frac{1}{t} \log P_x^w \left(\frac{l_t}{t} \in A \right) = - \inf_{\lambda \in A} I(\lambda)\,.$$

This paper is organized as follow. After giving preliminaries in Section 1, we shall prove that a large deviation principle for the positive continuous additive functional A_t^{μ} in the Revuz correspondence with μ in the Green-tight Kato class in Section 2. Finally, We shall give an example for our theorem to the 1-dimensional Brownian motion with a positive drift k in Section 3.

1. Preliminaries. Let X be a locally compact separable metric space and m a positive Radon measure on X with full topological support. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be an m-symmetric regular irreducible Dirichlet form on $L^2(X; m)$. It is known that a regular Dirichlet form $\mathcal{E}$ has the Beurling-Deny decomposition ([5, Theorem 3.2.1]) : for $u \in \mathcal{D}(\mathcal{E})$

$$\mathcal{E}(u, u) = \frac{1}{2}\int_X d\mu^c_{\langle u \rangle} + \iint_{X\times X\setminus diag} (u(x) - u(y))^2 J(dxdy) + \int_X u^2 dk\,. \tag{2}$$

Here $\mu^c_{\langle u \rangle}$, J and k are the energy measure of the strongly local part, the jumping measure and the killing measure with respect to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, respectively.

We assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is transient, that is, there exists a strictly positive, bounded function $g \in L^1(X; m)$ such that for $u \in \mathcal{D}(\mathcal{E})$

$$\int_X |u| g dm \leq \sqrt{\mathcal{E}(u, u)}$$

(cf. [5, p.40]).

We denote by $u \in \mathcal{D}_{loc}(\mathcal{E})$ if for any relatively compact open set D there exists a function $v \in \mathcal{D}(\mathcal{E})$ such that $u = v$ m-a.e. on D. We denote by $\mathcal{D}_e(\mathcal{E})$ the family of m-measurable functions u on X such that $|u| < \infty$ m-a.e. and there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\}$ of functions in $\mathcal{D}(\mathcal{E})$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. We call $\mathcal{D}_e(\mathcal{E})$ the *extended Dirichlet space* of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Let $\mathbf{M} = (\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \{P_x\}_{x\in X}, \{X_t\}_{t\geq 0}, \zeta)$ be the m-symmetric Hunt process generated by $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, where $\{\mathscr{F}_t\}_{t\geq 0}$ is the augmented filtration and ζ is the lifetime of $\mathbf{M}$. Denote by $\{p_t\}_{t\geq 0}$ and $\{G_\alpha\}_{\alpha\geq 0}$ the semigroup and resolvent of $\mathbf{M}$:

$$p_t f(x) = E_x(f(X_t)), \qquad G_\alpha f(x) = \int_0^\infty e^{-\alpha t} p_t f(x) dt\,.$$

We assume that $\mathbf{M}$ satisfies the next conditions:

Irreducibility (I). If a Borel set A is p_t-invariant, i.e., $p_t(1_A f)(x) = 1_A p_t f(x)$ m-a.e. for any $f \in L^2(X; m) \cap \mathscr{B}_b(X)$ and $t > 0$, then A satisfies either $m(A) = 0$ or $m(X \setminus A) = 0$. Here $\mathscr{B}_b(X)$ is the space of bounded Borel functions on X.

Conservativeness (C). $P_x(\zeta = \infty) = 1$ for each $x \in X$.

Doubly Feller Property (DF). For each $t > 0$, $p_t(C_\infty(X)) \subset C_\infty(X)$, $\lim_{t\to 0} \|p_t f - f\|_\infty = 0$ for any $f \in C_\infty(X)$ and $p_t(\mathscr{B}_b(X)) \subset C_b(X)$, where $C_\infty(X)$ (resp. $C_b(X)$) is the space of continuous functions on X vanishing at infinity (resp. the space of bounded continuous functions on X).

Local Ultra-contractivity (LU). Let $\{p_t^G\}$ be the semigroup defined by $p_t^G f(x) = E_x(f(X_t); t < \tau_G)$ for any $f \in \mathscr{B}_b(X)$, where τ_G is the first exit time from G. Then for any relatively compact open set G, the semigroup $\{p_t^G\}$ is ultra-contractive, $\|p_t^G f\|_\infty \leq C(t)\|f\|_1$, where $C(t)$ is the operator norm of p_t^G from $L^1(G; m)$ to $L^\infty(G; m)$.

REMARK 1.1. $C(t)$ is non-increasing. Indeed, for $t > s$

$$\|p_t f\|_\infty = \|p_s \cdot p_{t-s} f\|_\infty \leq \|p_s\|_{1,\infty}\|p_{t-s} f\|_{1,1} \leq \|p_s\|_{1,\infty}\|p_{t-s}\|_{1,1}\|f\|_1$$

and $\|p_{t-s}\|_{1,1} \leq 1$, we have $\|p_t\|_{1,\infty} \leq \|p_s\|_{1,\infty}$.

We remark that **(DF)** implies

Absolute Continuity Condition (AC). The transition probability of $\mathbf{M}$ is absolutely continuous with respect to m, $p(t, x, dy) = p(t, x, y)m(dy)$ for each $t > 0$ and $x \in X$.

