Tohoku Math. J.
70 (2018), 139–174

A COUPLING OF BROWNIAN MOTIONS IN THE $\mathcal{L}_0$-GEOMETRY

TAKAFUMI AMABA AND KAZUMASA KUWADA

(Received April 28, 2015)

Abstract. Under a complete Ricci flow, we construct a coupling of two Brownian motions such that their $\mathcal{L}_0$-distance is a supermartingale. This recovers a result of Lott [J. Lott, Optimal transport and Perelman's reduced volume, Calc. Var. Partial Differential Equations 36 (2009), no. 1, 49–84.] on the monotonicity of $\mathcal{L}_0$-distance between heat distributions.

1. Introduction. Since Perelman's pioneering article [21], there are several attempts to study Ricci flow in connection with heat distributions. As one of them, J. Lott [18] provides several monotonicity formulae from the viewpoint of optimal transportation by extending Topping's approach [20, 25]. His argument is based on the Eulerian calculus, which can be rigorous if everything is sufficiently regular. On the other hand, sometimes it is not easy to verify the required regularity. For instance, we must take much care on it if the manifold is non-compact.

Among results in [18], Lott introduced $\mathcal{L}_0$-functional on the space of (space-time) curves and the associated $\mathcal{L}_0$-distance as an analog of $\mathcal{L}$-distance in [21, 25]. He proved the monotonicity of transportation cost given in terms of $\mathcal{L}_0$-distance between heat distributions. By using it, he gave an alternative proof of the monotonicity of $\mathcal{F}$-functional in [21]. The main purpose of this paper is to prove the monotonicity of $\mathcal{L}_0$-transportation cost by a probabilistic approach using a coupling of Brownian motions. As an advantage of our approach, we just require much weaker regularity assumptions and we can extend the result to more general transportation costs.

The organization of this paper is as follows. In the next subsection, we will state our framework and results more precisely. We also review Lott's result on the monotonicity of $\mathcal{L}_0$-transportation cost there. In Subsection 1.2, we give a review of historical background and related known results in more detail. All necessary calculations, formulae and properties on $\mathcal{L}_0$ are summarized in Section 2. The reader, who wants to grasp only heuristics or a rough story of this paper, can skip Section 2 except for Proposition 2.8, where we give a Hessian estimate for the $\mathcal{L}_0$-distance. The most part of Section 2 (e.g., the $\mathcal{L}_0$-cut locus, the $\mathcal{L}_0$-exponential and so on) are analogous to ones for Riemannian distance or $\mathcal{L}$-distance.

In Sections 3 and 4, we will construct a coupling of $g(\tau)$-Brownian motions which satisfies a requirement of our main theorem (Theorem 1.1 below). In Section 3 we will discuss

2010 *Mathematics Subject Classification.* Primary 53C21; Secondary 53C44, 58J65, 60J05.

Key words and phrases. $\mathcal{L}_0$-geometry, coupling of Brownian motions, approximation by geodesic random walks.

The first author was supported by JSPS KAKENHI Grant Number 24·5772. The second author is partially supported by the Grant-in-Aid for Young Scientist (B) 22740083.

it under a strong regularity assumption on $\mathcal{L}_0$. More precisely, we assume that $\mathcal{L}_0$-cut locus is empty there (see Section 2 for the definition of $\mathcal{L}_0$-cut locus; It is a concept analogous to the cut locus for the Riemannian distance function). While it is very restrictive, we believe that the argument given in that section will be insightful and that it helps us to understand the rigorous argument given in Section 4. We employ an approximation of Brownian motions by geodesic random walks there, as in the previous result [15] (see Subsection 1.2 for known results as well as the reason why we choose it).

1.1. Framework and main results. Let $(g(t))_{0\le t\le T}$ be a Ricci flow on a connected manifold M without boundary with $d := \dim M \ge 2$. That is, $g(t)$ solves the following evolution equation:

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}_{g(t)}\,. \tag{1}$$

In the sequel, we always assume that $(M, g(t))$ is a complete Riemannian manifold for each t.

