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# TEICHMÜLLER SPACES AND TAME QUASICONFORMAL MOTIONS

In memory of Professor Clifford J. Earle

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**Abstract.** The concept of "quasiconformal motion" was first introduced by Sullivan and Thurston (in [24]). Theorem 3 of that paper asserted that any quasiconformal motion of a set in the sphere over an interval can be extended to the sphere. In this paper, we give a counter-example to that assertion. We introduce a new concept called "tame quasiconformal motion" and show that their assertion is true for tame quasiconformal motions. We prove a much more general result that, any tame quasiconformal motion of a closed set in the sphere, over a simply connected Hausdorff space, can be extended as a quasiconformal motion of the sphere. Furthermore, we show that this extension can be done in a conformally natural way. The fundamental idea is to show that the Teichmüller space of a closed set in the sphere is a "universal parameter space" for tame quasiconformal motions of that set over a simply connected Hausdorff space.

**1. Introduction.** Throughout this paper, we will use $\mathbb{C}$ for the complex plane, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ for the Riemann sphere, $I = [0, 1]$ for the closed unit interval and $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ for the open unit disk.

When we write $V$ is "simply connected", we mean that $V$ is a path-connected topological space and that its fundamental group is trivial (see, for example, [13] or [18]).

In their famous paper [24], Sullivan and Thurston introduced the idea of "quasiconformal motion". Theorem 3 of their paper claimed that every quasiconformal motion of a set in $\widehat{\mathbb{C}}$ over $I$, can be extended to a quasiconformal motion of $\widehat{\mathbb{C}}$. The first result in our paper is to give a counter-example to that claim. We introduce a new concept, called "tame quasiconformal motion". We show that the claim of Theorem 3 in [24] is correct for tame quasiconformal motions of a set in $\widehat{\mathbb{C}}$. More generally, we show that every tame quasiconformal motion of a set in $\widehat{\mathbb{C}}$ over a simply connected Hausdorff space (with a basepoint) can be extended to a quasiconformal motion of $\widehat{\mathbb{C}}$. We also show that this extension can be done in a conformally natural way. The main idea is to show that the Teichmüller space of a closed set $E$ in $\widehat{\mathbb{C}}$ is a "universal parameter space" for tame quasiconformal motions of $E$ over a simply connected Hausdorff space $V$.

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**1.1. Basic definitions.** We begin with some definitions.

DEFINITION 1. Let $E$ be a subset of $\widehat{\mathbb{C}}$, and let $X$ be a connected Hausdorff space with basepoint $x_0$. A *motion of $E$ over $X$* is a map $\phi : X \times E \to \widehat{\mathbb{C}}$ satisfying

(i) $\phi(x_0, z) = z$ for all $z \in E$, and
(ii) for all $x \in X$, the map $\phi(x, \cdot) : E \to \widehat{\mathbb{C}}$ is injective.

We say that $X$ is the *parameter space* of the motion $\phi$.

We will assume that 0, 1, and $\infty$ belong to $E$ and that the motion $\phi$ is *normalized*, i.e. 0, 1, and $\infty$ are fixed points of the map $\phi(x, \cdot)$ for every $x$ in $X$.

Let $E \subset \widehat{E}$, $\phi : X \times E \to \widehat{\mathbb{C}}$ and $\widehat{\phi} : X \times \widehat{E} \to \widehat{\mathbb{C}}$ be two motions. We say that $\widehat{\phi}$ *extends* $\phi$ if $\widehat{\phi}(x, z) = \phi(x, z)$ for all $(x, z) \in X \times E$.

For any motion $\phi : X \times E \to \widehat{\mathbb{C}}$, $x$ in $X$, and any quadruplet of distinct points $a, b, c, d$ of points in $E$, let $\phi_x(a, b, c, d)$ denote the cross-ratio of the values $\phi(x, a)$, $\phi(x, b)$, $\phi(x, c)$, and $\phi(x, d)$. We will often write $\phi(x, z)$ as $\phi_x(z)$ for $x$ in $X$ and $z$ in $E$. So we have:

$$\phi_x(a, b, c, d) = \frac{(\phi_x(a) - \phi_x(c))(\phi_x(b) - \phi_x(d))}{(\phi_x(a) - \phi_x(d))(\phi_x(b) - \phi_x(c))} \tag{1.1}$$

for each $x$ in $X$.

It is obvious that condition (ii) in Definition 1 holds if and only if $\phi_x(a, b, c, d)$ is a well-defined point in the thrice-punctured sphere $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ for all $x$ in $X$ and all quadruplets $a, b, c, d$ of distinct points in $E$.

Let $\rho$ be the Poincaré distance on $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$. In their paper [24], Sullivan and Thurston introduced the following definition.

DEFINITION 2. A *quasiconformal motion* is a motion $\phi : X \times E \to \widehat{\mathbb{C}}$ of $E$ over $X$ with the following additional property:

(iii) given any $x$ in $X$ and any $\varepsilon > 0$, there exists a neighborhood $U_x$ of $x$ such that for any quadruplet of distinct points $a, b, c, d$ in $E$, we have

$$\rho\big(\phi_y(a, b, c, d), \phi_{y'}(a, b, c, d)\big) < \varepsilon \quad \text{for all } y \text{ and } y' \text{ in } U_x.$$

We also need the definition of a *continuous motion*.

DEFINITION 3. A *continuous motion* of $\widehat{\mathbb{C}}$ over $X$ is a motion $\phi : X \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that the map $\phi$ is continuous.

Recall that all motions in this paper are normalized. If $\phi$ is a continuous motion of $\widehat{\mathbb{C}}$, then each $\phi_x$, $x$ in $X$, is a map from $\widehat{\mathbb{C}}$ to itself that fixes 0, 1, and $\infty$. Since $\phi_x$ is injective and continuous, it is a homeomorphism of $\widehat{\mathbb{C}}$ onto itself, by invariance of domain.

Now we recall the definition of a holomorphic motion.

DEFINITION 4. Let $W$ be a connected complex manifold with basepoint $x_0$. A *holomorphic motion of $E$ over $W$* is a motion $\phi \colon W \times E \to \widehat{\mathbb{C}}$ of $E$ over $W$ such that the map $\phi(\cdot, z) \colon W \to \widehat{\mathbb{C}}$ is holomorphic for each $z$ in $E$.

REMARK 1. Suppose $\phi\colon W \times E \to \widehat{\mathbb{C}}$ is a holomorphic motion. For any quadruplet of points $a, b, c, d$ in $E$, the map $x \mapsto \phi_x(a, b, c, d)$ from $W$ into $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ is holomorphic. Therefore, it is distance-decreasing with respect to the Kobayashi metrics on $W$ and $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$. It easily follows that $\phi$ is also a quasiconformal motion.

DEFINITION 5. Let $X$ and $Y$ be connected Hausdorff spaces with basepoints, and $f$ be a continuous basepoint preserving map of $X$ into $Y$. If $\phi$ is a motion of $E$ over $Y$ its *pullback* by $f$ is the motion

$$f^*(\phi)(x, z) = \phi(f(x), z) \qquad \forall (x, z) \in X \times E \tag{1.2}$$

of $E$ over $X$.

REMARK 2. If the motion $\phi$ is quasiconformal or continuous, then $f^*(\phi)$ has the same property. If $X$ and $Y$ are complex manifolds, $f$ is holomorphic, and $\phi$ is a holomorphic motion, then so is $f^*(\phi)$.

A natural question is:

If $\phi : V \times E \to \widehat{\mathbb{C}}$ is a quasiconformal motion, where $V$ is simply connected, does there exist a quasiconformal motion $\widetilde{\phi} : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\widetilde{\phi}$ extends $\phi$?

The answer is affirmative when $E$ is a finite set. We shall discuss this in §6. However, Theorem I of our paper shows that the answer is negative for an infinite closed set, where $V = I$. This gives a counter-example to Theorem 3 in [24], where the authors claim that any quasiconformal motion of $E$ over an interval can be extended to a quasiconformal motion of $\widehat{\mathbb{C}}$.

For this reason, we introduce the new concept of a “tame quasiconformal motion”.

DEFINITION 6. Let $X$ be a connected Hausdorff space with a basepoint $x_0$, and $E$ be a set in $\widehat{\mathbb{C}}$ (containing the points 0, 1, and $\infty$). A *tame quasiconformal motion* is a motion $\phi : X \times E \to \widehat{\mathbb{C}}$ of $E$ over $X$ with the additional property:

(iii) Given any $x$ in $X$, there exists a quasiconformal map $w : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, a neighborhood $N(x)$, with basepoint $x$, and a quasiconformal motion $\psi : N(x) \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ over $N(x)$ such that $\phi(y, z) = \psi(y, w(z))$ for all $(y, z) \in N(x) \times E$.

Let $X$ and $Y$ be connected Hausdorff spaces with basepoints, and $f$ be a continuous basepoint preserving map of $X$ into $Y$. If $\phi$ is a tame quasiconformal motion of $E$ over $Y$ its pullback $f^*(\phi)$ is a tame quasiconformal motion of $E$ over $X$.

DEFINITION 7. Let $\phi : X \times E \to \widehat{\mathbb{C}}$ be a (normalized) motion. Let $G$ be a group of Möbius transformations, such that $E$ is invariant under $G$ (which means $g(E) = E$ for all $g$ in $G$). We say that $\phi$ is *G-equivariant* if and only if for each $g$ in $G$, and $x$ in $X$, there is a Möbius transformation $\theta_x(g)$ such that

$$\phi(x, g(z)) = (\theta_x(g))(\phi(x, z)) \qquad \text{for all } z \in E. \tag{1.3}$$

**1.2. Statements of the main results.** The main purpose in this paper is to prove the following theorems.

THEOREM I. *There exist a closed set $E$ (in $\widehat{\mathbb{C}}$), with $\#(E) = \infty$, and a quasiconformal motion $\phi : I \times E \to \widehat{\mathbb{C}}$, such that $\phi$ can be extended to a continuous motion of $\widehat{\mathbb{C}}$ over $I$. However, for any neighborhood $U$ about* 0*, $\phi$ CANNOT be extended to a quasiconformal motion of $\widehat{\mathbb{C}}$ over $U$.*

REMARK 3. We will show that a tame quasiconformal motion of a set (over a simply connected parameter space) can always be extended to $\widehat{\mathbb{C}}$.

For the next theorem, we assume that the set $E$ is closed; (as usual, the points 0, 1, and $\infty$ belong to $E$). Associated to each closed set $E$ in $\widehat{\mathbb{C}}$, there is a contractible complex Banach manifold which we call the Teichmüller space of the closed set $E$, denoted by $T(E)$. This was first studied by G. Lieb in his doctoral dissertation [12]. We will give precise definitions of $T(E)$ and a tame quasiconformal motion

$$\Psi_E : T(E) \times E \to \widehat{\mathbb{C}}$$

of $E$ over the parameter space $T(E)$ in §4 and §5.