Under **(AC)**, there exists a non-negative, jointly measurable α-resolvent kernel $G_\alpha(x, y)$ on $X \times X$:

$$G_\alpha f(x) = \int_X G_\alpha(x, y) f(y) m(dy),\ x \in X,\ f \in \mathscr{B}_b(X)\,.$$

Moreover, $G_\alpha(x, y)$ is α-excessive in x and in y ([5, Lemma 4.2.4]). We simply write $G(x, y)$ for $G_0(x, y)$. For a measure μ, we define the α-potential of μ by

$$G_\alpha \mu(x) = \int_X G_\alpha(x, y)\mu(dy)\,.$$

We define the (1-)*capacity* Cap associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ as follows: for an open set $O \subset X$,

$$\mathrm{Cap}(O) = \inf\{\mathcal{E}_1(u,u) : u \in \mathcal{D}(\mathcal{E}), u \geq 1,\ m\text{-a.e. on } O\},$$

where $\mathcal{E}_1(u,u) = \mathcal{E}(u,u) + (u,u)_m$, for a Borel set $A \subset X$,

$$\mathrm{Cap}(A) = \inf\{\mathrm{Cap}(O) : O \text{ is open, } O \supset A\}.$$

A statement depending on $x \in X$ is said to hold q.e. on X if there exists a set $N \subset X$ of zero capacity such that the statement is true for every $x \in X \setminus N$. The notation "q.e." is an abbreviation of "quasi-everywhere". A real valued function u defined q.e. on X is said to be *quasi-continuous* if for any $\varepsilon > 0$ there exists an open set $G \subset X$ such that $\mathrm{Cap}(G) < \varepsilon$ and $u|_{X\setminus G}$ is finite and continuous. Here, $u|_{X\setminus G}$ denotes the restriction of u to $X \setminus G$. It is known that each function u in $\mathcal{D}_e(\mathcal{E})$ admits a quasi-continuous version $\tilde{u}$, that is, $u = \tilde{u}$ m-a.e.([5, Theorem 2.1.7]). In the sequel, we always assume that every function $u \in \mathcal{D}_e(\mathcal{E})$ is represented by its quasi-continuous version.

Let S_{00} be the set of positive Borel measures μ such that $\mu(X) < \infty$ and $G_1\mu$ is bounded. We call a Borel measure μ on X *smooth* if there exists a sequence $\{E_n\}$ of Borel sets increasing to X such that $1_{E_n} \cdot \mu \in S_{00}$ for each n and

$$P_x(\lim_{n\to\infty} \sigma_{X\setminus E_n} \geq \zeta) = 1, \quad \forall x \in X.$$

Here $\sigma_{X\setminus E_n}$ is the hitting time of $X \setminus E_n$ by $\mathbf{M}$, $\sigma_{X\setminus E_n} = \inf\{t > 0 : X_t \in X \setminus E_n\}$. We denote by S the set of positive smooth Borel measures. In [5], a measure in S is called a *smooth measure in the strict sense*. Here we omit the adjective phrase "in the strict sense".

A stochastic process $\{A_t\}_{t\geq 0}$ is said to be an *additive functional* (AF in abbreviation) if the following conditions hold:

(i) $A_t(\cdot)$ is $\mathscr{F}_t$-measurable for all $t \geq 0$.

(ii) There exists a set $\Lambda \in \mathscr{F}_\infty = \sigma\left(\bigcup_{t\geq 0} \mathscr{F}_t\right)$ such that $P_x(\Lambda) = 1$, for all $x \in X$, $\theta_t\Lambda \subset \Lambda$ for all $t > 0$, and for each $\omega \in \Lambda$, $A_\cdot(\omega)$ is right continuous and has the left limit on $[0, \zeta(\omega))$, $A_0(\omega) = 0$, $|A_t(\omega)| < \infty$ for $t < \zeta(\omega)$, $A_t(\omega) = A_{\zeta(\omega)}(\omega)$ for $t \geq \zeta$, and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for $s, t \geq 0$.

If an AF $\{A_t\}_{t\geq 0}$ is positive and continuous with respect to t for each $\omega \in \Lambda$, the AF is called a *positive continuous additive functional* (PCAF in abbreviation). The set of all PCAF's is denoted by $\mathbf{A}_c^+$. The family S and $\mathbf{A}_c^+$ are in one-to-one correspondence (Revuz correspondence) as follows: for each smooth measure μ, there exists a unique PCAF $\{A_t\}_{t\geq 0}$ such that for any $f \in \mathscr{B}^+(X)$ and γ-excessive function h,

$$\lim_{t\to 0} \frac{1}{t} E_{h\cdot m}\left(\int_0^t f(X_s)dA_s\right) = \int_X f(x)h(x)\mu(dx) \tag{3}$$

([5, Theorem 5.1.7]). Here, $E_{h\cdot m}(\,\cdot\,) = \int_X E_x(\,\cdot\,)h(x)m(dx)$. We denote the PCAF A_t^μ by A_t^μ to emphasize the correspondence between μ and $\{A_t\}_{t\geq 0}$.

We define some classes of smooth measures.

DEFINITION 1.2. Suppose that $\mu \in S$ is a positive Radon measure.

(1) A measure μ is said to be in the *Kato class* of **M** ($\mathcal{K}$ in abbreviation) if

$$\lim_{\alpha\to\infty} \|G_\alpha \mu\|_\infty = 0\,.$$

A measure μ is said to be in the *local Kato class* of **M** ($\mathcal{K}_{loc}$ in abbreviation) if $1_K \cdot \mu \in \mathcal{K}$ for any relatively compact open set K. Here 1_K is the indicator function of K.

(2) A measure μ is said to be in the class $\mathcal{K}_\infty$ if $\mu \in \mathcal{K}$ and for any $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon)$

$$\sup_{x\in X} \int_{K^c} G(x, y)\mu(dy) < \varepsilon\,.$$

A measure μ in $\mathcal{K}_\infty$ is called *Green-tight*.