For stating Lott's result and ours, we introduce $\mathcal{L}_0$-functional and some notions concerning with it. Under the Ricci flow $g(t)$ on d-dimensional manifold M, $\mathcal{L}_0$-functional is given by

$$\mathcal{L}_0(\gamma) := \frac{1}{2}\int_{t'}^{t''} \left\{|\dot\gamma(t)|^2_{g(t)} + R_{g(t)}(\gamma(t))\right\}\mathrm{d}t$$

for each piecewise smooth curve $\gamma : [t', t''] \to M$, where $R_{g(t)}$ is the scalar curvature with respect to $g(t)$. As a minimal value of $\mathcal{L}_0$-functional with the fixed endpoints (in space-time), we define $\mathcal{L}_0$-distance L_0. That is, for $0 \le t' < t'' \le T$ and $m', m'' \in M$, $L_0^{t',t''}(m', m'')$ is given by

$$L_0^{t',t''}(m', m'') := \inf_{\gamma} \mathcal{L}_0(\gamma)\,, \quad m',\ m'' \in M\,,$$

where the infimum is taken over piecewise smooth curves $\gamma : [t', t''] \to M$ such that $\gamma(t') = m'$ and $\gamma(t'') = m''$.

We denote by $P(M)$ the space of Borel probability measures on M and $P^\infty(M)$ the subspace of $P(M)$ whose element has smooth density. For $0 \le t' < t'' \le T$ and $\mu', \mu'' \in P(M)$, the (optimal) $\mathcal{L}_0$-transportation cost $C_0^{t',t''}(\mu', \mu'')$ is defined by

$$C_0^{t',t''}(\mu', \mu'') := \inf_{\pi\in\Pi(\mu',\mu'')} \int_{M\times M} L_0^{t',t''}(m', m'')\pi(\mathrm{d}m', \mathrm{d}m'')\,,$$

where $\Pi(\mu', \mu'')$ is the set of couplings of μ' and μ''.

In this framework, Lott proved the following: Assume that M is closed. If c', $c'' : [t_0, t_1] \to P^\infty(M)$ are solutions to the backward heat equation

$$\frac{\mathrm{d}\mu_t}{\mathrm{d}t} = -\triangle_{g(t)}\mu_t, \tag{2}$$

and if they satisfy some technical assumptions (see Corollary 5 in [18] for details, although we will mention it partially in the sequel), then

$$u \mapsto C_0^{t'+u,t''+u}\big(c'(t'+u), c''(t''+u)\big) \tag{3}$$

is non-decreasing (Proposition 13 in [18]).

Let us turn to state our result. Until the end of the paper, we fix two time intervals

$$0 \le t_0' < t_1' \le T \quad \text{and} \quad 0 \le t_0'' < t_1'' \le T$$

with $t_0' < t_0''$ and $t_1' - t_0' = t_1'' - t_0'' =: S$. We denote by (t', t'') the coordinate on

$$\left\{(t', t'') \in [t_0', t_1'] \times [t_0'', t_1''] : t' < t''\right\}.$$

Since it looks awkward to work with backward heat equation (2), we shall reverse the time by setting

$$\tau' = \tau'(s) := t_1' - s \quad \text{and} \quad \tau'' = \tau''(s) := t_1'' - s$$

for $0 \le s \le S$. By *$g(\tau')$-Brownian motion (resp. $g(\tau'')$-Brownian motion)*, we mean a time-inhomogeneous diffusion process on M associated with $\triangle_{g(\tau'(s))}$ (resp. $\triangle_{g(\tau''(s))}$), where s stands for the time-parameter of the process.