THEOREM II. *Let $\phi : V \times E \to \widehat{\mathbb{C}}$ be a tame quasiconformal motion. If $V$ is a simply connected Hausdorff space with a basepoint $x_0$, there exists a unique basepoint preserving continuous map $F : V \to T(E)$ such that $F^*(\Psi_E) = \phi$.*

COROLLARY 1 (Extension to the Riemann Sphere). *Let $V$ be a simply connected Hausdorff space with a basepoint, and $\phi : V \times E \to \widehat{\mathbb{C}}$ be a tame quasiconformal motion. Then, there exists a quasiconformal motion $\widetilde{\phi} : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\widetilde{\phi}$ extends $\phi$.*

Let $G$ be a group of Möbius transformations, such that the closed set $E$ is invariant under $G$.

COROLLARY 2 (Group Equivariance). *Let $V$ be a simply connected Hausdorff space with a basepoint, and $\phi : V \times E \to \widehat{\mathbb{C}}$ be a $G$-equivariant tame quasiconformal motion. Then, there exists a $G$-equivariant quasiconformal motion $\widetilde{\phi} : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\widetilde{\phi}$ extends $\phi$.*

This is the analogue of Theorem 1 in [4] for tame quasiconformal motions.

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## 2. Some properties of tame quasiconformal motions.

Recall that a homeomorphism of $\widehat{\mathbb{C}}$ is called *normalized* if it fixes the points 0, 1, and $\infty$.

We use $M(\mathbb{C})$ to denote the open unit ball of the complex Banach space $L^\infty(\mathbb{C})$. Each $\mu$ in $M(\mathbb{C})$ is the Beltrami coefficient of a unique normalized quasiconformal homeomorphism $w^\mu$ of $\widehat{\mathbb{C}}$ onto itself. The basepoint of $M(\mathbb{C})$ is the zero function.

We will need the following properties of quasiconformal motions of $\widehat{\mathbb{C}}$, proved in [16].

PROPOSITION 1. *A motion $\phi\colon X \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is quasiconformal if and only if it satisfies*

(a) *the map* $\phi_x : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *is quasiconformal for each* $x$ *in* $X$, *and*
(b) *the map from* $X$ *to* $M(\mathbb{C})$ *that sends* $x$ *to the Beltrami coefficient of* $\phi_x$ *for each* $x$ *in* $X$ *is continuous.*

Part (b) means that the map $x \mapsto \mu_x = \frac{(\phi_x)_{\bar{z}}}{(\phi_x)_z}$, $x \in$X, is continuous.

PROPOSITION 2. *Every quasiconformal motion of* $\widehat{\mathbb{C}}$ *is a continuous motion.*

The following useful lemma is an immediate consequence of Definition 6.

LEMMA 1. *A motion* $\phi : X \times E \to \widehat{\mathbb{C}}$ *is a tame quasiconformal motion if and only if given any* $x$ *in* $X$, *there exists a neighborhood* $N(x)$, *and a continuous map* $g_x : N(x) \to M(\mathbb{C})$ *such that* $\phi(y, z) = w^{g_x(y)}(z)$ *for all* $(y, z) \in N(x) \times E$.

PROOF. Let $\phi : X \times E \to \widehat{\mathbb{C}}$ be a motion. Suppose, for each $x$ in $X$, there exists a neighborhood $N(x)$, and a continuous map $g_x : N(x) \to M(\mathbb{C})$ such that $\phi(y, z) = w^{g_x(y)}(z)$ for all $(y, z) \in N(x) \times E$. Set $w = w^{g_x(x)}$ and $\psi(y, z) = w^{g_x(y)}(w^{-1}(z))$ in $N(x) \times \widehat{\mathbb{C}}$. It now follows that $\phi$ is a tame quasiconformal motion of $E$ over $X$.

Conversely, if $\phi : X \times E \to \widehat{\mathbb{C}}$ is a tame quasiconformal motion, then by Proposition 1, the condition of our lemma immediately follows. □

LEMMA 2. *Let* $\phi : X \times E \to \widehat{\mathbb{C}}$ *be a tame quasiconformal motion. Then,* $\phi$ *is a quasiconformal motion.*

PROOF. The proof follows immediately from Lemma 1 and the quasi-invariance of cross ratios (see Theorem 1 in [11]). □

LEMMA 3. *Let* $\phi : X \times E \to \widehat{\mathbb{C}}$ *be a tame quasiconformal motion, where* $X$ *is a connected Hausdorff space with a basepoint. Then, for each fixed* $x$ *in* $X$, *the map* $\phi(x, \cdot) : E \to \widehat{\mathbb{C}}$ *is continuous.*

PROOF. The proof follows easily from Lemma 1. □

**3. Proof of theorem I.** Let $I = [0, 1]$ with 0 as the base point. We take $1 < r_1 < r_2 < \cdots < r_n < r_{n+1} < \cdots$ so that $r_{n+1}/r_n \to \infty$ as $n \to \infty$. Put $X = \hat{\mathbb{C}} \setminus (\bigcup_{n=1}^{\infty} r_n \cup \{\infty\})$ and $E = \bigcup_{n=1}^{\infty} C_n \cup \{0, 1, \infty\}$, where $C_n = \{|z| = r_n\}$. Let $\alpha_n$ $(n \in \mathbb{N})$ be a simple closed curve in $X$ only surrounding $r_{2n}$ and $r_{2n+1}$ as Figure 1.

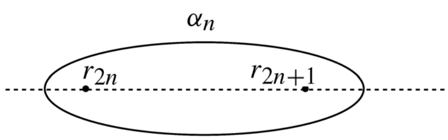


FIGURE 1.

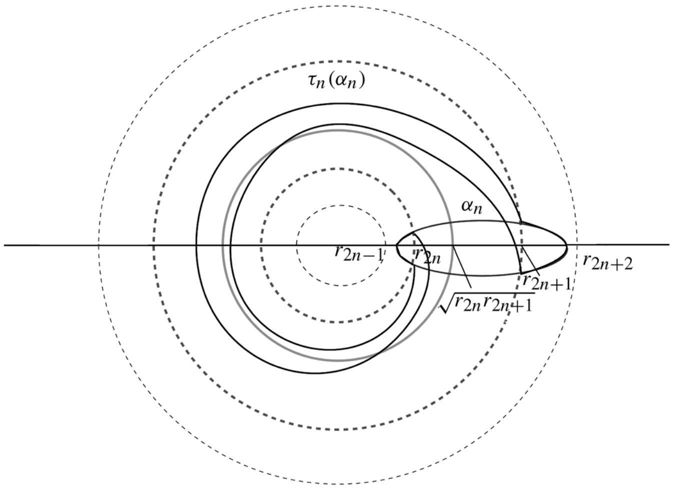


FIGURE 2.

Let $A_n := \{r_{2n} \le |z| \le r_{2n+1}\}$ and $B_n := \{r_{2n+1} \le |z| \le r_{2n+2}\}$. We take $p_n \in \mathbb{N}$ so large that

$$\lim_{n\to\infty} \frac{\ell_X(\tau_n^{p_n}(\alpha_n))}{\ell_X(\alpha_n)} = \infty, \tag{3.1}$$

where $\tau_n$ is the right Dehn twist in $A_n$ about $D_n := \{|z| = \sqrt{r_{2n}r_{2n+1}}\}$ (see Figure 2) and $\ell_X(c)$ stands for the hyperbolic length of the geodesic on $X$ homotopic to a closed curve $c$ in $X$.

For each $n \in \mathbb{N}$, we define $\Phi_n : I \times E \to \widehat{\mathbb{C}}$ by

$$\Phi_n(t, z) = z \exp\{2\pi i n(n+1)(t - (n+1)^{-1})p_n\}$$

for $(t, z) \in [(n+1)^{-1}, n^{-1}] \times C_{2n+1}$ and $\Phi_n(t, z) = z$ elsewhere. Note that $n(n+1)(t - (n+1)^{-1})p_n \uparrow p_n$ as $t \uparrow n^{-1}$ and $\Phi_n((n+1)^{-1}, z) = \Phi_n(n^{-1}, z) = z$. Thus $\Phi_n(t, z)$ is continuous at $n^{-1}$ and $(n+1)^{-1}$. Now, we define $\phi : I \times E \to \widehat{\mathbb{C}}$ by

$$\phi(t, z) = \lim_{n\to\infty} \Phi_n \circ \cdots \circ \Phi_1(t, z)$$

for every $(t, z) \in I \times E$. Obviously, $\phi$ is a continuous motion of $E$ over $I$.

CLAIM 1. $\phi : I \times E \to \widehat{\mathbb{C}}$ can be extended to a continuous motion $\Phi : I \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

PROOF. We extend $\Phi_n(t, z)$ to $I \times \widehat{\mathbb{C}}$ by

$$\Phi_n(t, re^{2\pi i\theta}) = z \exp\left\{2\pi i \frac{\log r - \log r_{2n}}{\log r_{2n+1} - \log r_{2n}} n(n+1)p_n(t - (n+1)^{-1})\right\},$$

for $(t, re^{2\pi i\theta}) \in [(n+1)^{-1}, n^{-1}] \times A_n$,

$$\Phi_n(t, re^{2\pi i\theta}) = z \exp\left\{2\pi i \frac{\log r - \log r_{2n+2}}{\log r_{2n+1} - \log r_{2n+2}} n(n+1)p_n(t-(n+1)^{-1})\right\},$$

for $(t, re^{2\pi i\theta}) \in [(n+1)^{-1}, n^{-1}] \times B_n$, and $\Phi_n(t, z) = z$ elsewhere. Then, we define

$$\Phi(t, z) = \lim_{n\to\infty} \Phi_n \circ \cdots \circ \Phi_1(t, z)$$

for $(t, z) \in I \times \widehat{\mathbb{C}}$. Clearly, this is an extension of $\phi$. It is also clear that $\Phi$ is continuous in $I \times \mathbb{C}$. Since the annuli $A_n$ shrink to $\infty$ on $\widehat{\mathbb{C}}$ in the spherical metric as $n \to \infty$, $\Phi : I \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is continuous. This implies that $\Phi$ is a continuous motion of $\widehat{\mathbb{C}}$ which extends $\phi$. □

CLAIM 2. $\phi$ is a quasiconformal motion of $E$ over $I$.

PROOF. We define quasiconformal homeomorphisms $f^+_{t,n}$ and $f^-_{t,n}$ of $\widehat{\mathbb{C}}$ as follows:

For any $t \in [(n+1)^{-1}, n^{-1}]$, let $\theta_n(t)$ be in $[0, 1)$ with $n(n+1)p_nt - \theta_n(t) \in \mathbb{N}$. The function $\theta_n$ is not continuous at $T_{n,m} := (np_n + m)\{n(n+1)p_n\}^{-1}$ $(m = 0, \ldots, p_n)$. Indeed, $\lim_{t\uparrow T_{n,m}} \theta_n(t) = 1$, while $\theta_n(T_{n,m}) = 0$.

