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PSEUDO-HERMITIAN MANIFOLDS WITH AUTOMORPHISM GROUP OF MAXIMAL DIMENSION

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Abstract. This paper concerns a local characterization of 5-dimensional pseudo-Hermitian manifolds with maximal automorphism group in the case the underlying almost CR structures are not integrable. We also present examples of globally homogeneous model of maximal dimensional automorphism group.

1. Introduction. One of classical and fundamental problems in differential geometry is to characterize homogeneous models in terms of their automorphism groups. This characterization is tightly related with local equivalence problems for the given geometric structure. The local equivalence problem between Levi non-degenerate CR manifolds was first established by E. Cartan in dimension 3 (cf. [3]). His idea on the absolute parallelism was developed further by N. Tanaka (cf. [11]) and Chern-Moser (cf. [4]). Tanaka's theory on the graded Lie algebra has led the development of the canonical connection theory for the parabolic geometry. A. Čap and H. Schichl proved in [2] that if an almost CR structure is partially integrable, then the homogeneous model is the standard Heisenberg group and hence one can construct a canonical Cartan connection by the connection theory for the parabolic geometry.

On the other hand, if we do not assume the partial integrability, then we can construct another homogeneous model of strongly pseudoconvex almost CR manifolds. K.-H. Lee and the author introduced in [7] a family of homogeneous models with almost CR structures that are not partially integrable. (They call the homogeneous models *the generalized Heisenberg groups*. See also Section 3 for the construction.) They also developed Webster's pseudo-Hermitian theory ([13]) in almost CR cases to characterize the generalized Heisenberg groups by the non-proper action of the automorphism group in low dimensional cases.

The aim of this paper is to characterize the local structure of the generalized Heisenberg groups by the local action of the automorphism group. The main result is Theorem 3.4 on the local characterization of 5-dimensional pseudo-Hermitian manifolds with maximal automorphism group whose underlying almost CR structures are not integrable.

In Section 2, we briefly discuss about almost CR structure and pseudo-Hermitian structure equations. In Section 3, we define a family of homogeneous strongly pseudoconvex

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almost CR manifolds, the generalized Heisenberg groups. After reviewing the basic properties for the generalized Heisenberg groups, we present Theorem 3.4, the main theorem of this paper for the local characterization of homogeneous models. We also introduce another homogeneous models in Theorem 3.5 which do not satisfy the hypothesis of Theorem 3.4. The proof of Theorem 3.4 is based on the method of prolongation of local automorphisms to frame bundles. In Section 4, we discuss about the prolongation argument that is required in the proof, and we will complete the proof of Theorem 3.4 in Section 5.

Notation and Convention. Throughout this paper, the summation convention for duplicated indices is always assumed. We denote the complex conjugate of a tensor by taking the bar on the indices, that is, $\overline{Z_\alpha} = Z_{\bar{\alpha}}$, $\overline{W^{\alpha}_{\bar{\beta}}} = W^{\bar{\alpha}}_{\beta}$ and so on. We will also use the matrix of the Levi form $(g_{\alpha\bar{\beta}})$ and its inverse matrix $(g^{\bar{\beta}\alpha})$ to raise and lower indices: $\theta_\alpha = g_{\alpha\bar{\beta}}\theta^{\bar{\beta}}$, $R_{\beta}{}^{\alpha}{}_{\mu\bar{\nu}} = g^{\bar{\gamma}\alpha}R_{\beta\bar{\gamma}\mu\bar{\nu}}$. In this paper, we consider frame bundles of a base manifold. We distinguish tensors on the base manifold from those on the frame bundle by adding $'$.

2. Preliminaries: Basic definitions for almost CR and pseudo-Hermitian structures.

2.1. Almost CR manifolds.

Let M be a $(2n+1)$-dimensional manifold for a positive integer n. An *almost CR structure* on M is by definition, a sub-bundle $H = \bigcup_{p\in M} H_p \subset TM$ of fiber dimension $2n$ with a smooth family of endomorphisms $J = \{J_p : H_p \to H_p\}$ such that $J_p \circ J_p = -I_p$. Here I_p denotes the identity transform of H_p for every $p \in M$. We call the triple (M, H, J) an *almost CR manifold*. We may abbreviate (M, H, J) by (M, J) or simply by M, if there is no danger of confusion. We denote by $H_{1,0}$ and $H_{0,1}$ sub-bundles of $\mathbb{C} \otimes H$ whose fibers are eigenspaces for eigenvalues i and $-i$ of J, respectively. Both $H_{1,0}$ and $H_{0,1}$ are complex sub-bundles of $\mathbb{C} \otimes H$ of fiber dimension n with

$$H_{0,1} = \overline{H_{1,0}} \quad \text{and} \quad \mathbb{C} \otimes H = H_{1,0} \oplus H_{0,1}\,.$$

A CR structure (H, J) is said to be *integrable* if the sub-bundle $H_{0,1}$ is involutive when it is regarded as a smooth distribution. That is, (H, J) is integrable if for every pair of local sections $\overline{Z'}$ and $\overline{W'}$ of $H_{0,1}$, their Lie bracket $[\overline{Z'}, \overline{W'}]$ is also a local section of $H_{0,1}$.

The *Levi form* of an almost CR manifold (M, H, J) is a Hermitian form $L : H_{1,0} \times H_{0,1} \to \mathbb{C} \otimes TM/\mathbb{C} \otimes H$ defined by

$$L(v, \bar{w}) \equiv i[V', \overline{W'}] \mod \mathbb{C} \otimes H$$

for $v, w \in H_{1,0}$ at a point $p \in M$, where V' and W' are local sections of $H_{1,0}$ such that $V'(p) = v$ and $W'(p) = w$. Suppose that M is oriented. Since H has a natural orientation given by J, orientation of M yields an orientation on TM/H. Choose a 1-form θ such that $\theta|_H \equiv 0$ and is compatible with the orientation of TM/H. If we regard θ as a 1-form on $\mathbb{C} \otimes TM$, it can be also regarded as a form on $\mathbb{C} \otimes TM/\mathbb{C} \otimes H$. Let

$$L_\theta(v, \bar{w}) := \theta(L(v, \bar{w}))\,.$$

The almost CR manifold M is said to be *strongly pseudoconvex* if the Hermitian form L_θ is positive definite for a 1-form compatible with the orientation of M. L_θ is called the Levi form of M with respect to θ.

Suppose that an almost CR manifold M is strongly pseudoconvex. Let θ be a fixed 1-form satisfying that $\theta|_H \equiv 0$ and L_θ is positive definite. Then it turns out that M is a contact manifold with the contact distribution H and that θ is a contact 1-form. The quadruple (M, H, J, θ) is called a *pseudo-Hermitian manifold*. We will abbreviate it by (M, θ), if there is no danger of confusion.

By a *CR automorphism* of M, we mean a diffeomorphism $\varphi : M \to M$ preserving the almost CR structure. That is a diffeomorphism φ is a CR automorphism if and only if $\varphi_* H_{1,0} = H_{1,0}$. We denote by $\mathrm{Aut}(M)$ the group of all CR automorphisms of M. If (M, θ) is a pseudo-Hermitian manifold, we denote by $\mathrm{Aut}(M, \theta)$ the group of pseudo-Hermitian automorphisms, that is,

$$\mathrm{Aut}(M, \theta) = \{\varphi \in \mathrm{Aut}(M) : \varphi^*\theta = \theta\} .$$

For a given point $p \in M$, we denote by $\mathrm{Aut}^p(M)$ and $\mathrm{Aut}^p(M, \theta)$ the isotropy subgroups of $\mathrm{Aut}(M)$ and $\mathrm{Aut}(M, \theta)$ at p, respectively.

2.2. Pseudo-Hermitian structure equations. We recall the construction of the pseudo-Hermitian structure equations and the canonical connection. For some details, one may refer [7]. Let (M, θ) be a pseudo-Hermitian manifold of dimension $(2n+1)$. We call the vector field X' on M determined uniquely by

$$\theta(X') = 1 \quad \text{and} \quad \iota_{X'} d\theta = 0$$

the *characteristic vector field* for (M, θ). A *moving frame* is a set $\{X'_1, \ldots, X'_n\}$ of local sections of $H_{1,0}$ such that $X', X'_1, \ldots, X'_n, X'_{\bar 1}, \ldots, X'_{\bar n}$ form a basis for $\mathbb{C} \otimes TM$ at each point. We call a set of 1-forms $\{\theta'^\alpha : \alpha = 1, \ldots, n\}$ a *coframe* of $\{X'_\alpha\}$ if

$$\theta'^\alpha(X'_\beta) = \delta^\alpha_\beta, \quad \theta'^\alpha(X'_{\bar\beta}) = 0 .$$

A coframe $\{\theta'^\alpha\}$ is said to be *admissible* to θ if it satisfies in addition that $\theta'^\alpha(X') = 0$ for every $\alpha = 1, \ldots, n$. An admissible coframe is determined uniquely by the choice of moving frame, since $\{\theta'^\alpha\}$ is admissible to θ if and only if $\theta, \theta'^\alpha, \theta'^{\bar\alpha}$ are dual to $X', X'_\alpha, X'_{\bar\alpha}$.

Let $\{\theta'^\alpha\}$ be an admissible coframe of (M, θ). Since θ is real, there exist complex valued functions $g'_{\alpha\bar\beta}, h'_{\alpha\beta}$ such that

$$d\theta = i g'_{\alpha\bar\beta} \theta'^\alpha \wedge \theta'^{\bar\beta} + h'_{\alpha\beta} \theta'^\alpha \wedge \theta'^\beta + h'_{\bar\alpha\bar\beta} \theta'^{\bar\alpha} \wedge \theta'^{\bar\beta} , \tag{2.1}$$

$$g'_{\alpha\bar\beta} = g'_{\bar\beta\alpha} \quad \text{and} \quad h'_{\alpha\beta} = -h'_{\beta\alpha} .$$

The matrix $(g'_{\alpha\bar\beta})$ determines the Levi form L_θ and hence it is positive definite.

PROPOSITION 2.1 ([7]). *Let* $\{\theta'^{\alpha}\}$ *be an admissible coframe for* θ. *Then there are functions* ${T'}_{\beta}{}^{\alpha}{}_{\gamma}$, ${N'}_{\bar\beta}{}^{\alpha}{}_{\bar\gamma}$, ${A'}^{\alpha}{}_{\bar\beta}$, ${B'}^{\alpha}{}_{\beta}$ *and* 1*-form* $({\omega'}_{\beta}{}^{\alpha})$ *uniquely determined by the following equations:*

$$d\theta'^{\alpha} = \theta'^{\beta}\wedge{\omega'}_{\beta}{}^{\alpha} + {T'}_{\beta}{}^{\alpha}{}_{\gamma}\theta'^{\beta}\wedge\theta'^{\gamma} + {N'}_{\bar\beta}{}^{\alpha}{}_{\bar\gamma}\theta'^{\bar\beta}\wedge\theta'^{\bar\gamma} + {A'}^{\alpha}{}_{\bar\beta}\theta\wedge\theta'^{\bar\beta} + {B'}^{\alpha}{}_{\beta}\theta\wedge\theta'^{\beta}\,, \tag{2.2}$$

$$dg'_{\alpha\bar\beta} - \omega'_{\alpha\bar\beta} - \omega'_{\bar\beta\alpha} = 0\,, \tag{2.3}$$

$${T'}_{\beta}{}^{\alpha}{}_{\gamma} = -{T'}_{\gamma}{}^{\alpha}{}_{\beta}\,,\quad {N'}_{\bar\beta}{}^{\alpha}{}_{\bar\gamma} = -{N'}_{\bar\gamma}{}^{\alpha}{}_{\bar\beta}\,, \tag{2.4}$$

and

$$B'_{\alpha\bar\beta} = B'_{\bar\beta\alpha}\,. \tag{2.5}$$

Here, we lower indices by $(g'_{\alpha\bar\beta})$, *that is,*

$$\omega'_{\alpha\bar\beta} = {\omega'}_{\alpha}{}^{\gamma}g'_{\gamma\bar\beta},\quad B'_{\bar\gamma\beta} = {B'}^{\alpha}{}_{\beta}g'_{\alpha\bar\gamma}\,.$$

We call the 1-form $\omega' = ({\omega'}_{\beta}{}^{\alpha})$ the *canonical connection form* and the functions ${T'}_{\beta}{}^{\alpha}{}_{\gamma}$, ${N'}_{\bar\beta}{}^{\alpha}{}_{\bar\gamma}$, ${A'}^{\alpha}{}_{\bar\beta}$, ${B'}^{\alpha}{}_{\beta}$ and $h'_{\alpha\beta}$ the *coefficients of torsion tensors*. The equations (2.1) and (2.2) are called the *structure equations* for (M,θ) with respect to the coframe $\{\theta'^{\alpha}\}$.

