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THE STRUCTURE OF THE SPACE OF POLYNOMIAL SOLUTIONS TO THE CANONICAL CENTRAL SYSTEMS OF DIFFERENTIAL EQUATIONS ON THE BLOCK HEISENBERG GROUPS: A GENERALIZATION OF A THEOREM OF KORÁNYI

ANTHONY C. KABLE

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Abstract. A result of Korányi that describes the structure of the space of polynomial solutions to the Heisenberg Laplacian operator is generalized to the canonical central systems on the block Heisenberg groups. These systems of differential operators generalize the Heisenberg Laplacian and, like it, admit large algebras of conformal symmetries. The main result implies that in most cases all polynomial solutions can be obtained from a single one by the repeated application of conformal symmetry operators.

1. Introduction. The theorem of Korányi that is referred to in the title concerns the structure of the space of polynomial solutions to a certain partial differential equation. The genealogy of results of this type traces back to observations made by mathematicians such as Maxwell, Sylvester, and Clebsch in their investigations of harmonic polynomials. Dowker has recently written a beautiful account of this work, with numerous references to the original sources [1]. In order to provide context for Korányi's result and the results reported here, we shall briefly describe some of the simplest aspects of this classical story.

Let $\Delta = \partial_1^2 + \cdots + \partial_n^2$ be the Laplacian acting on the polynomial ring $A = \mathbb{C}[x_1, \dots, x_n]$ and let $\mathcal{H} \subset A$ denote the subspace of solutions to the equation $\Delta \bullet \varphi = 0$. (Here and below, we use $\bullet$ to denote the action of a differential operator on a function.) The operators $x_i\partial_j - x_j\partial_i$ for $1 \leq i < j \leq n$ act on A and preserve $\mathcal{H}$. They span a Lie algebra isomorphic to $\mathfrak{so}(n, \mathbb{C})$ (we shall work throughout with complex coefficients, since the real form makes no essential difference when we are considering polynomial solutions). Let $\mathbb{E} = x_1\partial_1 + \cdots + x_n\partial_n$ be the Euler operator and define $D_i = x_i(2 - n - 2\mathbb{E}) + \|x\|^2\partial_i$ for $1 \leq i \leq n$. The operators D_i are obtained by conjugating the operators ∂_i by the Kelvin transform $\mathbb{K}$ given by $(\mathbb{K}\varphi)(x) = \|x\|^{2-n}\varphi(x/\|x\|^2)$; in particular, they commute with one another. Together with the copy of $\mathfrak{so}(n, \mathbb{C})$ already described, the operators $\partial_1, \dots, \partial_n, D_1, \dots, D_n$, and $2 - n - 2\mathbb{E}$ span a Lie algebra isomorphic to $\mathfrak{so}(n + 2, \mathbb{C})$. We shall denote this algebra by $\mathfrak{r}$. The action of $\mathfrak{r}$ on A preserves $\mathcal{H}$ so that $\mathcal{H}$ becomes a module for the universal enveloping algebra $\mathcal{U}(\mathfrak{r})$. If $n = 1$ or $n \geq 3$ then this module is irreducible. Let us assume that $n \geq 3$. Then any

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non-zero harmonic polynomial is a $\mathcal{U}(\mathfrak{r})$-cyclic vector for $\mathcal{H}$. In particular, 1 is a $\mathcal{U}(\mathfrak{r})$-cyclic vector and it follows from this and the PBW Theorem that we have $\mathcal{H} = \mathbb{C}[D_i]\bullet 1$. That is, every harmonic polynomial may be obtained by applying some polynomial in the operators $D_1, \ldots, D_n$ to 1. With modern tools, this assertion can be proved in a few lines. A dimension count shows that the representation of $\varphi \in \mathcal{H}$ in the form $\varphi = \psi(D)\bullet 1$ cannot be unique and the desire to make it so in an interesting and useful way is the starting point of the Maxwell-Sylvester theory of poles. The statement that $\mathcal{H} = \mathbb{C}[D_i]\bullet 1$, which is essentially equivalent to the statement that $\mathcal{H}$ is an irreducible $\mathcal{U}(\mathfrak{r})$-module, is what we would describe as the Korányi theorem for this situation.

The Heisenberg Laplacian $\square_w$, where w is a complex parameter, is a differential operator on the Heisenberg group. It has been extensively studied, initially because of its importance in complex analysis [2]. It may be considered as a perturbation of the Euclidean Laplacian along the center of the Heisenberg group. Korányi [11] considered the polynomial solutions to the Heisenberg Laplacian. In this situation there are analogues of the Kelvin transform and of the Lie algebra $\mathfrak{r}$ described for the Laplacian in the previous paragraph. Korányi showed that the polynomial 1 is a $\mathcal{U}(\mathfrak{r})$-cyclic vector for the module of polynomial solutions to $\square_w$, provided that w avoids a countable set of bad values. The proof is analytic, relying crucially on the fact that $\square_w$ is hypoelliptic, and is substantially harder than in the case of the Laplacian. Subsequently, the author gave an algebraic proof of Korányi's theorem, and also determined the structure of the $\mathcal{U}(\mathfrak{r})$-module of polynomial solutions when w takes one of the bad values [6]. Korányi's theorem provides one approach to understanding the space of polynomial solutions to the Heisenberg Laplacian. It is also possible to write an explicit basis of polynomial solutions expressed in terms of Jacobi polynomials [4].

Both the Laplacian and the Heisenberg Laplacian admit a large algebra of conformal symmetries, although both were first considered for reasons unconnected with this property. From the perspective of Lie theory, it is natural to begin with a candidate algebra of conformal symmetries and then systematically to construct operators that admit this algebra. This procedure will, of course, yield known examples such as the Laplacian and the Heisenberg Laplacian, but will often place these examples into families most of whose members were previously unknown. It emerges that it is most natural to consider systems of differential operators, without restricting the number of operators that may occur, so that systems consisting of a single operator become the exception rather than the rule. The author undertook the task of finding a general Lie-theoretic framework for the Heisenberg Laplacian [7] and, as a consequence, found what are here called the canonical central systems. One may attempt to generalize the methods that have been used to study the polynomial solutions of the Heisenberg Laplacian to these more general systems. It currently appears that the explicit approach that was mentioned above succeeds only for a limited class of the new systems [8]. The purpose of the current work is to show that, in contrast, a Korányi theorem can be obtained for one of the three infinite families of canonical central systems. It is expected that this will also be true for the other two infinite families and for the exceptional system. The main results are Theorem 5.1, which is the Korányi theorem for the systems here considered, and Theorem 5.2, which gives a converse to the Korányi theorem for these systems.

To prove the two results described in the previous paragraph we use a method which differs both from the analytic method of Korányi [11] and the algebraic method previously used by the author in a special case [6], neither of which appears to generalize. It is based on a model for the solution space of the canonical central system that we call the initial model. If the reader is prepared to countenance multidimensional time then the initial model may be thought of as the space of initial conditions for solutions to the canonical central system. The initial model exists provided that the parameter w does not lie in a certain set of bad values. The determination of this set of bad values required a substantial amount of work [7, 9], as did the demonstration that the canonical central system is free of integrability conditions [9] so that the initial model contains all polynomials on the initial hypersurface. Although it is still somewhat complicated, the initial model is sufficiently tractable that we may establish irreducibility and reducibility results by studying the actions of subalgebras of the conformal algebra on the initial model directly.

In Section 2 we review the general construction of the canonical central systems and basic facts about their initial models. Section 3 contains specific information about the canonical central systems on the block Heisenberg groups. Section 4 is devoted to deriving what we call a dual b-function identity, which plays an essential role in the proof of the main results. It may be helpful to set this identity in context and to explain the choice of nomenclature. The starting point for deriving a dual b-function identity is a b-function identity associated with a prehomogeneous vector space. This identity will take the familiar form

$$Q_j \bullet P_1^{s_1} \cdots P_n^{s_n} = b_j(s_1, \ldots, s_n) P_1^{s_1} \cdots P_j^{s_j - 1} \cdots P_n^{s_n},$$

where $P_1, \ldots, P_n$ are relatively invariant polynomials on the prehomogeneous vector space in question, Q_j is a constant coefficient differential operator derived from a relatively invariant polynomial on the dual space, and b_j is a polynomial. To obtain a dual b-function identity from this, we conjugate the b-function identity by a suitable equivariant rational map from the prehomogeneous vector space to its dual space. The resulting identity has the form

$$D_j(s) \bullet P_1^{s_1} \cdots P_n^{s_n} = c_j(s, s_1, \ldots, s_n) P_1^{s_1} \cdots P_j^{s_j + 1} \cdots P_n^{s_n}, \tag{1.1}$$

where s is an additional parameter, $D_j(s)$ is a differential operator (no longer having constant coefficients), and c_j is a polynomial. Of course, the simplest identity of this form is

$$P_j \cdot P_1^{s_1} \cdots P_n^{s_n} = P_1^{s_1} \cdots P_j^{s_j + 1} \cdots P_n^{s_n} \tag{1.2}$$

and, indeed, (1.1) is a perturbation of (1.2), in the sense that we have

$$\lim_{s \to \infty} s^p D_j(s) = P_j$$

for a suitable value of p. In the present context, the operators $D_j(s)$ that appear in the dual b-function identity are related to raising operators in initial models of the modules whose irreducibility we wish to investigate. The main results are then proved in Section 5. The proofs rest mainly on the existence of the initial model and on the dual b-function identity.

The reader who is familiar with the theory of Hermitian symmetric spaces will recognize that the dual b-function identity that we make use of here is closely related to the evaluation of a generalized gamma function for a certain Hermitian symmetric space. (The reader might consult Wilson's work [15] for a compressed account of this theory in the context of the study of reducibility questions.) The information contained in the dual b-function identity is usually obtained analytically in that context, whereas here it is obtained by algebraic means. In addition, there are many prehomogeneous vector spaces besides those connected to the Hermitian symmetric spaces that support dual b-function identities. It would be interesting to investigate the significance of these identities from the current perspective.