We note that every measure treated in this paper is supposed to be Radon. Thus we see from [1, Theorem 3.9] that $\mu \in \mathcal{K}$ if and only if

$$\lim_{t\downarrow 0} \sup_{x\in X} E_x(A_t^\mu) = \lim_{t\downarrow 0} \sup_{x\in X} \int_0^t \int_X p(s, x, y)\mu(dy)ds = 0. \tag{4}$$

Chen [2] defined the Green-tight class in slightly different way, however two definitions are equivalent under the strong Feller property ([6, Lemma 4.1]). We see from [8] that for $\alpha \geq 0$ and $\mu \in \mathcal{K}$

$$\int_X u^2 d\mu \leq \|G_\alpha \mu\|_\infty \cdot \mathcal{E}_\alpha(u, u) \quad \text{for any } u \in \mathcal{D}(\mathcal{E})\,. \tag{5}$$

Let $\mu \in \mathcal{K}$. We define the Schrödinger form by

$$\begin{cases} \mathcal{E}^\mu(u, u) = \mathcal{E}(u, u) - \int_X u^2 d\mu \\ \mathcal{D}(\mathcal{E}^\mu) = \mathcal{D}(\mathcal{E})\,. \end{cases} \tag{6}$$

We denote by $\mathcal{L}^\mu = \mathcal{L} + \mu$ the self-adjoint operator associated with the closed symmetric form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$, that is, $(-\mathcal{L}^\mu u, v)_m = \mathcal{E}^\mu(u, v)$ for any $u, v \in \mathcal{D}(\mathcal{E})$.

We define the *Feynman-Kac semigroup* $\{p_t^\mu\}_{t\geq 0}$ by

$$p_t^\mu f(x) = E_x(\exp(A_t^\mu) f(X_t)), \;\; x \in X, \;\; f \in \mathscr{B}_b(X)\,.$$

THEOREM 1.3 ([12]). *Let $\mu \in \mathcal{K}$. For any $\varepsilon > 0$ there exists $M(\varepsilon) > 0$ such that for any $u \in \mathcal{D}(\mathcal{E})$*

$$\int_X u^2 d\mu \leq \varepsilon \mathcal{E}(u, u) + M(\varepsilon) \int_X u^2 dm\,.$$

THEOREM 1.4 ([12]). *Let $\mu \in \mathcal{K}_\infty$. Then for any $u \in \mathcal{D}(\mathcal{E})$*

$$\int_X u^2 d\mu \leq \|G\mu\|_\infty \cdot \mathcal{E}(u, u)\,.$$

2. Large deviation principle. Let $G \subset X$ be a relatively compact open set. We set

$$\mathcal{D}(\mathcal{E}^G) = \{u \in \mathcal{D}(\mathcal{E}) : u = 0 \text{ q.e. on } X \setminus G\}.$$

Here $\mathcal{E}^G$ is the part of the Dirichlet form $\mathcal{E}$ on G. $\mathcal{D}(\mathcal{E}^G)$ is a closed subspace of the Hilbert space $(\mathcal{D}(\mathcal{E}), \mathcal{E}_1)$. $(\mathcal{E}^G, \mathcal{D}(\mathcal{E}^G))$ is a regular Dirichlet form on $L^2(G; m)$. Let $\mathbf{M}^G$ be the associated the Markov process of $(\mathcal{E}^G, \mathcal{D}(\mathcal{E}^G))$, namely, the part process of $\mathbf{M}$ on G ([5, A.2]). Indeed, $\mathbf{M}^G$ is an absorbing Markov process on G with an m-symmetric transition function p_t^G on $(G, \mathcal{B}(G))$ defined by $p_t^G(x, B) = P_x(X_t \in B; t < \tau_G)$, where τ_G is the first exit time of G.

For $\theta \in \mathbb{R}^1$ let

$$\mathcal{E}^{\theta\mu,G}(u,u) = \mathcal{E}^G(u,u) - \theta \int_X u^2 d\mu, \quad u \in \mathcal{D}(\mathcal{E}^G) \tag{7}$$

and

$$C^G(\theta) = -\inf\left\{\mathcal{E}^{\theta\mu,G}(u,u) : u \in \mathcal{D}(\mathcal{E}^G), \int_G u^2 dm = 1\right\}. \tag{8}$$

Let I^G be the Legendre transform of C^G:

$$I^G(\lambda) = \sup_{\theta \in \mathbb{R}^1} \left\{\lambda\theta - C^G(\theta)\right\}, \quad \lambda \in \mathbb{R}^1.$$

LEMMA 2.1. *For $u_1, u_2 \in \mathcal{D}(\mathcal{E})$ and $0 \le \alpha \le 1$, $u := \sqrt{\alpha u_1^2 + (1-\alpha)u_2^2} \in \mathcal{D}(\mathcal{E})$ and*

$$\mathcal{E}(u,u) \le \alpha\mathcal{E}(u_1,u_1) + (1-\alpha)\mathcal{E}(u_2,u_2).$$

PROOF. First, we consider the energy measure of the strongly local part of (2). By Theorem 5.6.2 in [5], for any $\Phi \in C^1(\mathbb{R}^d)$ and $v_1, \ldots, v_d \in \mathcal{D}(\mathcal{E})_b := \mathcal{D}(\mathcal{E}) \cap L^\infty(X; m)$, the composite function $\Phi(v) = \Phi(v_1, \ldots, v_d)$ with $\Phi(0) = 0$ is in $\mathcal{D}(\mathcal{E})_b$ and

$$d\mu^c_{\langle \Phi(v), w\rangle} = \sum_{i=1}^d \Phi_{x_i}(v) d\mu^c_{\langle v_i, w\rangle}, \quad \text{for any } \omega \in \mathcal{D}(\mathcal{E})_b,$$

where Φ_{x_i} is the partial derivative of Φ with respect to x_i.