REMARK 2.2. Comparing with the structure equations in integrable case (see [13]), we have more torsion tensors $h'_{\alpha\beta}$, ${T'}_{\beta}{}^{\alpha}{}_{\gamma}$, ${N'}_{\bar\beta}{}^{\alpha}{}_{\bar\gamma}$ and ${B'}^{\alpha}{}_{\beta}$. Although these tensors arise from the non-integrability of the almost CR structure, they are not mutually independent. Non-integrability tensor of the almost CR structure is given by

$$[X'_{\bar\alpha}, X'_{\bar\beta}] \mod \Gamma(H_{0,1}) \equiv -h'_{\bar\alpha\bar\beta}X' - {N'}_{\bar\alpha}{}^{\gamma}{}_{\bar\beta}X'_{\gamma}$$

by Cartan's lemma and the structure equations (2.1) and (2.2). That is, ${N'}_{\bar\beta}{}^{\alpha}{}_{\bar\gamma}$ measures the part of non-integrability tensor lying on the complex direction and $h'_{\bar\alpha\bar\beta}$ measures that escaping to the real direction. The relations between torsion tensors can be obtained from the Bianchi identity (2.19) below.

DEFINITION 2.3. We say the almost CR structure of M is *integrable* at $p\in M$ if ${N'}_{\bar\beta}{}^{\alpha}{}_{\bar\gamma}(p) = h'_{\bar\alpha\bar\beta}(p) = 0$ for every $\alpha,\beta,\gamma = 1,\dots,n$. We say that the almost CR structure is *partially integrable* at $p\in M$ if $h'_{\bar\alpha\bar\beta}(p) = 0$ for every $\alpha,\beta = 1,\dots,n$. One should notice that the definitions for the integrability and the partial integrability do not depend on the choice of contact 1-form θ.

Let $\{\theta^{\alpha}\}$ be another admissible coframe for (M,θ). Then there exists a local $GL(n,\mathbb{C})$-valued function $U = ({U_{\beta}}^{\alpha})$ such that

$$\theta^{\alpha} = \theta'^{\beta}{U_{\beta}}^{\alpha}\,. \tag{2.6}$$

Let $g_{\alpha\bar{\beta}}$ be the coefficients of the Levi form, let $({\omega_\beta}^\alpha)$ be the canonical connection form and let $h_{\alpha\beta}$, ${T_\beta}^\alpha{}_\gamma$, ${N_{\bar\beta}}^\alpha{}_{\bar\gamma}$, $A^\alpha{}_{\bar\beta}$, $B^\alpha{}_\beta$ be the coefficients of torsion tensors with respect to the coframe $\{\theta^\alpha\}$. Then it turns out that

$$g_{\alpha\bar{\beta}} = {(U^{-1})_\alpha}^\gamma {(U^{-1})_{\bar\beta}}^{\bar\sigma} g'_{\gamma\bar\sigma}\,, \quad h_{\alpha\beta} = {(U^{-1})_\alpha}^\gamma {(U^{-1})_\beta}^\sigma h'_{\gamma\sigma}\,, \tag{2.7}$$

$${\omega_\beta}^\alpha = {(U^{-1})_\beta}^\gamma {\omega'_\gamma}^\sigma {U_\sigma}^\alpha - {(U^{-1})_\beta}^\gamma d{U_\gamma}^\alpha\,, \tag{2.8}$$

$${T_\beta}^\alpha{}_\gamma = {(U^{-1})_\beta}^\mu {(U^{-1})_\gamma}^\nu {T'_\mu}^\sigma{}_\nu {U_\sigma}^\alpha\,, \quad {N_{\bar\beta}}^\alpha{}_{\bar\gamma} = {(U^{-1})_{\bar\beta}}^{\bar\mu} {(U^{-1})_{\bar\gamma}}^{\bar\nu} {N'_{\bar\mu}}^\sigma{}_{\bar\nu} {U_\sigma}^\alpha\,, \tag{2.9}$$

$$A^\alpha{}_{\bar\beta} = {(U^{-1})_{\bar\beta}}^{\bar\sigma} A'^\gamma{}_{\bar\sigma} {U_\gamma}^\alpha\,, \quad B^\alpha{}_\beta = {(U^{-1})_\beta}^\sigma B'^\gamma{}_\sigma {U_\gamma}^\alpha\,. \tag{2.10}$$

Let $\pi : \mathcal{F} \to M$ be the bundle of coframes admissible to θ over M. Then $\mathcal{F}$ is a principal fibre bundle with structure group $GL(n, \mathbb{C})$. Exploiting ${U_\beta}^\alpha$ as a vertical coordinates of $\mathcal{F}$, we may assume that θ^α, $({\omega_\beta}^\alpha)$, $g_{\alpha\bar\beta}$, $h_{\alpha\beta}$, ${T_\beta}^\alpha{}_\gamma$, ${N_{\bar\beta}}^\alpha{}_{\bar\gamma}$, $A^\alpha{}_{\bar\beta}$, $B^\alpha{}_\beta$ are defined on $\mathcal{F}$ from the equations (2.6)–(2.10). Moreover, if we denote $\pi^*\theta$ by θ, then from (2.1)–(2.5), we see that

$$d\theta = i g_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta} + h_{\alpha\beta}\theta^\alpha \wedge \theta^\beta + h_{\bar\alpha\bar\beta}\theta^{\bar\alpha} \wedge \theta^{\bar\beta}\,, \tag{2.11}$$

$$d\theta^\alpha = \theta^\beta \wedge {\omega_\beta}^\alpha + {T_\beta}^\alpha{}_\gamma \theta^\beta \wedge \theta^\gamma + {N_{\bar\beta}}^\alpha{}_{\bar\gamma}\theta^{\bar\beta} \wedge \theta^{\bar\gamma} + A^\alpha{}_{\bar\beta}\theta \wedge \theta^{\bar\beta} + B^\alpha{}_\beta\theta \wedge \theta^\beta\,, \tag{2.12}$$

$$dg_{\alpha\bar\beta} - \omega_{\alpha\bar\beta} - \omega_{\bar\beta\alpha} = 0\,, \tag{2.13}$$

$$g_{\alpha\bar\beta} = g_{\bar\beta\alpha}, \quad h_{\alpha\beta} = -h_{\beta\alpha}\,, \quad {T_\beta}^\alpha{}_\gamma = -{T_\gamma}^\alpha{}_\beta\,, \quad {N_{\bar\beta}}^\alpha{}_{\bar\gamma} = -{N_{\bar\gamma}}^\alpha{}_{\bar\beta}, \tag{2.14}$$

and

$$B_{\alpha\bar\beta} = B_{\bar\beta\alpha} \tag{2.15}$$

on $\mathcal{F}$. Notice that we lower indices by $g_{\alpha\bar\beta}$ on $\mathcal{F}$, that is, $\omega_{\alpha\bar\beta} = {\omega_\alpha}^\gamma g_{\gamma\bar\beta}$ and so on. We also call (2.11) and (2.12) the structure equations for (M, θ).

Let

$$\Omega'_\beta{}^\alpha = d\omega'_\beta{}^\alpha - \omega'_\beta{}^\gamma \wedge \omega'_\gamma{}^\alpha$$

on M and let

$${\Omega_\beta}^\alpha = d{\omega_\beta}^\alpha - {\omega_\beta}^\gamma \wedge {\omega_\gamma}^\alpha \tag{2.16}$$

on $\mathcal{F}$. By a straightforward computation, it turns out that

$${\Omega_\beta}^\alpha = (U^{-1})^\gamma_\beta\, \Omega'_\gamma{}^\sigma\, U^\alpha_\sigma. \tag{2.17}$$

To simplify the notation, we introduce the following forms on $\mathcal{F}$:

$$\begin{aligned}\Theta &= d\theta - i g_{\alpha\bar\beta}\,\theta^\alpha \wedge \theta^{\bar\beta} = d\theta - i\theta^\alpha \wedge \theta_\alpha \\ &= h_{\alpha\beta}\,\theta^\alpha \wedge \theta^\beta + h_{\bar\alpha\bar\beta}\,\theta^{\bar\alpha} \wedge \theta^{\bar\beta}, \\ \Theta^\alpha &= d\theta^\alpha - \theta^\beta \wedge {\omega_\beta}^\alpha\end{aligned}$$

$$= T_{\beta}{}^{\alpha}{}_{\gamma}\theta^{\beta} \wedge \theta^{\gamma} + N_{\bar{\beta}}{}^{\alpha}{}_{\bar{\gamma}}\theta^{\bar{\beta}} \wedge \theta^{\bar{\gamma}} + A^{\alpha}{}_{\bar{\beta}}\theta \wedge \theta^{\bar{\beta}} + B^{\alpha}{}_{\beta}\theta \wedge \theta^{\beta} .$$

We define matrix-valued forms ω and Ω by

$$\omega = \begin{pmatrix} 0 & \theta^{\alpha} & \theta \\ 0 & \omega_{\beta}{}^{\alpha} & i\theta_{\beta} \\ 0 & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & \Theta^{\alpha} & \Theta \\ 0 & \Omega_{\beta}{}^{\alpha} & i\Theta_{\beta} \\ 0 & 0 & 0 \end{pmatrix}.$$

Then the equations (2.11), (2.12) and (2.16) are components of

$$d\omega - \omega \wedge \omega = \Omega . \tag{2.18}$$

Differentiating (2.18), we obtain the *Bianchi identity*:

$$d\Omega - \omega \wedge \Omega + \Omega \wedge \omega = 0 . \tag{2.19}$$

A differential form φ on $\mathcal{F}$ is said to be *semi-basic* if $\iota_V \varphi = 0$ for every vertical vector V on $\mathcal{F}$. By (2.17), we see that $\Omega_{\beta}{}^{\alpha}$ is semi-basic. Therefore,

$$\Omega_{\alpha\bar{\beta}} = R_{\alpha\bar{\beta}\mu\bar{\nu}}\theta^{\mu} \wedge \theta^{\bar{\nu}} + P_{\alpha\bar{\beta}\mu\nu}\theta^{\mu} \wedge \theta^{\nu} + Q_{\alpha\bar{\beta}\bar{\mu}\bar{\nu}}\theta^{\bar{\mu}} \wedge \theta^{\bar{\nu}} + S_{\alpha\bar{\beta}\mu}\theta \wedge \theta^{\mu} + W_{\alpha\bar{\beta}\bar{\mu}}\theta \wedge \theta^{\bar{\mu}}$$

for some functions $R_{\alpha\bar{\beta}\mu\bar{\nu}}$, $P_{\alpha\bar{\beta}\mu\nu}$, $Q_{\alpha\bar{\beta}\bar{\mu}\bar{\nu}}$ and $W_{\alpha\bar{\beta}\bar{\mu}}$ on $\mathcal{F}$, where $P_{\alpha\bar{\beta}\mu\nu} = -P_{\alpha\bar{\beta}\nu\mu}$ and $Q_{\alpha\bar{\beta}\bar{\mu}\bar{\nu}} = -Q_{\alpha\bar{\beta}\bar{\nu}\bar{\mu}}$. We call these functions the *coefficients of the curvature tensor*. Differentiating (2.13), we see that

$$\Omega_{\alpha\bar{\beta}} + \Omega_{\bar{\beta}\alpha} = 0 .$$

Equivalently,

$$R_{\alpha\bar{\beta}\mu\bar{\nu}} = R_{\bar{\beta}\alpha\bar{\nu}\mu}\,, \quad Q_{\alpha\bar{\beta}\bar{\mu}\bar{\nu}} = -P_{\bar{\beta}\alpha\bar{\mu}\bar{\nu}} \quad \text{and} \quad W_{\alpha\bar{\beta}\bar{\mu}} = -S_{\bar{\beta}\alpha\bar{\mu}}\,. \tag{2.20}$$

REMARK 2.4. Let

$$R_{\mu\bar{\nu}} = R_{\alpha}{}^{\alpha}{}_{\mu\bar{\nu}}\,, \quad S_{\mu\bar{\nu}} = R_{\mu}{}^{\alpha}{}_{\alpha\bar{\nu}}$$

and let

$$R = R_{\mu\bar{\nu}}g^{\mu\bar{\nu}}\,, \quad S = S_{\mu\bar{\nu}}g^{\mu\bar{\nu}}$$

where $(g^{\mu\bar{\nu}}) = (g_{\mu\bar{\nu}})^{-1}$. We call $R_{\mu\bar{\nu}}$ and $S_{\mu\bar{\nu}}$ the coefficients of the *Ricci curvatures of the first kind and the second kind*, respectively. We also call R and S the *scalar curvatures of the first and the second kind*, respectively. Note that the curvatures of the first and the second kind coincide with each other if the almost CR structure is integrable, since in integrable case, we have an additional symmetry relation for the curvature tensor that

$$R_{\alpha\bar{\beta}\mu\bar{\nu}} = R_{\mu\bar{\beta}\alpha\bar{\nu}}\,.$$

See [13] for details.