**(i):** For $z = re^{2\pi i\theta} \in A_n$,

$$f^+_{t,n}(z) = z \exp\left\{2\pi i \frac{\log r - \log r_{2n}}{\log r_{2n+1} - \log r_{2n}} \theta_n(t)\right\}$$

and for $z = re^{2\pi i\theta} \in B_n$,

$$f^+_{t,n}(z) = z \exp\left\{2\pi i \frac{\log r - \log r_{2n+2}}{\log r_{2n+1} - \log r_{2n+2}} \theta_n(t)\right\},$$

**(ii):** For $z = re^{2\pi i\theta} \in A_n$,

$$f^-_{t,n}(z) = z \exp\left\{2\pi i \frac{\log r - \log r_{2n}}{\log r_{2n+1} - \log r_{2n}} (\theta_n(t) - 1)\right\}$$

and for $z = re^{2\pi i\theta} \in B_n$,

$$f^-_{t,n}(z) = z \exp\left\{2\pi i \frac{\log r - \log r_{2n+2}}{\log r_{2n+1} - \log r_{2n+2}} (\theta_n(t) - 1)\right\},$$

**(iii):** $f^+_{t,n}(z) = f^-_{t,n}(z) = z$ for $z \notin A_n \cup B_n$.

Since $\lim_{n\to\infty}(\log r_{2n} - \log r_{2n-1}) = \lim_{n\to\infty}(\log r_{2n+1} - \log r_{2n}) = \infty$ and $\theta_n(t) \in [0, 1)$, we see that

$$\lim_{n\to\infty} \sup\left(K(f^{\pm}_{t,n}) : (n+1)^{-1} \le t \le n^{-1}\right) = 1\,. \tag{3.2}$$

and

$$\lim_{t\downarrow T_{n,m}} K(f^+_{t,n}) = \lim_{t\uparrow T_{n,m}} K(f^-_{t,n}) = 1\,, \tag{3.3}$$

where $K(f)$ denotes the maximal dilatation of a quasiconformal map $f$.

We also see that $f^+_{t,n}(z) = f^-_{t,n}(t, z) = \phi(t, z)$ for $z \in E$ and $t \in [(n+1)^{-1}, n^{-1}]$. Moreover, $f^+_{T_{n,m},n}(z) = z$ on $\widehat{\mathbb{C}}$ because $\theta_n(T_{n,m}) = 0$.

Now, we are ready to show that $\phi : I \times E \to \widehat{\mathbb{C}}$ is a quasiconformal motion. Let $t_0 \in I$ and $\varepsilon > 0$. If $t_0 \neq 0$, then choose a positive integer $n$ such that $t_0 \in [(n+1)^{-1}, n^{-1})]$.

CASE 1. $t_0 \neq T_{n,m}, (m = 1, \dots, p_n - 1)$ and $t_0 \neq 0$: Since $K(f^+_{t,n} \circ (f^+_{t_0,n})^{-1}) \to 1$ as $t \to t_0$, it follows from the uniform continuity of cross ratios under quasiconformal deformation (see [11], the proof of the "only if" part of Theorem 1) that there exists a $\delta > 0$ such that if $|t - t_0| < \delta$, then

$$\begin{aligned}\rho(\phi_t(a,b,c,d), \phi_{t_0}(a,b,c,d)) &= \rho(f^+_{t,n}(a,b,c,d), f^+_{t_0,n}(a,b,c,d)) \\ &= \rho(f^+_{t,n} \circ (f^+_{t_0,n})^{-1}(a_{t_0}, b_{t_0}, c_{t_0}, d_{t_0}), (a_{t_0}, b_{t_0}, c_{t_0}, d_{t_0})) < \varepsilon\end{aligned}$$

for any four distinct points $a, b, c, d$ in $E$, where $a_{t_0} = f^+_{t_0,n}(a)$, $b_{t_0} = f^+_{t_0,n}(b)$, $c_{t_0} = f^+_{t_0,n}(c)$ and $d_{t_0} = f^+_{t_0,n}(d)$.

CASE 2. $t_0 = T_{n,m}$ for some $m$ $(1 \leq m \leq p_n - 1)$: Note that $\phi(t_0, z) = z$ for any $z \in E$. By using (3.3) and the uniform continuity of cross ratios as above, we may find a $\delta > 0$ such that for any four distinct points $a, b, c, d$ in $E$,

$$\rho(\phi_t(a,b,c,d), \phi_{t_0}(a,b,c,d)) = \rho(f^+_{t,n}(a,b,c,d), (a,b,c,d)) < \varepsilon$$

if $t_0 < t < t_0 + \delta$ and

$$\rho(\phi_t(a,b,c,d), \phi_{t_0}(a,b,c,d)) = \rho(f^-_{t,n}(a,b,c,d), (a,b,c,d)) < \varepsilon$$

if $t_0 - \delta < t < t_0$.

CASE 3. $t_0 = n^{-1}$: In this case, $\phi(t_0, \cdot)$ is still the identity on $E$. By the same argument as in Case 2, we see that there exists $\delta > 0$ such that for any four distinct points $a, b, c, d$ in $E$,

$$\rho(\phi_t(a,b,c,d), \phi_{t_0}(a,b,c,d)) = \rho(f^+_{t,n}(a,b,c,d), (a,b,c,d)) < \varepsilon$$

if $t_0 < t < t_0 + \delta$ and

$$\rho(\phi_t(a,b,c,d), \phi_{t_0}(a,b,c,d)) = \rho(f^-_{t,n+1}(a,b,c,d), (a,b,c,d)) < \varepsilon$$

if $t_0 - \delta < t < t_0$.

CASE 4. $t_0 = 0$: By the definition, $\phi(0, z) = z$ on $E$. Using the uniform continuity of cross ratios again, we see from (3.2) that

$$\rho(\phi_t(a,b,c,d), \phi_0(a,b,c,d)) = \rho(f^+_{t,n}(a,b,c,d), (a,b,c,d)) < \varepsilon$$

holds for sufficiently small $t > 0$ and large $n \in \mathbb{N}$.

Therefore, we conclude that $\phi : I \times E \to \widehat{\mathbb{C}}$ is a quasiconformal motion. □

CLAIM 3. $\phi$ cannot be extended to a quasiconformal motion of $\widehat{\mathbb{C}}$ over any neighbourhood $U \subset I$ about 0

PROOF. Suppose that there exists a quasiconformal motion $\hat{\phi}$ of $\widehat{\mathbb{C}}$ over $U$ which extends $\phi$. It follows from Proposition 1 that $\hat{\phi}_t : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasiconformal map for $t \in U$ and $U \ni t \mapsto \mu_{\hat{\phi}_t} \in M(\mathbb{C})$ is continuous. Hence, there exists $K \geq 1$ such that $\hat{\phi}_t$ is $K$-quasiconformal for any $t \in U$ (taking $U$ smaller if it is necessary).

Let $N > 0$ such that $1/N \in U$. For any $n > N$, we consider $F_t := \hat{\phi}_t \circ (\hat{\phi}_{(n+1)^{-1}})^{-1}$ for $t \in [(n+1)^{-1}, n^{-1}]$. Since $F_{(n+1)^{-1}} = id$ and $F_{n^{-1}} = \lim_{t \uparrow n^{-1}} F_t$, we verify that $F_{n^{-1}}(\alpha_n) = \lim_{t \uparrow n^{-1}} F_t(\alpha_n)$ is homotopic to $\tau_n^{p_n}(\alpha_n)$ in $X$. (Indeed, $F_{n^{-1}}|_{A_n}$ is a homeomorphism of the annulus $A_n$ which keeps each boundary point fixed. It gives a $p_n$-times rotation on $A_n$. Since $F_t$ is a family of homeomorphisms of $\widehat{\mathbb{C}}$ continuously depending on $t$, so when $t$ changes from $(n+1)^{-1}$ to $n^{-1}$, $\alpha_n$ moves continuously to $\tau_n^{p_n}(\alpha_n)$.)

Now, we use the following lemma by Wolpert (see [22], [23], [25]).

LEMMA 4 (Wolpert). *Let $X$, $Y$ be hyperbolic Riemann surfaces and $f : X \to Y$ be a $K$-quasiconformal map from $X$ onto $Y$. Then, for any non-trivial and non-peripheral closed curve $\alpha$ on $X$,*

$$\frac{1}{K}\ell_X(\alpha) \leq \ell_Y(f(\alpha)) \leq K\ell_X(\alpha)$$

*holds, where $\ell_X(\alpha)$ is the hyperbolic length of the geodesic on $X$ homotopic to $\alpha$.*

Since $\hat{\phi}_t$ is $K$-quasiconformal, we see from Lemma 4 that

$$\ell_X(\tau_n^{p_n}(\alpha_n)) = \ell_X(F_{n^{-1}}(\alpha_n)) \leq K^2\ell_X(\alpha_n)\,.$$

This contradicts (3.1). Thus, we have shown that $\phi$ cannot be extended to a quasiconformal motion of $\widehat{\mathbb{C}}$ over $U$. □

**4. Teichmüller space of a closed set in the sphere.** By Lemma 2, every tame quasiconformal motion is a quasiconformal motion. In the Appendix of our paper, we show that a quasiconformal motion of set $E$ in $\widehat{\mathbb{C}}$, over a connected Hausdorff space, can be extended to the closure of $E$. This fact is also proved in the paper [24], where the parameter space is an interval. It therefore follows that every tame quasiconformal motion of a set can be extended to its closure.

Henceforth, we will always assume that $E$ is a closed set in $\widehat{\mathbb{C}}$ (as usual, 0, 1, and $\infty$ are in $E$).

One of our goals in this paper is to study the "universal property" for tame quasiconformal motions of a closed set $E$ in $\widehat{\mathbb{C}}$, over $\Delta$. For that, we need some basic facts about the Teichmüller space of $E$, which is related to the "universal" holomorphic motion of $E$.

**4.1. $T(E)$ as a complex manifold.** Two normalized quasiconformal self-mappings $f$ and $g$ of $\widehat{\mathbb{C}}$ are said to be $E$-equivalent if and only if $f^{-1} \circ g$ is isotopic to the identity rel $E$. The *Teichmüller space* $T(E)$ is the set of all $E$-equivalence classes of normalized quasiconformal self-mappings of $\widehat{\mathbb{C}}$.

An analytic description of $T(E)$ will be more useful for our purposes. Let $M(\mathbb{C})$ be the open unit ball of the complex Banach space $L^\infty(\mathbb{C})$. Each $\mu$ in $M(\mathbb{C})$ is the Beltrami coefficient of a unique normalized quasiconformal homeomorphism $w^{\mu}$ of $\widehat{\mathbb{C}}$ onto itself. The basepoint of $M(\mathbb{C})$ is the zero function. We define the quotient map

$$P_E : M(\mathbb{C}) \to T(E)$$

by setting $P_E(\mu)$ equal to the $E$-equivalence class of $w^{\mu}$, written as $[w^{\mu}]_E$. Clearly, $P_E$ maps the basepoint of $M(\mathbb{C})$ to the basepoint of $T(E)$.

In his doctoral dissertation ([12]), G. Lieb proved that $T(E)$ is a complex Banach manifold such that the projection map $P_E$ from $M(\mathbb{C})$ to $T(E)$ is a holomorphic split submersion. For details, the reader is referred to the paper [5].