By applying the formula above to $v = (u_1, u_2)$ and $\Phi(v) = \sqrt{\alpha u_1^2 + (1-\alpha)u_2^2}$, we have

$$d\mu^c_{\langle u\rangle} = \frac{\alpha^2 u_1^2}{\alpha u_1^2 + (1-\alpha)u_2^2} d\mu^c_{\langle u_1\rangle} + 2\frac{\alpha(1-\alpha)u_1u_2}{\alpha u_1^2 + (1-\alpha)u_2^2} d\mu^c_{\langle u_1,u_2\rangle} + \frac{(1-\alpha)^2 u_2^2}{\alpha u_1^2 + (1-\alpha)u_2^2} d\mu^c_{\langle u_2\rangle}.$$

Since

$$\begin{aligned}
&\int_X \frac{\alpha(1-\alpha)u_1u_2}{\alpha u_1^2 + (1-\alpha)u_2^2} d\mu^c_{\langle u_1,u_2\rangle} \\
&\le \left(\int_X \frac{\alpha(1-\alpha)u_2^2}{\alpha u_1^2 + (1-\alpha)u_2^2} d\mu^c_{\langle u_1\rangle}\right)^{1/2} \left(\int_X \frac{\alpha(1-\alpha)u_1^2}{\alpha u_1^2 + (1-\alpha)u_2^2} d\mu^c_{\langle u_2\rangle}\right)^{1/2}
\end{aligned}$$

$$\leq \int_X \frac{\alpha(1-\alpha)u_2^2}{\alpha u_1^2+(1-\alpha)u_2^2} d\mu^c_{\langle u_1\rangle} + \int_X \frac{\alpha(1-\alpha)u_1^2}{\alpha u_1^2+(1-\alpha)u_2^2} d\mu^c_{\langle u_2\rangle},$$

by Lemma 5.6.1 in [5], we have

$$\int_X d\mu^c_{\langle u\rangle} \leq \int_X \frac{\alpha(\alpha u_1^2+(1-\alpha)u_2^2)}{\alpha u_1^2+(1-\alpha)u_2^2} d\mu^c_{\langle u_1\rangle} + \int_X \frac{(1-\alpha)(\alpha u_1^2+(1-\alpha)u_2^2)}{\alpha u_1^2+(1-\alpha)u_2^2} d\mu^c_{\langle u_2\rangle}$$

$$\leq \alpha \int_X d\mu^c_{\langle u_1\rangle} + (1-\alpha)\int_X d\mu^c_{\langle u_2\rangle}.$$

Moreover, noting

$$u(x)u(y) = \sqrt{\alpha u_1^2(x)+(1-\alpha)u_2^2(x)}\sqrt{\alpha u_1^2(y)+(1-\alpha)u_2^2(y)}$$
$$\geq \alpha u_1(x)u_1(y) + (1-\alpha)u_2(x)u_2(y),$$

we have

$$(u(x)-u(y))^2 \leq \alpha(u_1(x)-u_1(y))^2 + (1-\alpha)(u_2(x)-u_2(y))^2$$

and thus $\mathcal{E}^j(u,u) \leq \alpha\mathcal{E}^j(u_1,u_1) + (1-\alpha)\mathcal{E}^j(u_2,u_2)$. The proof of this lemma is completed. □

Define

$$\tilde{J}^G(\lambda) := \inf\left\{\mathcal{E}^G(u,u) : u \in \mathcal{D}(\mathcal{E}^G),\ \int_G u^2 d\mu = \lambda,\ \int_G u^2 dm = 1\right\},\quad \lambda \in \mathbb{R}^1$$

and

$$J^G(\lambda) = \lim_{\varepsilon\to 0} \inf_{|\lambda'-\lambda|<\varepsilon} \tilde{J}^G(\lambda').$$

J^G is the lower semi-continuous modification of $\tilde{J}^G$. From Lemma 2.1, we have

LEMMA 2.2. *The function $\tilde{J}^G$ is convex: for $0 \leq \alpha \leq 1$ and $\lambda_1, \lambda_2 \in \mathbb{R}^1$*

$$\tilde{J}^G(\alpha\lambda_1 + (1-\alpha)\lambda_2) \leq \alpha\tilde{J}^G(\lambda_1) + (1-\alpha)\tilde{J}^G(\lambda_2).$$

PROOF. For any $u_1, u_2 \in \mathcal{D}(\mathcal{E}^G)$ such that

$$\int_G u_i^2 d\mu = \lambda_i,\ \int_G u_i^2 dm = 1,\ i = 1, 2,$$

let $u := \sqrt{\alpha u_1^2 + (1-\alpha)u_2^2},\ 0 \leq \alpha \leq 1$. Then u belongs to $\mathcal{D}(\mathcal{E}^G)$,

$$\int_G u^2 d\mu = \alpha\lambda_1 + (1-\alpha)\lambda_2 \text{ and } \int_G u^2 dm = 1.$$

We see by the definition of $\tilde{J}^G(\lambda)$ and Lemma 2.1 that for any $u_1, u_2 \in \mathcal{D}(\mathcal{E}^G)$ satisfying above conditions,

$$\tilde{J}^G(\alpha\lambda_1 + (1-\alpha)\lambda_2) \leq \mathcal{E}(u,u)$$
$$\leq \alpha\mathcal{E}(u_1,u_1) + (1-\alpha)\mathcal{E}(u_2,u_2).$$