3. Generalized Heisenberg groups and the main theorem. We first recall the definition and fundamental properties of a family of homogeneous models with non-integrable almost CR structure which was introduced in [7]. We denote by $(t, z) = (t, z^1, \dots, z^n)$ the standard coordinates of $\mathbb{R} \times \mathbb{C}^n$. ($n \geq 2$.) For any $P = (P_{\alpha\beta}) \in \mathfrak{so}(n, \mathbb{C})$, the space of $n \times n$ skew-symmetric complex matrices, we endow $\mathbb{R} \times \mathbb{C}^n$ with a Lie group structure $* = *_P$ defined by

$$(t, z) * (t', z') = (t + t' + 2\mathrm{Im}\left\langle z, z' \right\rangle - 2\mathrm{Re}\, P(z, z'), z + z')$$

where $\left\langle z, z' \right\rangle$ is the usual Hermitian inner product of $\mathbb{C}^n$ and $P(z, z')$ is the skew-symmetric bilinear operator defined by $P = (P_{\alpha\beta})$:

$$\left\langle z, z' \right\rangle = \delta_{\alpha\bar{\beta}} z^{\alpha} z'^{\bar{\beta}} \ , \quad P(z, z') = P_{\alpha\beta} z^{\alpha} z'^{\beta} \ .$$

We call $\mathbf{H}_P = (\mathbb{R} \times \mathbb{C}^n, *_P)$ the *generalized Heisenberg group* associated to P. Let

$$X'_{\alpha} = \frac{\partial}{\partial z^{\alpha}} + \left(i z^{\bar{\alpha}} + P_{\alpha\beta} z^{\beta} \right) \frac{\partial}{\partial t} \, .$$

Let $J = J_P$ be the almost CR structure defined by $H_{1,0}$ bundle spanned by $\{X'_{\alpha} : \alpha = 1, \dots, n\}$. Then it turns out that:

(i) the vector field X'_{α} is left invariant with respect to the group operation $*_P$ for every $\alpha = 1, \dots, n$ and the almost CR structure J_P is strongly convex,
(ii) the almost CR structure J_P is not integrable unless $(P_{\alpha\beta}) = 0$, while $\mathbf{H}_0$ is the classic Heisenberg group,
(iii) if

$$\theta_P = dt + 2\mathrm{Re} \left(i z^{\alpha} dz^{\bar{\alpha}} + P_{\alpha\beta} z^{\alpha} dz^{\beta} \right) ,$$

then θ_P is a contact form for the distribution $H = \mathrm{Re}\, H_{1,0}$ and is also left invariant.

From (i), (ii) and (iii), we see that $(\mathbf{H}_P, \theta_P)$ is a homogeneous pseudo-Hermtian manifold for every $P \in \mathfrak{so}(n, \mathbb{C})$.

Taking exterior differentiation for θ_P, w have

$$d\theta_P = 2i \delta_{\alpha\bar{\beta}} dz^{\alpha} \wedge dz^{\bar{\beta}} + P_{\alpha\beta} dz^{\alpha} \wedge dz^{\beta} + P_{\bar{\alpha}\bar{\beta}} dz^{\bar{\alpha}} \wedge dz^{\bar{\beta}} \, .$$

Therefore, $\{\theta'^{\alpha} = \sqrt{2} dz^{\alpha}\}$ forms an admissible unitary coframe for the pseudo-Hermitian manifold $(\mathbf{H}_P, \theta_P)$. Since $d\theta'^{\alpha} = ddz^{\alpha} = 0$, we see that all torsion coefficients except $h'_{\alpha\beta} (= P_{\alpha\beta})$ vanish identically, and that the connection 1-form and hence the curvature tensor also vanish.

In fact, every $\mathbf{H}_P$ has the map Λ_{ε} defined by $\Lambda_{\varepsilon}(t, z) := (\varepsilon^2 t, \varepsilon z)$ as its CR automorphism, for every $\varepsilon > 0$. In particular, the CR automorphism group of $\mathbf{H}_P$ acts on $\mathbf{H}_P$ non-properly. In [7], we have characterized $\mathbf{H}_P$ with this property in low dimensional cases, as a generalization of Schoen's theorem [10].

THEOREM 3.1 ([7]). *Let M be a strongly pseudoconvex almost CR manifold of dimension $2n+1=5$ or 7. Suppose that the CR automorphism group of M acts on M non-properly. Then M is CR equivalent with $\mathbf{H}_P$ for some $P \in \mathfrak{so}(n,\mathbb{C})$ in case M is noncompact. If M is compact, then the almost CR structure of M is integrable and M is CR equivalent with the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$.*

Although every $\mathbf{H}_P$ has Λ_ε as its CR automorphism, the group of CR automorphisms of $\mathbf{H}_P$ for nonzero P is quite different with that of $\mathbf{H}_0$. It is very well-known (cf. [11]) that the Lie algebra of CR automorphism group of $\mathbf{H}_0$ has a graded Lie algebra structure

$$\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

where, $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is the subalgebra generating the translation which is isomorphic to the Lie algebra of $\mathbf{H}_0$, $\mathfrak{g}_0$ consists of vector fields generating Λ_ε and the pseudo-Hermitian isotropy group of $(\mathbf{H}_0, \theta_0)$ which is isomorphic to $U(n)$, and $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ generates CR isotropic automorphisms determined by the second order jet at the origin. ($\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is the Lie algebra of $\mathrm{Aut}^0_{CR}(\mathbf{H}_0)$.) On the other hand, if $P \neq 0$, then the group of CR automorphism fixing the origin does not contain any CR automorphism determined by the second order jet. In fact, it has been shown in [7] that the CR isotropy subgroup of $\mathbf{H}_P$ is generated by Λ_ε and the pseudo-Hermitian isotropy subgroup of $(\mathbf{H}_P, \theta_P)$ which is given by

$$\mathrm{Aut}^0(\mathbf{H}_P, \theta_P) = \{U \in U(n) : U^t\, P\, U = P\}\,. \tag{3.1}$$

This difference between CR automorphism groups of $\mathbf{H}_0$ and $\mathbf{H}_P$ is in fact a consequence of the difference between pseudo-Hermitian automorphism groups of $(\mathbf{H}_0, \theta_0)$ and $(\mathbf{H}_P, \theta_P)$. To see this difference locally, we introduce a notion of local infinitesimal automorphisms.

DEFINITION 3.2. A local vector field V' on a pseudo-Hermitian manifold (M, H, J, θ) is called an *infinitesimal pseudo-Hermitian automorphism* if it generates a 1-parameter family of local pseudo-Hermitian automorphisms, that is, V' is an infinitesimal pseudo-Hermitian automorphism if and only if

$$\mathcal{L}_{V'}\theta = 0\,, \quad \mathcal{L}_{V'}J = 0$$

($\mathcal{L}$ denotes the Lie derivative). For $p \in M$, we denote by $\mathfrak{aut}_p(M,\theta)$ the Lie algebra of germs of infinitesimal pseudo-Hermitian automorphisms at p and by $\mathrm{Aut}^0_p(M,\theta)$ the subalgebra of germs of infinitesimal pseudo-Hermitian automorphisms vanishing at p.

Note that $\mathcal{L}_{V'}J = 0$ if and only if $\mathcal{L}_{V'}\theta'^\alpha \equiv 0 \mod \{\theta'^\beta\}$ for an admissible coframe $\{\theta'^\alpha\}$.

In 1969, S. Tanno proved that only integrable models can admit maximal dimensional space of infinitesimal pseudo-Hermitian automorphisms.

THEOREM 3.3 ([12]). *Let (M,θ) be a $(2n+1)$-dimensional pseudo-Hermitian manifold. Then $\dim \mathfrak{aut}_p(M,\theta) \le (n+1)^2$. Moreover, if $\dim \mathfrak{aut}_p(M,\theta) = (n+1)^2$, then all torsion coefficients and all coefficients of curvature tensors except $R_{\alpha\bar\beta\mu\bar\nu}$ vanish on $\pi^{-1}(p)$,* *and*

$$R_{\alpha\bar{\beta}\mu\bar{\nu}} = \frac{R}{n(n+1)}(g_{\alpha\bar{\beta}}g_{\mu\bar{\nu}} + g_{\alpha\bar{\nu}}g_{\mu\bar{\beta}})$$

on $\pi^{-1}(p)$ for some real constant R. (R is the scalar curvature.) In other words, the germ of (M, θ) is equivalent to one of the integrable homogeneous models of constant curvature.

Therefore, if the underlying almost CR structure of a $(2n+1)$-dimensional pseudo-Hermtian manifold (M, θ) is not integrable, then $\dim \mathfrak{aut}_p(M, \theta) < (n+1)^2$. However, even for the generalized Heisenberg group $(\mathbf{H}_P, \theta_P)$, the dimension of pseudo-Hermitian automorphism group does not depend only on the dimension of $\mathbf{H}_P$ but on the shape of P from (3.1). This ambiguity disappears in case $2n+1=5$, since in this case, P is determined by a single complex number P_{12}. Note that if $2n+1=5$ and $P \neq 0$, then $\mathrm{Aut}(\mathbf{H}_P, \theta_P)$ is 8-dimensional and is generated by

$$\mathrm{Aut}^0(\mathbf{H}_P, \theta_P) = \{U \in \mathrm{U}(2) : U^t\, P\, U = P\} = \mathrm{SU}(2)$$

and translations by the group operation of $\mathbf{H}_P$. With this observation, we state the main theorem of this paper.

THEOREM 3.4 (Main Theorem). *Let (M, θ) be a 5-dimensional pseudo-Hermitian manifold with almost CR structure which is not integrable at p. Assume that* $\dim \mathfrak{aut}_p(M, \theta) = 8$. *Then*

(i) *the almost CR structure of M is not partially integrable at p, and*
(ii) $\mathfrak{aut}^0_p(M, \theta) \cong \mathfrak{su}(2)$ *as Lie algebras.*

If we assume more that the characteristic vector field X' is an element of $\mathfrak{aut}_p(M, \theta)$, then

(iii) *all the coefficients of torsion tensors but $(h_{\alpha\beta})$ vanish on $\pi^{-1}(p)$, and*
(iv) *all the coefficients of curvature tensors but $R_{\alpha\bar{\beta}\mu\bar{\nu}}$ vanish on $\pi^{-1}(p)$. Moreover,*

$$R_{\alpha\bar{\beta}\mu\bar{\nu}} = \frac{R}{6}(g_{\alpha\bar{\beta}}g_{\mu\bar{\nu}} + g_{\alpha\bar{\nu}}g_{\mu\bar{\beta}})$$

on $\pi^{-1}(p)$ for some real constant R.

Here, $\mathfrak{su}(2)$ stands for the Lie algebra of $SU(2)$, which is the algebra of trace-free 2×2 skew-Hermitian matrices. Note that $\mathbf{H}_P$ is the flat model of Theorem 3.4.

We are focusing on 5-dimensional case, since this is the most fundamental case of non-integrable model. But we are also interested in 5-dimensional case, since we can find another model structure of 8-dimensional automorphism group. In case when the characteristic vector field belongs to $\mathfrak{aut}_p(M, \theta)$, we call it the *characteristic automorphism*. In the proof of (iii) and (iv) in Theorem 3.4, the existence of characteristic automorphism is crucial. Then one may ask if there exists a homogeneous model of pseudo-Hermitian manifold with 8-dimensional automorphism group which does not contain the characteristic automorphism. The following theorem is an answer to the question.