The last substantive thing that remains is briefly to set our results in the context of the theory of modules in category $\mathcal{O}_{\mathfrak{p}}$. We have avoided this language in the body of the paper since we do not need it and its use might be an unnecessary obstacle to some readers. The module $\mathcal{P}_w$ of polynomial solutions to the canonical central system is a submodule of a dual scalar generalized Verma module $M^{\vee}$ attached to the parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{sl}(l+1)$ whose associated Dynkin diagram is displayed near the beginning of Section 3. The block Heisenberg algebra is the nilradical of this parabolic. The conformal invariance of the canonical central system guarantees that the generalized Verma module M, and hence also its dual, is reducible. In fact, $\mathcal{P}_w$ is always a non-zero proper submodule of $M^{\vee}$. Thus when $\mathcal{P}_w$ is irreducible, it affords a model of the unique irreducible submodule $L^{\vee} \subset M^{\vee}$. The module $L^{\vee}$ is always equal to the submodule of $\mathcal{P}_w$ generated by $1 \in \mathcal{P}_w$. In this way, our results might be interpreted as extending the results of Suga [13] to a particular family of prehomogeneous vector spaces of non-commutative parabolic type. From this perspective, the existence of the initial model of $\mathcal{P}_w$ amounts to a vector space isomorphism $\mathcal{P}_w \cong M_1^{\vee} \otimes M_2^{\vee}$, where M_1 and M_2 are scalar generalized Verma modules for the two standard maximal parabolic subalgebras of $\mathfrak{sl}(l+1)$ that contain $\mathfrak{p}$. However, this isomorphism is far from an isomorphism of $\mathcal{U}(\mathfrak{sl}(l+1))$-modules and seems to the author to be hard to interpret in the context of generalized Verma modules, whereas it is extremely natural in the context of systems of partial differential equations.

We wish to comment on an issue that has been raised by the referee, that of notational persistence from section to section below. As far as possible, each section is written in a self-contained way. In particular, incidental notation (such as dummy variables in summations) is chosen to enhance local readability and does not persist from one section to another. Our thanks are due to the referee for his or her careful reading of the paper, and for corrections and suggestions for improving the exposition.

2. The canonical central systems. Canonical central systems of differential equations are defined on several families of nilpotent groups. Our main focus here is on the systems that are defined on the block Heisenberg groups, but in this section we shall summarize the construction and some of the properties of these systems in general. As often happens, generality promotes simplicity in describing structural features. In addition, this section will be a useful reference for later work on the canonical central systems on other groups.

Let $\mathbf{R}$ be a simple reduced root system, $\mathbf{R}^+ \subset \mathbf{R}$ a positive system, and $\mathbf{R}^s \subset \mathbf{R}^+$ the corresponding set of simple roots. Normalize the inner product $(\cdot\,,\,\cdot)$ on the ambient space of $\mathbf{R}$ so that the long roots have squared-length 2. Let $\gamma \in \mathbf{R}^+$ be the highest root and write $\gamma = \sum_{\mu\in\mathbf{R}^s} n_\mu \mu$. Suppose that $\alpha, \beta \in \mathbf{R}^s$ are distinct and that $n_\alpha = n_\beta = 1$. Let $d(\alpha, \beta)$ be the distance between α and β in the Dynkin graph of $\mathbf{R}$. Denote by ϖ_α and ϖ_β the fundamental weights dual to α and β, respectively. Let

$$\lambda_w = -w\varpi_\alpha + (w + d(\alpha, \beta))\varpi_\beta$$

where w is a complex parameter. Let $\mathfrak{g}$ be a complex simple Lie algebra of type $\mathbf{R}$, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, and identify $\mathbf{R}$ with $\mathbf{R}(\mathfrak{g}, \mathfrak{h})$. Let $\mathbb{B}$ be a non-degenerate invariant bilinear form on $\mathfrak{g}$. The form $\mathbb{B}$ induces a bilinear form on $\mathfrak{h}^*$ which restricts to a scalar multiple of the inner product $(\cdot\,,\,\cdot)$ on the ambient space of $\mathbf{R}$. We normalize $\mathbb{B}$ so that the scalar in question is 1.

There are unique elements H_0 and Z_0 in $\mathfrak{h}$ such that $\alpha(H_0) = 1$, $\beta(H_0) = 1$, $\alpha(Z_0) = 1$, $\beta(Z_0) = -1$, $\mu(H_0) = 0$, and $\mu(Z_0) = 0$ for all $\mu \in \mathbf{R}^s \setminus \{\alpha, \beta\}$. These elements induce a bigrading $\mathfrak{g} = \oplus\mathfrak{g}(j, k)$ of $\mathfrak{g}$, where

$$\mathfrak{g}(j, k) = \{X \in \mathfrak{g} \mid [H_0, X] = jX,\ [Z_0, X] = kX\},$$

and $\mathfrak{g}(j, k) = \{0\}$ unless $(j, k) \in \{(\pm 2, 0), (\pm 1, \pm 1), (0, 0)\}$. There is a corresponding partition $\mathbf{R} = \cup\mathbf{R}(j, k)$, where $\mu \in \mathbf{R}(j, k)$ if $\mathfrak{g}_\mu \subset \mathfrak{g}(j, k)$. The subspace $\mathfrak{l} = \mathfrak{g}(0, 0)$ is a reductive subalgebra of $\mathfrak{g}$. The center of $\mathfrak{l}$ is the span of H_0 and Z_0, and there is a decomposition $\mathfrak{l} = \mathfrak{z}(\mathfrak{l}) \oplus \mathfrak{l}^{ss}$ with $\mathfrak{l}^{ss} = [\mathfrak{l}, \mathfrak{l}]$ semisimple. Each space $\mathfrak{g}(j, k)$ with $(j, k) \neq (0, 0)$ is an irreducible $\mathfrak{l}$-module. In addition, $\mathfrak{n} = \mathfrak{g}(1, 1) \oplus \mathfrak{g}(1, -1) \oplus \mathfrak{g}(2, 0)$ and $\bar{\mathfrak{n}} = \mathfrak{g}(-1, 1) \oplus \mathfrak{g}(-1, -1) \oplus \mathfrak{g}(-2, 0)$ are two-step nilpotent subalgebras of $\mathfrak{g}$. We may regard λ_w as a functional on $\mathfrak{h}$. As such, it vanishes on $\mathfrak{h} \cap \mathfrak{l}^{ss}$ and so it may be extended uniquely to a functional on $\mathfrak{l}$ that vanishes on $\mathfrak{l}^{ss}$.

Let $\mathfrak{h}_0 \subset \mathfrak{g}_0$ be a split real form of $\mathfrak{h} \subset \mathfrak{g}$. If $E \subset \mathfrak{g}$ is an $\mathrm{ad}(\mathfrak{h})$-invariant subspace of $\mathfrak{g}$ then $E_0 = E \cap \mathfrak{g}_0$ is a real form of E. Let N be a real Lie group whose real Lie algebra is $\mathfrak{n}_0$. The exponential map $\exp : \mathfrak{n}_0 \to N$ is a diffeomorphism. We shall regard N as a subgroup of the real Lie group $\mathrm{Aut}(\mathfrak{g}_0)$ via the embedding

$$\exp(X) \mapsto \sum_{j=0}^{\infty} \frac{1}{j!}\big(\mathrm{ad}(X)\big)^j .$$

The group N is nilpotent of class 2 and $\mathfrak{g}(2, 0)_0$ is the real Lie algebra of its center. In order to introduce coordinates on N, we define

$$n(X, Y, T) = \exp\big(X + Y + T - \frac{1}{2}[X, Y]\big)$$

for $X \in \mathfrak{g}(1, 1)_0$, $Y \in \mathfrak{g}(1, -1)_0$, and $T \in \mathfrak{g}(2, 0)_0$. With this definition, N is the set of all $n(X, Y, T)$, the product on N is

$$n(X, Y, T)n(X', Y', T') = n(X + X', Y + Y', T + T' + [X, Y']),$$

and the inverse on N is

$$n(X, Y, T)^{-1} = n(-X, -Y, -T + [X, Y]) .$$

We refer to a function on N as a polynomial function if it is a polynomial when expressed in the coordinates X, Y, and T. To each $Z \in \mathfrak{n}_0$ is associated a first-order differential operator $R(Z)$ on N that is determined by

$$\big(R(Z)\bullet\varphi\big)(n) = \frac{d}{d\tau}\bigg|_{\tau=0} \varphi(n \exp(\tau Z))$$

for $\varphi \in C^\infty(N)$. The map $Z \mapsto R(Z)$ is a Lie algebra homomorphism from $\mathfrak{n}_0$ into the algebra of differential operators on N. It extends uniquely to an algebra homomorphism from $\mathfrak{U}(\mathfrak{n})$, the universal enveloping algebra of $\mathfrak{n}$, into the algebra of differential operators on N.

It will sometimes be helpful to have more explicit expressions for operators such as $R(Z)$. When this is so, we assume that root vectors X_μ have been chosen in the root spaces in $\mathfrak{n}$ and introduce the corresponding structure constants by $[X_\mu, X_\nu] = N_{\mu,\nu} X_{\mu+\nu}$, with $N_{\mu,\nu} = 0$ when $\mu + \nu$ is not a root. For μ a root in $\mathfrak{n}$ we let ∂_μ be the corresponding directional derivative. For example,

$$(\partial_\mu\bullet\varphi)(n(X, Y, T)) = \frac{d}{d\tau}\bigg|_{\tau=0} \varphi\big(n(X + \tau X_\mu, Y, T)\big)$$

when $\mu \in \mathbf{R}(1, 1)$. We also introduce coordinates dual to the chosen root vectors. Specifically, we take x_μ to be dual to X_μ when $\mu \in \mathbf{R}(1, 1)$, y_ν to be dual to X_ν when $\nu \in \mathbf{R}(1, -1)$, and t_ζ to be dual to X_ζ when $\zeta \in \mathbf{R}(2, 0)$. The operators ∂_μ are then the partial derivative operators with respect to these coordinates. With this notation in place, we have $R(X_\mu) = \partial_\mu$ for $\mu \in \mathbf{R}(1, 1) \cup \mathbf{R}(2, 0)$ and

$$R(X_\nu) = \partial_\nu + \sum_{\mu \in \mathbf{R}(1,1)} x_\mu N_{\mu,\nu} \partial_{\mu+\nu}$$

for $\nu \in \mathbf{R}(1, -1)$.