**4.2. The finite case.** Let $E$ be a finite set. Its complement $E^c = \Omega$ is the Riemann sphere with punctures at the points of $E$. Since $T(E)$ and the classical Teichmüller space $Teich(\Omega)$ are quotients of $M(\mathbb{C})$ by the same equivalence relation, $T(E)$ can be naturally identified with $Teich(\Omega)$ (see Example 3.1 in [15]). For references on standard Teichmüller theory, see [8] or [20].

**4.3. Forgetful maps.** Let $E$ and $\widehat{E}$ be two closed sets such that $E \subset \widehat{E}$; as usual, 0, 1, and $\infty$ belong to both $E$ and $\widehat{E}$. If $\mu$ is in $M(\mathbb{C})$, then the $\widehat{E}$-equivalence class of $w^{\mu}$ is contained in the $E$-equivalence class of $w^{\mu}$. Therefore, there is a well-defined 'forgetful map' $p_{\widehat{E},E}$ from $T(\widehat{E})$ to $T(E)$ such that $P_E = p_{\widehat{E},E} \circ P_{\widehat{E}}$. It is easy to see that this forgetful map is a basepoint preserving holomorphic split submersion.

**4.4. Teichmüller metric on $T(E)$.** Teichmüller distance $d_M(\mu, \nu)$ between $\mu$ and $\nu$ on $M(\mathbb{C})$ is defined by

$$d_M(\mu, \nu) = \tanh^{-1} \left\| \frac{\mu - \nu}{1 - \overline{\mu}\nu} \right\|_{\infty}.$$

The *Teichmüller metric* on $T(E)$ is the quotient metric

$$d_{T(E)}(s, t) = \inf\{d_M(\mu, \nu) : \mu \text{ and } \nu \in M(\mathbb{C}),\, P_E(\mu) = s \text{ and } P_E(\nu) = t\}.$$

It is proved in [5] that the Teichmüller metric on $T(E)$ is the same as its Kobayashi metric.

**4.5. Douady-Earle section.** The following fact will be useful in our paper.

PROPOSITION 3. *There is a continuous basepoint preserving map $s$ from $T(E)$ to $M(\mathbb{C})$ such that $P_E \circ s$ is the identity map on $T(E)$.*

See [5] for a proof. It immediately follows that

COROLLARY 3. *The Teichmüller space $T(E)$ is contractible.*

Let $t \in T(E)$ and $P_E(\mu) = t$ for $\mu \in M(\mathbb{C})$. If $\|\mu\|_{\infty} = k$, then $\|s(t)\|_{\infty} \leq \max(k, c(k))$ where $c(k)$ is a constant that depends only on $k$ and $0 \leq c(k) < 1$. The existence of $c(k)$ follows from Proposition 7 in [3]. For details see Sections 3.2 and 3.3 (and especially Remark 3.6) in [9].

DEFINITION 8. The map $s$ from $T(E)$ to $M(\mathbb{C})$ is called the *Douady-Earle section* of $P_E$ for the Teichmüller space $T(E)$.

Let $G$ be a group of Möbius transformations that map $E$ onto itself. For each $g$ in $G$, there exists a biholomorphic map $\rho_g : T(E) \to T(E)$ which is defined as follows: for each $\mu$ in $M(\mathbb{C})$,

$$\rho_g([w^\mu]_E) = [\widehat{g} \circ w^\mu \circ g^{-1}]_E \tag{4.1}$$

where $\widehat{g}$ is the unique Möbius transformation such that $\widehat{g} \circ w^\mu \circ g^{-1}$ fixes the points 0, 1, and $\infty$.

It follows from the definition that, for each $g$ in $G$, $\rho_g$ is basepoint preserving.

DEFINITION 9. We define $M(\mathbb{C})^G$ and $T(E)^G$ as follows:

$$M(\mathbb{C})^G := \{\mu \in M(\mathbb{C}) : (\mu \circ g)\frac{\bar{g'}}{g'} = \mu \text{ a.e. on } \mathbb{C} \text{ for each } g \in G\}$$

and

$$T(E)^G := \{t \in T(E) : \rho_g(t) = t \text{ for each } g \in G\}\,.$$

The next proposition shows the conformal naturality of the Douady-Earle section $s : T(E) \to M(\mathbb{C})$.

PROPOSITION 4. *If $t \in T(E)^G$, then $s(t) \in M(\mathbb{C})^G$.*

See [9] or [10] for a proof.

**5. Universal holomorphic motion.** The *universal holomorphic motion $\Psi_E$* of $E$ over $T(E)$ is defined as follows:

$$\Psi_E(P_E(\mu), z) = w^\mu(z) \text{ for } \mu \in M(\mathbb{C}) \text{ and } z \in E\,.$$

The definition of $P_E$ in §4.1 guarantees that $\Psi_E$ is well-defined. It is a holomorphic motion since $P_E$ is a holomorphic split submersion and $\mu \mapsto w^\mu(z)$ is a holomorphic map from $M(\mathbb{C})$ to $\widehat{\mathbb{C}}$ for every fixed $z$ in $\widehat{\mathbb{C}}$ (by Theorem 11 in [1]). This holomorphic motion is "universal" in the following sense:

THEOREM 1. *Let $\phi : V \times E \to \widehat{\mathbb{C}}$ be a holomorphic motion. If $V$ is a simply connected complex Banach manifold with a basepoint $x_0$, there is a unique basepoint preserving holomorphic map $f : V \to T(E)$ such that $f^*(\Psi_E) = \phi$.*

For a proof see Section 14 in [15].

Note that if $E = \widehat{\mathbb{C}}$, then $T(E) = M(\mathbb{C})$, and the universal holomorphic motion $\Psi_{\widehat{\mathbb{C}}} : M(\mathbb{C}) \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is given by:

$$\Psi_{\widehat{\mathbb{C}}}(\mu, z) = w^\mu(z) \qquad \text{for all } (\mu, z) \in M(\mathbb{C}) \times \widehat{\mathbb{C}}\,.$$

We also have the following (see Corollary 6.1 in [16]). Here, $V$ is a simply connected complex Banach manifold with a basepoint, and $E$ is a closed set in $\widehat{\mathbb{C}}$ (as usual, 0, 1, $\infty$ are in $E$).

PROPOSITION 5. *Let $\phi : V \times E \to \widehat{\mathbb{C}}$ be a holomorphic motion. Then, there exists a quasiconformal motion $\widetilde{\phi} : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\widetilde{\phi}$ extends $\phi$.*

PROPOSITION 6. *Let $\phi : X \times E \to \widehat{\mathbb{C}}$ be a holomorphic motion where $X$ is a connected complex Banach manifold with a basepoint $x_0$. Then, $\phi$ is a tame quasiconformal motion.*

PROOF. It is sufficient to consider a simply connected neighborhood $N(x_0)$ of the basepoint $x_0$. By Proposition 5, there exists a quasiconformal motion $\widetilde{\phi} : N(x_0) \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\widetilde{\phi}(x, z) = \phi(x, z)$ for all $(x, z) \in N(x_0) \times E$. Our assertion now follows by Proposition 1, and Lemma 1. □

By the above proposition, $\Psi_E : T(E) \times E \to \widehat{\mathbb{C}}$ is also a tame quasiconformal motion. Theorem II claims that this is the universal tame quasiconformal motion of the closed set $E$ over a simply connected Hausdorff space.

Let $B$ be a path-connected Hausdorff space with a basepoint $x_0$.

LEMMA 5. *If the continuous maps $f$ and $g$ from $B$ to $T(E)$ satisfy:*

(i) *$\Psi_E(f(x), z) = \Psi_E(g(x), z)$ for all $x$ in $B$, and for all $z$ in $E$, and*

(ii) *$f(p) = g(p)$ for some $p$ in $B$,*

*then $f(x) = g(x)$ for all $x$ in $B$.*

See Lemma 12.2 in [15].

Suppose $E_1$ and $E_2$ are closed subsets of $\widehat{\mathbb{C}}$ such that $E_1 \subset E_2$ and 0, 1, and $\infty$ are in $E_1$. We have the standard projections $P_{E_1} : M(\mathbb{C}) \to T(E_1)$ and $P_{E_2} : M(\mathbb{C}) \to T(E_2)$. Recall from §4.3 that there is a well-defined 'forgetful map' $p_{E_2,E_1}$ from $T(E_2)$ to $T(E_1)$ such that $P_{E_1} = p_{E_2,E_1} \circ P_{E_2}$, and that $p_{E_2,E_1}$ is a basepoint preserving holomorphic split submersion. Furthermore, both $\Psi_1 : T(E_1) \times E_1 \to \widehat{\mathbb{C}}$ and $\Psi_2 : T(E_2) \times E_2 \to \widehat{\mathbb{C}}$ are tame quasiconformal motions.

PROPOSITION 7. *Let $f_1$ and $f_2$ be basepoint preserving continuous maps from $B$ into $T(E_1)$ and $T(E_2)$ respectively. Then $p_{E_2,E_1} \circ f_2 = f_1$ if and only if $f_2^*(\Psi_{E_2})$ extends $f_1^*(\Psi_{E_1})$.*

See Proposition 4.7 in [10] for a proof.

In Proposition 7, if $E_1 = E$ and $E_2 = \widehat{\mathbb{C}}$, we get the following

COROLLARY 4. *Let $f_1$ and $f_2$ be basepoint preserving continuous maps from $B$ into $T(E)$ and $M(\mathbb{C})$ respectively. Then $P_E \circ f_2 = f_1$ if and only if $f_2^*(\Psi_{\widehat{\mathbb{C}}})$ extends $f_1^*(\Psi_E)$.*

**6. Quasiconformal motion of a finite set.** Let $X$ be a connected Hausdorff space with a basepoint $x_0$ and $E$ be a closed set in $\widehat{\mathbb{C}}$ (as usual, 0, 1, and $\infty$ are in $E$).

LEMMA 6. *If $\phi : X \times E \to \widehat{\mathbb{C}}$ is a quasiconformal motion, for each $z$ in $E$, $\phi(\cdot, z) : X \to \widehat{\mathbb{C}}$ is continuous.*

See Lemma 4.4 in [10].

REMARK 4. Let $\phi : X \times E \to \widehat{\mathbb{C}}$ be a tame quasiconformal motion. By Lemma 2, $\phi$ is also a quasiconformal motion. Therefore, by Lemma 6, it follows that, for each $z$ in $E$, the map $\phi(\cdot, z) : X \to \widehat{\mathbb{C}}$ is continuous.

For the rest of this section, we assume that $E = \{0, 1, \infty, \zeta_1, \dots, \zeta_n\}$ where $n \geq 1$ and $\zeta_i \neq \zeta_j$ for $1 \leq i \neq j \leq n$ and $\zeta_i \neq 0, 1, \infty$ for $1 \leq i \leq n$. Recall from §4.2 that $T(E)$ is naturally identified with $Teich(\widehat{\mathbb{C}} \setminus E)$.

PROPOSITION 8 (Nag). *Given $n > 0$, let*

$$Y_n = \{z \in \mathbb{C}^n : z_i \neq z_j \text{ for } 1 \leq i \neq j \leq n \text{ and } z_i \neq 0, 1 \text{ for all } i = 1, \dots, n\}.$$

*There is a holomorphic universal covering $\widehat{p} : T(E) \to Y_n$ such that*

$$\widehat{p}([w^\mu]_E) = (w^\mu(\zeta_1), \dots, w^\mu(\zeta_n)) \qquad \text{for all } \mu \in M(\mathbb{C}).$$

See [19]. A proof is also given in [2].