THEOREM 3.5 (Spherical SU(3)-model). *There exists a homogeneous pseudo-Hermitian structure on the* 5*-dimensional sphere* S^5 *with* 8*-dimensional automorphism group, which does not admit the characteristic automorphism.*

PROOF. The Cayley number system on $\mathbb{R}^8$ induces a cross product structure $\times$ on $\mathbb{R}^7$ and an almost complex structure $\widetilde{J}$ on the six sphere $S^6 \subset \mathbb{R}^7$. The almost complex structure $\widetilde{J}$ on S^6 is defined by

$$\widetilde{J}_x(Y) = x \times Y$$

for every $x \in S^6$ and $Y \in T_x S^6$. See for instance [1, 6] for the fundamental geometric properties of $(S^6, \widetilde{J})$. An expression of $\widetilde{J}$ in terms of the Euclidean coordinates of $\mathbb{R}^7$ is given as follows:

$$\widetilde{J}_x = \begin{pmatrix} 0 & -x^4 & -x^7 & x^2 & -x^6 & x^5 & x^3 \\ x^4 & 0 & -x^5 & -x^1 & x^3 & -x^7 & x^6 \\ x^7 & x^5 & 0 & -x^6 & -x^2 & x^4 & -x^1 \\ -x^2 & x^1 & x^6 & 0 & -x^7 & -x^3 & x^5 \\ x^6 & -x^3 & x^2 & x^7 & 0 & -x^1 & -x^4 \\ -x^5 & x^7 & -x^4 & x^3 & x^1 & 0 & -x^2 \\ -x^3 & -x^6 & x^1 & -x^5 & x^4 & x^2 & 0 \end{pmatrix}$$

where $x = (x^1, \dots, x^7) \in S^6$. For $0 \le r < 1$, let

$$S_r = \{x = (x^1, \dots, x^7) \in S^6 : x^7 = r\}.$$

It is known that the holomorphic automorphism group of $(S^6, \widetilde{J})$ is isomorphic to the 14-dimensional exceptional Lie subgroup G_2 of SO(7) and the isotropy subgroup at $x_0 = (0, \dots, 0, 1)$ is $\mathrm{SU}(3) = G_2 \cap \mathrm{SO}(6)$. Note that this holomorphic isotropy subgroup acts on S_r transitively for every r. Therefore, S_r is a homogeneous almost CR manifold for every $0 \le r < 1$. Lehmann and Feldmueller ([8]) proved that a homogeneous integrable CR structure on the sphere of dimension ≥ 5 is the standard CR structure induced from the complex structure of $\mathbb{C}^n$ and hence it is strongly pseudoconvex. On the other hand, the 5-sphere S_0 contains S^2 as a Riemann surface. Therefore, it turns out that the almost CR structures S_0 and hence S_r are not integrable for sufficiently small r, since the non-integrability is stable under a small perturbation of the structure.

We first prove that in fact, the almost CR structure of S_r is non-integrable and strongly pseudoconvex for every $0 < r < 1$ by an explicit computation.

Let $x = (x^1, \dots, x^6, r) \in S_r$ and let $s = \sqrt{\sum_{i=1}^6 (x^i)^2} = \sqrt{1 - r^2}$. Let

$$Y = \left(\frac{r}{s}x^1, \dots, \frac{r}{s}x^6, -s\right)^t \in T_x S^6$$

be the unit tangent vector perpendicular to S_r and let

$$X' = \widetilde{J}_x Y = s^{-1}(-x^3, -x^6, x^1, -x^5, x^4, x^2, 0)^t\ .$$

We denote by J the induced almost CR structure on S_r from $\widetilde{J}$. Let

$$X_1 = (v^1, x^1, v^3, 0, 0, 0, 0)^t \quad \text{and} \quad X_2 = (u^1, 0, u^3, 0, x^1, 0, 0)^t$$

where

$$v^1 = -\frac{(x^1)^2x^2 + x^1x^3x^6}{(x^1)^2 + (x^3)^2}, \quad v^3 = \frac{(x^1)^2x^6 - x^1x^2x^3}{(x^1)^2 + (x^3)^2}$$

and

$$u^1 = \frac{x^1x^3x^4 - (x^1)^2x^5}{(x^1)^2 + (x^3)^2}, \quad u^3 = -\frac{x^1x^3x^5 + (x^1)^2x^4}{(x^1)^2 + (x^3)^2}.$$

Then X_1 and X_2 are in T_xS_r that are perpendicular to X'. Therefore,

$$Z_1' = \frac{1}{2}(X_1 - iJX_1) \quad \text{and} \quad Z_2' = \frac{1}{2}(X_2 - iJX_2)$$

form a local frame for (S_r, J). Let

$$\theta := \langle X', \cdot \rangle$$

on S_r, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product of $\mathbb{R}^7$. Then $\theta(Z_\alpha') = \theta(Z_{\bar{\alpha}}') = 0$, since the Euclidean inner product is invariant under J-transformation. Let $x_R = (s, 0, 0, 0, 0, 0, r) \in S_r$ be a reference point. By a straightforward computation, we see that

$$\begin{aligned}
[X_1, JX_1] = [X_2, JX_2] &= -2sr\frac{\partial}{\partial x^3}, \\
[X_1, X_2] = [JX_1, JX_2] &= 0, \\
[X_1, JX_2] = [JX_1, X_2] &= -2s^2\frac{\partial}{\partial x^3}
\end{aligned}$$

at x_R. Therefore,

$$[Z_1', Z_{\bar{1}}'] = -isr\frac{\partial}{\partial x^3} = [Z_2', Z_{\bar{2}}']$$

and $[Z_1', Z_{\bar{2}}'] = [Z_2', Z_{\bar{1}}'] = 0$ at x_R. Since $X' = \partial/\partial x^3$ at x_R, this yields that S_r is strongly pseudoconvex at x_R if $r > 0$ and the Levi form $g_{\alpha\bar{\beta}}'$ for θ is

$$g_{1\bar{1}}' = g_{2\bar{2}}' = sr, \quad \text{and} \quad g_{1\bar{2}}' = 0.$$

Note that S_r and θ is invariant under the action of $\mathrm{SU}(3) = G_2 \cap \mathrm{SO}(6)$, since $\widetilde{J}$ is invariant under G_2 action. Therefore, we conclude that (S_r, θ) is a pseudo-Hermitian manifold which is homogeneous by the 8-dimensional Lie group SU(3) action for every $0 < r < 1$. Moreover,

$$\begin{aligned}
[Z_1', Z_2'] &= \frac{1}{4}[X_1 - iJX_1, X_2 - iJX_2] \\
&= \frac{1}{4}([X_1, X_2] - [JX_1, JX_2]) - \frac{i}{4}([X_1, JX_2] + [JX_1, X_2]) \\
&= is^2\frac{\partial}{\partial x^3}
\end{aligned}$$

at x_R. Therefore,

$$h'_{12}(x_R) = -\theta[Z'_1, Z'_2] = -is^2\,,$$

and this implies that the almost CR structure J is not partially integrable. Since $\dim \mathfrak{aut}_{x_R}(S_r, \theta) < 9$ by Theorem 3.3, we see that

$$\mathfrak{aut}_{x_R}(S_r, \theta) \simeq \mathfrak{su}(3)\,.$$

To see that the characteristic vector field X' is not an infinitesimal automorphism, it suffices to show that $({A'^{\alpha}}_{\bar\beta}) \neq 0$ by Theorem 3.4. Note that at x_R, X_1, JX_1, X_2, JX_2 form an orthogonal set of norm s with respect to the Euclidean inner product of $\mathbb{R}^7$. Therefore, if we denote the admissible coframe of Z'_1, Z'_2 for θ by θ'^1, θ'^2, then

$$\theta'^{\alpha} = \frac{2}{s^2}\langle Z'_{\bar\alpha}, \cdot\rangle$$

at x_R. From the structure equation,

$${A'^{\alpha}}_{\bar\beta} = d\theta'^{\alpha}(X, Z'_{\bar\beta}) = -\theta'^{\alpha}[X, Z'_{\bar\beta}] = -\frac{2}{s^2}\langle Z'_{\bar\alpha}, [X, Z'_{\bar\beta}]\rangle$$

at x_R for every $\alpha, \beta = 1, 2$. Then by a straightforward computation, we can see that

$$({A'^{\alpha}}_{\bar\beta}) = \frac{3i}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

at x_R. This yields the conclusion that the pseudo-Hermitian manifold (S_r, θ) does not admit local automorphisms generated by the characteristic vector field for every $0 < r < 1$. □

4. Prolongation of infinitesimal automorphisms.

4.1. Canonical lift. Let V' is a vector field on M. A vector field V on $\mathcal{F}$ is called a *lift* of V' if

$$\pi_*(V) = V'\,.$$

PROPOSITION 4.1. *Let V' be an infinitesimal automorphism. Then there exists a lift V uniquely determined by*

$$\mathcal{L}_V\theta = 0, \quad \mathcal{L}_V\theta^{\alpha} = 0\,. \tag{4.1}$$

PROOF. Let $\{X', X'_\alpha, X'_{\bar\alpha}\}$ be the dual frame of $\{\theta, \theta'^{\alpha}, \theta'^{\bar\alpha}\}$ on M and let $\{X, X_\alpha, X_{\bar\alpha}, {E_\alpha}^{\beta}, {E_{\bar\alpha}}^{\bar\beta}\}$ be the dual frame for $\{\theta, \theta^\alpha, \theta^{\bar\alpha}, {\omega_\beta}^{\alpha}, {\omega_{\bar\beta}}^{\bar\alpha}\}$ on $\mathcal{F}$. Let

$$V' = v'X' + v'^{\alpha}X'_\alpha + v'^{\bar\alpha}X'_{\bar\alpha}$$

and let

$$V = vX + v^\alpha X_\alpha + v^{\bar\alpha}X_{\bar\alpha} + {v_\beta}^{\alpha}{E_\alpha}^{\beta} + {v_{\bar\beta}}^{\bar\alpha}{E_{\bar\alpha}}^{\bar\beta}\,.$$

From the fact that $d\pi(V) = V'$, we see that

$$v = v', \quad \text{and} \quad v^\alpha = v'^{\beta}{U_\beta}^{\alpha}\,. \tag{4.2}$$

Therefore, it can be immediately seen that $\mathcal{L}_V\theta = 0$ on $\mathcal{F}$ if and only if $\mathcal{L}_{V'}\theta = 0$ on M.

Moreover, $\mathcal{L}_V\theta^\alpha = 0$ if and only if

$$E_\gamma{}^\beta(v^\alpha) + v^\beta\delta^\alpha_\gamma = 0\,, \quad E_{\bar\gamma}{}^{\bar\beta}(v^\alpha) = 0\,, \tag{4.3}$$

$$X(v^\alpha) - A^\alpha{}_{\bar\beta}v^{\bar\beta} - B^\alpha{}_\beta v^\beta = 0\,, \tag{4.4}$$

$$X_{\bar\beta}(v^\alpha) - 2N_{\bar\beta}{}^\alpha{}_{\bar\gamma}v^{\bar\gamma} + A^\alpha{}_{\bar\beta}v = 0\,, \tag{4.5}$$

and

$$v_\beta{}^\alpha = X_\beta(v^\alpha) - 2T_\beta{}^\alpha{}_\gamma v^\gamma + B^\alpha{}_\beta v\,. \tag{4.6}$$

The equation (4.3) follows immediately from (4.2) and the fact that

$$E_\beta{}^\alpha = -U_\gamma{}^\alpha\frac{\partial}{\partial U_\gamma{}^\beta}\,.$$

Equations (4.4) and (4.5) are the consequences of $\mathcal{L}_{V'}\theta'^\alpha \equiv 0 \mod \{\theta'^\beta\}$ on M. Therefore, the lift V satisfying (4.1) is uniquely determined by (4.2) and (4.6). □

We call the lift V of an infinitesimal automorphism V' determined by (4.1) the *canonical lift* of V'.

COROLLARY 4.2. *Let V be the canonical lift of an infinitesimal automorphism V' of (M,θ). Then*

$$\mathcal{L}_V\omega = 0 \tag{4.7}$$

on $\mathcal{F}$.