Choose a basis $\{X_i\}$ for $\mathfrak{g}(1, 1)$ and let $\{\bar{X}_i\}$ be the $\mathbb{B}$-dual basis of $\mathfrak{g}(-1, -1)$. Also choose a basis $\{Y_j\}$ of $\mathfrak{g}(1, -1)$ and let $\{\bar{Y}_j\}$ be the $\mathbb{B}$-dual basis of $\mathfrak{g}(-1, 1)$. Define a map $\omega_w : \mathfrak{g}(2, 0) \to \mathfrak{U}(\mathfrak{n})$ by

$$\omega_w(T) = \big(w + d(\alpha, \beta)\big)T + \sum_{i,j} \mathbb{B}\big(T, [\bar{X}_i, \bar{Y}_j]\big) X_i Y_j ,$$

where w is a complex parameter. Observe that the dependence of this map on the choice of bases for $\mathfrak{g}(1, 1)$ and $\mathfrak{g}(1, -1)$ is only apparent. We now set

$$\Omega_w(T) = R\big(\omega_w(T)\big)$$

and thus obtain a differential operator $\Omega_w(T)$ on N for each $T \in \mathfrak{g}(2, 0)$ and $w \in \mathbb{C}$. The systems of differential operators that are of interest here are $\{\Omega_w(T) \mid T \in \mathfrak{g}(2, 0)\}$ for each fixed $w \in \mathbb{C}$. They may be replaced by finite systems of differential operators with the same solutions by restricting T to run through a given basis for $\mathfrak{g}(2, 0)$.

We refer to the systems constructed in the previous paragraph as the canonical central systems on N. The operators in these systems are indexed by elements of the center of $\mathfrak{n}$, although the nomenclature was actually motivated by the role that the central element Z_0 plays in obtaining them in the context of the staircase method, a more general method for constructing conformally invariant systems on the nilradicals of parabolic subgroups in semisimple Lie groups. Their construction is canonical from the datum $(\mathbf{R}, \{\alpha, \beta\})$. In fact, there are also canonical central systems associated to such data even when α and β do not satisfy the restriction $n_\alpha = n_\beta = 1$, although the operators in the resulting systems have order $n_\alpha + n_\beta$. As far as the author is aware, these higher-order canonical central systems have not yet received any attention.

If D is an operator on functions on N then we denote the functional $\varphi \mapsto (D \bullet \varphi)(n)$ by D_n. We have the decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{l} \oplus \bar{\mathfrak{n}}$ and associated projection operators $\mathrm{pr}_{\mathfrak{n}}$, $\mathrm{pr}_{\mathfrak{l}}$, and $\mathrm{pr}_{\bar{\mathfrak{n}}}$. For $Z \in \mathfrak{g}$ there is a differential operator $\Pi_w(Z)$ on N such that

$$\Pi_w(Z)_n = -R\big(\mathrm{pr}_{\mathfrak{n}}(\mathrm{Ad}(n^{-1})Z)\big) - \lambda_w\big(\mathrm{pr}_{\mathfrak{l}}(\mathrm{Ad}(n^{-1})Z)\big) \tag{2.1}$$

for all $n \in N$. The map Π_w affords a representation of $\mathfrak{g}$ in the space of differential operators on N in the sense that $\Pi_w([Z_1, Z_2]) = [\Pi_w(Z_1), \Pi_w(Z_2)]$ for all $Z_1, Z_2 \in \mathfrak{g}$. It will be useful to have explicit expressions for $\Pi_w(Z)$ with $Z \in \mathfrak{n}$. These may be derived from the general expression for $\Pi_w(Z)$ given above or by noting that when $Z \in \mathfrak{n}$ the operator $\Pi_w(Z)$ coincides with the natural left action of Z on functions on N. By either method, it emerges that $\Pi_w(X_\nu) = -\partial_\nu$ for $\nu \in \mathbf{R}(1, -1) \cup \mathbf{R}(2, 0)$ and

$$\Pi_w(X_\mu) = -\partial_\mu - \sum_{\nu \in \mathbf{R}(1,-1)} y_\nu N_{\mu,\nu} \partial_{\mu+\nu}$$

for $\mu \in \mathbf{R}(1, 1)$.

Each $\Pi_w(Z)$, $Z \in \mathfrak{g}$, is a conformal symmetry of the canonical central system. To express this generalized symmetry property as an equation, let $\{T_k\}$ be a basis for $\mathfrak{g}(2, 0)$. Then, for all $Z \in \mathfrak{g}$, we have

$$[\Pi_w(Z), \Omega_w(T_k)] = \sum_p C_w(Z)_k^p \Omega_w(T_p)\,,$$

where the coefficients $C_w(Z)_k^p$ are polynomial functions on N. For present purposes, the main consequence of this property is that the operators $\Pi_w(Z)$ preserve the subspace of solutions to the canonical central system in any suitable space of smooth functions on N. In this way, the solution space of the canonical central system becomes a module for $\mathcal{U}(\mathfrak{g})$ via Π_w.

Let $\mathcal{P}_w$ denote the subspace of solutions to the canonical central system inside the space of polynomials on N. We regard $\mathcal{P}_w$ as a $\mathcal{U}(\mathfrak{g})$-module via Π_w as explained above. Note that the constant polynomial 1 belongs to $\mathcal{P}_w$.

PROPOSITION 2.1. *The submodule of $\mathcal{P}_w$ generated by* 1 *is irreducible and large. In particular, $\mathcal{P}_w$ is indecomposable.*

PROOF. It suffices to show that if $\varphi \in \mathcal{P}_w \setminus \{0\}$ then the submodule of $\mathcal{P}_w$ generated by φ contains 1. This is clear from the expressions for $\Pi_w(Z)$, $Z \in \mathfrak{n}$, that were given above. The operators $\Pi_w(X_\zeta)$, $\zeta \in \mathbf{R}(2,0)$, may be applied to φ as necessary to obtain a non-zero polynomial independent of all t_ζ. Then the operators $\Pi_w(X_\nu)$, $\nu \in \mathbf{R}(1,-1)$, and $\Pi_w(X_\mu)$, $\mu \in \mathbf{R}(1,1)$, may be used to obtain a non-zero constant polynomial. □

Let res : $\mathcal{P}_w \to \mathbb{C}[\mathfrak{g}(1,1) \oplus \mathfrak{g}(1,-1)]$ be defined by

$$\operatorname{res}(\varphi)(X,Y) = \varphi(n(X,Y,0))$$

and denote the image of res by $\tilde{\mathcal{P}}_w$. We regard the variables t_ζ, $\zeta \in \mathbf{R}(2,0)$, as (multiple) times, so that the map res sends a polynomial solution to the canonical central system to its initial data. We say that w lies in the uniqueness set if res is injective and otherwise that w lies in the non-uniqueness set. The non-uniqueness set is always a subset of $\mathbb{Z}$. It has been determined for each canonical central system (see [7] and [9]) and will be recalled below for the systems on the block Heisenberg groups.

Assume now that w lies in the uniqueness set. Then we may define an action $\tilde{\Pi}_w$ of $\mathcal{U}(\mathfrak{g})$ on $\tilde{\mathcal{P}}_w$ by

$$\tilde{\Pi}_w(Z)\bullet\psi = \operatorname{res}\big(\Pi_w(Z)\bullet\varphi\big),$$

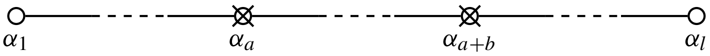
where $\varphi \in \mathcal{P}_w$ is chosen to satisfy $\operatorname{res}(\varphi) = \psi$. Note that $\tilde{\Pi}_w(Z)$ need not be (the restriction to $\tilde{\mathcal{P}}_w$ of) a differential operator on the space of polynomials on $\mathfrak{g}(1,1) \oplus \mathfrak{g}(1,-1)$. In fact, $\tilde{\Pi}_w(Z)$ is a differential operator for $Z \in \bar{\mathfrak{n}} \oplus \mathfrak{l}$ but not for general $Z \in \mathfrak{g}$. The operator $\tilde{\Pi}_w(Z)$ does always belong to a suitable localization of the algebra of differential operators on $\mathfrak{g}(1,1) \oplus \mathfrak{g}(1,-1)$. When w lies in the uniqueness set, this construction gives a model of the module $\mathcal{P}_w$ in the space $\tilde{\mathcal{P}}_w$. We refer to this as the initial model of this module.

It is expected that we always have $\tilde{\mathcal{P}}_w = \mathbb{C}[\mathfrak{g}(1,1) \oplus \mathfrak{g}(1,-1)]$ when w lies in the uniqueness set, although this equality definitely fails in some cases when w lies in the non-uniqueness set. One may express this by saying that the canonical central system is free of integrability conditions when w lies in the uniqueness set. This expectation has been confirmed for the canonical central systems on the block Heisenberg groups [9, Theorem 6.5]. We shall use this fact below without further comment.

The existence of roots α and β with the property assumed here implies that $\mathbf{R}$ is of type A, D, or E_6. In particular, $\mathbf{R}$ is always simply laced. Any two roots may be chosen for α and β when $\mathbf{R}$ is of type A. When $\mathbf{R}$ is of type D, there are three choices of α and β available. When $\mathbf{R}$ is of type E_6, there is a unique choice of α and β available. The block Heisenberg groups arise when $\mathbf{R}$ is of type A.

3. The systems of Type A. Fix $l \geq 2$ and choose $a, b \geq 1$ such that $a + b \leq l$. Let $c = l - a - b + 1$. We work in the Lie algebra $\mathfrak{g} = \mathfrak{sl}(l+1)$ with the standard choice of Cartan subalgebra and standard model for the root system. In this model, the simple roots are $\alpha_j = \varepsilon_j - \varepsilon_{j+1}$ and we choose $\alpha = \alpha_a$ and $\beta = \alpha_{a+b}$ in the general construction. This choice of simple roots may be indicated as usual by striking out the roots α and β in the

Dynkin diagram, as shown below.



Note that $d(\alpha, \beta) = b$.