PROPOSITION 9. *Let $\phi : V \times E \to \widehat{\mathbb{C}}$ be a quasiconformal motion. If $V$ is simply connected, there exists a basepoint preserving continuous map $f : V \to T(E)$ such that $f^*(\Psi_E) = \phi$.*

PROOF. For $x$ in $V$, let

$$F(x) = (\phi(x, \zeta_1), \dots, \phi(x, \zeta_n)).$$

Note that the basepoint of $Y_n$ is

$$\big(\phi(x_0, \zeta_1), \dots, \phi(x_0, \zeta_n)\big) = (\zeta_1, \dots, \zeta_n).$$

By Lemma 6, $F : V \to Y_n$ is a basepoint preserving continuous map. Since $V$ is simply connected, by Proposition 8, there exists a basepoint preserving continuous map $f : V \to T(E)$, such that $\widehat{p} \circ f = F$. Let $f(x) = P_E(\mu)$ for $\mu$ in $M(\mathbb{C})$. It immediately follows (by Proposition 8) that $f^*(\Psi_E) = \phi$. □

THEOREM 2. *Let $V$ be simply connected and let $\phi : V \times E \to \widehat{\mathbb{C}}$ be a quasiconformal motion. There exists a quasiconformal motion $\widetilde{\phi} : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\widetilde{\phi}$ extends $\phi$.*

PROOF. By Proposition 9, there exists a basepoint preserving continuous map $f : V \to T(E)$ such that $f^*(\Psi_E) = \phi$. By Proposition 3, there exists a basepoint preserving continuous map $s$ from $T(E)$ to $M(\mathbb{C})$ such that $P_E \circ s$ is the identity map on $T(E)$. Let $\widetilde{f} = s \circ f$. Define $\widetilde{\phi} : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ as follows:

$$\widetilde{\phi}(x, z) = w^{\widetilde{f}(x)}(z) \qquad \text{for all } (x, z) \in V \times \widehat{\mathbb{C}}.$$

Since $\widetilde{f}$ is continuous, it follows by Proposition 1 that $\widetilde{\phi}$ is a quasiconformal motion.

Finally, for all $(x, z) \in V \times E$, we have

$f^*(\Psi_E)(x, z) = \Psi_E(f(x), z) = \Psi_E(P_E(s(f(x)), z) = \Psi_E(P_E(\widetilde{f}(x)), z)$
$= w^{\widetilde{f}(x)}(z) = \widetilde{\phi}(x, z)$ which shows that $\widetilde{\phi}$ extends $\phi$. □

## 7. Proof of theorem II.

**7.1. A construction.** Henceforth we assume that $E$ is an infinite closed set in $\widehat{\mathbb{C}}$ such that 0, 1, and $\infty$ are in $E$. Let $E_1, E_2, \dots, E_n, \dots$ be a sequence of finite subsets of $E$ such that

$$\{0, 1, \infty\} \subset E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots$$

and $\bigcup_{n=1}^{\infty} E_n$ is dense in $E$.

For each $n \geq 1$, let $S_n = \widehat{\mathbb{C}} \setminus E_n$. We saw in Subsection 4.2 that $T(E_n)$ and $Teich(S_n)$ are naturally identified. Let $0_n$ be the basepoint of $Teich(S_n)$, and let $d_n$ be the Teichmüller metric on $Teich(S_n)$.

Let $S = \coprod_n S_n$ be the disjoint union of the $S_n$. The *product Teichmüller space* $Teich(S)$ is the set of sequences $t = \{t_n\}_{n=1}^{\infty}$ such that $t_n$ belongs to $Teich(S_n)$ for each $n$ and

$$\sup\{d_n(0_n, t_n) : n \geq 1\} < \infty .$$

The basepoint of $Teich(S)$ is the sequence $0 = \{0_n\}$ whose $n$th term is the basepoint of $Teich(S_n)$. It is well-known that $Teich(S)$ is a complex Banach manifold. The Teichmüller distance on $Teich(S)$, denoted by $d_T$ is given by:

$$d_T(t, s) = \sup_n \{d_n(t_n, s_n)\}$$

where $t = \{t_n\}$ and $s = \{s_n\}$ are two points in $Teich(S)$. For more details about product Teichmüller space, see §7 in [5] or §5 in [15]. For the reader's convenience we note the following fact, which will be useful in our discussion.

LEMMA 7. *Let $X$ be a connected complex Banach manifold and, for each $n \geq 1$, let $f_n$ be a holomorphic map of $X$ into $Teich(S_n)$. For each $x$ in $X$, let $f(x)$ be the sequence $\{f_n(x)\}$. If $f(x_0)$ belongs to $Teich(S)$ for some $x_0$ in $X$, then $f(x)$ also belongs to $Teich(S)$ for all $x$ in $X$, and the map $x \mapsto f(x)$ from $X$ to $Teich(S)$ is holomorphic.*

For a proof see Corollary 7.6 in [5] or Corollary 5.5 in [15].

For each $n \geq 1$, let $\pi_n$ be the forgetful map $p_{E,E_n}$ from $T(E)$ to $Teich(S_n)$ and let $p_n$ be the forgetful map $p_{E_{n+1},E_n}$ from $Teich(S_{n+1})$ to $Teich(S_n)$. (The map $p_n$ is the same as the puncture-forgetting map in classical Teichmüller theory.)

It is clear that

$$\pi_n = p_n \circ \pi_{n+1} \quad \text{for all } n \geq 1. \tag{7.1}$$

Since each forgetful map $\pi_n$ preserves basepoints, Lemma 7 implies that the sequence $\{\pi_n(\tau)\}$ belongs to $Teich(S)$ for each $\tau$ in $T(E)$ and that the map $\pi : T(E) \to Teich(S)$ defined by setting

$$\pi(\tau) = (\pi_1(\tau), \dots, \pi_n(\tau), \dots) \quad \text{for all } \tau \in T(E)$$

is holomorphic. Equation (7.1) implies that $\pi$ maps $T(E)$ into the closed subset

$$T' = \{x = (x_1, x_2, \dots) \in Teich(S) : p_n(x_{n+1}) = x_n \quad \text{for all } n \geq 1\}$$

of $Teich(S)$.

PROPOSITION 10. *The map $\pi$ is a homeomorphism from $T(E)$ onto $T'$.*

See Theorem 7.1 in [15].

**7.2. A proposition.** Let $\phi : V \times E \to \widehat{\mathbb{C}}$ be a tame quasiconformal motion, where $V$ is a simply connected Hausdorff space with a basepoint $x_0$. We assume that $E$ is an infinite closed set in $\widehat{\mathbb{C}}$ such that 0, 1, and $\infty$ are in $E$. Let $E_1, E_2, \ldots, E_n, \ldots$ be a sequence of finite subsets of $E$ such that

$$\{0, 1, \infty\} \subset E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots$$

and $\bigcup_{n=1}^{\infty} E_n$ is dense in $E$. For each $n \geq 1$, let $S_n = \widehat{\mathbb{C}} \setminus E_n$. Let $S = \coprod_n S_n$ be the disjoint union of the $S_n$, and $Teich(S)$ denote its product Teichmüller space. Let $\phi_n : V \times E_n \to \widehat{\mathbb{C}}$ be $\phi$ restricted to $V \times E_n$. So, $\phi_n : V \times E_n \to \widehat{\mathbb{C}}$ is a tame quasiconformal motion of the finite set $E_n$. By Lemma 2, $\phi_n$ is also a quasiconformal motion. Therefore, by Proposition 9, each $\phi_n$ gives a unique basepoint preserving continuous map $f_n : V \to T(E_n)$ such that $f_n^*(\Psi_{E_n}) = \phi_n$. Note that each $T(E_n)$ is naturally identified with $Teich(S_n)$. Define $f = (f_n)$. Then the following proposition shows that $f$ is a map of $V$ to $Teich(S)$.

PROPOSITION 11. *For each $x$ in $V$, $f(x)$ is in $Teich(S)$ and the map $f : V \to Teich(S)$ is continuous.*

PROOF. There exists a neighborhood $N(x_0)$, and a continuous map $g_{x_0} : N(x_0) \to M(\mathbb{C})$ such that $\phi(x, z) = w^{g_{x_0}(x)}(z)$ for all $x$ in $N(x_0)$ and for all $z$ in $E$ (and therefore, for $z_n$ in $E_n$ for each $n \geq 1$). Note that $g_{x_0}$ maps $x_0$ to 0 in $M(\mathbb{C})$. For each $n \geq 1$, there exists a basepoint preserving continuous map $f_n : V \to T(E_n)$ such that $f_n^*(\Psi_{E_n}) = \phi_n$.

We see that $P_{E_n} \circ g_{x_0} = f_n$ for all $n \geq 1$. Indeed, $\widehat{p}_n : T(E_n) \to Y_n$ is the holomorphic universal covering (Proposition 8) and it follows from $f_n^*(\Psi_{E_n}) = \phi_n$ that $\widehat{p}_n(P_{E_n} \circ g_{x_0}(x)) = \widehat{p}_n(f_n(x))$ for any $x \in N(x_0)$. Thus, for a curve $\gamma \subset N(x_0)$ connecting $x_0$ and $x$, $P_{E_n} \circ g_{x_0}(\gamma)$ and $f_n(\gamma)$ are lifts of the same curve $\widehat{p}_n(f_n(\gamma))$ in $Y_n$. Furthermore, $P_{E_n} \circ g_{x_0}(x_0) = f_n(x_0)$ because both $P_{E_n} \circ g_{x_0}$ and $f_n$ are basepoint preserving maps. It follows from the monodromy theorem of coverings (cf. [21] Chapter 2) that $P_{E_n} \circ g_{x_0}(x) = f_n(x)$ and we obtain that $P_{E_n} \circ g_{x_0} = f_n$ on $N(x_0)$.

Since the quasiconformal map $w^{g_{x_0}(x)}$ determines the point $f_n(x)$, we have

$$d_n(f_n(x_0), f_n(x)) \leq \log K(w^{g_{x_0}(x)}) \quad (n \in \mathbb{N})$$

from the definition of the Teichmüller distance. Therefore, we conclude that

$$\sup_n\{d_n(f_n(x_0), f_n(x))\} \leq \log K(w^{g_{x_0}(x)}) < \infty.$$

This implies that $f(x)$ is in $Teich(S)$ for any $x$ in $N(x_0)$.

From the same argument as above, we see that

$$d_n(f_n(x), f_n(x')) \leq \log K(w^{g_{x_0(x)}} \circ (w^{g_{x_0}(x')})^{-1}),$$

for every $n \in \mathbb{N}$. Since $g_{x_0} : N(x_0) \to M(\mathbb{C})$ is continuous, we see that $f = (f_n)$ is continuous in $N(x_0)$.