PROOF. Let φ_t be the 1-parameter family generated by V. From (4.1), we see

$$\varphi_t^*\theta = \theta\,, \quad \varphi_t^*\theta^\alpha = \theta^\alpha\,. \tag{4.8}$$

Differentiating (4.8),

$$\begin{aligned} d\theta &= i(\varphi_t^*g_{\alpha\bar\beta})\theta^\alpha\wedge\theta^{\bar\beta} + (\varphi_t^*h_{\alpha\beta})\theta^\alpha\wedge\theta^\beta + (\varphi_t^*h_{\bar\alpha\bar\beta})\theta^{\bar\alpha}\wedge\theta^{\bar\beta}\,,\\ d\theta^\alpha &= \theta^\beta\wedge\phi_t^*\omega_\beta{}^\alpha + (\varphi_t^*T_\beta{}^\alpha{}_\gamma)\theta^\beta\wedge\theta^\gamma + (\varphi_t^*N_{\bar\beta}{}^\alpha{}_{\bar\gamma})\theta^{\bar\beta}\wedge\theta^{\bar\gamma}\\ &\quad + (\varphi_t^*A^\alpha{}_{\bar\beta})\theta\wedge\theta^{\bar\beta} + (\varphi_t^*B^\alpha{}_\beta)\theta\wedge\theta^\beta\,. \end{aligned}$$

Since the coefficients of the Levi form and the connection 1-form are determined uniquely by the structure equations, we have

$$\varphi_t^*g_{\alpha\bar\beta} = g_{\alpha\bar\beta}\,, \qquad \varphi_t^*\omega_\beta{}^\alpha = \omega_\beta{}^\alpha\,. \tag{4.9}$$

Therefore, $\varphi_t^*\theta_\alpha = \varphi_t^*(g_{\bar\beta\alpha}\theta^{\bar\beta}) = \theta_\alpha$, and hence $\varphi_t^*\omega = \omega$. Differentiating this equation, we achieve the conclusion. □

Let

$$F_V = \iota_V \omega = \begin{pmatrix} 0 & v^\alpha & v \\ 0 & v_\beta{}^\alpha & i v_\beta \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$0 = d(\iota_V \omega) + \iota_V(\omega \wedge \omega + \Omega) = dF_V + [F_V, \omega] + \iota_V \Omega \tag{4.10}$$

where the bracket $[\cdot, \cdot]$ denotes the commutator between matrices.

Since (4.10) is a complete system of PDE's for V in the sense of [5], the germ of an infinitesimal automorphism V' at $p \in M$ is completely determined by the value of V, the canonical lift of V' at a point $u \in \pi^{-1}(p)$. In fact, the map

$$\mathfrak{aut}_p(M, \theta) \ni [V'] \to V(u) \in T_u \mathcal{F}$$

is a linear injective homomorphism.

4.2. Fundamental properties for canonical lifts of infinitesimal automorphisms. We call an admissible coframe $\{\theta'^\alpha\}$ *unitary* if the corresponding coefficients of the Levi form is $(\delta_{\alpha\bar\beta})$. We denote by $\widetilde{\mathcal{F}}$ the bundle of admissible unitary coframes. Then $\pi : \widetilde{\mathcal{F}} \to M$ is the $U(n)$-reduction of $\pi : \mathcal{F} \to M$. Then the structure equation (2.18) and the Bianchi identity (2.19) are still valid for $\widetilde{\mathcal{F}}$. Moreover, the canonical lift V of an infinitesimal automorphism V' to $\widetilde{\mathcal{F}}$ is also uniquely determined by (4.1) or (4.7) which are equivalent to (4.10). From now on, we make use of this $U(n)$-reduction for simplicity. Then we always have

$$g_{\alpha\bar\beta} = \delta_{\alpha\bar\beta}$$

and

$$\omega_\beta{}^\alpha = \omega_{\beta\bar\alpha} = -\omega_{\bar\alpha\beta} = -\omega_{\bar\alpha}{}^{\bar\beta}.$$

Moreover, if we denote by $X, X_\alpha, X_{\bar\alpha}, E_\alpha{}^\beta$ the dual vectors of $\theta, \theta^\alpha, \theta^{\bar\alpha}, \omega_\beta{}^\alpha$ on $\widetilde{\mathcal{F}}$, then obviously, $E_\alpha{}^\beta = -E_{\bar\beta}{}^{\bar\alpha}$ and a tangent vector $V \in T_u\widetilde{\mathcal{F}}$ has a unique expression $V = vX + v^\alpha X_\alpha + v^{\bar\alpha} X_{\bar\alpha} + v_\beta{}^\alpha E_\alpha{}^\beta$ for some $v \in \mathbb{R}$, $v^\alpha \in \mathbb{C}$ and $(v_\beta{}^\alpha) \in \mathfrak{u}(n)$.

PROPOSITION 4.3. *Let V be the canonical lift of an infinitesimal automorphism V' to $\widetilde{\mathcal{F}}$. Then*

$$V h_{\alpha\beta} = V T_\beta{}^\alpha{}_\gamma = V N_{\bar\beta}{}^\alpha{}_{\bar\gamma} = V A^\alpha{}_{\bar\beta} = V B^\alpha{}_\beta = 0,$$

and

$$V R_\beta{}^\alpha{}_{\mu\bar\nu} = V P_\beta{}^\alpha{}_{\mu\nu} = V S_\beta{}^\alpha{}_\mu = 0.$$

PROOF. Since $\mathcal{L}_V\omega = 0$,

$$\mathcal{L}_V\Omega = \mathcal{L}_V(\omega\wedge\omega + \Omega) = \mathcal{L}_V d\omega = d\mathcal{L}_V\omega = 0\,.$$

Equivalently, we have

$$\mathcal{L}_V\Theta = 0\,, \quad \mathcal{L}_V\Theta^\alpha = 0\,, \quad \text{and} \quad \mathcal{L}_V\Omega_\beta{}^\alpha = 0\,.$$

Therefore, $V\,h_{\alpha\beta} = 0$ since

$$0 = \mathcal{L}_V\Theta = \mathcal{L}_V(h_{\alpha\beta}\theta^\alpha\wedge\theta^\beta + h_{\bar\alpha\bar\beta}\theta^{\bar\alpha}\wedge\theta^{\bar\beta}) = (V\,h_{\alpha\beta})\theta^\alpha\wedge\theta^\beta + (V\,h_{\bar\alpha\bar\beta})\theta^{\bar\alpha}\wedge\theta^{\bar\beta}\,.$$

(Recall that $\mathcal{L}_V\theta^\alpha = 0$.) Other identities follow from $\mathcal{L}_V\Theta^\alpha = 0$ and $\mathcal{L}_V\Omega_\beta{}^\alpha = 0$. □

We introduce an equivalence relation $\sim$: Let Φ and Ψ be differential forms on $\widetilde{\mathcal{F}}$. We write $\Phi\sim\Psi$ if $\Phi-\Psi$ is semi-basic, that is, the interior product of $\Phi-\Psi$ with $E_\beta{}^\alpha$ vanishes identically for every $\alpha, \beta = 1,\dots,n$.

Let us consider the component

$$d\Theta = i\theta^\alpha\wedge\Theta_\alpha - i\Theta^\alpha\wedge\theta_\alpha \tag{4.11}$$

of the Bianchi identity (2.19). Since the right hand side is semi-basic, we have

$$\begin{aligned}0\sim d\Theta &= d\big(h_{\alpha\beta}\theta^\alpha\wedge\theta^\beta + h_{\bar\alpha\bar\beta}\theta^{\bar\alpha}\wedge\theta^{\bar\beta}\big)\\ &\sim \big(\nabla h_{\alpha\beta}\big)\wedge\theta^\alpha\wedge\theta^\beta + \Big(\nabla h_{\bar\alpha\bar\beta}\Big)\wedge\theta^{\bar\alpha}\wedge\theta^{\bar\beta}\end{aligned}$$

where

$$\nabla h_{\alpha\beta} = dh_{\alpha\beta} - \omega_\alpha{}^\gamma h_{\gamma\beta} - \omega_\beta{}^\gamma h_{\alpha\gamma}$$

the covariant derivative of $h_{\alpha\beta}$. Therefore, we see that $\nabla h_{\alpha\beta}$ is semi-basic. Let

$$\nabla h_{\alpha\beta} = h_{\alpha\beta;0}\theta + h_{\alpha\beta;\gamma}\theta^\gamma + h_{\alpha\beta;\bar\gamma}\theta^{\bar\gamma}$$

and let

$$V = vX + v^\gamma X_\gamma + v^{\bar\gamma}X_{\bar\gamma} + v_\beta{}^\alpha E_\alpha{}^\beta$$

be the canonical lift of an infinitesimal automorphism V' to $\widetilde{\mathcal{F}}$ where $v\in\mathbb{R}$, $v^\gamma\in\mathbb{C}$ and $v_\beta{}^\alpha = v_{\beta\bar\alpha} = -v_{\bar\alpha\beta} = -v_{\bar\alpha}{}^{\bar\beta}$. Since $Vh_{\alpha\beta} = 0$,

$$\begin{aligned}0 = Vh_{\alpha\beta} &= \big(\nabla h_{\alpha\beta} + \omega_\alpha{}^\gamma h_{\gamma\beta} + \omega_\beta{}^\gamma h_{\alpha\gamma}\big)(V)\\ &= (v\,h_{\alpha\beta;0} + v^\gamma h_{\alpha\beta;\gamma} + v^{\bar\gamma}h_{\alpha\beta;\bar\gamma}) + v_\alpha{}^\gamma h_{\gamma\beta} + v_\beta{}^\gamma h_{\alpha\gamma}\,.\end{aligned} \tag{4.12}$$

From (2.19), we also have

$$d\Theta^\alpha - \theta^\beta\wedge\Omega_\beta{}^\alpha + \Theta^\beta\wedge\omega_\beta{}^\alpha = 0\,. \tag{4.13}$$

Since $\Omega_\beta{}^\alpha$ is semi-basic,

$$0\sim d\Theta^\alpha + \Theta^\beta\wedge\omega_\beta{}^\alpha$$

$$\sim \left(\nabla T_{\beta}{}^{\alpha}{}_{\gamma}\right) \wedge \theta^{\beta} \wedge \theta^{\gamma} + \left(\nabla N_{\bar{\beta}}{}^{\alpha}{}_{\bar{\gamma}}\right) \wedge \theta^{\bar{\beta}} \wedge \theta^{\bar{\gamma}} + \left(\nabla A^{\alpha}{}_{\bar{\beta}}\right) \wedge \theta \wedge \theta^{\bar{\beta}} + \left(\nabla B^{\alpha}{}_{\beta}\right) \wedge \theta \wedge \theta^{\beta}$$

where

$$\begin{aligned}
\nabla T_{\beta}{}^{\alpha}{}_{\gamma} &= dT_{\beta}{}^{\alpha}{}_{\gamma} - \omega_{\beta}{}^{\sigma} T_{\sigma}{}^{\alpha}{}_{\gamma} + \omega_{\sigma}{}^{\alpha} T_{\beta}{}^{\sigma}{}_{\gamma} - \omega_{\gamma}{}^{\sigma} T_{\beta}{}^{\alpha}{}_{\sigma}\,, \\
\nabla N_{\bar{\beta}}{}^{\alpha}{}_{\bar{\gamma}} &= dN_{\bar{\beta}}{}^{\alpha}{}_{\bar{\gamma}} - \omega_{\bar{\beta}}{}^{\bar{\sigma}} N_{\bar{\sigma}}{}^{\alpha}{}_{\bar{\gamma}} + \omega_{\sigma}{}^{\alpha} N_{\bar{\beta}}{}^{\sigma}{}_{\bar{\gamma}} - \omega_{\bar{\gamma}}{}^{\bar{\sigma}} N_{\bar{\beta}}{}^{\alpha}{}_{\bar{\sigma}}\,, \\
\nabla A^{\alpha}{}_{\bar{\beta}} &= dA^{\alpha}{}_{\bar{\beta}} + \omega_{\sigma}{}^{\alpha} A^{\sigma}{}_{\bar{\beta}} - \omega_{\bar{\beta}}{}^{\bar{\sigma}} A^{\alpha}{}_{\bar{\sigma}}\,, \\
\nabla B^{\alpha}{}_{\beta} &= dB^{\alpha}{}_{\beta} + \omega_{\sigma}{}^{\alpha} B^{\sigma}{}_{\beta} - \omega_{\beta}{}^{\sigma} B^{\alpha}{}_{\sigma}\,,
\end{aligned}$$