THEOREM 1.1. *Assume that $(g(t))_{0\le t\le T}$ satisfies*

$$\inf_{(t,m)\in[0,T]\times M} \inf_{\substack{V\in T_mM \\ \|V\|_{g(t)}=1}} \mathrm{Ric}_{g(t)}(V, V) > -\infty. \tag{4}$$

Then for each $(m', m'') \in M \times M$, there exists a coupling of $g(\tau')$-Brownian motion $X = (X_s)_{0\le s\le S}$ starting from m' and $g(\tau'')$-Brownian motion $Y = (Y_s)_{0\le s\le S}$ starting from m'' such that

$$s \mapsto L_0^{\tau'(s),\tau''(s)}(X_s, Y_s)$$

is a supermartingale and the map

$$(m', m'') \mapsto \textit{the law of } (X, Y) \textit{ with } (X_0, Y_0) = (m', m'')$$

is measurable.

Theorem 1.1 provides us a probabilistic interpretation of the monotonicity of (3). That is, roughly speaking, we can show the monotonicity by taking an expectation of $L_0^{\tau'(s),\tau''(s)}(X_s, Y_s)$. Actually, we can say more: Let $\varphi : \mathbb{R} \to \mathbb{R}$ be concave and non-decreasing. We define a new transportation cost $C_{0,\varphi}^{t',t''}(\mu', \mu'')$ by

$$C_{0,\varphi}^{t',t''}(\mu', \mu'') := \inf_{\pi\in\Pi(\mu',\mu'')} \int_{M\times M} \varphi\big(L_0^{t',t''}(m', m'')\big)\pi(\mathrm{d}m', \mathrm{d}m'').$$

COROLLARY 1.2. *Assume that our Ricci flow satisfies the condition* (4)*. Then for any two families $c(t')$ and $c(t'')$ of probability measures satisfying* (2)*, $C_{0,\varphi}^{t'+s,t''+s}\big(c'(t'+s), c''(t''+s)\big)$ is nondecreasing in s,*

Note that Lott's result can be regarded as a special case of Corollary 1.2, that is, the case $\varphi(x) = x$ and M is compact. Moreover, in order to make Otto's calculus rigorous, Lott further assumed that the curves c', c'' and the "E_0-minimizing geodesics" (see (4.9) in [18] for the definition of E_0) which interpolate $c'(t_0' + s)$ and $c''(t_0'' + s)$ lie in $P^\infty(M)$. In [18], it is claimed that the last extra condition can be relaxed by giving an alternative proof which is analogous to Topping's approach in [25]. In this paper, we give a proof based on the theory of stochastic calculus, and we do not only relax the extra regularity assumption but weaken the compactness assumption on M to the curvature condition (4). Recall that, under a well-known sufficient condition to the unique existence of a Ricci flow in [4, 23], the condition (4) is automatically satisfied. As a matter of fact, stochastic calculus is already used in [15] to extend Topping's result [25] (see below for more details). Thus our result provides an additional evidence that stochastic calculus is an efficient tool to study this sort of problems.

From technical point of view, some people might think that our result is not interesting by the reason that our argument goes along the same way as the one in [15]. We believe that we can study geometric analysis of Ricci flow deeper from stochastic analytic techniques combined with optimal transport, and that our result would support a further development at some stage. For instance, as a by-product of our approach, we have established $\mathcal{L}_0$-geometry, an analog of Riemannian geometry associated with $\mathcal{L}_0$-functional instead of energy functional of curves, as a counterpart of $\mathcal{L}$-geometry. Though it is similar to $\mathcal{L}$-geometry, we think that making a rigorous framework of $\mathcal{L}_0$-geometry would contribute a further study of $\mathcal{L}_0$-functional to some extent.

1.2. Historical background and related results. Recall that the Ricci flow is a solution to the evolution equation (1). This equation was introduced by Hamilton in [9] and used to find an Einstein metric (i.e., a metric g such that $\mathrm{Ric}(g) = \mathrm{const.}g$) by deforming any given Riemannian metric g_0 with positive Ricci curvature on a compact 3-manifold.

Inspired by quantum field theory, such as nonlinear σ-models, Perelman [21] interprets the Ricci flow as a *gradient flow*; At least formally, the Ricci flow can be regarded as the gradient flow of the so-called Perelman's $\mathcal{F}$*-functional*. This interpretation naturally leads to the *monotonicity formula* for $\mathcal{F}$: The functional $\mathcal{F}$ is nondecreasing along the Ricci flow. Additionally, the so-called $\mathcal{W}$*-functional* is also shown to be non-decreasing along the Ricci flow. As is well-known, these monotonicity formulae are effectively used in the resolution of Thurston's geometrization conjecture by Perelman.