The Lie algebra $\mathfrak{l}$ consists of block-diagonal matrices of the form $\operatorname{diag}(A_1, A_2, A_3)$ with $A_1 \in \operatorname{Mat}(a)$, $A_2 \in \operatorname{Mat}(b)$, and $A_3 \in \operatorname{Mat}(c)$ and $\operatorname{tr}(A_1) + \operatorname{tr}(A_2) + \operatorname{tr}(A_3) = 0$. Here $\operatorname{Mat}(m, n)$ denotes the space of m-by-n matrices and $\operatorname{Mat}(n)$ is an abbreviation of $\operatorname{Mat}(n, n)$. The weights ϖ_α and ϖ_β are given by

$$\begin{aligned} \varpi_\alpha &= \frac{1}{l+1}\big((b+c)(\varepsilon_1 + \cdots + \varepsilon_a) - a(\varepsilon_{a+1} + \cdots + \varepsilon_{l+1})\big)\,, \\ \varpi_\beta &= \frac{1}{l+1}\big(c(\varepsilon_1 + \cdots + \varepsilon_{a+b}) - (a+b)(\varepsilon_{a+b+1} + \cdots + \varepsilon_{l+1})\big)\,. \end{aligned}$$

When interpreted as characters of $\mathfrak{l}$, these weights have the simpler expressions

$$\begin{aligned} \varpi_\alpha(\operatorname{diag}(A_1, A_2, A_3)) &= \operatorname{tr}(A_1)\,, \\ \varpi_\beta(\operatorname{diag}(A_1, A_2, A_3)) &= -\operatorname{tr}(A_3)\,. \end{aligned}$$

The non-uniqueness set for these systems was determined in [9, Theorem 2.1], which completed the partial determination in [7, Theorem 5.4]. For consistency, we denote the non-uniqueness set by $\sigma(F)$. (It is, in fact, the spectrum of a locally finite operator on a free module over a polynomial ring, and this is the origin of the notation.) We have

$$\sigma(F) = \begin{cases} -(a-1) + \mathbb{N} & \text{if } b \geq a, \\ -b + \mathbb{N} & \text{if } b < a. \end{cases}$$

Recall that if $w \notin \sigma(F)$ then the initial model $\tilde{\mathcal{P}}_w$ of the module $\mathcal{P}_w$ is available.

The elements of the Lie algebra $\mathfrak{n}$ have the form

$$\begin{bmatrix} 0 & X & T \\ 0 & 0 & Y \\ 0 & 0 & 0 \end{bmatrix}$$

with $X \in \operatorname{Mat}(a, b)$, $Y \in \operatorname{Mat}(b, c)$, and $T \in \operatorname{Mat}(a, c)$. The Lie bracket is

$$\left[\begin{bmatrix} 0 & X & T \\ 0 & 0 & Y \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & X' & T' \\ 0 & 0 & Y' \\ 0 & 0 & 0 \end{bmatrix}\right] = \begin{bmatrix} 0 & 0 & XY' - X'Y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and a brief calculation reveals that

$$n(X, Y, T) = \begin{pmatrix} I_a & X & T \\ 0 & I_b & Y \\ 0 & 0 & I_c \end{pmatrix}\,.$$

We refer to the group N consisting of all $n(X, Y, T)$ as a block Heisenberg group. If we take $a = c = 1$ then we obtain a group isomorphic to the standard real Heisenberg group. Let

$X = [x_{ij}]$, $Y = [y_{jk}]$, and $T = [t_{ik}]$, so that x_{ij}, y_{jk}, and t_{ik} afford a system of coordinates on N.

We can write out the operators in the canonical central system on the block Heisenberg group explicitly in the coordinates that were just introduced. To this end, define

$$\Delta_{ik} = \sum_{j=1}^{b} \frac{\partial^2}{\partial x_{ij} \partial y_{jk}}$$

for $1 \leq i \leq a$ and $1 \leq k \leq c$. For i and k in the same ranges, let

$$\Omega_{ik}^{[w]} = \Delta_{ik} - w \frac{\partial}{\partial t_{ik}} + \sum_{\substack{1 \leq p \leq a \\ 1 \leq q \leq b}} x_{pq} \frac{\partial^2}{\partial x_{iq} \partial t_{pk}},$$

where w is the parameter introduced above. The canonical central system on the block Heisenberg group N consists of the operators $\Omega_{ik}^{[w]}$ for all $1 \leq i \leq a$ and $1 \leq k \leq c$. The parameter w may be omitted from the notation when it does not need to be emphasized. When $a = c = 1$ (and only then) the canonical central system consists of a single operator, the Heisenberg ultrahyperbolic operator [6]. It is a real form of the Heisenberg Laplacian operator studied by Folland and Stein [2] and many subsequent authors.

For $1 \leq i \leq a$ and $1 \leq k \leq c$ define

$$\varphi_{ik} = \sum_{1 \leq j \leq b} x_{ij} y_{jk}.$$

Let

$$\mathbb{E}_x = \sum_{\substack{1 \leq i \leq a \\ 1 \leq j \leq b}} x_{ij} \frac{\partial}{\partial x_{ij}}$$

and

$$\mathbb{E}_y = \sum_{\substack{1 \leq j \leq b \\ 1 \leq k \leq c}} y_{jk} \frac{\partial}{\partial y_{jk}}$$

be the Euler operators associated to $\mathfrak{g}(1, 1)$ and $\mathfrak{g}(1, -1)$, respectively. If $1 \leq r \leq a$ and $1 \leq s \leq b$ then let

$$\bar{X}_{rs} = \begin{bmatrix} 0 & 0 & 0 \\ E_{sr} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where E_{sr} denotes the elementary matrix with 1 in the (s, r)-place and zeros elsewhere. If $1 \leq r \leq b$ and $1 \leq s \leq c$ then let

$$\bar{Y}_{rs} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & E_{sr} & 0 \end{bmatrix}.$$

The matrices $\bar{X}_{rs}$ are a basis for $\mathfrak{g}(-1,-1)$ and the matrices $\bar{Y}_{rs}$ are a basis for $\mathfrak{g}(-1,1)$.

LEMMA 3.1. *For $1 \le r \le a$ and $1 \le s \le b$ we have*

$$\Pi_w(\bar{X}_{rs}) = -wx_{rs} + \sum_{\substack{1\le m\le a\\1\le n\le b}} x_{rn}x_{ms}\frac{\partial}{\partial x_{mn}} - \sum_{1\le q\le c} t_{rq}\frac{\partial}{\partial y_{sq}}\,.$$

If in addition $w \notin \sigma(F)$ then

$$\tilde{\Pi}_w(\bar{X}_{rs}) = -wx_{rs} + \sum_{\substack{1\le m\le a\\1\le n\le b}} x_{rn}x_{ms}\frac{\partial}{\partial x_{mn}}\,.$$

For $1 \le r \le b$ and $1 \le s \le c$ we have

$$\Pi_w(\bar{Y}_{rs}) = (w+b)y_{rs} + \sum_{\substack{1\le m\le b\\1\le n\le c}} y_{rn}y_{ms}\frac{\partial}{\partial y_{mn}} - \sum_{1\le i\le a}(\varphi_{is}-t_{is})\frac{\partial}{\partial x_{ir}} + \sum_{\substack{1\le i\le a\\1\le k\le c}} y_{rk}t_{is}\frac{\partial}{\partial t_{ik}}\,.$$

If in addition $w \notin \sigma(F)$ then

$$\tilde{\Pi}_w(\bar{Y}_{rs}) = (w+b)y_{rs} + \sum_{\substack{1\le m\le b\\1\le n\le c}} y_{rn}y_{ms}\frac{\partial}{\partial y_{mn}} - \sum_{1\le i\le a}\varphi_{is}\frac{\partial}{\partial x_{ir}}\,.$$

PROOF. The proof is based on the general expression (2.1) for $\Pi_w(Z)$ with $Z \in \mathfrak{g}$. The two evaluations are obtained similarly, but the second is a little more elaborate and so we shall present that one. Let $n = n(X, Y, T)$. A calculation shows that

$$n^{-1}\bar{Y}_{rs}n = \begin{bmatrix} 0 & (XY-T)E_{sr} & (XY-T)E_{sr}Y \\ 0 & -YE_{sr} & -YE_{sr}Y \\ 0 & E_{sr} & E_{sr}Y \end{bmatrix}$$

and so $\mathrm{pr}_{\mathfrak{l}}(n^{-1}\bar{Y}_{rs}n) = \mathrm{diag}(0, -YE_{sr}, E_{sr}Y)$ and

$$\mathrm{pr}_{\mathfrak{n}}(n^{-1}\bar{Y}_{rs}n) = \begin{bmatrix} 0 & (XY-T)E_{sr} & (XY-T)E_{sr}Y \\ 0 & 0 & -YE_{sr}Y \\ 0 & 0 & 0 \end{bmatrix}.$$

Now $\mathrm{tr}(E_{sr}Y) = y_{rs}$, and so

$$\lambda_w\big(\mathrm{pr}_{\mathfrak{l}}(n^{-1}\bar{Y}_{rs}n)\big) = -(w+b)y_{rs}\,.$$

We have

$$(XY-T)E_{sr} = \sum_{1\le i\le a}(\varphi_{is}-t_{is})E_{ir}\,,$$

which may be regarded as an element of $\mathfrak{g}(1,1)$. The operator associated to this element by the right action map R is

$$\sum_{1\le i\le a}(\varphi_{is}-t_{is})\frac{\partial}{\partial x_{ir}}\,. \tag{3.1}$$

Similarly,

$$(XY - T)E_{sr}Y = \sum_{\substack{1 \le p \le a \\ 1 \le n \le c}} (\varphi_{ps} - t_{ps}) y_{rn} E_{pn}$$

may be regarded as an element of $\mathfrak{g}(2, 0)$ and the associated operator is

$$\sum_{\substack{1 \le p \le a \\ 1 \le n \le c}} (\varphi_{ps} - t_{ps}) y_{rn} \frac{\partial}{\partial t_{pn}} . \tag{3.2}$$

Finally,

$$-YE_{sr}Y = -\sum_{\substack{1 \le j \le b \\ 1 \le n \le c}} y_{js} y_{rn} E_{jn}$$

may be regarded as an element of $\mathfrak{g}(1, -1)$ and the associated operator is

$$\begin{aligned} &-\sum_{\substack{1 \le j \le b \\ 1 \le n \le c}} y_{js} y_{rn} \left(\frac{\partial}{\partial y_{jn}} + \sum_{1 \le p \le a} x_{pj} \frac{\partial}{\partial t_{pn}} \right) \\ &= -\sum_{\substack{1 \le j \le b \\ 1 \le n \le c}} y_{js} y_{rn} \frac{\partial}{\partial y_{jn}} - \sum_{\substack{1 \le p \le a \\ 1 \le n \le c}} \varphi_{ps} y_{rn} \frac{\partial}{\partial t_{pn}} . \end{aligned} \tag{3.3}$$