Therefore, we have the lemma. □

LEMMA 2.3. *The function J^G is convex.*

PROOF. Let $\lambda_1, \lambda_2 \in \mathbb{R}^1$. For λ' and λ'' with $|\lambda' - \lambda_1| < \varepsilon$ and $|\lambda'' - \lambda_2| < \varepsilon$,

$$\inf_{|\lambda-(\alpha\lambda_1+(1-\alpha)\lambda_2)|<\varepsilon} \tilde{J}^G(\lambda) \le \tilde{J}^G(\alpha\lambda' + (1-\alpha)\lambda'')$$
$$\le \alpha\tilde{J}^G(\lambda') + (1-\alpha)\tilde{J}^G(\lambda'')$$

by Lemma 2.2, and thus

$$\inf_{|\lambda-(\alpha\lambda_1+(1-\alpha)\lambda_2)|<\varepsilon} \tilde{J}^G(\lambda) \le \alpha \inf_{|\lambda'-\lambda_1|<\varepsilon} \tilde{J}^G(\lambda') + (1-\alpha) \inf_{|\lambda''-\lambda_2|<\varepsilon} \tilde{J}^G(\lambda'') .$$

The proof is completed by letting $\varepsilon \to 0$. □

LEMMA 2.4. *The function C^G is the Legendre conjugate of J^G,*

$$C^G(\theta) = \sup_{\lambda\in\mathbb{R}^1} \{\theta\lambda - J^G(\lambda)\} .$$

PROOF. Let

$$\mathcal{A} = \left\{u \in \mathcal{D}(\mathcal{E}^G) : \int_G u^2 dm = 1\right\}$$
$$\mathcal{A}_\lambda = \left\{u \in \mathcal{D}(\mathcal{E}^G) : \int_G u^2 d\mu = \lambda,\ \int_G u^2 dm = 1\right\},\quad \lambda \in \mathbb{R}^1 .$$

For any $\varepsilon > 0$, set

$$\mathcal{A}_{\lambda,\varepsilon} = \left\{u \in \mathcal{D}(\mathcal{E}^G) : \lambda - \varepsilon < \int_G u^2 d\mu < \lambda + \varepsilon,\ \int_G u^2 dm = 1\right\} .$$

Then

$$\inf_{u\in\mathcal{A}} \mathcal{E}^{\theta\mu,G}(u,u) \le \inf_{u\in\mathcal{A}_{\lambda,\varepsilon}} \mathcal{E}^{\theta\mu,G}(u,u) \le \lim_{\varepsilon\to 0} \inf_{u\in\mathcal{A}_{\lambda,\varepsilon}} \mathcal{E}^{\theta\mu,G}(u,u) \le \inf_{u\in\mathcal{A}_\lambda} \mathcal{E}^{\theta\mu,G}(u,u)$$

and thus

$$\inf_{u\in\mathcal{A}} \mathcal{E}^{\theta\mu,G}(u,u) \le \inf_\lambda \lim_{\varepsilon\to 0} \inf_{u\in\mathcal{A}_{\lambda,\varepsilon}} \mathcal{E}^{\theta\mu,G}(u,u) \le \inf_\lambda \inf_{u\in\mathcal{A}_\lambda} \mathcal{E}^{\theta\mu,G}(u,u) = \inf_{u\in\mathcal{A}} \mathcal{E}^{\theta\mu,G}(u,u) .$$

Hence we have

$$C^G(\theta) = -\inf_\lambda \lim_{\varepsilon\to 0} \inf_{u\in\mathcal{A}_{\lambda,\varepsilon}} \mathcal{E}^{\theta\mu,G}(u,u)$$
$$= -\inf_\lambda \lim_{\varepsilon\to 0} \inf_{|\lambda'-\lambda|<\varepsilon} \left(\inf_{u\in\mathcal{A}_{\lambda'}} \mathcal{E}^{\theta\mu,G}(u,u)\right)$$
$$= -\inf_\lambda \lim_{\varepsilon\to 0} \inf_{|\lambda'-\lambda|<\varepsilon} \left(\tilde{J}^G(\lambda') - \theta\lambda'\right) .$$

Noting

$$\lim_{\varepsilon\to 0} \inf_{|\lambda'-\lambda|<\varepsilon} \left(\tilde{J}^G(\lambda') - \theta\lambda'\right) = J^G(\lambda) - \theta\lambda ,$$

we have

$$C^G(\theta) = -\inf_{\lambda}\{J^G(\lambda) - \theta\lambda\} = \sup_{\lambda}\{\theta\lambda - J^G(\lambda)\}\,.$$

□

As a result, we see that

LEMMA 2.5.

$$I^G = J^G\,.$$

PROOF. The function J^G is lower semi-continuous, convex and not identically infinite. Hence, it follows from Lemma 2.4 and [4, Theorem 2.2.15] that $J^G = I^G$. □

We use the notations J (resp. $\widetilde{J}$) for J^G (resp. $\widetilde{J}^G$) when $G = X$.

LEMMA 2.6. *Let $\{G_n\}$ be an increasing sequence of relatively compact open sets with $\bigcup_{n=1}^{\infty} G_n = X$. Then for an open set $O \subset \mathbb{R}^1$*

$$\inf_{\lambda\in O} J(\lambda) = \inf_n \inf_{\lambda\in O} J^{G_n}(\lambda)\,.$$

PROOF. By the regularity of the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$,

$$\begin{aligned}
\inf_{\lambda\in O} \widetilde{J}(\lambda) &= \inf\left\{\mathcal{E}(u,u) : u \in \mathcal{D}(\mathcal{E}), \int_X u^2 d\mu \in O, \int_X u^2 dm = 1\right\} \\
&= \inf\left\{\mathcal{E}(u,u) : u \in \mathcal{D}(\mathcal{E}) \cap C_0(X), \int_X u^2 d\mu \in O, \int_X u^2 dm = 1\right\} \\
&= \inf_n \inf\left\{\mathcal{E}(u,u) : u \in \mathcal{D}(\mathcal{E}) \cap C_0(G_n), \int_X u^2 d\mu \in O, \int_X u^2 dm = 1\right\} \\
&= \inf_n \inf_{\lambda\in O} \widetilde{J}^{G_n}(\lambda)\,.
\end{aligned}$$