Recently, alternative approaches to those formulae have been initiated on the basis of optimal transportation. For the monotonicity of $\mathcal{W}$, Topping [25] gave an alternative proof by using the $\mathcal{L}$-transportation cost via so-called Lagrangian calculus. More precisely, he consider the optimal transportation cost whose cost function is given by a renormalization of Perelman's $\mathcal{L}$-distance. He proved the monotonicity of this transportation cost between (time-rescaled) heat distributions and derived the monotonicity of $\mathcal{W}$-functional by taking a sort of time-derivative of the optimal cost. For studying the monotonicity of $\mathcal{F}$, Lott [18] showed the monotonicity of (3) as explained in Subsection 1.1. Then he recovered the monotonicity of $\mathcal{F}$-functional (Corollary 6 in [18]) again by taking a sort of time-derivative. As mentioned,

Lott's argument is based on so-called Eulerian calculus (see e.g. [26] for a comparison with Lagrangian calculus).

Such monotonicity formulae for optimal transportation costs between heat distributions are also studied in a slightly different context. For instance, on a complete Riemannian manifold with a fixed metric, the same sort of monotonicity of L^p-Wasserstein distance is equivalent to non-negative Ricci curvature or an L^q-gradient estimate for the heat semigroup (see [22] and references therein). For time-dependent metrics, the same sort of monotonicity of L^2-Wasserstein distance is shown to be equivalent to the property that the metric evolves as a super Ricci flow by McCann and Topping [20]. On one hand, the latter result is a natural extension of the former one since the latter recovers the former when the metric does not depend on time. On the other hand, this result can be regarded as a primitive form of the results [18, 25] (This observation is also addressed in [18, 25] themselves). These former results indicate that monotonicity formulae in the optimal transportation should be connected with the geometry of the space in a deeper way and that more studies are expected in this direction.

From its definition, optimal transportation cost is strongly related with the notion of coupling of random variables or stochastic processes. Thus it is natural to consider the above-mentioned problem by using a coupling method of stochastic processes. Even only in stochastic differential geometry, there are several results in coupling method. Traditionally, they paid much attention to estimating the time that two particle meets, while it does not match with our present purpose (see [7, 11, 17]; see [12, 27] also). By using a similar idea as ones in those studies, we can construct a coupling by parallel transport on a complete Riemannian manifold with a lower Ricci curvature bound and it is tightly connected with the monotonicity of Wasserstein distances (see [22] and references therein). Extensions of those kind of coupling to the time-dependent metric case are achieved by Kuwada [16] and by Arnaudon, Coulibaly and Thalmaier [1]. A typical example of the time-dependent metric is backward (super) Ricci flow. To construct a coupling, the former used an approximation by coupled geodesic random walks and the latter construct a one-parameter family of coupled particles which consists of a string and moves continuously as time evolves. As a result, they recover the monotonicity formula in [20] and extend it to non-compact spaces. Topping's monotonicity formula is also proved and extended by Kuwada and Philipowski [15] and discussed later by Cheng [5]. The former used the same method as in [16] and the latter uses an argument studied in [27].

In all those couplings, we are interested in the behavior of distance-like functions (e.g. distance itself or $\mathcal{L}$-distance) between a coupling of diffusion processes (X_s, Y_s). In those cases, the main technical difficulty arises at singular points of the functions (e.g. cut locus). Roughly speaking, there are two obstructions: Firstly, the construction of the coupling itself depends on a regularity of the function. Secondly, we can not apply Itô's formula directly when the coupled process lies on the singular points. Thus we require some indirect arguments as mentioned above to overcome these obstructions. In this paper, we follow the argument used in [15, 16]. More precisely, we consider a coupling of geodesic random walks $(X^\varepsilon_s, Y^\varepsilon_s)$. The construction of it requires less regularity and this fact works well to avoid the first obstruction. Then, instead of Itô's formula for $L_0^{\tau'(s),\tau''(s)}(X_s, Y_s)$, we can employ a *"dif-*