By adding (3.1), (3.2), and (3.3) we conclude that

$$R\big(\mathrm{pr}_{\mathfrak{n}}(n^{-1}\bar{Y}_{rs}n)\big) = -\sum_{\substack{1 \le j \le b \\ 1 \le n \le c}} y_{js} y_{rn} \frac{\partial}{\partial y_{jn}} + \sum_{1 \le i \le a} (\varphi_{is} - t_{is}) \frac{\partial}{\partial x_{ir}} - \sum_{\substack{1 \le p \le a \\ 1 \le n \le c}} y_{rn} t_{ps} \frac{\partial}{\partial t_{pn}} .$$

We now have all the evaluations required to obtain the formula for $\Pi_w(\bar{Y}_{rs})$ given in the statement. To obtain $\tilde{\Pi}_w(\bar{Y}_{rs})$ when $w \notin \sigma(F)$, note that the operators $t_{ps}\partial/\partial t_{pn}$ and $t_{is}\partial/\partial x_{ir}$ give zero on restriction to $T = 0$ and so

$$\tilde{\Pi}_w(\bar{Y}_{rs}) = (w + b) y_{rs} + \sum_{\substack{1 \le j \le b \\ 1 \le n \le c}} y_{js} y_{rn} \frac{\partial}{\partial y_{jn}} - \sum_{1 \le i \le a} \varphi_{is} \frac{\partial}{\partial x_{ir}}$$

as claimed. □

We next wish to recall the notion of conjugacy for canonical central systems [9, Section 5]. This provides a relation between the canonical central systems on the block Heisenberg group N with sizes (a, b, c) and the block Heisenberg group $\check{N}$ with sizes (c, b, a). It is based on the existence of the automorphism $g \mapsto \check{g}$ of $\mathrm{SL}(l + 1)$ given by $\check{g} = J_1 (g^{-1})^\top J_1^{-1}$, where $\top$ denotes the transpose and

$$J_1 = \begin{pmatrix} 0 & 0 & I_c \\ 0 & -I_b & 0 \\ I_a & 0 & 0 \end{pmatrix} .$$

This automorphism maps N to $\check{N}$; indeed, a calculation shows that

$$\check{n}(X, Y, T) = \begin{pmatrix} I_c & Y^\top & (XY - T)^\top \\ 0 & I_b & X^\top \\ 0 & 0 & I_a \end{pmatrix} .$$

It induces automorphisms of the various objects associated to $\mathrm{SL}(l + 1)$. We denote all of these automorphisms by the same symbol. If $Z = \mathrm{diag}(A_1, A_2, A_3) \in \mathfrak{l}$ then $\check{Z} = \mathrm{diag}(-A_3^\top, -A_2^\top, -A_1^\top) \in \check{\mathfrak{l}}$ and so

$$\begin{aligned} \check{\lambda}_w(\check{Z}) &= -w\mathrm{tr}(-A_3^\top) - (w + b)\mathrm{tr}(-A_1^\top) \\ &= (w + b)\mathrm{tr}(A_1) + w\mathrm{tr}(A_3) \\ &= \lambda_{-w-b}(Z) . \end{aligned}$$

This motivates us to define $\check{w} = -w - b$ so that $\check{\lambda}_{\check{w}}(\check{Z}) = \lambda_w(Z)$ for all w and all $Z \in \mathfrak{l}$.

Given a smooth function φ on N, we define a smooth function $\check{\varphi}$ on $\check{N}$ by $\check{\varphi}(\check{n}) = \varphi(n)$. It follows from the intinsic nature of the constructions (and may be verified by routine calculation) that

$$\check{\Pi}_{\check{w}}(\check{Z})\bullet\check{\varphi} = \big(\Pi_w(Z)\bullet\varphi\big)^\vee \tag{3.4}$$

and

$$\check{\Omega}_{ki}^{[\check{w}]}\bullet\check{\varphi} = \big(\Omega_{ik}^{[w]}\bullet\varphi\big)^\vee. \tag{3.5}$$

Note that $\check{\varphi}$ is a polynomial on $\check{N}$ if and only if φ is a polynomial on N. It follows from this and (3.5) that the map $\varphi \mapsto \check{\varphi}$ is a linear isomorphism from $\mathcal{P}_w$ to $\check{\mathcal{P}}_{\check{w}}$. It then follows from (3.4) that this linear isomorphism is a $\mathcal{U}(\mathfrak{g})$-module isomorphism along the automorphism $u \mapsto \check{u}$ of $\mathcal{U}(\mathfrak{g})$. In particular, $\mathcal{U}(\mathfrak{g})$-submodules of $\mathcal{P}_w$ correspond to $\mathcal{U}(\mathfrak{g})$-submodules of $\check{\mathcal{P}}_{\check{w}}$ under this isomorphism, and the lattice of submodules in $\mathcal{P}_w$ is isomorphic to the lattice of submodules in $\check{\mathcal{P}}_{\check{w}}$. It may be worth mentioning that the initial models of $\mathcal{P}_w$ and $\check{\mathcal{P}}_{\check{w}}$ do not correspond under this isomorphism. In fact, restriction to the set $T = 0$ in $\check{\mathcal{P}}_{\check{w}}$ corresponds to restriction to the set $T = XY$ in $\mathcal{P}_w$.

4. A Dual *b*-Function Identity. Let $a, b \geq 1$ and $n = \min\{a, b\}$. Let u_{ij} with $1 \leq i \leq a$ and $1 \leq j \leq b$ be variables, and denote by ∂_{ij} the partial derivative with respect to u_{ij}. Let $U = [u_{ij}]$ and $\partial = [\partial_{ij}]$ be the indicated a-by-b matrices. If M is any matrix, I is a subset of the set of row indices of M, and J is a subset of the set of column indices of M, then we let $M[I, J]$ denote the submatrix of M formed with the rows from I and the columns from J. We abbreviate $M[I, I]$ to $M[I]$. For $1 \leq j \leq n$, define $j' = n - j + 1$. For $1 \leq i \leq n$, let

$$\Delta_i = \det U[\{1, \dots, i\}]$$

and

$$\Delta_i^* = \det \partial[\{1', \dots, i'\}] .$$

It will be convenient to extend the definition of Δ_i by setting $\Delta_0 = 1$. We shall make use of formal powers P^s with $P \in \mathbb{C}[u_{rs}]$. As usual, the action of the partial derivatives is extended to such formal powers by defining $\partial_{ij} \bullet P^s = sP^s(\partial_{ij} \bullet P)/P$.

The identity stated in Theorem 4.1 has been attributed to Mikio Sato (see [14, p. 150], for example). The usual reference that is quoted for it is the notes prepared by Takuro Shintani of a course given by Sato [12].

THEOREM 4.1 (*b*-Function Identity). *For* $1 \le m \le n$ *we have*

$$\Delta_m^* \bullet (\Delta_1^{s_1} \cdots \Delta_n^{s_n}) = s_n(s_n + s_{n-1} + 1) \cdots (s_n + \cdots + s_{m'} + m - 1)\Delta_1^{s_1} \cdots \Delta_{n-1}^{s_{n-1}} \Delta_n^{s_n - 1} \Delta_{n-m} .$$

Let s be a parameter. For $1 \le p \le a$ and $1 \le q \le b$ we define

$$D_{pq} = su_{pq} + \sum_{\substack{1 \le i \le a \\ 1 \le j \le b}} u_{pj} u_{iq} \partial_{ij} . \tag{4.1}$$

The following could of course be proved by direct computation. The alternative proof we offer should partially explain our interest in the operators D_{pq}.

LEMMA 4.2. *The operators* D_{pq} *commute with one another.*

PROOF. First suppose that $s \notin \mathbb{Z}$. After substituting $u_{ij} = x_{ij}$, we have $D_{pq} = \tilde{\Pi}_{-s}(\bar{X}_{pq})$ by Lemma 3.1. Now

$$[\mathfrak{g}(-1, -1), \mathfrak{g}(-1, -1)] \subset \mathfrak{g}(-2, -2) = \{0\}$$

and so $\mathfrak{g}(-1, -1)$ is an abelian subalgebra of $\mathfrak{g}$. Since $\tilde{\Pi}_{-s}$ is a representation of $\mathfrak{g}$, it follows that the operators D_{pq} mutually commute. We may remove the restriction that $s \notin \mathbb{Z}$ by noting that $[D_{p_1 q_1}, D_{p_2 q_2}]$ is polynomial in s and vanishes when $s \notin \mathbb{Z}$. Thus it is identically zero. □

In order to prepare for the proof of the Dual *b*-Function Identity, we digress to discuss the relationship between the operators ∂_{pq} and D_{pq} when $a = b = n$. In order to describe this relationship it is convenient to work with the algebra $\mathcal{A} = \mathbb{C}[s, u_{ij}, \Delta_n^{-1}, \Delta_n^{-s}, \Delta_n^s]$, which is also a module for the Weyl algebra of $\mathbb{C}[u_{ij}]$ in the usual way. Let

$$J = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}$$

and for an invertible n-by-n matrix M define $\mathring{M} = J(M^{-1})^\top J$. In particular, $U = [u_{ij}] \in \mathrm{Mat}(n, \mathcal{A})$ is invertible and so we obtain a matrix $\mathring{U} \in \mathrm{Mat}(n, \mathcal{A})$. We write this matrix as $\mathring{U} = [\mathring{u}_{ij}]$. The entries in $U^{-1} = [\tilde{u}_{ij}]$ are given by

$$\tilde{u}_{ij} = (-1)^{i+j} \Delta_n^{-1} \det U[\{j\}^c, \{i\}^c] , \tag{4.2}$$

where $I^c = \{1, \dots, n\} \setminus I$ for $I \subset \{1, \dots, n\}$, and $\mathring{u}_{ij} = \tilde{u}_{j'i'}$. The map $u_{ij} \mapsto \mathring{u}_{ij}$ extends to an algebra automorphism of $\mathcal{A}$ such that $\mathring{s} = s$, $(\Delta_n^{-s})^\circ = \Delta_n^s$, and $(\Delta_n^s)^\circ = \Delta_n^{-s}$. Note that we have $\varphi^{\circ\circ} = \varphi$ for all $\varphi \in \mathcal{A}$.