Next, we will show that $f(x)$ is in $Teich(S)$ for any $x \in V$. We take a curve $\gamma : [0, 1] \to V$ with $\gamma(0) = x_0$ and $\gamma(1) = x$. For each $\gamma(t)$ ($t \in [0, 1]$), there exists a neighborhood $N(\gamma(t))$ of $\gamma(t)$ and a continuous map $\tilde{g}_t : N(\gamma(t)) \to M(\mathbb{C})$ such that

$$\phi(y, z) = w^{\tilde{g}_t(y)}(z) \tag{7.2}$$

for each $y \in N(\gamma(t))$ and $z \in E$.

Since $\gamma : [0, 1] \to V$ is continuous, we may take an open covering $I_0, I_1, \ldots, I_k$ of $[0, 1]$ such that $I_{i-1} \cap I_i$ is a subinterval of $[0, 1]$, and $\gamma(I_i) \subset N(\gamma(s_i))$ for some $s_i \in I_i$ ($i = 0, 1, \ldots, k$). Put $g_i := \tilde{g}_{s_i}$, then the map $\varphi_{i,n}$ defined by

$$I_i \ni t \mapsto [g_i(t)]_{E_n} \in T(E_n) \tag{7.3}$$

is continuous. Now, we compare $f_n \circ \gamma | I_0$ and $\varphi_{1,n}$ on $I_0 \cap I_1$.

We use the space $Y_n$ given in Proposition 8 and the holomorphic universal covering $\widehat{p}_n : T(E_n) \to Y_n$ again. Because of (7.2), we have $\widehat{p}_n(f_n \circ \gamma(t)) = \widehat{p}_n(\varphi_{1,n}(t))$ for every $t \in I_0 \cap I_1$. It means that $f_n \circ \gamma(I_0 \cap I_1)$ and $\varphi_{1,n}(I_0 \cap I_1)$ are lifts of the same curve in $Y_n$. Therefore, there exists an element $\chi$ of the mapping class group of the surface $\widehat{\mathbb{C}} \setminus E_n$ such that

$$\chi \circ \varphi_{1,n} = f_n \circ \gamma \quad \text{on } I_0 \cap I_1 .$$

Thus, a map $F_1 : I_0 \cup I_1 \to T(E_n)$ defined by

$$F_1 = \begin{cases} f_n \circ \gamma & \text{on} \quad I_0 \\ \chi \circ \varphi_{1,n} & \text{on} \quad I_1 \end{cases}$$

is continuous on $I_0 \cup I_1$. Furthermore, $\widehat{p}_n(F_1(t)) = \widehat{p}_n(f_n \circ \gamma(t))$ for any $t \in I_0 \cup I_1$ and we conclude that $F_1 = f_n \circ \gamma$ on $I_0 \cup I_1$ from the monodromy theorem.

Now, we take points $t_1$ in $I_0 \cap I_1$ and $t_2$ in $I_1 \cap I_2$. Since $\chi$ is an isometry with respect to the Teichmüller distance, we have

$$\begin{aligned} d_n(F_1(t_1), F_1(t_2)) &= d_n(\chi(\varphi_{1,n}(t_1)), \chi(\varphi_{1,n}(t_2))) \\ &= d_n(\varphi_{1,n}(t_1), \varphi_{1,n}(t_2)) \\ &= d_n([g_1(t_1)]_{E_n}, [g_1(t_2)]_{E_n}) . \end{aligned}$$

Noting that $g_1$ is independent of $n$, we see that there exists a constant $d_{12} > 0$ not depending on $n$ such that

$$d_n(f_n(\gamma(t_1)), f_n(\gamma(t_2))) = d_n(F_1(t_1), F_1(t_2)) \leq d_{12} .$$

By continuing the same argument for $t_i \in I_{i-1} \cap I_i$ ($i = 3, 4, \ldots, k$), we have

$$d_n(f_n(\gamma(t_{i-1}), f_n(\gamma(t_i))) \leq d_{(i-1)i}$$

for some constant $d_{(i-1)i} > 0$. Therefore, we conclude that

$$d_n(f_n(x_0), f_n(x)) = d_n(f_n(\gamma(0)), f_n(\gamma(1)))$$

$$\leq \sum_{i=1}^{k+1} d_n(f_n(\gamma(t_{i-1})), f_n(\gamma(t_i))) \quad (t_0 = 0, t_{k+1} = 1)$$
$$\leq \sum_{i=1}^{k+1} d_{(i-1)i} < \infty \quad \text{for } n \geq 1 .$$

This implies that $f(x) = (f_n(x))$ belongs to $Teich(S)$. Similarly, we can prove the continuity of $f$. □

**7.3. Proof of Theorem II.** Let $\phi : V \times E \to \widehat{\mathbb{C}}$ be a tame quasiconformal motion, where $V$ is a simply connected Hausdorff space with a basepoint $x_0$.

First, observe that if $F$ and $G$ are two basepoint preserving continuous maps from $V$ into $T(E)$ such that $F^*(\Psi_E) = G^*(\Psi_E) = \phi$, then by Lemma 5 it follows that $F = G$. Thus, if a basepoint preserving continuous map $F : V \to T(E)$ exists with $F^*(\Psi_E) = \phi$, then it must be unique.

We now show the existence of such a map. For each $n \geq 1$, the restriction $\phi_n$ of $\phi$ to $V \times E_n$ is a tame quasiconformal motion of the finite set $E_n$ (as in §7.2). By Lemma 2, $\phi_n$ is also a quasiconformal motion. By Proposition 9, each $\phi_n$ gives a unique basepoint preserving continuous map $f_n : V \to T(E_n)$ such that $f_n^*(\Psi_{E_n}) = \phi_n$ for each $n \geq 1$. Let $f = (f_n)$. By Proposition 11, $f$ is a basepoint preserving continuous map from $V$ into $Teich(S)$. It is clear that $\phi_{n+1}$ extends $\phi_n$. Therefore, by Proposition 7, we have $p_n \circ f_{n+1} = f_n$ for all $n \geq 1$. Therefore, $f$ maps $V$ into $T'$. By Proposition 10, $\pi$ maps $T(E)$ homeomorphically onto $T'$. Hence, there exists a unique map $F : V \to T(E)$ such that $f = \pi \circ F$. The map $F$ clearly preserves basepoints, and is also continuous.

Next, observe that $\pi_n \circ F = f_n$ for each $n \geq 1$. It follows by Proposition 7 that $F^*(\Psi_E)$ extends $f_n^*(\Psi_{E_n}) = \phi_n$ for each $n$. Therefore, $F^*(\Psi_E) = \phi$ on $V \times \bigcup_{n=1}^{\infty} E_n$. Since $\bigcup_n E_n$ is dense in $E$, it follows by Lemma 3 that $F^*(\Psi_E) = \phi$ on $V \times E$. □

**7.4. Corollaries.** We give the proofs of the Corollaries of Theorem II.

PROOF OF COROLLARY 1. By Theorem II, there exists a (unique) basepoint preserving continuous map $F : V \to T(E)$ such that $F^*(\Psi_E) = \phi$. Consider the Douady-Earle section $s : T(E) \to M(\mathbb{C})$ given in Definition 8. By Proposition 3, the map $s$ is basepoint preserving and is continuous. Let $\widetilde{F} = s \circ F$. Define $\widetilde{\phi} : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ as follows:

$$\widetilde{\phi}(x, z) = w^{\widetilde{F}(x)}(z) \qquad \text{for all } (x, z) \in V \times \widehat{\mathbb{C}} .$$

Since $\widetilde{F}$ is a basepoint preserving continuous map, it follows by Proposition 1 that $\widetilde{\phi}$ is a quasiconformal motion.

Finally, for all $(x, z) \in V \times E$, we have
$F^*(\Psi_E)(x, z) = \Psi_E(F(x), z) = \Psi_E(P_E(s(F(x)), z) = \Psi_E(P_E(\widetilde{F}(x)), z)$
$= w^{\widetilde{F}(x)}(z) = \widetilde{\phi}(x, z)$. This shows that $\widetilde{\phi}$ extends $\phi$. □

As usual, $E$ is an infinite closed set in $\widehat{\mathbb{C}}$, and 0, 1, and $\infty$ belong to $E$. Let $G$ be a group of Möbius transformations such that $E$ is invariant under the action of $G$. For each $g$ in $G$, there exists a biholomorphic map $\rho_g : T(E) \to T(E)$ which is defined as follows: for each $\mu$ in $M(\mathbb{C})$,

$$\rho_g([w^\mu]_E) = [\widehat{g} \circ w^\mu \circ g^{-1}]_E \tag{7.4}$$

where $\widehat{g}$ is the unique Möbius transformation such that $\widehat{g} \circ w^\mu \circ g^{-1}$ fixes the points 0, 1, and $\infty$.

It follows from the definition that, for each $g$ in $G$, $\rho_g$ is basepoint preserving.

DEFINITION 10. We define $M(\mathbb{C})^G$ and $T(E)^G$ as follows:

$$M(\mathbb{C})^G := \{\mu \in M(\mathbb{C}) : (\mu \circ g)\frac{\bar{g'}}{g'} = \mu \text{ a.e. on } \mathbb{C} \text{ for each } g \in G\}$$

and

$$T(E)^G := \{t \in T(E) : \rho_g(t) = t \text{ for each } g \in G\}\,.$$

The next proposition shows the conformal naturality of the Douady-Earle section $s : T(E) \to M(\mathbb{C})$.

PROPOSITION 12. *If $t \in T(E)^G$, then $s(t) \in M(\mathbb{C})^G$.*

See [9] or [10] for a proof.

In the next proposition, $B$ is a path-connected Hausdorff space with a basepoint $x_0$. The proof is exactly the same as in the proof of Proposition 4.10 in [10], where it was proved for quasiconformal motions. We include it for the reader's convenience.

PROPOSITION 13. *Let $\phi : B \times E \to \widehat{\mathbb{C}}$ be a tame quasiconformal motion, where $B$ is a path-connected Hausdorff space with a basepoint. Suppose there exists a basepoint preserving continuous map $f : B \to T(E)$ such that $f^*(\Psi_E) = \phi$. Then, $\phi : B \times E \to \widehat{\mathbb{C}}$ is $G$-equivariant if and only if $f$ maps $B$ into $T(E)^G$.*

PROOF. Suppose $f$ maps $B$ into $T(E)^G$. Let $g \in G$, $x \in V$, and $f(x) = P_E(\mu)$. So, $\phi(x, z) = \Psi_E(f(x), z) = w^\mu(z)$ for all $z$ in $E$, and $\phi(x, g(z)) = w^\mu(g(z))$ for all $z$ in $E$.

Now, $\rho_g(f(x)) = f(x)$ implies that

$$[w^\mu]_E = [\theta_x(g) \circ w^\mu \circ g^{-1}]_E$$

where $\theta_x(g)$ is the unique Möbius transformation such that $\theta_x(g) \circ w^\mu \circ g^{-1}$ fixes 0, 1, and $\infty$. This means that $\theta_x(g) \circ w^\mu \circ g^{-1} = w^\mu$ on $E$. Therefore, we have

$$\theta_x(g)\big(w^\mu(z)\big) = w^\mu\big(g(z)\big) \quad \text{for all } z \in E\,.$$

We conclude that $\phi(x, g(z)) = \theta_x(g)(\phi(x, z))$ for all $z$ in $E$, and so, $\phi$ satisfies Equation (1.3).