ference" (rather than differential) *inequality* for $L_0^{\tau'(s),\tau''(s)}(X_s^\varepsilon, Y_s^\varepsilon)$ at each approximation step ε (see Proposition 4.2), up to an well-controlled error. Along this idea, we can avoid the second obstruction. Since geodesic random walks converge to (time-inhomogeneous) Brownian motions, we can obtain an estimate for a coupling of Brownian motions as the limit. For those who are interested in other approaches, it is worth mentioning that a comparison with other approaches is discussed in [15, 16].

2. $\mathcal{L}_0$-geometry along Ricci flow. In the rest of the paper, we always assume (4).

2.1. Differential calculus of $\mathcal{L}_0$ and L_0. The main aim in this subsection is to give an estimate of (a contraction of) the Hessian for $L_0^{t',t''}$, which we will use in the subsequent sections. Let $\gamma : [t', t''] \to M$ be a piecewise smooth curve. For each variation of γ with a variational vector field V, we denote by $(\delta_V \mathcal{L}_0)(\gamma)$ and $(\delta_V \delta_V \mathcal{L}_0)(\gamma)$ the first and second variation of $\mathcal{L}_0$. We omit all proofs in this subsection except for Proposition 2.8, because all proofs are routine (see e.g., [13, Sections 17, 18 and 19]).

PROPOSITION 2.1. *For any smooth variation (not being necessarily proper) of a smooth curve $\gamma : [t', t''] \to M$ with a variational vector field V, we have*

$$(\delta_V \mathcal{L}_0)(\gamma) = \langle V(t), \dot\gamma(t)\rangle_{g(t)}\Big|_{t=t'}^{t=t''} + \frac{1}{2}\int_{t'}^{t''} \langle \mathscr{G}_t(\gamma), V(t)\rangle_{g(t)} \mathrm{d}t$$

where $\mathscr{G}_t(\gamma) := \nabla^{g(t)} R_{g(t)} - 2\nabla^{g(t)}_{\dot\gamma(t)}\dot\gamma(t) + 4\mathrm{Ric}_{g(t)}(\dot\gamma(t), \cdot)$. In particular, $(\delta_V \mathcal{L}_0)(\gamma)$ is independent of the choice of a variation which realizes the variational vector field V as its infinitesimal variation. Hence we call

$$\nabla^{g(t)}_{\dot\gamma(t)}\dot\gamma(t) - \frac{1}{2}\nabla^{g(t)} R_{g(t)} - 2\mathrm{Ric}_{g(t)}(\dot\gamma(t), \cdot) = 0 \tag{5}$$

the $\mathcal{L}_0$-geodesic equation, where the $\mathrm{Ric}_{g(t)}(\dot\gamma(t), \cdot)$ is naturally understood as a $(1,0)$-tensor by the metric $g(t)$. We call any solution of $\mathcal{L}_0$-geodesic equation an $\mathcal{L}_0$-geodesic.

PROPOSITION 2.2. *Assume that L_0 is smooth around $(t', m'; t'', m'')$ and that there exists a unique $\mathcal{L}_0$-minimizing curve γ joining (t', m') and (t'', m''). Then we have*

$$\begin{cases} \dfrac{\partial L_0^{t',t''}}{\partial t'}(m', m'') = -\dfrac{1}{2}\big\{|\dot\gamma(t')|^2_{g(t')} + R_{g(t')}(m')\big\} - \nabla^{g(t')}_{\dot\gamma(t')} L_0^{t',t''}(\cdot, m''), \\ \dfrac{\partial L_0^{t',t''}}{\partial t''}(m', m'') = \dfrac{1}{2}\big\{|\dot\gamma(t'')|^2_{g(t'')} + R_{g(t'')}(m'')\big\} - \nabla^{g(t'')}_{\dot\gamma(t'')} L_0^{t',t''}(m', \cdot). \end{cases}$$