Noting that $\inf_{\lambda\in O} \widetilde{J}^G(\lambda) = \inf_{\lambda\in O} J^G(\lambda)$ for any open set $O \subset \mathbb{R}^1$, we have the lemma. □

Let $\mu \in \mathcal{K}_{loc}$. Let G be a relatively compact open set of X. Denote by $\{G_\alpha^G\}_{\alpha\geq 0}$ the resolvent of the part process $\mathbf{M}^G$ of $\mathbf{M}$ on G. Then the part process $\mathbf{M}^G$ is *tight* in the sense that for any $\varepsilon > 0$, there exists a compact set $K \subset G$ such that

$$\sup_{x\in G} G_1^G 1_{K^c}(x) \leq \varepsilon\,.$$

Here 1_{K^c} is the indicator function of $G \setminus K$. In fact, note that for $x \in G$,

$$G_1^G 1_{K^c}(x) = \int_0^\infty e^{-t} p_t^G 1_{K^c}(x)dt = \int_0^\delta e^{-t} p_t^G 1_{K^c}(x)dt + \int_\delta^\infty e^{-t} p_t^G 1_{K^c}(x)dt\,.$$

We see from (**LU**) and Remark 1.1 that the right hand side is dominated by

$$\begin{aligned}
\int_0^\delta e^{-t}dt + \int_\delta^\infty e^{-t}\|p_t^G\|_{1,\infty} m(G\setminus K)dt &\leq 1 - e^{-\delta} + \int_\delta^\infty e^{-t} C(\delta) m(G\setminus K)dt \\
&\leq 1 - e^{-\delta} + e^{-\delta} C(\delta) m(G\setminus K)\,.
\end{aligned}$$

For any $\varepsilon > 0$, we choose $\delta > \log(1 - \frac{\varepsilon}{2})$ and a compact set $K \subset G$ satisfying $m(G \setminus K) < \frac{e^{\delta}\varepsilon}{2c(\delta)}$, and obtain the tightness of $\mathbf{M}^G$.

Let $\{p_t^{\mu,G}\}_{t>0}$ be the semigroup defined by

$$p_t^{\mu,G} f(x) = E_x\left(e^{A_t^{\mu}} f(X_t); t < \tau_G\right), \quad \text{for } f \in \mathscr{B}_b(G).$$

Define the L^p-spectral bounds of $\{p_t^{\mu,G}\}_{t>0}$ by

$$\lambda_p^G(\mu) = -\lim_{t\to\infty} \frac{1}{t} \log \| p_t^{\mu,G} \|_{p,p}, \quad 1 \le p \le \infty,$$

where $\| p_t^{\mu,G} \|_{p,p}$ is the operator norm of $p_t^{\mu,G}$ from $L^p(G; m)$ to $L^p(G; m)$. We omit 'G' from $\lambda_p^G(\mu)$ when $G = X$.

The L^p-independence of the spectral bounds of $\{p_t^{\mu,G}\}_{t>0}$ means that

$$\lambda_p^G(\mu) = \lambda_2^G(\mu), \quad 1 \le p \le \infty.$$

As mentioned above, the Markov process $\mathbf{M}^G$ is tight, so $\lambda_p^G(\theta\mu)$ is independent of p by [11, Theorem 4.1]. We easily see the following inequality

$$\begin{aligned}
-\lambda_2^G(\theta\mu) \le \liminf_{t\to\infty} \frac{1}{t} \log E_x\left(e^{\theta A_t^{\mu}}; t < \tau_G\right) &\le \limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in G} E_x\left(e^{\theta A_t^{\mu}}; t < \tau_G\right) \\
&= \limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in G} p_t^{\theta\mu,G} 1(x) \\
&= \limsup_{t\to\infty} \frac{1}{t} \log \| p_t^{\theta\mu,G} \|_{\infty} \\
&= -\lambda_{\infty}^G(\theta\mu).
\end{aligned}$$

By combining the L^p-independence of the spectral bounds of $\{p_t^{\theta\mu,G}\}_{t>0}$ and the variational formula for $\lambda_2^G(\theta\mu)$,

$$\lim_{t\to\infty} \frac{1}{t} \log E_x\left(e^{\theta A_t^{\mu}}; t < \tau_G\right) = C^G(\theta). \tag{9}$$

By using (**LU**), the transition function $p_t^{\theta\mu,G}(x, y)$ of $p_t^{\theta\mu,G}$ is bounded for each $t > 0$ and $x, y \in X$, and thus $p_t^{\theta\mu,G}$ is a Hilbert-Schmidt integral operator, in particular, a compact operator. Hence, we see that $C^G(\theta)$ is an analytic function in θ because it is nothing but the eigenvalue of $\mathcal{L}^{\mu}$. Then, combining (9) with the Gärtner-Ellis theorem ([3, Section 2.3]), we obtain the next lower estimate: For any open set $O \subset \mathbb{R}^1$,

$$\liminf_{t\to\infty} \frac{1}{t} \log P_x\left(\frac{A_t^{\mu}}{t} \in O; t < \tau_G\right) \ge -\inf_{\lambda\in O} I^G(\lambda), \tag{10}$$

where I^G is the Legendre transform of C^G.