Next, suppose the tame quasiconformal motion $\phi$ satisfies Equation (1.3). Let $x \in B$ and $f(x) = [w^\mu]_E$. For $x \in B$, and $g \in G$, there exists a Möbius transformation $\theta_x(g)$ such that

$$\phi(x, g(z)) = \theta_x(g)(\phi(x, z)) \quad \text{for all } z \in E\,.$$

Since $f(x) = [w^\mu]_E$, we have $\phi(x, g(z)) = w^\mu(g(z))$ for all $z$ in $E$. Therefore, $w^\mu(g(z)) = \theta_x(g)(w^\mu(z))$ for all $z \in E$. We conclude that $w^\mu = \theta_x(g) \circ w^\mu \circ g^{-1}$ on $E$. Since the quasiconformal map $w^\mu$ fixes 0, 1, and $\infty$, it follows that $\theta_x(g) \circ w^\mu \circ g^{-1}$ fixes 0, 1, and $\infty$.

By definition of $\rho_g$, we have

$$\rho_g([w^\mu]_E) = [\widehat{g} \circ w^\mu \circ g^{-1}]_E$$

where $\widehat{g}$ is the unique Möbius transformation such that $\widehat{g} \circ w^\mu \circ g^{-1}$ fixes 0, 1, and $\infty$. It follows that $\widehat{g} = \theta_x(g)$. Therefore, we have

$$f(x) = [w^\mu]_E \quad \text{and} \quad \rho_g(f(x)) = [\theta_x(g) \circ w^\mu \circ g^{-1}]_E \,.$$

Since $f$ is continuous, and $\rho_g$ is holomorphic for each $g$ in $G$, it follows that $\rho_g \circ f$ is a continuous map for each $g$ in $G$. Also, since $f$ and $\rho_g$ are both basepoint preserving, we have $f(x_0) = \rho_g(f(x_0))$. And since $w^\mu = \theta_x(g) \circ w^\mu \circ g^{-1}$ on $E$, we have $\Psi_E(f(x), z) = \Psi_E(\rho_g(f(x)), z)$ for all $z$ in $E$. It follows by Lemma 5 that $f(x) = \rho_g(f(x))$ for any $x$ in $B$. This means, that $f$ maps $B$ into $T(E)^G$. □

PROOF OF COROLLARY 2. We use the arguments in the proof of Theorem 2. By Theorem II, there exists a basepoint preserving continuous map $F : V \to T(E)$ such that $F^*(\Psi_E) = \phi$. By Proposition 3, there exists a basepoint preserving continuous map $s$ from $T(E)$ to $M(\mathbb{C})$ such that $P_E \circ s$ is the identity map on $T(E)$. Let $\widetilde{F} = s \circ F$. Define $\widetilde{\phi} : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ as follows:

$$\widetilde{\phi}(x, z) = w^{\widetilde{F}(x)}(z) \quad \text{for all } (x, z) \in V \times \widehat{\mathbb{C}} \,.$$

As in the proof of Theorem 2 it is clear that $\widetilde{\phi}$ extends $\phi$, and $\widetilde{\phi}$ is a quasiconformal motion. Since $\phi$ is $G$-equivariant, it follows by Proposition 13 that $F : V \to T(E)^G$. By Proposition 12, $\widetilde{F} : V \to M(\mathbb{C})^G$. This shows that $\widetilde{\phi}$ is $G$-equivariant. □

**8. Appendix.** In the following discussion, let $E$ be any set (not necessarily closed) in $\widehat{\mathbb{C}}$. The blanket assumption that 0, 1, and $\infty$ belong to $E$ holds. Following Definition 3, we can introduce the concept of continuous motion of $E$ (also given in [17]).

DEFINITION 11. Let $X$ be a connected Hausdorff space with a basepoint $x_0$, and let $E$ be a set in $\widehat{\mathbb{C}}$ such that $E$ contains the points 0, 1, and $\infty$. A *normalized continuous motion* of $E$ over $X$ is a continuous map $\phi : X \times E \to \widehat{\mathbb{C}}$ such that:

(i) $\phi(x_0, z) = z$ for all $z$ in $E$, and

(ii) for each $x$ in $X$, the map $\phi(x, \cdot)$ is a homeomorphism of $E$ onto its image, that fixes the points 0, 1 and $\infty$.

PROPOSITION 14. *Let $\phi : X \times E \to \widehat{\mathbb{C}}$ be a quasiconformal motion of $E$ where $X$ is a connected Hausdorff space with a basepoint $x_0$. Then $\phi$ can be extended to a quasiconformal motion of the closure $\overline{E}$ over $X$. Furthermore, $\phi : X \times E \to \widehat{\mathbb{C}}$ is a continuous motion.*

PROOF. The idea of the proof given here is inspired by the proof of the $\lambda$-lemma in [14]. However, our proof is quite modified, since the parameter space here is any connected Hausdorff space.

The proof is divided into four steps.

We first show that $\phi$ is jointly continuous on $X \times E$. In the second step, we prove that for any $x \in X$, $\phi_x(\cdot) = \phi(x, \cdot)$ is locally uniformly continuous on $E$. Thus, $\phi_x$ can be extended to a continuous function $\overline{\phi}_x$ on $\overline{E}$. In the third step, we prove that

$$\overline{\phi}(x, z) = \overline{\phi}_x(z) : X \times \overline{E} \to \widehat{\mathbb{C}}$$

is a quasiconformal motion extending $\phi$. From the first step we know that $\overline{\phi}$ is jointly continuous on $X \times \overline{E}$. Since $\overline{\phi}_x$ is injective and continuous on $\overline{E}$, which is a compact subset in $\widehat{\mathbb{C}}$, it is a homeomorphism from $\overline{E}$ onto $\overline{\phi}_x(\overline{E})$. This implies that $\overline{\phi}$ is a continuous motion, and thus $\phi$ is also a continuous motion. For the reader's convenience, we include all details.

STEP 1: $\phi$ is a jointly continuous map on $X \times E$. For each $x \in X$, there exists a neighborhood $U_x$ of $x$ such that

$$\rho(\phi_x(a, b, c, d), \phi_y(a, b, c, d)) < 1$$

holds for any $y \in U_x$ and for any quadruple $(a, b, c, d)$ of distinct points in $E$. Since $\phi$ is normalized and $(z, 1, 0, \infty) = z$, we have

$$\rho(\phi_x(z), \phi_y(z)) < 1$$

for any $z (\neq 0, 1, \infty) \in E$ and $y \in U_x$. Therefore, for any $z \in E \setminus \{0, \infty\}$, there exists a constant $C = C(|\phi_x(z)|) > 0$ such that

$$0 < C^{-1} \leq |\phi_y(z)| \leq C\,, \tag{8.1}$$

holds for any $y \in U_x$, since $\phi_y(1) = 1$.

Now, we divide $X$ into two parts $X_0$ and $X_1 = X \setminus X_0$, where

$$X_0 = \{x \in X \mid \phi_x(\cdot) \text{ is continuous on } E\}\,.$$

We will show that $X = X_0$. First, we show that $X_0$ is open. We show that $U_x \subset X_0$ for $x \in X_0$. Since $\phi_x$ is continuous on $E$, for each $z \in E \setminus \{\infty\}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $|\phi_x(z) - \phi_x(z')| < \varepsilon$ if $|z - z'| < \delta$. From (8.1), we have for the constant $C = C(|\phi_x(z)|)$ above,

$$|\phi_x(z', 0, z, \infty)| = \left|\frac{\phi_x(z) - \phi_x(z')}{\phi_x(z)}\right| \leq C|\phi_x(z) - \phi_x(z')| < C\varepsilon\,,$$

when $z$ is in $E \setminus \{0, \infty\}$. Since $\rho(\phi_x(z', 0, z, \infty), \phi_y(z', 0, z, \infty)) < 1$ for $y \in U_x$ and $\phi_x(z', 0, z, \infty) \to 0$ as $\varepsilon \to 0$, there exists a constant $D_1 = D_1(C, \varepsilon) > 0$ such that

$$\left|\frac{\phi_y(z) - \phi_y(z')}{\phi_y(z)}\right| = |\phi_y(z', 0, z, \infty)| \leq D_1\,, \tag{8.2}$$

and $D_1 \to 0$ as $\varepsilon \to 0$. It is because the hyperbolic metric $\rho(z)|dz|$ on $\mathbb{C} \setminus \{0, 1\}$ diverges as $z \to 0$.

It follows from (8.1) and (8.2) that

(8.3) $$|\phi_y(z) - \phi_y(z')| \le CD_1 \to 0 \quad (\varepsilon \to 0)\,.$$

Therefore, $\phi_y$ is continuous on $E \setminus \{0, \infty\}$ for $y \in U_x$. Permuting the role in $\{0, 1, \infty\}$, we see that $\phi_y$ is continuous on $E$ for $y \in U_x$ and $X_0$ is an open set.

Next, we will show that $X_1$ is open. For $x \in X_1$, we show that $U_x \subset X_1$.

Take $z \in E$ where $\phi_x$ is not continuous. By the same reason as above, we may assume that $z$ is in $E \setminus \{0, \infty\}$. Since $\phi_x$ is not continuous on $E$, there exist a constant $\varepsilon_0 > 0$ and a sequence $\{z_n\}_{n=1}^{\infty} \subset E$ converging to $z$ such that

$$|\phi_x(z) - \phi_x(z_n)| \ge \varepsilon_0 \quad (n = 1, 2, \dots)\,.$$

Thus, from (8.1) we have

$$|\phi_x(z_n, 0, z, \infty)| = \left|\frac{\phi_x(z) - \phi_x(z_n)}{\phi_x(z)}\right| \ge C^{-1}\varepsilon_0\,.$$

Since $\rho(\phi_x(z_n, 0, z, \infty), \phi_y(z_n, 0, z, \infty)) < 1$, there exists a constant $D_2 = D_2(C, \varepsilon_0) > 0$ such that

$$|\phi_y(z_n, 0, z, \infty)| = \left|\frac{\phi_y(z) - \phi_x(z_n)}{\phi_y(z)}\right| \ge D_2\,.$$

By using (8.1) again, we obtain

$$|\phi_y(z) - \phi_y(z_n)| = |\phi_y(z_n, 0, z, \infty)||\phi_y(z)| \ge C^{-1}D_2 > 0\,.$$

Hence, $\phi_y$ is not continuous at $z$ and $X_1$ is open. Therefore, we conclude that $X = X_0$ because $x_0 \in X_0$.

Finally, we show that $\phi : V \times E \to \widehat{\mathbb{C}}$ is jointly continuous. Take a point $(x, z) \in V \times E$ and $\varepsilon > 0$. We may assume that $z \ne 0, \infty$ by the same reason as above. We take a point $z_0 (\ne 0, \infty, z)$ in $E$ and fix it. We also take $\varepsilon' > 0$ sufficiently small so that $|\phi_x(z) - w| < \varepsilon$ if $\rho((\phi_x(z), 0, \phi_x(z_0), \infty), (w, 0, \phi_x(z_0), \infty)) < \varepsilon'$, where $(a, b, c, d)$ is the cross-ratio of distinct 4 points $a, b, c$ and $d$.