PROPOSITION 2.3. *Under the assumption in Proposition 2.2, we have*

$$\nabla^{g(t')}_{m'} L_0^{t',t''}(\cdot, m'') = -\dot\gamma(t') \quad \text{and} \quad \nabla^{g(t'')}_{m''} L_0^{t',t''}(m', \cdot) = \dot\gamma(t'').$$

In particular, by combining with Proposition 2.2,

$$\begin{cases} \dfrac{\partial L_0^{t',t''}}{\partial t'}(m', m'') = \dfrac{1}{2}\left\{|\dot\gamma(t')|^2_{g(t')} - R_{g(t')}(m')\right\}, \\ \dfrac{\partial L_0^{t',t''}}{\partial t''}(m', m'') = -\dfrac{1}{2}\left\{|\dot\gamma(t'')|^2_{g(t'')} - R_{g(t'')}(m'')\right\}. \end{cases}$$

We denote the curvature tensor with respect to $g(t)$ by $\mathcal{R}_{g(t)}$. It appears in the following second variation formula for $\mathcal{L}_0$. For a piecewise smooth curve $\gamma : [t', t''] \to M$ and two vector fields V, W along γ, we define the $\mathcal{L}_0$-index form $\mathcal{L}_0 I_\gamma(V, W)$ as follows:

$$\begin{aligned} \mathcal{L}_0 I_\gamma(V, W) := \int_{t'}^{t''} \Big\{ & \langle \nabla^{g(t)}_{\dot\gamma(t)} V(t), \nabla^{g(t)}_{\dot\gamma(t)} W(t)\rangle_{g(t)} + \langle \mathcal{R}_{g(t)}(V(t), \dot\gamma(t))W(t), \dot\gamma(t)\rangle \\ & + \frac{1}{2}\mathrm{Hess} R_{g(t)}(V(t), W(t)) + (\nabla^{g(t)}_V \mathrm{Ric}_{g(t)})(\dot\gamma(t), W(t)) \\ & + (\nabla^{g(t)}_W \mathrm{Ric}_{g(t)})(\dot\gamma(t), V(t)) - (\nabla^{g(t)}_{\dot\gamma(t)} \mathrm{Ric}_{g(t)})(V(t), W(t))\Big\} \mathrm{d}t. \end{aligned}$$

By definition, $\mathcal{L}_0 I_\gamma(V, W)$ is symmetric in V and W.

PROPOSITION 2.4. *For any smooth variation* (*not being necessarily proper*) *of an* $\mathcal{L}_0$*-geodesic* $\gamma : [t', t''] \to M$ *with a variational vector field* V*, we have*

$$(\delta_V \delta_V \mathcal{L}_0)(\gamma) = \left\langle \nabla^{g(t)}_{V(t)} V(t), \dot\gamma(t)\right\rangle_{g(t)}\Big|_{t=t'}^{t=t''} + \mathcal{L}_0 I_\gamma(V, V). \tag{6}$$

REMARK 2.5. In (6), the second term on the right hand side is independent of the choice of a variation of γ which realizes the variational vector field V as its infinitesimal variation. On the other hand, since γ is $\mathcal{L}_0$-geodesic, the first term can be written as follows:

$$\left\langle \nabla^{g(t)}_{V(t)} V(t), \dot\gamma(t)\right\rangle_{g(t)}\Big|_{t=t'}^{t=t''} = (\delta_{\nabla_V V}\mathcal{L}_0)(\gamma).$$

The next formula is derived from Proposition 2.4.