LEMMA 4.3. *For $1 \le m \le n$ we have $\mathring{\Delta}_m = \Delta_{n-m}\Delta_n^{-1}$.*

PROOF. From the definition of Δ_m we obtain

$$\mathring{\Delta}_m = \det \mathring{U}[\{1, \dots, m\}] = \det U^{-1}[\{1', \dots, m'\}] .$$

We now appeal to Jacobi's Identity for the minors of an inverse matrix (see [3, p. 21]) to conclude that

$$\mathring{\Delta}_m = \Delta_n^{-1} \det U[\{1', \dots, m'\}^c] = \Delta_n^{-1} \det U[\{1, \dots, n-m\}] = \Delta_{n-m}\Delta_n^{-1} ,$$

as required. □

We next define an $\tau : \mathcal{A} \to \mathcal{A}$ by $\tau(\varphi) = \Delta_n^{-s}\mathring{\varphi}$. The map τ is a linear automorphism (but not an algebra automorphism) and $\tau \circ \tau$ is the identity. Suppose that $E : \mathcal{A} \to \mathcal{A}$ is a linear map. Then we may define a linear map $\tau(E) : \mathcal{A} \to \mathcal{A}$ by $\tau(E) = \tau \circ E \circ \tau$. This definition is chosen so that $\tau(E)\tau(\varphi) = \tau(E\varphi)$ for all $\varphi \in \mathcal{A}$. As we observed above, the Weyl algebra of $\mathbb{C}[u_{ij}]$ acts on $\mathcal{A}$ and so for each element δ in the Weyl algebra we obtain a linear map $\tau(\delta) : \mathcal{A} \to \mathcal{A}$. Note that we have $\tau(\delta_1\delta_2) = \tau(\delta_1)\tau(\delta_2)$ for all δ_1 and δ_2 in the Weyl algebra.

LEMMA 4.4. *For $1 \le i, j \le n$ we have $\tau(\partial_{ij}) = -D_{i'j'}$.*

PROOF. The entries of $U^{-1} = [\tilde{u}_{ij}]$ are a coordinate system on the space of invertible n-by-n matrices and so we may consider the corresponding partial derivatives $\tilde{\partial}_{ij}$. By the Chain Rule, we have

$$\partial_{ij} = \sum_{1 \le p,q \le n} \frac{\partial \tilde{u}_{pq}}{\partial u_{ij}} \tilde{\partial}_{pq} .$$

By Laplace's Expansion and (4.2), we have

$$\partial_{ij} \bullet \Delta_n = (-1)^{i+j} \det U[\{i\}^c, \{j\}^c] = \tilde{u}_{ji}\Delta_n .$$

It will briefly be convenient to make use of the Iverson bracket [10] to express certain relationships; recall that if Φ is a boolean expression then $[\Phi]$ is defined to be 1 when Φ evalutes to True and to be 0 when Φ evaluates to False. With this notation, we have

$$\partial_{ij} \bullet \det U[\{q\}^c, \{p\}^c] = [i \ne q][j \ne p](-1)^{i+j+[i>q]+[j>p]} \det U[\{i, q\}^c, \{j, p\}^c] .$$

The extra sign factors arise because the index of the i^{th} row is decreased by 1 in the submatrix $U[\{q\}^c, \{p\}^c]$ if $i > q$, and similarly with the j^{th} column. By Jacobi's Identity, we have

$$\det U[\{i, q\}^c, \{j, p\}^c] = (-1)^{i+j+p+q} \Delta_n \det U^{-1}[\{j, p\}, \{i, q\}]$$

and so

$$\partial_{ij} \bullet \det U[\{q\}^c, \{p\}^c] = [i \ne q][j \ne p](-1)^{p+q+[i>q]+[j>p]} \Delta_n \det U^{-1}[\{j, p\}, \{i, q\}]$$

$$= (-1)^{p+q}\Delta_n \begin{vmatrix} \tilde{u}_{ji} & \tilde{u}_{jq} \\ \tilde{u}_{pi} & \tilde{u}_{pq} \end{vmatrix} .$$

Note that the 2-by-2 determinant is identically zero if $i = q$ or if $j = p$, and that sign changes appear when we switch rows and columns to write $\det U^{-1}[\{j, p\}, \{i, q\}]$ in this form if $i > q$ or if $j > p$. This accounts for the disappearance of the Iverson brackets in the final expression. By combining these evaluations with (4.2) and simplifying, we obtain

$$\begin{aligned} \frac{\partial \tilde{u}_{pq}}{\partial u_{ij}} &= -\tilde{u}_{ji}\tilde{u}_{pq} + \begin{vmatrix} \tilde{u}_{ji} & \tilde{u}_{jq} \\ \tilde{u}_{pi} & \tilde{u}_{pq} \end{vmatrix} \\ &= -\tilde{u}_{pi}\tilde{u}_{jq} \end{aligned}$$

so that

$$\partial_{ij} = -\sum_{1\le p,q\le n} \tilde{u}_{pi}\tilde{u}_{jq}\tilde{\partial}_{pq} .$$

The fact that $(U^{-1})^{-1} = U$ then implies that

$$\tilde{\partial}_{ij} = -\sum_{1\le p,q\le n} u_{pi}u_{jq}\partial_{pq}$$

and hence that

$$\mathring{\partial}_{ij} = -\sum_{1\le p,q\le n} u_{pj'}u_{i'q}\partial_{pq} . \tag{4.3}$$

For $\varphi \in \mathcal{A}$ we have

$$\begin{aligned} \tau(\partial_{ij})\bullet\varphi &= (\tau\partial_{ij}\tau)(\varphi) \\ &= \tau\big(\partial_{ij}\bullet(\Delta_n^{-s}\mathring{\varphi})\big) \\ &= \tau\big(-s\Delta_n^{-s}\tilde{u}_{ji}\mathring{\varphi} + \Delta_n^{-s}\partial_{ij}\bullet\mathring{\varphi}\big) \\ &= \Delta_n^{-s}\big(-s\Delta_n^{s}(\tilde{u}_{ji})^{\circ}\varphi + \Delta_n^{s}(\partial_{ij}\bullet\mathring{\varphi})^{\circ}\big) \\ &= -su_{i'j'}\varphi + \mathring{\partial}_{ij}\bullet\varphi \\ &= -D_{i'j'}\bullet\varphi , \end{aligned}$$

and so $\tau(\partial_{ij}) = -D_{i'j'}$, as claimed. Note that in this last calculation we have used the fact that $(U^{-1})^{\circ} = JU^{\top}J$, so that $(\tilde{u}_{ji})^{\circ} = u_{i'j'}$, the evaluation of $\mathring{\partial}_{ij}$ given in (4.3), and the definition of $D_{i'j'}$ given in (4.1). □

This ends the digression relating the operators ∂_{ij} and D_{ij}, so we now drop the assumption that $a = b$ and return to the general case. Since the operators D_{ij} commute with one another, by Lemma 4.2, it makes sense to define an operator $\Delta_m(D)$ by

$$\Delta_m(D) = \begin{vmatrix} D_{11} & \dots & D_{1m} \\ \vdots & & \vdots \\ D_{m1} & \dots & D_{mm} \end{vmatrix}$$

for $1 \le m \le n$.

THEOREM 4.5 (Dual b-Function Identity). *Let $1 \le m \le n$. Then we have*

$$\Delta_m(D)\bullet\big(\Delta_1^{s_1}\cdots\Delta_n^{s_n}\big) = c_m(s)\Delta_1^{s_1}\cdots\Delta_m^{s_m+1}\cdots\Delta_n^{s_n}\,,$$

where

$$c_m(s) = \prod_{j=1}^{m}(s + s_j + \cdots + s_n - j + 1)\,.$$

PROOF. As usual, it suffices to derive the identity under the restriction that $s_1, \dots, s_n$ are natural numbers and s is a negative integer satisfying $s \le -(s_1+\cdots+s_n)$. This is because the dependence of both sides on $s_1, \dots, s_n, s$ is polynomial and a polynomial that vanishes on the specified set must be identically zero. It also suffices to assume that $a = b = n$. This is because only the variables in the n-by-n upper-rightmost submatrix of U occur in the product $\Delta_1^{s_1}\cdots\Delta_n^{s_n}$. Thus if $1 \le i_1, j_1 \le n$ then $D_{i_1 j_1}\bullet\big(\Delta_1^{s_1}\cdots\Delta_n^{s_n}\big)$ contains only these variables. It follows that if $1 \le i_2, j_2 \le n$ then $D_{i_2 j_2}D_{i_1 j_1}\bullet\big(\Delta_1^{s_1}\cdots\Delta_n^{s_n}\big)$ contains only these variables, and so on. We conclude that the operators D_{ij} may be replaced by what they would be if $a = b = n$ without changing the value of the left-hand side of the proposed identity, as claimed.