the covariant derivatives of torsion coefficients. Therefore, $\nabla T_{\beta}{}^{\alpha}{}_{\gamma}$, $\nabla N_{\bar{\beta}}{}^{\alpha}{}_{\bar{\gamma}}$, $\nabla A^{\alpha}{}_{\bar{\beta}}$ and $\nabla B^{\alpha}{}_{\beta}$ are all semi-basic. We denote the coefficients of the covariant derivatives by adding subscripts after semicolon as before. From $VT_{\beta}{}^{\alpha}{}_{\gamma} = VN_{\bar{\beta}}{}^{\alpha}{}_{\bar{\gamma}} = VA^{\alpha}{}_{\bar{\beta}} = VB^{\alpha}{}_{\beta} = 0$ we also have

$$0 = (T_{\beta}{}^{\alpha}{}_{\gamma;0} v + T_{\beta}{}^{\alpha}{}_{\gamma;\mu} v^{\mu} + T_{\beta}{}^{\alpha}{}_{\gamma;\bar{\mu}} v^{\bar{\mu}}) + v_{\beta}{}^{\sigma} T_{\sigma}{}^{\alpha}{}_{\gamma} - v_{\sigma}{}^{\alpha} T_{\beta}{}^{\sigma}{}_{\gamma} + v_{\gamma}{}^{\sigma} T_{\beta}{}^{\alpha}{}_{\sigma}\,, \tag{4.14}$$

$$0 = (N_{\bar{\beta}}{}^{\alpha}{}_{\bar{\gamma};0} v + N_{\bar{\beta}}{}^{\alpha}{}_{\bar{\gamma};\mu} v^{\mu} + N_{\bar{\beta}}{}^{\alpha}{}_{\bar{\gamma};\bar{\mu}} v^{\bar{\mu}}) + v_{\bar{\beta}}{}^{\bar{\sigma}} N_{\bar{\sigma}}{}^{\alpha}{}_{\bar{\gamma}} - v_{\sigma}{}^{\alpha} N_{\bar{\beta}}{}^{\sigma}{}_{\bar{\gamma}} + v_{\bar{\gamma}}{}^{\bar{\sigma}} N_{\bar{\beta}}{}^{\alpha}{}_{\bar{\sigma}}\,, \tag{4.15}$$

$$0 = (A^{\alpha}{}_{\bar{\beta};0} v + A^{\alpha}{}_{\bar{\beta};\gamma} v^{\gamma} + A^{\alpha}{}_{\bar{\beta};\bar{\gamma}} v^{\bar{\gamma}}) - v_{\sigma}{}^{\alpha} A^{\sigma}{}_{\bar{\beta}} + v_{\bar{\beta}}{}^{\bar{\sigma}} A^{\alpha}{}_{\bar{\sigma}}\,, \tag{4.16}$$

$$0 = (B^{\alpha}{}_{\beta;0} v + B^{\alpha}{}_{\beta;\gamma} v^{\gamma} + B^{\alpha}{}_{\beta;\bar{\gamma}} v^{\bar{\gamma}}) - v_{\sigma}{}^{\alpha} B^{\sigma}{}_{\beta} + v_{\beta}{}^{\sigma} B^{\alpha}{}_{\sigma}\,. \tag{4.17}$$

Finally, from

$$d\Omega_{\beta}{}^{\alpha} - \omega_{\beta}{}^{\gamma} \wedge \Omega_{\gamma}{}^{\alpha} + \Omega_{\beta}{}^{\gamma} \wedge \omega_{\gamma}{}^{\alpha} = 0\,, \tag{4.18}$$

the component of (2.19), we see that $\nabla R_{\beta}{}^{\alpha}{}_{\mu\bar{\nu}}$, $\nabla P_{\beta}{}^{\alpha}{}_{\mu\nu}$ and $\nabla S_{\beta}{}^{\alpha}{}_{\mu}$ are also semi-basic, where

$$\begin{aligned}
\nabla R_{\beta}{}^{\alpha}{}_{\mu\bar{\nu}} &= dR_{\beta}{}^{\alpha}{}_{\mu\bar{\nu}} - \omega_{\beta}{}^{\sigma} R_{\sigma}{}^{\alpha}{}_{\mu\bar{\nu}} + \omega_{\sigma}{}^{\alpha} R_{\beta}{}^{\sigma}{}_{\mu\bar{\nu}} - \omega_{\mu}{}^{\sigma} R_{\beta}{}^{\alpha}{}_{\sigma\bar{\nu}} - \omega_{\bar{\nu}}{}^{\bar{\sigma}} R_{\beta}{}^{\alpha}{}_{\mu\bar{\sigma}}\,, \\
\nabla P_{\beta}{}^{\alpha}{}_{\mu\nu} &= dP_{\beta}{}^{\alpha}{}_{\mu\nu} - \omega_{\beta}{}^{\sigma} P_{\sigma}{}^{\alpha}{}_{\mu\nu} + \omega_{\sigma}{}^{\alpha} P_{\beta}{}^{\sigma}{}_{\mu\nu} - \omega_{\mu}{}^{\sigma} P_{\beta}{}^{\alpha}{}_{\sigma\nu} - \omega_{\nu}{}^{\sigma} P_{\beta}{}^{\alpha}{}_{\mu\sigma}\,, \\
\nabla S_{\beta}{}^{\alpha}{}_{\mu} &= dS_{\beta}{}^{\alpha}{}_{\mu\nu} - \omega_{\beta}{}^{\sigma} S_{\sigma}{}^{\alpha}{}_{\mu} + \omega_{\sigma}{}^{\alpha} S_{\beta}{}^{\sigma}{}_{\mu} - \omega_{\mu}{}^{\sigma} S_{\beta}{}^{\alpha}{}_{\sigma}\,.
\end{aligned}$$

Therefore, from $VR_{\beta}{}^{\alpha}{}_{\mu\bar{\nu}} = VP_{\beta}{}^{\alpha}{}_{\mu\nu} = VS_{\beta}{}^{\alpha}{}_{\mu} = 0$, we have

$$\begin{aligned}
0 = (R_{\beta}{}^{\alpha}{}_{\mu\bar{\nu};0} v &+ R_{\beta}{}^{\alpha}{}_{\mu\bar{\nu};\gamma} v^{\gamma} + R_{\beta}{}^{\alpha}{}_{\mu\bar{\nu}\gamma} v^{\bar{\gamma}}) \\
&+ v_{\beta}{}^{\sigma} R_{\sigma}{}^{\alpha}{}_{\mu\bar{\nu}} - v_{\sigma}{}^{\alpha} R_{\beta}{}^{\sigma}{}_{\mu\bar{\nu}} + v_{\mu}{}^{\sigma} R_{\beta}{}^{\alpha}{}_{\sigma\bar{\nu}} + v_{\bar{\nu}}{}^{\bar{\sigma}} R_{\beta}{}^{\alpha}{}_{\mu\bar{\sigma}}\,,
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
0 = (P_{\beta}{}^{\alpha}{}_{\mu\nu;0} v &+ P_{\beta}{}^{\alpha}{}_{\mu\nu;\gamma} v^{\gamma} + P_{\beta}{}^{\alpha}{}_{\mu\nu;\bar{\gamma}} v^{\bar{\gamma}}) \\
&+ v_{\beta}{}^{\sigma} P_{\sigma}{}^{\alpha}{}_{\mu\nu} - v_{\sigma}{}^{\alpha} P_{\beta}{}^{\sigma}{}_{\mu\nu} + v_{\mu}{}^{\sigma} P_{\beta}{}^{\alpha}{}_{\sigma\nu} + v_{\nu}{}^{\sigma} P_{\beta}{}^{\alpha}{}_{\mu\sigma}\,,
\end{aligned} \tag{4.20}$$

$$0 = (S_{\beta}{}^{\alpha}{}_{\mu;0} v + S_{\beta}{}^{\alpha}{}_{\mu;\mu} v^{\mu} + S_{\beta}{}^{\alpha}{}_{\mu;\bar{\mu}} v^{\bar{\mu}}) + v_{\beta}{}^{\sigma} S_{\sigma}{}^{\alpha}{}_{\mu} - v_{\sigma}{}^{\alpha} S_{\beta}{}^{\sigma}{}_{\mu} + v_{\mu}{}^{\sigma} S_{\beta}{}^{\alpha}{}_{\sigma}\,. \tag{4.21}$$

We have proved the following proposition.

PROPOSITION 4.4. *Let V be the canonical lift of an infinitesimal automorphism V'. Related with the coefficients of torsion and curvature tensors, V satisfies the linear equations* (4.12), (4.14), (4.15), (4.16), (4.17), (4.19), (4.20) *and* (4.21).

5. Proof of the main theorem. In this section, we complete the proof of Theorem 3.4. Let (M, θ) be a pseudo-Hermitian manifold and let $p \in M$. We have already seen that for a fixed $u \in \pi^{-1}(p)$, the map

$$\mathfrak{aut}_p(M, \theta) \ni [V'] \to V(u) \in T_u\widetilde{\mathcal{F}}$$

is a linear injective homomorphism, where V is the canonical lift of V'. Therefore, we may regard $\mathfrak{aut}_p(M, \theta)$ as a linear subspace of $T_u\widetilde{\mathcal{F}}$. Under this identification, $\mathfrak{aut}^0_p(M, \theta)$ is a subspace of the vertical space

$$\mathcal{V}_u = \{Z \in T_u\widetilde{\mathcal{F}} : d\pi(Z) = 0\}\,.$$

Now assume that $\dim M = 5$ and $\dim \mathfrak{aut}_p(M, \theta) = 8$. Then since $\dim \mathfrak{aut}^0_p(M, \theta) \leq \dim \mathcal{V}_u = 4$, either $\dim \mathfrak{aut}^0_p(M, \theta) = 4$, or $\dim \mathfrak{aut}^0_p(M, \theta) = 3$ in the case $\mathfrak{aut}_p(M, \theta)$ is transverse to $\mathcal{V}_u$.

5.1. CASE 1: $\dim \mathfrak{aut}^0_p(M, \theta) = 4$. We show that this case does not happen by proving the next proposition.

PROPOSITION 5.1. *Let (M, θ) be a pseudo-Hermitian manifold of dimension $2n + 1$. If $\dim \mathfrak{aut}^0_p(M, \theta) = n^2$, then the almost CR structure of M is integrable at p.*

PROOF. Since $\mathfrak{aut}^0_p(M, \theta)$ is a subspace of $\mathcal{V}_u \cong \mathfrak{u}(n)$, we see that $\mathfrak{aut}^0_p(M, \theta) = \mathfrak{u}(n)$ from the assumption. Therefore, $V = v_\beta{}^\alpha E_\alpha{}^\beta \in \mathfrak{aut}^0_p(M, \theta) \subset \mathfrak{aut}_p(M, \theta)$ for every $(v_\beta{}^\alpha) \in \mathfrak{u}(n)$ and hence

$$v_\sigma{}^\gamma(\delta^\sigma_\alpha\, h_{\gamma\beta} + \delta^\sigma_\beta\, h_{\alpha\gamma}) = 0$$

at u by (4.12). Since $(v_\beta{}^\alpha) \in \mathfrak{u}(n)$ is arbitrary, this implies that

$$\delta^\sigma_\alpha\, h_{\gamma\beta} + \delta^\sigma_\beta\, h_{\alpha\gamma} = 0\,.$$