Since $\phi : X \times E \to \widehat{\mathbb{C}}$ is a quasiconformal motion of $E$, there exists a neighborhood $U$ of $x$ in $X$ such that

$$\rho(\phi_x(z, 0, z_0, \infty), \phi_y(z, 0, z_0, \infty)) < \varepsilon'$$

for any $y \in U$. Thus, we have

$$|\phi_x(z) - \phi_y(z)| < \varepsilon\,.$$

By the same argument as in (8.3), we see that

$$|\phi_y(z) - \phi_y(z')| < \varepsilon$$

if $z'$ belongs to a sufficiently small neighborhood $N$ of $z$. Therefore, for $(y, z') \in U \times N$, we have

$$|\phi_x(z) - \phi_y(z')| \le |\phi_x(z) - \phi_y(z)| + |\phi_y(z) - \phi_y(z')| < 2\varepsilon.$$

Hence, we conclude that $\phi$ is a jointly continuous map on $X \times E$.

STEP 2: For each $x \in X$, $\phi_x$ is locally uniformly continuous and thus can be continuously extended to $\overline{E}$. Consider

$$E_N := E \cap \left\{ \frac{1}{N} \leq |z| \leq N \right\}$$

for every positive integer $N$. Since $\phi_x$ is continuous on $E$ and $\phi_x(0) = 0$ and $\phi_x(\infty) = \infty$, there exists a constant $\widetilde{C} = \widetilde{C}(x, N) > 0$ such that we have

$$0 < \widetilde{C}^{-1} \leq |\phi_x(z)| \leq \widetilde{C} \tag{8.4}$$

for every $z \in E_N$. Hence, we see that there exists a constant $C' = C'(x, N) > 0$ such that

$$0 < C'^{-1} \leq |\phi_y(z)| \leq C' \tag{8.5}$$

holds for any $y \in U_x$ for any $z \in E_N$.

Now, we divide $X$ into two parts $X_0'$ and $X_1' = X \setminus X_0'$, where

$$X_0' = \{x \in X \mid \phi_x(\cdot) \text{ is uniformly continusous on } E_N\}.$$

We will show that $X = X_0'$. First, we show that $X_0'$ is open.

Since $\phi_x$ is uniformly continuous on $E_N$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|\phi_x(z) - \phi_x(z')| < \varepsilon$ whenever $|z - z'| < \delta$ for two points $z, z' \in E_N$. From (8.5), we have

$$|\phi_x(z', 0, z, \infty)| = \left| \frac{\phi_x(z) - \phi_x(z')}{\phi_x(z)} \right| \leq C'|\phi_x(z) - \phi_x(z')| < C'\varepsilon .$$

Since $\rho(\phi_x(z', 0, z, \infty), \phi_y(z', 0, z, \infty)) < 1$ for $y \in U_x$, there exists a constant $D_1' = D_1'(C', \varepsilon) > 0$ such that

$$|\phi_y(z', 0, z, \infty)| = \left| \frac{\phi_y(z) - \phi_y(z')}{\phi_y(z)} \right| \leq D_1' , \tag{8.6}$$

and $D' \to 0$ as $\varepsilon \to 0$. It follows from (8.5) and (8.6) that

$$|\phi_y(z) - \phi_y(z')| \leq C'D_1' \to 0 \quad (\varepsilon \to 0) .$$

Therefore, $\phi_y$ is uniformly continuous on $E_N$ for $y \in U_x$ and $X_0'$ is an open set.

Next, we will show that $X_1'$ is open. For $x \in X_1'$, we show that $U_x \subset X_1'$. Since $\phi_x$ is not uniformly continuous on $E_N$, there exist a constant $\varepsilon_0 > 0$ and two sequences $\{z_n\}_{n=1}^{\infty}, \{z_n'\}_{n=1}^{\infty} \subset E_N$ such that

$$|z_n - z_n'| \to 0 \quad (n \to \infty)$$

but

$$|\phi_x(z_n) - \phi_x(z_n')| \geq \varepsilon_0 \quad (n = 1, 2, \dots) .$$

Thus, from (8.5) we have

$$|\phi_x(z_n', 0, z_n, \infty)| = \left| \frac{\phi_x(z_n) - \phi_x(z_n')}{\phi_x(z_n)} \right| \geq C'^{-1}\varepsilon_0 .$$

Since $\rho(\phi_x(z_n', 0, z_n, \infty), \phi_y(z_n', 0, z_n, \infty)) < 1$, there exists a constant $D_2' = D_2'(C', \varepsilon_0) > 0$ such that

$$|\phi_y(z_n', 0, z_n, \infty)| = \left|\frac{\phi_y(z_n) - \phi_x(z_n')}{\phi_y(z_n)}\right| \geq D_2' .$$

By using (8.5) again, we obtain

$$|\phi_y(z_n) - \phi_y(z_n')| = |\phi_y(z_n', 0, z_n, \infty)||\phi_y(z_n)| \geq C'^{-1} D_2' > 0 .$$

Hence, $\phi_y$ is not uniformly continuous on $E_N$ and $X_1$ is open. Therefore, we conclude that $X = X_0'$ because $x_0 \in X_0'$.

Letting $N \to \infty$, we see that $\phi_x$ is locally uniformly continuous on $E \setminus \{0, \infty\}(= \bigcup_{N=1}^{\infty} E_N)$. Since we may permute the role in $\{0, 1, \infty\}$, $\phi_x$ is locally uniformly continuous on $E$.

Since $\phi_x$ is locally uniformly continuous on $E$, it can be continuously extended to $\overline{\phi}_x$ on $\overline{E}$. Define a map

$$\overline{\phi} : X \times \overline{E} \to \widehat{\mathbb{C}}$$

by

$$\overline{\phi}(x, z) = \overline{\phi}_x(z) .$$

STEP 3: $\overline{\phi} : X \times \overline{E} \to \widehat{\mathbb{C}}$ is a quasiconformal motion. We first show that $\overline{\phi}_x$ is injective on $\overline{E}$ for every $x \in X$. The proof is done by the same technique as in Steps 1 and 2. Moreover, it suffices to show the claim only for $\overline{E} \setminus \{0, \infty\}$ because the argument works on $\overline{E}$ by permuting the role in $\{0, 1, \infty\}$ as before.

We set

$$X_0'' = \{x \in X \mid \overline{\phi}_x \text{ is injective on } \overline{E}\}$$

and $X_1'' = X \setminus X_0''$. We show that $U_x \subset X_0''$ for $x \in X_0''$ as before. Take any $y \in U_x$ and two distinct points $z, z' \in \overline{E}$. It suffices to show that $\overline{\phi}_y(z) \neq \overline{\phi}_y(z')$ when $z$ or $z' \in \overline{E} \setminus E$. Suppose that $z \in \overline{E} \setminus E$ and $z' \in E$. Then, there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset E \setminus \{z'\}$ converging to $z$. Since $\overline{\phi}_x$ is injective, there exists a constant $\varepsilon_0 > 0$ such that

$$|\overline{\phi}_x(z') - \overline{\phi}_x(z_n)| = |\phi_x(z') - \phi_x(z_n)| \geq \varepsilon_0$$

for any $n \in \mathbb{N}$. Hence, we may use the same argument in proving the openness of $X_1$ in Step 1 and we obtain

$$|\phi_y(z') - \phi_y(z_n)| = |\phi_y(z_n, 0, z', \infty)||\phi_y(z')| \geq C^{-1} D_2 > 0 ,$$

for some constants $C$, $D_2$ which are independent of $n$. Thus, by taking the limit, we conclude that $\overline{\phi}_y(z) \neq \overline{\phi}_y(z')$. The same argument shows that $\overline{\phi}_y(z) \neq \overline{\phi}_y(z')$ for two distinct points $z, z'$ in $\overline{E} - E$.

The openness of $X_1''$ is shown by the same way. For $x \in X_1''$, we take $y \in U_x$. Since $\overline{\phi}_x$ is not injective, we have two distinct points $z, z' \in \overline{E}$ with $\overline{\phi}_x(z) = \overline{\phi}_x(z')$. Suppose that

$z \in \overline{E} \setminus E$ and $z' \in E$. Then, there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset E$ converging to $z$. Since $\overline{\phi}_x$ is continuous on $\overline{E}$, we have

$$|\overline{\phi}_x(z_n) - \overline{\phi}_x(z')| = |\phi_x(z_n) - \phi_x(z')| \to 0 \quad (n \to \infty)\,.$$

Now, we use the same argument in proving the openness of $X_0$ in Step 1 and we obtain

$$|\overline{\phi}_y(z_n) - \overline{\phi}_y(z')| = |\phi_y(z_n) - \phi_y(z')| \to 0 \quad (n \to \infty)\,.$$

Therefore, $y \in X_1''$ and $X_1''$ is open. Since $x_0 \in X_0''$, we have $X = X_0''$ as desired.

Let $z_i \in \overline{E}$ $(i = 1, 2, 3, 4)$ be four distinct points. Then, there exists sequences $\{z_i^n\}_{n=1}^{\infty} \subset E$ converging to $z_i$. Since $\phi$ is a quasiconformal motion of $E$ over $X$, for any $\varepsilon > 0$ and for any $x \in X$, there exists a neighborhood $U_x(\varepsilon)$ such that

$$\rho(\phi_x(z_1^n, z_2^n, z_3^n, z_4^n), \phi_y(z_1^n, z_2^n, z_3^n, z_4^n)) < \frac{\varepsilon}{2}$$

holds for any $y \in U_x(\varepsilon)$ and for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, we obtain

$$\rho(\overline{\phi}_x(z_1, z_2, z_3, z_4), \overline{\phi}_y(z_1, z_2, z_3, z_4)) \leq \frac{\varepsilon}{2} < \varepsilon\,.$$

We have shown that $\overline{\phi}$ is a quasiconformal motion of $\overline{E}$ over $X$.

STEP 4: $\overline{\phi}$ and $\phi$ are both continuous motions. Since $\overline{E} \subseteq \widehat{\mathbb{C}}$ is closed and thus compact and since $\overline{\phi}_x : \overline{E} \to \widehat{\mathbb{C}}$ is continuous for any $x \in X$, the image $\overline{\phi}_x(\overline{E}) \subseteq \widehat{\mathbb{C}}$ is closed and thus compact. Since $\overline{\phi}_x$ is also injective on $\overline{E}$,

$$\overline{\phi}_x^{-1} : \overline{\phi}_x(\overline{E}) \to \overline{E}$$

is continuous. We conclude that

$$\overline{\phi}_x : \overline{E} \to \overline{\phi}_x(\overline{E})$$

is a homeomorphism. From Steps 1 and 3, we know that $\overline{\phi}$ is jointly continuous on $X \times \overline{E}$, thus $\overline{\phi}$ is a continuous motion. Since it is an extension of $\phi$, we conclude that $\phi$ is also a continuous motion. □

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