To derive the identity under the assumptions identified in the previous paragraph, we apply the operator τ to both sides of the b-Function Identity given in Theorem 4.1. By Lemma 4.4 we have $\tau(\Delta_m^*) = (-1)^m\Delta_m(D)$ and from Lemma 4.3 we have

$$\begin{aligned}&\tau(\Delta_1^{s_1}\cdots\Delta_n^{s_n})\\ &= \Delta_n^{-s}(\Delta_{n-1}\Delta_n^{-1})^{s_1}\cdots(\Delta_1\Delta_n^{-1})^{s_{n-1}}\Delta_n^{-s_n}\\ &= \Delta_1^{s_{n-1}}\cdots\Delta_{n-1}^{s_1}\Delta_n^{-t}\,,\end{aligned}$$

where $t = s + s_1 + \cdots + s_n$. Thus the left-hand side of the b-Function Identity becomes

$$(-1)^m\Delta_m(D)\bullet\big(\Delta_1^{s_{n-1}}\cdots\Delta_{n-1}^{s_1}\Delta_n^{-t}\big)\,. \tag{4.4}$$

By a similar calculation, the right-hand side of the b-Function Identity becomes

$$s_n(s_n + s_{n-1} + 1)\cdots(s_n + \cdots + s_{m'} + m - 1)\Delta_1^{s_{n-1}}\cdots\Delta_m^{s_{n-m}+1}\cdots\Delta_{n-1}^{s_1}\Delta_n^{-t} \tag{4.5}$$

and so (4.4) and (4.5) are equal. In this equality, we replace s_j by s_{n-j} for $1 \le j \le n-1$, s_n by $-(s_1 + \cdots + s_n + s)$, and multiply both sides by $(-1)^m$. The result is that

$$\Delta_m(D)\bullet\big(\Delta_1^{s_1}\cdots\Delta_n^{s_n}\big) = c_m(s)\Delta_1^{s_1}\cdots\Delta_m^{s_m+1}\cdots\Delta_n^{s_n}\,,$$

where

$$\begin{aligned}c_m(s) &= (-1)^m\big(-(s_1 + \cdots + s_n + s)\big)\big(-(s_2 + \cdots + s_n + s - 1)\big)\\ &\qquad\cdots\big(-(s_m + \cdots + s_n + s - m + 1)\big)\\ &= (s + s_1 + \cdots + s_n)(s + s_2 + \cdots + s_n - 1)\cdots(s + s_m + \cdots + s_n - m + 1)\,,\end{aligned}$$

as required. □

5. A Korányi Theorem and Its Converse. Let $a, b, c \geq 1$ and define $m = \min\{a, b\}$ and $n = \min\{b, c\}$. The group $H = \mathrm{SL}(a) \times \mathrm{SL}(b) \times \mathrm{SL}(c)$ acts on the space $\mathrm{Mat}(a, b) \oplus \mathrm{Mat}(b, c)$ by $(g_1, g_2, g_3)(X, Y) = (g_1 X g_2^{-1}, g_2 Y g_3^{-1})$. As in Section 3 we write x_{ij} and y_{jk} with $1 \leq i \leq a$, $1 \leq j \leq b$, and $1 \leq k \leq c$ for the standard coordinates on $\mathrm{Mat}(a, b)$ and $\mathrm{Mat}(b, c)$, respectively. Our first task is to describe certain features of the action of H on $\mathbb{C}[x_{ij}, y_{jk}] \cong \mathbb{C}[x_{ij}] \otimes \mathbb{C}[y_{jk}]$. To this end, let

$$\Delta_i(x) = \begin{vmatrix} x_{11} & \dots & x_{1i} \\ \vdots & & \vdots \\ x_{i1} & \dots & x_{ii} \end{vmatrix}$$

for $1 \leq i \leq m$ and

$$\Delta_k(y) = \begin{vmatrix} y_{11} & \dots & y_{1k} \\ \vdots & & \vdots \\ y_{k1} & \dots & y_{kk} \end{vmatrix}$$

for $1 \leq k \leq n$. For $s = (s_1, \dots, s_m) \in \mathbb{N}^m$ let

$$\Delta(x)^s = \Delta_1(x)^{s_1} \cdots \Delta_m(x)^{s_m}$$

and for $t = (t_1, \dots, t_n) \in \mathbb{N}^n$ let

$$\Delta(y)^t = \Delta_1(y)^{t_1} \cdots \Delta_n(y)^{t_n} .$$

Also let

$$\Delta(x, y)^{(s,t)} = \Delta(x)^s \Delta(y)^t .$$

The structure of $\mathbb{C}[x_{ij}]$ as a representation of $\mathrm{SL}(a) \times \mathrm{SL}(b)$ may be deduced from the $\mathrm{GL}(a)$-$\mathrm{GL}(b)$ Duality Theorem [5, Section 2.1]. There is a decomposition

$$\mathbb{C}[x_{ij}] = \bigoplus_{s \in \mathbb{N}^m} \Gamma_x(s) ,$$

where $\Gamma_x(s) \cong U_x(s) \boxtimes V_x(s)$ is the outer tensor product of an irreducible representation of $\mathrm{GL}(a)$ having lowest weight $-(\varpi(s), 0_{a-m})$ and an irreducible representation of $\mathrm{GL}(b)$ having highest weight $(\varpi(s), 0_{b-m})$. Here we have written

$$\varpi(s) = (s_1 + \cdots + s_m, s_2 + \cdots + s_m, \dots, s_m)$$

and are using the standard choices of torus and positive system in $\mathrm{GL}(a)$ and in $\mathrm{GL}(b)$. The representations $U_x(s)$ and $V_x(s)$ remain irreducible on restriction to $\mathrm{SL}(a)$ and $\mathrm{SL}(b)$, respectively. The polynomial $\Delta(x)^s$ lies in $\Gamma_x(s)$ and is simultaneously a lowest weight vector for $\mathrm{SL}(a)$ and a highest weight vector for $\mathrm{SL}(b)$. Similarly, there is a decomposition

$$\mathbb{C}[y_{jk}] = \bigoplus_{t \in \mathbb{N}^n} \Gamma_y(t) ,$$

where $\Gamma_y(t) \cong V_y(t) \boxtimes W_y(t)$ is the outer tensor product of an irreducible representation of $\mathrm{GL}(b)$ having lowest weight $-(\varpi(t), 0_{b-n})$ and an irreducible representation of $\mathrm{GL}(c)$

having highest weight $(\varpi(t), 0_{c-n})$. The representations $V_y(t)$ and $W_y(t)$ remain irreducible on restriction to $\mathrm{SL}(b)$ and $\mathrm{SL}(c)$, respectively. The polynomial $\Delta(y)^t$ is simultaneously a lowest weight vector for $\mathrm{SL}(b)$ and a highest weight vector for $\mathrm{SL}(c)$.

By combining the decompositions described in the previous paragraph we obtain a decomposition

$$\mathbb{C}[x_{ij}, y_{jk}] = \bigoplus_{s\in\mathbb{N}^m,\, t\in\mathbb{N}^n} \Gamma_x(s) \otimes \Gamma_y(t)\,.$$

The representation

$$\Gamma_x(s) \otimes \Gamma_y(t) \cong U_x(s) \boxtimes \big(V_x(s) \otimes V_y(t)\big) \boxtimes W_y(t)$$

of H is generally reducible, since the inner tensor product $V_x(s) \otimes V_y(t)$ is generally so. However, $\Gamma_x(s) \otimes \Gamma_y(t)$ is a cyclic $\mathfrak{U}(\mathfrak{h})$-module and $\Delta(x, y)^{(s,t)}$ is a cyclic vector in this module. This follows from the observations that we have made above together with two well-known general facts. The first is that the outer tensor product of cyclic vectors is cyclic in the outer tensor product of their respective modules. The second is that if $\mathfrak{r}$ is a simple Lie algebra, a Cartan subalgebra and positive system are chosen for $\mathfrak{r}$, V_1 and V_2 are irreducible $\mathfrak{r}$-modules, v_1^+ is a highest weight vector in V_1, and v_2^- is a lowest weight vector in V_2 then $v_1^+ \otimes v_2^-$ is a $\mathfrak{U}(\mathfrak{r})$-cyclic vector in $V_1 \otimes V_2$. We apply the second fact to $\mathfrak{r} = \mathfrak{sl}(b)$ and the representations $V_x(s)$ and $V_y(t)$, and then the first fact to the representations $U_x(s)$, $V_x(s) \otimes V_y(t)$, and $W_y(t)$.

Recall that the initial model of $\mathcal{P}_w$ exists provided that $w \notin \sigma(F)$, where

$$\sigma(F) = \begin{cases} -(a-1) + \mathbb{N} & \text{if } b \geq a\,, \\ -b + \mathbb{N} & \text{if } b < a\,. \end{cases}$$

Similarly, the initial model of the space $\check{\mathcal{P}}_{\check{w}}$ of polynomial solutions to the conjugate system exists provided that $\check{w} \notin \sigma(\check{F})$, where $\check{w} = -w - b$ and

$$\sigma(\check{F}) = \begin{cases} -(c-1) + \mathbb{N} & \text{if } b \geq c\,, \\ -b + \mathbb{N} & \text{if } b < c\,. \end{cases}$$

The set $S = \{w \mid w \in \sigma(F) \text{ and } \check{w} \in \sigma(\check{F})\}$ is always a finite interval in $\mathbb{Z}$ and so, except for a finite number of w values, we may use either the initial model of $\mathcal{P}_w$ or the initial model of $\check{\mathcal{P}}_{\check{w}}$. As we noted at the end of Section 3, the lattice of submodules in $\mathcal{P}_w$ is isomorphic to the lattice of submodules in $\check{\mathcal{P}}_{\check{w}}$. It follows that we may use initial models to decide on the reducibility of $\mathcal{P}_w$ unless $w \in S$. This is done in the next two results.

THEOREM 5.1. *Suppose that either*

(1) $w \notin \mathbb{Z}$ *or*
(2) $w \in \mathbb{Z} \setminus \sigma(F)$ *and* $w > n - 1 - b$ *or*
(3) $\check{w} \in \mathbb{Z} \setminus \sigma(\check{F})$ *and* $\check{w} > m - 1 - b$.

Then $\mathcal{P}_w$ *is an irreducible* $\mathfrak{U}(\mathfrak{g})$*-module.*

PROOF. We may assume that either Condition (1) or Condition (2) holds. If, instead, Condition (3) holds then we may apply the following argument to the conjugate system. It follows from Proposition 2.1 that it is sufficient to show that 1 is a cyclic vector for $\mathcal{P}_w$ as a $\mathcal{U}(\mathfrak{g})$-module. Since $w \notin \sigma(F)$, the initial model $\tilde{\mathcal{P}}_w$ is available and so it suffices to show that 1 is a cyclic vector for $\tilde{\mathcal{P}}_w$ as a $\mathcal{U}(\mathfrak{g})$-module. It then suffices to show that $\Delta(x, y)^{(s,t)}$ lies in the $\mathcal{U}(\mathfrak{g})$-submodule of $\tilde{\mathcal{P}}_w$ generated by 1 for all $s \in \mathbb{N}^m$ and $t \in \mathbb{N}^n$. If this is so then, by the discussion above, $\Gamma_x(s) \otimes \Gamma_y(t)$ is contained in the $\mathcal{U}(\mathfrak{g})$-submodule of $\tilde{\mathcal{P}}_w$ generated by 1 for all $s \in \mathbb{N}^m$ and $t \in \mathbb{N}^n$, and so this submodule is equal to $\tilde{\mathcal{P}}_w$, as required.