Contracting σ and γ, we deduce that

$$h_{\alpha\beta} + h_{\alpha\beta} = 2h_{\alpha\beta} = 0$$

at u. Therefore, $h_{\alpha\beta} = 0$ on $\pi^{-1}(p)$. Similarly, by (4.15), we see that

$$\delta^\beta_\sigma N_{\bar\rho}{}^\alpha{}_{\bar\gamma} + \delta^\gamma_\sigma N_{\bar\beta}{}^\alpha{}_{\bar\rho} + \delta^\alpha_\sigma N_{\bar\beta}{}^\rho{}_{\bar\gamma} = 0$$

on $\pi^{-1}(p)$. Contracting ρ and σ, we have

$$N_{\bar\beta}{}^\alpha{}_{\bar\gamma} + N_{\bar\beta}{}^\alpha{}_{\bar\gamma} + N_{\bar\beta}{}^\alpha{}_{\bar\gamma} = 3N_{\bar\beta}{}^\alpha{}_{\bar\gamma} = 0$$

on $\pi^{-1}(p)$. Therefore the almost CR structure of M is integrable at p. □

5.2. CASE 2: $\dim \mathfrak{aut}^0_p(M, \theta) = 3$.

PROPOSITION 5.2. *Let (M,θ) be a $(2n+1)$-dimensional pseudo-Hermitian manifold. Then the restriction of the linear homomorphism $\mathfrak{aut}_p(M,\theta) \ni [V'] \to V(u) \in T_u\widetilde{\mathcal{F}}$ to $\mathfrak{aut}^0_p(M,\theta)$ induces a Lie algebra homomorphism into $\mathfrak{u}(n) \cong \mathcal{V}_u$.*

PROOF. Let $V' = vX' + v'^{\alpha}X'_{\alpha} + v'^{\bar\alpha}X'_{\bar\alpha}$ be an infinitesimal automorphism with $V'(p) = 0$. Then from (4.6),

$$v_{\beta}{}^{\alpha} = X_{\beta}(v^{\alpha})$$

on $\pi^{-1}(p)$, where $V = vX + v^{\alpha}X_{\alpha} + v^{\bar\alpha}X_{\bar\alpha} + v_{\beta}{}^{\alpha}E_{\alpha}{}^{\beta}$, since $v = v^{\alpha} = 0$ on $\pi^{-1}(p)$. Therefore, if we denote by $W' = wX' + w'^{\alpha}X'_{\alpha} + w'^{\bar\alpha}X'_{\bar\alpha}$ the Lie bracket $[U', V']$ for a pair U', V' of infinitesimal automorphisms vanishing at p, it suffices to show that

$$X_{\beta}(w^{\alpha}) = X_{\beta}(u^{\gamma})X_{\gamma}(v^{\alpha}) - X_{\beta}(v^{\gamma})X_{\gamma}(u^{\alpha}) \tag{5.1}$$

on $\pi^{-1}(p)$. By straightforward computations, we see that

$$X_{\beta}(v) = (U^{-1})_{\beta}{}^{\gamma}X'_{\gamma}(v)\,, \tag{5.2}$$

$$X_{\beta}(v^{\alpha}) = (U^{-1})_{\beta}{}^{\rho}X'_{\rho}(v'^{\sigma})U_{\sigma}{}^{\alpha}, \quad X_{\bar\beta}(v^{\alpha}) = (U^{-1})_{\bar\beta}{}^{\bar\rho}X'_{\bar\rho}(v'^{\sigma})U_{\sigma}{}^{\alpha} \tag{5.3}$$

on $\pi^{-1}(p)$ for every infinitesimal automorphism V' with $V'(p) = 0$. By the first equation of (5.3), the equation (5.1) is equivalent to

$$X'_{\beta}(w'^{\alpha}) = X'_{\beta}(u'^{\gamma})X'_{\gamma}(v'^{\alpha}) - X'_{\beta}(v'^{\gamma})X'_{\gamma}(u'^{\alpha}) \tag{5.4}$$

at p. To prove (5.4), we first observe that

$$\begin{aligned} w'^{\alpha} &= uX'(v'^{\alpha}) - vX'(u'^{\alpha}) + u'^{\beta}X'_{\beta}(v'^{\alpha}) - v'^{\beta}X'_{\beta}(u'^{\alpha}) \\ &\quad + u'^{\bar\beta}X'_{\bar\beta}(v'^{\alpha}) - v'^{\bar\beta}X'_{\bar\beta}(u'^{\alpha}) + \mathcal{Q}\,, \end{aligned}$$

where $\mathcal{Q}$ consists of quadratic terms in u', v', u'^{α}, v'^{α}. This yields that

$$\begin{aligned} X'_{\beta}(w'^{\alpha}) &= X'_{\beta}(u)X'(v'^{\alpha}) - X'_{\beta}(v)X'(u'^{\alpha}) \\ &\quad + X'_{\beta}(u'^{\gamma})X'_{\gamma}(v'^{\alpha}) - X'_{\beta}(v'^{\gamma})X'_{\gamma}(u'^{\alpha}) + X'_{\beta}(u'^{\bar\gamma})X'_{\bar\gamma}(v'^{\alpha}) - X'_{\beta}(v'^{\bar\gamma})X'_{\bar\gamma}(u'^{\alpha}) \end{aligned}$$

at p, since $u(p) = v(p) = u'^{\alpha}(p) = v'^{\alpha}(p) = 0$. Therefore, the equation (5.4) will follow, if we show that

$$X'_{\beta}(v) = X'_{\bar\beta}(v'^{\alpha}) = 0 \tag{5.5}$$

at p for every infinitesimal automorphism V' with $V'(p) = 0$. From (4.7), we see that the canonical lift V of V' satisfies

$$dv + iv^{\alpha}\theta_{\alpha} - i\theta^{\alpha}v_{\alpha} + \iota_V\Theta = 0 \tag{5.6}$$

and

$$dv^{\alpha} + v^{\beta}\omega_{\beta}{}^{\alpha} - \theta^{\beta}v_{\beta}{}^{\alpha} + \iota_V\Theta^{\alpha} = 0\,. \tag{5.7}$$

From (5.6) and (5.7),

$$X_\beta(v) = i v_\beta - 2h_{\alpha\beta} v^\alpha = 0$$

and

$$X_{\bar\beta}(v^\alpha) = 2N_{\bar\beta}{}^{\alpha}{}_{\bar\gamma} v^{\bar\gamma} - A^\alpha{}_{\bar\beta} v = 0\,,$$

since $v = v^\alpha = 0$ on $\pi^{-1}(p)$. The above identities yield (5.5) by (5.2) and (5.3). □

Therefore, if $\dim M = 5$ and $\dim \mathfrak{aut}_p^0(M, \theta) = 3$, then $\mathfrak{aut}_p^0(M, \theta)$ can be regarded as a 3-dimensional subalgebra of $\mathfrak{u}(2)$. Note that $\mathfrak{u}(2)$ contains only one 3-dimensional subalgebra, which is $\mathfrak{su}(2)$. This implies the assertion (ii) of Theorem 3.4.

Equation (4.12) implies that

$$v_\alpha{}^\gamma h_{\gamma\beta} + v_\beta{}^\gamma h_{\alpha\gamma} = 0$$

for every $(v_\beta{}^\alpha) \in \mathfrak{su}(2) = \mathfrak{aut}_p^0(M, \theta)$. This is equivalent to

(5.8) $$(v_1{}^1 + v_2{}^2)h = 0$$

where $h = h_{12} = -h_{21}$. But this is always satisfied for every $h \in \mathbb{C}$, since $(v_\beta{}^\alpha) \in \mathfrak{su}(2)$.

LEMMA 5.3.

(5.9) $$T_\beta{}^\alpha{}_\gamma = N_{\bar\beta}{}^\alpha{}_{\bar\gamma} = 0$$

for every $\alpha, \beta, \gamma = 1, 2$.

PROOF. Equations (4.14) and (4.15) imply that

$$v_\beta{}^\sigma T_\sigma{}^\alpha{}_\gamma - v_\sigma{}^\alpha T_\beta{}^\sigma{}_\gamma + v_\gamma{}^\sigma T_\beta{}^\alpha{}_\sigma = 0$$

and

$$v_{\bar\beta}{}^{\bar\sigma} N_{\bar\sigma}{}^\alpha{}_{\bar\gamma} - v_\sigma{}^\alpha N_{\bar\beta}{}^\sigma{}_{\bar\gamma} + v_{\bar\gamma}{}^{\bar\sigma} N_{\bar\beta}{}^\alpha{}_{\bar\sigma} = 0$$

for every $(v_\beta{}^\alpha) \in \mathfrak{su}(2)$. Then we see that

$$v_1{}^1 T_1{}^1{}_2 + v_2{}^1 T_1{}^2{}_2 = 0\,, \quad v_1{}^1 N_{\bar 1}{}^1{}_{\bar 2} + v_2{}^1 N_{\bar 1}{}^2{}_{\bar 2} = 0$$

for every $v_1{}^1 \in i\mathbb{R}$ and $v_2{}^1 \in \mathbb{C}$. This yields the conclusion. □

Since we assume that the almost CR structure of M is not integrable at p, we see that $h \neq 0$ at u and hence on $\pi^{-1}(p)$. This implies the assertion (i) that the almost CR structure is not partially integrable at p.

From now on, we assume that the characteristic vector field is an infinitesimal automorphism, that is, $V = vX \in \mathfrak{aut}_p(M, \theta)$ for every $v \in \mathbb{R}$.

LEMMA 5.4.

(5.10) $$A^\alpha{}_{\bar\beta} = B^\alpha{}_\beta = 0$$

on $\pi^{-1}(p)$ *for every* $\alpha, \beta = 1, 2$.

PROOF. From (4.16) and (4.17),

$$v_{\bar{\beta}}{}^{\bar{\sigma}} A^{\alpha}{}_{\bar{\sigma}} = v_{\sigma}{}^{\alpha} A^{\sigma}{}_{\bar{\beta}}\,, \quad v_{\beta}{}^{\sigma} B^{\alpha}{}_{\sigma} = v_{\sigma}{}^{\alpha} B^{\sigma}{}_{\beta}$$

for every $(v_{\beta}{}^{\alpha}) \in \mathfrak{su}(2)$. This is equivalent to

$$A^1{}_{\bar{1}} = A^2{}_{\bar{2}} = 0\,, \quad A^1{}_{\bar{2}} = -A^2{}_{\bar{1}} =: A$$

and

$$B^1{}_2 = B^2{}_1 = 0\,, \quad B^1{}_1 = B^2{}_2 =: B$$

for some $A \in \mathbb{C}$ and $B \in \mathbb{R}$.

Let $V = vX$ for a given $v \in \mathbb{R}$. Then from (4.12), we see that $v\, h_{\alpha\beta;0} = 0$. Since v is arbitrary, this implies that

$$h_{\alpha\beta;0} = 0 \tag{5.11}$$

on $\pi^{-1}(p)$. Thanks to (5.9) and (5.11), we see that the equation (4.11) induces the following relations among h, A and B by comparing the type of each term.

$$B = i(\bar{h}\overline{A} - hA)\,, \quad \overline{A} = i(2hB + h_{12;0}) = 2ihB\,.$$

Plugging the second equation to the first one,

$$B = -4|h|^2 B\,,$$

which implies the assertion that $A = B = 0$. □

Lemma 5.3 and Lemma 5.4 yield the assertion (iii).

LEMMA 5.5. *All the covariant derivatives of $h_{\alpha\beta}$, $T_{\beta}{}^{\alpha}{}_{\gamma}$, $N_{\bar{\beta}}{}^{\alpha}{}_{\bar{\gamma}}$, $A^{\alpha}{}_{\bar{\beta}}$, $B^{\alpha}{}_{\beta}$ vanish on $\pi^{-1}(p)$.*

PROOF. Since $\mathfrak{aut}_p(M,\theta)$ is transverse to the vertical space $\mathcal{V}_u$, for every $v \in \mathbb{R}$, $v^{\alpha} \in \mathbb{C}$, there exists $(v_{\beta}{}^{\alpha}) \in \mathfrak{u}(2)$ such that $V = vX + v^{\alpha}X_{\alpha} + v^{\bar{\alpha}}X_{\bar{\alpha}} + v_{\beta}{}^{\alpha}E_{\alpha}{}^{\beta} \in \mathfrak{aut}_p(M,\theta)$. Then by (4.14) and Lemma 5.3, we have

$$T_{\beta}{}^{\alpha}{}_{\gamma;0}v + T_{\beta}{}^{\alpha}{}_{\gamma;\mu}v^{\mu} + T_{\beta}{}^{\alpha}{}_{\gamma;\bar{\mu}}v^{\bar{\mu}} = 0\,.$$

Since $v \in \mathbb{R}$ and $v^{\mu} \in \mathbb{C}$ are arbitrary, this yields that $\nabla T_{\beta}{}^{\alpha}{}_{\gamma} = 0$. Repeating this argument for $N_{\bar{\beta}}{}^{\alpha}{}_{\bar{\gamma}}$, $A^{\alpha}{}_{\bar{\beta}}$ and $B^{\alpha}{}_{\beta}$, we see that

$$\nabla N_{\bar{\beta}}{}^{\alpha}{}_{\bar{\gamma}} = \nabla A^{\alpha}{}_{\bar{\beta}} = \nabla B^{\alpha}{}_{\beta} = 0$$

on $\pi^{-1}(p)$.