THEOREM 2.7. *Let $\mu \in \mathcal{K}_{loc}$. Then, for any open set $O \subset \mathbb{R}^1$*

$$\liminf_{t\to\infty} \frac{1}{t} \log P_x\left(\frac{A_t^\mu}{t} \in O\right) \geq -\inf_{\lambda\in O} I(\lambda).$$

PROOF. Let $\{G_n\}$ be a sequence of relatively compact open sets such that $G_n \uparrow X$ and simply write I^n for I^{G_n}. Then we have from (10) that

$$\begin{aligned} &\liminf_{t\to\infty} \frac{1}{t} \log P_x\left(\frac{A_t^\mu}{t} \in O\right) \\ \geq\ &\sup_n \liminf_{t\to\infty} \frac{1}{t} \log P_x\left(\frac{A_t^\mu}{t} \in O; t < \tau_{G_n}\right) \\ \geq\ &-\inf_n \inf_{\lambda\in O} I^n(\lambda). \end{aligned}$$

Since

$$\inf_n \inf_{\lambda\in O} I^n(\lambda) = \inf_{\lambda\in O} I(\lambda),$$

we obtain the theorem. □

Define

$$\gamma(\theta) := \inf\left\{\mathcal{E}(u,u) : u \in \mathcal{D}(\mathcal{E}),\ \theta\int_X u^2 d\mu = 1\right\},\quad \theta \in \mathbb{R}^1. \tag{11}$$

LEMMA 2.8.

$$\gamma(\theta) \leq 1 \Longleftrightarrow \inf\left\{\mathcal{E}^{\theta\mu}(u,u) : \int_X u^2 dm = 1\right\} \leq 0. \tag{12}$$

PROOF. We can prove this lemma by the same argument as in [12, Lemma 2.2]. Assume that $\gamma(\theta) \leq 1$. Then there exists a $\varphi_0 \in C_0(X)$ with $\theta\int_X \varphi_0^2 d\mu = 1$ such that $\mathcal{E}(\varphi_0,\varphi_0) \leq 1$. Hence we see

$$\mathcal{E}(\varphi_0,\varphi_0) \leq \theta\int_X \varphi_0^2 d\mu.$$

Letting

$$u_0 = \frac{\varphi_0}{\sqrt{\int_X \varphi_0^2 dm}},$$

we have

$$\mathcal{E}^{\theta\mu}(u_0,u_0) \leq 0.$$

On the other hand, we assume that $\inf\left\{\mathcal{E}^{\theta\mu}(u,u) : \int_X u^2 dm = 1\right\} \leq 0$. Then there exists a $\psi_0 \in C_0(X)$ with $\int_X \psi_0^2 dm = 1$ such that $\mathcal{E}^{\theta\mu}(\psi_0,\psi_0) \leq 0$. Letting

$$u_0 = \frac{\psi_0}{\sqrt{\theta\int_X {\psi_0}^2 d\mu}},$$

we have

$$\mathcal{E}(u_0, u_0) \leq 1 .$$

□

Let $\theta_0 > 0$ be a unique value such that $\gamma(\theta_0) = 1$. Suppose that $\mu \in \mathcal{K}_\infty$. Under the assumptions (**C**) and (**DF**), if $\lambda_2(\mu) \leq 0$, $\lambda_p(\mu)$ is independent of p by [10, Theorem 3.1]. By combining Lemma 2.8, we can derive the following in a similar way of (9): for $\theta \geq \theta_0$

$$C(\theta) = \lim_{t\to\infty} \frac{1}{t} \log E_x \left(e^{\theta A_t^\mu}\right) .$$

On the other hand, by Lemma 2.8 and [2, Theorem 5.1] on the Schrödinger type operator, we see that $\gamma(\theta) > 1$ is equivalent to

$$\sup_{x\in X} E_x \left(e^{\theta A_\infty^\mu}\right) < \infty .$$

Since A_t^μ is positive, for $\theta < \theta_0$

$$\lim_{t\to\infty} \frac{1}{t} \log E_x \left(e^{\theta A_t^\mu}\right) \leq \lim_{t\to\infty} \frac{1}{t} \log E_x \left(e^{\theta A_\infty^\mu}\right) = 0 .$$

Hence we have

THEOREM 2.9. *Let $\mu \in \mathcal{K}_\infty$. Then*

$$\lim_{t\to\infty} \frac{1}{t} \log E_x \left(e^{\theta A_t^\mu}\right) = \widetilde{C}(\theta) ,$$

where $\widetilde{C}(\theta)$ is the function defined by

$$\widetilde{C}(\theta) = \begin{cases} C(\theta), & \theta \geq \theta_0 , \\ 0, & \theta < \theta_0 . \end{cases}$$

Let $\widetilde{I}$ be the Legendre transform of $\widetilde{C}(\theta)$,

$$\widetilde{I}(\lambda) = \sup_{\theta\in\mathbb{R}^1} \{\lambda\theta - \widetilde{C}(\theta)\} .$$

Then, combining Theorem 2.9 with the Gärtner-Ellis theorem ([3, Section 2.3]), we have the upper bound:

THEOREM 2.10. *Let $\mu \in \mathcal{K}_\infty$. Then for any closed set $K \subset \mathbb{R}^1$,*

$$\limsup_{t\to\infty} \frac{1}{t} \log P_x \left(\frac{A_t^\mu}{t} \in K\right) \leq - \inf_{\lambda\in K} \widetilde{I}(\lambda) .$$

The Legendre transform of $C(\theta)$ and $\widetilde{C}(\theta)$ are expressed as follows:

$$\begin{aligned} I(\lambda) &= \sup_{\theta\in\mathbb{R}^1} \{\lambda\theta - C(\theta)\} \\ &= \begin{cases} \lambda(C')^{-1}(\lambda) - C((C')^{-1}(\lambda)), & \lambda \geq C'(\theta_0+) \\ C(0), & 0 \leq \lambda < C'(\theta_0+) \\ \infty, & \lambda < 0 . \end{cases} \end{aligned} \tag{13}$$

$$
(14)\qquad \begin{aligned}\widetilde{I}(\lambda) &= \sup_{\theta\in\mathbb{R}^1}\{\lambda\theta-\widetilde{C}(\theta)\}\\ &= \begin{cases}\lambda(C')^{-1}(\lambda)-C((C')^{-1}(\lambda)), & \lambda\geq C'(\theta_0+)\\ \lambda\theta_0, & 0\leq\lambda<C'(\theta_0+)\\ \infty, & \lambda<0\,.\end{cases}\end{aligned}
$$

Hence, I equals $\widetilde{I}$ on $[C'(\theta_0+),\infty)$.