The operators $\tilde{\Pi}_w(\bar{Y}_{jk})$ commute with one another and so we may define

$$\Delta_q(\bar{Y}) = \begin{vmatrix} \tilde{\Pi}_w(\bar{Y}_{11}) & \dots & \tilde{\Pi}_w(\bar{Y}_{1q}) \\ \vdots & & \vdots \\ \tilde{\Pi}_w(\bar{Y}_{q1}) & \dots & \tilde{\Pi}_w(\bar{Y}_{qq}) \end{vmatrix}$$

for $1 \leq q \leq n$. Inspection of the formula for $\tilde{\Pi}_w(\bar{Y}_{jk})$ given in Lemma 3.1 shows that this operator is the sum of an operator of the same form as the operator D_{jk} introduced in Section 4 (with $w + b$ in place of s and y_{jk} in place of u_{jk}) and an operator that is identically zero on $\mathbb{C}[y_{jk}]$. It follows from this observation, Theorem 4.5, and an induction argument that

$$\Delta(\bar{Y})^t \bullet 1 = \prod_{j=1}^{n} \prod_{e=1}^{t_j+\cdots+t_n} (w + b + e - j) \cdot \Delta(y)^t .$$

Similarly, we may define

$$\Delta_q(\bar{X}) = \begin{vmatrix} \tilde{\Pi}_w(\bar{X}_{11}) & \dots & \tilde{\Pi}_w(\bar{X}_{1q}) \\ \vdots & & \vdots \\ \tilde{\Pi}_w(\bar{X}_{q1}) & \dots & \tilde{\Pi}_w(\bar{X}_{qq}) \end{vmatrix}$$

for $1 \leq q \leq m$. Inspection of the formula for $\tilde{\Pi}_w(\bar{X}_{ij})$ given in Lemma 3.1 shows that this operator is of the same form as the operator D_{ij} introduced in Section 4 (with $-w$ in place of s and x_{ij} in place of u_{ij}). Moreover, this operator commutes with multiplication by any element of $\mathbb{C}[y_{jk}]$ since it only involves derivatives with respect to the x_{ij}. It follows from these observations, Theorem 4.5, and an induction argument that

(5.1)

$$\begin{aligned} \Delta(\bar{X})^s \Delta(\bar{Y})^t \bullet 1 &= \prod_{j=1}^{n} \prod_{e=1}^{t_j+\cdots+t_n} (w + b + e - j) \cdot \Delta(\bar{X})^s \bullet \Delta(y)^t \\ &= \prod_{j=1}^{n} \prod_{e=1}^{t_j+\cdots+t_n} (w + b + e - j) \cdot \Delta(y)^t \Delta(\bar{X})^t \bullet 1 \\ &= \prod_{j=1}^{m} \prod_{e=1}^{s_j+\cdots+s_m} (-w + e - j) \cdot \prod_{j=1}^{n} \prod_{e=1}^{t_j+\cdots+t_n} (w + b + e - j) \cdot \Delta(x, y)^{(s,t)} . \end{aligned}$$

If Condition (1) holds then the factor preceding $\Delta(x, y)^{(s,t)}$ in (5.1) is evidently non-zero. Thus the $\mathcal{U}(\mathfrak{g})$-submodule of $\tilde{\mathcal{P}}_w$ generated by 1 contains $\Delta(x, y)^{(s,t)}$ for all $s \in \mathbb{N}^m$ and $t \in \mathbb{N}^n$, as required. Now suppose that Condition (2) holds. By assumption, $w \in \mathbb{Z} \setminus \sigma(F)$. If $b \geq a$ then this implies that $w < -(a-1) = -(m-1)$ and so $-w \geq m$. If, on the other hand, $b < a$ then it implies that $w < -b = -m$ and so $-w > m$. In either case we conclude that $-w \geq m$. The smallest factor in the product

$$\prod_{j=1}^{m} \prod_{e=1}^{s_j+\cdots+s_m} (-w + e - j) \tag{5.2}$$

is $-w + 1 - m$ and it follows from what we just observed that this is at least 1. Thus the product (5.2) is non-zero. The smallest factor in the product

$$\prod_{j=1}^{n} \prod_{e=1}^{t_j+\cdots+t_n} (w + b + e - j) \tag{5.3}$$

is $w + b + 1 - n$. We have assumed that $w > n - 1 - b$ and so $w + b + 1 - n \geq 1$. Thus the product (5.3) is non-zero. It follows that the factor preceding $\Delta(x, y)^{(s,t)}$ in (5.1) is non-zero. As above, this allows us to complete the proof. □

Note that the proof of Theorem 5.1 includes an independent proof that $\tilde{\mathcal{P}}_w = \mathbb{C}[\mathfrak{g}(1,1) \oplus \mathfrak{g}(1,-1)]$ for certain values of w. This proof is simpler than the earlier proof [9, Theorem 6.5] but cannot be extended to cover the remaining values of w that are covered by the earlier proof.

THEOREM 5.2. *Suppose that $w \in \mathbb{Z}$ and that either*

(1) *$w \notin \sigma(F)$ and $w \leq n - 1 - b$ or*

(2) *$\check{w} \notin \sigma(\check{F})$ and $\check{w} \leq m - 1 - b$.*

Then $\mathcal{P}_w$ is a reducible $\mathcal{U}(\mathfrak{g})$-module.

PROOF. We may assume without loss of generality that Condition (1) holds, since otherwise the following argument may be applied to the conjugate system instead. Let $\mathcal{N}$ be the $\mathcal{U}(\mathfrak{g})$-submodule of $\tilde{\mathcal{P}}_w = \mathbb{C}[x_{ij}, y_{jk}]$ generated by 1. It suffices to show that $\mathcal{N}$ is a proper submodule of $\tilde{\mathcal{P}}_w$. Define res : $\tilde{\mathcal{P}}_w \to \mathbb{C}[y_{jk}]$ by $\text{res}(\varphi)(y_{jk}) = \varphi(0, y_{jk})$. We have $\text{res}(\tilde{\mathcal{P}}_w) = \mathbb{C}[y_{jk}]$ and so it suffices to show that $\text{res}(\mathcal{N}) \neq \mathbb{C}[y_{jk}]$.

Since $\tilde{\Pi}_w(\mathfrak{n})\bullet 1 = \{0\}$ and $\tilde{\Pi}_w(\mathfrak{l})\bullet 1 = \mathbb{C} \cdot 1$, it follows from the PBW Theorem that we have $\mathcal{N} = \tilde{\Pi}_w(\mathcal{U}(\bar{\mathfrak{n}}))\bullet 1$. By inspection of the expressions given in Lemma 3.1 for $\tilde{\Pi}_w(\bar{X}_{ij})$ and for $\tilde{\Pi}_w(\bar{Y}_{jk})$ we observe that application of $\tilde{\Pi}_w(\bar{X}_{ij})$ raises the x-degree of a polynomial by 1, whereas application of $\tilde{\Pi}_w(\bar{Y}_{jk})$ leaves the x-degree unchanged. It follows that application of the operator $\tilde{\Pi}_w([\bar{X}_{ij}, \bar{Y}_{kl}])$ also raises the x-degree by 1. We conclude from this that

$$\text{res}\big(\tilde{\Pi}_w\big(\mathfrak{g}(-1,-1)\mathcal{U}(\bar{\mathfrak{n}}) + \mathfrak{g}(-2,0)\mathcal{U}(\bar{\mathfrak{n}})\big)\bullet 1\big) = \{0\}$$

and, since $\mathcal{U}(\bar{\mathfrak{n}}) = \mathcal{U}(\mathfrak{g}(-1,1)) + \mathfrak{g}(-1,-1)\mathcal{U}(\bar{\mathfrak{n}}) + \mathfrak{g}(-2,0)\mathcal{U}(\bar{\mathfrak{n}})$, it follows that

$$\text{res}(\mathcal{N}) = \text{res}\big(\tilde{\Pi}_w\big(\mathcal{U}(\mathfrak{g}(-1,1))\big)\bullet 1\big) = \text{res}\big(\mathbb{C}[\tilde{\Pi}_w(\bar{Y}_{jk})]\bullet 1\big) = \mathbb{C}[D_{jk}]\bullet 1\,,$$

where

$$D_{jk} = (w+b)y_{jk} + \sum_{\substack{1\le p\le b\\ 1\le q\le c}} y_{jq}y_{pk}\frac{\partial}{\partial y_{pq}}\,.$$

Thus we are reduced to showing that $\mathbb{C}[D_{jk}]\bullet 1 \neq \mathbb{C}[y_{jk}]$.

Let $r = \max\{1, w+b+1\}$ and $l = \max\{1, 1-w-b\}$. By definition, $r \geq 1$ and $l \geq 1$. The hypothesis that $w \leq n-1-b$ implies that $r \leq n$. By Theorem 4.5 we have

$$\Delta_r(D)^l\bullet 1 = \prod_{p=1}^{l}\prod_{j=1}^{r}(w+b+p-j)\cdot \Delta_r(y)^l\,. \tag{5.4}$$

The smallest numerical factor that appears in (5.4) is $w+b+1-r$ and from the definition of r we have $w+b+1-r \leq 0$. The largest numerical factor that appears in (5.4) is $w+b+l-1$ and from the definition of l we have $w+b+l-1 \geq 0$. Every integer between $w+b+1-r$ and $w+b+l-1$ also occurs in the product in (5.4) and it follows that $\Delta_r(D)^l\bullet 1 = 0$. The operators D_{jk} mutually commute and it follows from this that we have $\mathbb{C}[D_{jk}]\bullet 1 \subset \ker\big(\Delta_r(D)^l\big)$. We are thus reduced to showing that $\ker\big(\Delta_r(D)^l\big) \neq \mathbb{C}[y_{jk}]$. Let t be any integer such that $t > r-1-w-b$. By Theorem 4.5 we have

$$\Delta_r(D)^l\bullet\Delta_r(y)^t = \prod_{p=1}^{l}\prod_{j=1}^{r}(w+b+t+p-j)\cdot\Delta_r(y)^{t+l}\,. \tag{5.5}$$

The smallest numerical factor that appears in (5.5) is $w+b+t+1-r$ and, by the choice of t, this number is positive. Thus $\Delta_r(y)^t \notin \ker\big(\Delta_r(D)^l\big)$ and this completes the proof. □

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DEPARTMENT OF MATHEMATICS
OKLAHOMA STATE UNIVERSITY
STILLWATER OK 74078
USA

E-mail address: akable@math.okstate.edu