It remains to prove that $\nabla h_{\alpha\beta} = 0$ on $\pi^{-1}(p)$. We have already $h_{\alpha\beta;0} = 0$ as in (5.11). Note that from Equation (4.11) we can reduce

$$h_{\alpha\beta;\bar{\gamma}} = -2iT_{\alpha\bar{\gamma}\beta} - 2N_{\alpha}{}^{\bar{\sigma}}{}_{\beta}h_{\bar{\sigma}\bar{\gamma}}\,.$$

Lemma 5.3 implies that $h_{\alpha\beta;\bar\gamma} = 0$ on $\pi^{-1}(p)$. Thus it suffices to show that $h_{\alpha\beta;\gamma} = 0$ on $\pi^{-1}(p)$

Since $\mathfrak{aut}^0_p(M,\theta) = \mathfrak{su}(2)$ is the algebra of trace-free elements of $\mathfrak{u}(2)$, for each $(v^\alpha) \in \mathbb{C}^2$, there exists unique $V(v) = v^\gamma X_\gamma + v^{\bar\gamma} X_{\bar\gamma} + {v_\beta}^\alpha {E_\alpha}^\beta \in \mathfrak{aut}_p(M,\theta)$ such that both ${v_1}^2 = -{v_2}^1$ and ${v_2}^2$ are zero at u. We consider the value of ${v_1}^1$ at u as a $i\mathbb{R}$-valued function with variable $(v^\alpha) \in \mathbb{C}^2$. Applying $V(v)$ to (4.12),

$$v^\gamma h_{\alpha\beta;\gamma} = -{v_\alpha}^\sigma h_{\sigma\beta} - {v_\beta}^\sigma h_{\alpha\sigma}\,. \tag{5.12}$$

In case of $\alpha = \beta$, we have $v^\gamma h_{\alpha\alpha;\gamma} = 0$ for each $(v^\alpha) \in \mathbb{C}^2$ so that $h_{\alpha\alpha;\gamma} = 0$. For $\alpha \neq \beta$, we consider only

$$v^\gamma h_{12;\gamma} = -{v_1}^1 h_{12} = -{v_1}^1 h\,.$$

The value $v^\gamma h_{12;\gamma}$ at u is a complex linear functional of $(v^\alpha) \in \mathbb{C}^2$, but $-{v_1}^1 h$ at u is not complex linear because ${v_1}^1$ is $i\mathbb{R}$-valued. This means that ${v_1}^1 = 0$ and $v^\gamma h_{12;\gamma} = 0$ for each (v^α). Therefore we conclude that $h_{\alpha\beta;\gamma} = 0$. □

In the final step of proof above, we show that ${v_1}^1 = 0$ for V; thus

PROPOSITION 5.6. *For any $v \in \mathbb{R}$ and $(v^\alpha) \in \mathbb{C}^n$, $V = vX + v^\alpha X_\alpha + v^{\bar\alpha} X_{\bar\alpha}$ belongs to $\mathfrak{aut}_p(M,\theta)$.*

Applying this Proposition to (4.19)–(4.21), we get

LEMMA 5.7. *All the covariant derivatives of $R_{\alpha\bar\beta\mu\bar\nu}$, $P_{\alpha\bar\beta\mu\nu}$, $S_{\alpha\bar\beta\mu}$ vanish on $\pi^{-1}(p)$.*

Lemma 5.3, Lemma 5.4 and Lemma 5.5 imply that

$$\Theta^\alpha = 0\,, \quad d\Theta^\alpha = 0$$

on $\pi^{-1}(p)$. Therefore, (4.13) is reduced to

$$\theta^\beta \wedge {\Omega_\beta}^\alpha = 0\,.$$

Equivalently,

$$P_{\alpha\bar\beta\mu\nu} = S_{\alpha\bar\beta\mu} = 0\,, \quad \text{and} \quad R_{\alpha\bar\beta\mu\bar\nu} = R_{\mu\bar\beta\alpha\bar\nu}\,. \tag{5.13}$$

From (4.19), we see

$${v_\beta}^\sigma {R_\sigma}^\alpha{}_{\mu\bar\nu} - {v_\sigma}^\alpha {R_\beta}^\sigma{}_{\mu\bar\nu} + {v_\mu}^\sigma {R_\beta}^\alpha{}_{\sigma\bar\nu} + {v_{\bar\nu}}^{\bar\sigma} {R_\beta}^\alpha{}_{\mu\bar\sigma} = 0$$

for every $({v_\beta}^\alpha) \in \mathfrak{su}(2)$. Via component-wise computation, it turns out that

$$R_{1\bar11\bar2} = R_{1\bar12\bar1} = R_{1\bar21\bar1} = R_{2\bar11\bar1} = R_{1\bar22\bar2} = R_{2\bar12\bar2} = R_{2\bar21\bar2} = R_{2\bar22\bar1} = 0\,,$$
$$R_{1\bar21\bar2} = R_{2\bar12\bar1} = 0\,,$$
$$R_{1\bar22\bar1} = R_{2\bar11\bar2}, \quad R_{1\bar12\bar2} = R_{2\bar21\bar1}\,,$$
$$R_{1\bar11\bar1} = R_{2\bar22\bar2} = R_{1\bar22\bar1} + R_{1\bar12\bar2}\,.$$

From the last equation of (5.13), we also have

$$R_{1\bar{2}2\bar{1}} = R_{1\bar{1}2\bar{2}}\,.$$

Altogether, we conclude that

$$\Omega_{\alpha\bar{\beta}} = R_{\alpha\bar{\beta}\mu\bar{\nu}}\theta^{\mu} \wedge \theta^{\bar{\nu}}$$

where

$$R_{\alpha\bar{\beta}\mu\bar{\nu}} = \frac{R}{6}(\delta_{\alpha\bar{\beta}}\delta_{\mu\bar{\nu}} + \delta_{\alpha\bar{\nu}}\delta_{\mu\bar{\beta}})$$

on $\pi^{-1}(p)$. This completes the proof of the assertion (iv).

6. Remarks on the higher dimensional cases. One of key points for proving Lemma 5.3, Lemma 5.4 and Lemma 5.5 is that SU(2) is sufficiently large in the sense that this group acts on S^3 transitively. Thus we can generalize these lemmata.

PROPOSITION 6.1. *Let (M,θ) be a $(2n+1)$-dimensional pseudo-Hermitian manifold whose almost CR structure is not partially integrable at $p \in M$. If $\mathrm{Aut}^0_p(M,\theta)$ acts transitively on the holomorphic sphere $SH_p = \{v \in H_p : \|v\| = 1\}$, then $n = 2k$ and $\mathfrak{aut}^0_p(M,\theta) \cong \mathfrak{sp}(k) = \mathfrak{u}(2k) \cap \mathfrak{sp}(k,\mathbb{C})$. If we additionally assume that the characteristic vector field belongs to $\mathfrak{aut}_p(M,\theta)$, then*

(i) *all the coefficients of torsion tensors but $(h_{\alpha\beta})$ vanish on $\pi^{-1}(p)$, and*

(ii) *all the coefficients of curvature tensors but $R_{\alpha\bar{\beta}\mu\bar{\nu}}$ vanish on $\pi^{-1}(p)$. Moreover,*

$$R_{\alpha\bar{\beta}\mu\bar{\nu}} = \frac{R}{n(n+1)}(g_{\alpha\bar{\beta}}g_{\mu\bar{\nu}} + g_{\alpha\bar{\nu}}g_{\mu\bar{\beta}})$$

on $\pi^{-1}(p)$ for some real constant R.

PROOF. A smallest subgroup of U(n) acting transitively on the sphere S^{2n-1} is SU(n) if $n \neq 2k$ and Sp(k) if $n = 2k$ (see [9]). But the isotropy group $\mathrm{Aut}^0_p(M,\theta)$ should be a subgroup of $\{U = (U_{\alpha}{}^{\beta}) \in \mathrm{U}(n) : U_{\alpha}{}^{\mu}U_{\beta}{}^{\nu}h_{\mu\nu}(p) = h_{\alpha\beta}(p)\}$. Since $(h_{\alpha\beta}(p))$ is non-zero by assumption, the possible case is that skew-symmetric matrix $(h_{\alpha\beta}(p))$ has the same non-zero eigenvalues with respect to $(g_{\alpha\bar{\beta}}(p))$ so that $n = 2k$ and $\mathrm{Aut}^0_p(M,\theta) \cong \mathrm{Sp}(k)$.

Now we may let $(h_{\alpha\beta}(p)) = \lambda\left(\begin{smallmatrix} 0 & I \\ -I & 0\end{smallmatrix}\right)$ and

$$\mathfrak{aut}^0_p(M,\theta) = \mathfrak{sp}(k) = \left\{\begin{pmatrix} U & S \\ -\overline{S} & \overline{U}\end{pmatrix} : U \in \mathfrak{u}(k),\ S^t = S\right\}. \tag{6.1}$$

Taking $V = \left(\begin{smallmatrix} iI & 0 \\ 0 & -iI\end{smallmatrix}\right)$ and $V = \left(\begin{smallmatrix} 0 & I \\ -I & 0\end{smallmatrix}\right)$ in $\mathfrak{aut}^0_p(M,\theta)$, we get Lemma 5.3, Lemma 5.4 and Lemma 5.5, except the vanishing of $h_{\alpha\beta;\gamma}$ at $\pi^{-1}(p)$. By the concrete form of $\mathfrak{sp}(k)$ as in (6.1), for each $v = (v^{\alpha}) \in \mathbb{C}^{2k}$, there is unique $V(v) = v^{\gamma}X_{\gamma} + v^{\bar{\gamma}}Z_{\bar{\gamma}} + v_{\beta}{}^{\alpha}E_{\alpha}{}^{\beta} \in \mathfrak{aut}_p(M,\theta)$ with

$$(v_{\beta}{}^{\alpha}) = \begin{pmatrix} U & W \\ -\overline{W}^t & 0\end{pmatrix}$$

where $U \in \mathfrak{u}(k)$ and W is strictly upper-triangular. The matrix expression of the complex linear functional (5.12) is

$$\begin{aligned}(v^\gamma h_{\alpha\beta;\gamma}) &= -\lambda\begin{pmatrix} U & W \\ -\overline{W}^t & 0\end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix} - \lambda\begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix}\begin{pmatrix} U^t & -\overline{W} \\ W^t & 0\end{pmatrix} \\ &= \lambda\begin{pmatrix} W - W^t & -U \\ U^t & \overline{W}^t - \overline{W}\end{pmatrix}.\end{aligned}$$

Then all entries of both $W - W^t$ and $\overline{W}^t - \overline{W} = -\overline{(W - W^t)}$ are complex linear for $v \in \mathbb{C}^{2k}$; thus $W - W^t = 0$ for all v. Since each W is strictly upper-triangular, $W = 0$. Consider $\mathfrak{u}(k)$-valued complex linear functional U. The skew-Hermitian condition of U means that each entry of U can not be complex linear, so that $U = 0$. Therefore $v^\gamma h_{\alpha\beta;\gamma} = 0$ for every $v \in \mathbb{C}^{2k}$. This concludes Lemma 5.5 and Proposition 5.6 □

Note that this is not a local characterization of pseudo-Hermitian structure by the maximality of dimension of automorphism group, even in the case of $n = 2k$. Consider generalized Heisenberg group $(\mathbf{H}_P, \theta_P)$ of dimension $2n + 1$ with $n = 8$. If all $P_{\alpha\beta}$ are vanishing except $P_{12} = -P_{21}$, then $\mathrm{Aut}_0(\mathbf{H}_P, \theta_P) = \mathrm{Sp}(1) \oplus \mathrm{U}(6)$ and its dimension is 39. If all eigenvalues of $(P_{\alpha\beta})$ are the same non-zero constant, then $\mathrm{Aut}_0(\mathbf{H}_P, \theta_P) = \mathrm{Sp}(4)$ is of 36-dimensional. Hence the assumption in Proposition 6.1 does not imply the maximality of the automorphism group.

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