3. Example.

EXAMPLE 3.1. Let us consider the 1-dimensional Brownian motion (P^w_x, B_t) with a positive drift k. Then the process (P^w_x, B_t) is transient and its infinitesimal generator $\mathcal{L}$ is given by $\frac{1}{2}\frac{d^2}{dx^2}+k\frac{d}{dx}$. Let $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ be the Dirichlet form on $L^2(\mathbb{R}^1; e^{2kx}dx)$ generated by (P^w_x, B_t), that is,

$$
\begin{cases}\mathcal{E}(u,v)=\dfrac{1}{2}\displaystyle\int_{\mathbb{R}^1}\frac{du}{dx}\frac{dv}{dx}e^{2kx}dx, \quad u,\ v\in\mathcal{D}(\mathcal{E})\\ \mathcal{D}(\mathcal{E})=\text{the closure of } C_0^\infty(\mathbb{R}^1) \text{ with respect to } \mathcal{E}_1^{1/2}\,.\end{cases}
$$

By using integration by parts,

$$
\begin{aligned}\mathcal{E}(u,v) &= -\frac{1}{2}\int_{\mathbb{R}}\left(\frac{d^2u}{dx^2}+2k\frac{du}{dx}\right)ve^{2kx}dx\\ &= (-\mathcal{L}u,v)_{e^{2kx}dx}\ .\end{aligned}
$$

Then (P^w_x, B_t) satisfies the assumptions **(I), (DF), (C)** and **(LU)**.

Let μ be the Dirac measure at the origin. i.e., $\mu=\delta_0$. Then $\mu\in\mathcal{K}_\infty$. Let l_t be the local time at 0. Then l_t is the continuous additive functional corresponding to μ.

We define the functions $C(\theta)$ and $\widetilde{C}(\theta)$ by

$$
C(\theta)=-\inf\left\{\mathcal{E}(u,u)-\theta u^2(0): u\in C_0^\infty(\mathbb{R}^1),\ \int_{\mathbb{R}^1}u^2e^{2kx}dx=1\right\},
$$

$$
\widetilde{C}(\theta)=\begin{cases}C(\theta), & \theta\geq\theta_0\\ 0, & \theta<\theta_0\,.\end{cases}
$$

The function $C(\theta)$ is equal to the bottom of spectrum of the self-adjoint operator $\mathcal{L}^{\delta_0}:=\mathcal{L}+\delta_0$. We first consider $C(\theta)$ for $\theta\geq 0$. For $u\in C_0^\infty(\mathbb{R}^1)$, the boundary condition

$$
u'(0+)-u'(0-)=-2\theta u(0)
$$

must be satisfied. Since $u\in L^2(\mathbb{R}^1, e^{2kx}dx)$, the eigenfunction corresponding to an eigenvalue λ forms

$$
u(x)=\begin{cases}Ce^{-(k+\sqrt{k^2-2\lambda})x}, & x\geq 0\\ Ce^{-(k-\sqrt{k^2-2\lambda})x}, & x<0\,,\end{cases}
$$

where C is a constant. From the boundary condition, we have

$$\sqrt{k^2 - 2\lambda} = \theta .$$

Hence,

$$\lambda = \frac{k^2 - \theta^2}{2} .$$

Since $C(\theta) = C(0)$ for $\theta < 0$, we have

$$C(\theta) = \begin{cases} \frac{\theta^2}{2} - \frac{k^2}{2}, & \theta \geq 0 \\ -\frac{k^2}{2}, & \theta < 0 . \end{cases}$$

Moreover, $\theta_0 = k$, we have

$$\widetilde{C}(\theta) = \begin{cases} \frac{\theta^2}{2} - \frac{k^2}{2}, & \theta \geq k \\ 0, & \theta < k . \end{cases}$$

Let $I(\lambda)$ (resp. $\widetilde{I}(\lambda)$) be the Legendre transform of $C(\theta)$ (resp. $\widetilde{C}(\theta)$):

$$\begin{aligned} I(\lambda) &= \sup_{\theta \in \mathbb{R}^1} \{\lambda\theta - C(\theta)\} \\ &= \begin{cases} \frac{\lambda^2}{2} + \frac{k^2}{2}, & \lambda \geq 0 \\ \infty, & \lambda < 0 . \end{cases} \end{aligned}$$

$$\begin{aligned} \widetilde{I}(\lambda) &= \sup_{\theta \in \mathbb{R}^1} \{\lambda\theta - \widetilde{C}(\theta)\} \\ &= \begin{cases} \frac{\lambda^2}{2} + \frac{k^2}{2}, & \lambda \geq k \\ \lambda k, & 0 \leq \lambda < k \\ \infty, & \lambda < 0 . \end{cases} \end{aligned}$$

Finally, for $A \subset [k, \infty)$ with $\inf_{\lambda \in A^\circ} I(\lambda) = \inf_{\lambda \in \bar{A}} I(\lambda)$,

$$\lim_{t \to \infty} \frac{1}{t} \log P_x^w \left(\frac{l_t}{t} \in A \right) = - \inf_{\lambda \in A} I(\lambda) .$$

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