PROPOSITION 2.6. *Keeping the notations in Proposition* 2.4*, we have an alternative form of the second variational formula:*

$$\begin{aligned} (\delta_V \delta_V \mathcal{L}_0)(\gamma) = & \left\langle \nabla^{g(t)}_{V(t)} V(t), \dot\gamma(t)\right\rangle\Big|_{t=t'}^{t=t''} + \mathrm{Ric}_{g(t)}(V(t), V(t))\Big|_{t=t'}^{t=t''} \\ & + \frac{1}{2}\int_{t'}^{t''} \Big\{ \mathrm{Hess}_{g(t)} R_{g(t)}(V(t), V(t)) + 2\left\langle \mathcal{R}_{g(t)}(V(t), \dot\gamma(t))V(t), \dot\gamma(t)\right\rangle_{g(t)} \\ & - 2\frac{\mathrm{dRic}_{g(t)}}{\mathrm{d}t}(V(t), V(t)) \\ & + 4\left[\left(\nabla^{g(t)}_{V(t)} \mathrm{Ric}_{g(t)}\right)(V(t), \dot\gamma(t)) - \left(\nabla^{g(t)}_{\dot\gamma(t)} \mathrm{Ric}_{g(t)}\right)(V(t), V(t))\right] \\ & + 2\left|\nabla^{g(t)}_{\dot\gamma(t)} V(t) - \mathrm{Ric}_{g(t)}(V(t), \cdot)\right|^2_{g(t)} - 2|\mathrm{Ric}_{g(t)}(V(t), \cdot)|^2_{g(t)}\Big\} \mathrm{d}t. \end{aligned}$$

To deduce an estimate of (a contraction of) the Hessian for $L_0^{t',t''}$, we need a testing vector field. For this, we introduce the notion of space-time parallel transport. This notion will be used also to construct a coupling of two Brownian motions in subsequent sections.

DEFINITION 2.7 (Space-time parallel transport along an $\mathcal{L}_0$-minimizing curve). Let $m', m'' \in M$ and $t' < t''$. Let γ be *an* $\mathcal{L}_0$-minimizing curve joining (t', m') and (t'', m''). We define the *space-time parallel transport* $/\!\!/_{m',m''}^{t',t''} : T_{m'}M \to T_{m''}M$ along γ as

$$/\!\!/_{m',m''}^{t',t''}(v) := V(t'')$$

by solving the linear differential equation

$$\begin{cases} \nabla_{\dot\gamma(t)}^{g(t)} V(t) = \mathrm{Ric}_{g(t)}(V(t), \cdot)\,, & t' \le t \le t''\,, \\ V(t') = v\,. \end{cases}$$

One can check easily that $/\!\!/_{m',m''}^{t',t''}$ gives a linear isometry from $(T_{m'}M,\, g(t'))$ to $(T_{m''}M,\, g(t''))$. Note that the space-time parallel transport can be defined as an isometry for more general time-dependent metrics; see [15, Remark 5].

The main result in this subsection is the following.

PROPOSITION 2.8. *Let $u_1', \dots, u_d'$ be an orthonormal basis of $(T_{m'}M,\, g(t'))$. Under the assumption in Proposition* 2.2*, Then it holds that*

$$\sum_{i=1}^{d} \big[\mathrm{Hess}_{g(t')\oplus g(t'')} L_0^{t',t''}\big]\big(u_i' \oplus /\!\!/_{m',m''}^{t',t''} u_i',\, u_i' \oplus /\!\!/_{m',m''}^{t',t''} u_i'\big) \le \Big\{\frac{\partial L_0^{t',t''}}{\partial t'} + \frac{\partial L_0^{t',t''}}{\partial t''}\Big\}(m', m'')\,.$$

For the proof, we gather formulae for geometric quantities along the Ricci flow. For the proof, see [24, equation (2.1.9) and Subsection 2.5].

PROPOSITION 2.9. *Along the Ricci flow $\frac{\mathrm{d}g}{\mathrm{d}t}(t) = -2\mathrm{Ric}_{g(t)}$, one has*

(i) $\dfrac{\partial R_{g(t)}}{\partial t} = \Delta_{g(t)} R_{g(t)} + 2|\mathrm{Ric}_{g(t)}|_{g(t)}^2$,

(ii) $\mathrm{tr}\dfrac{\mathrm{dRic}_{g(t)}}{\mathrm{d}t} = \Delta_{g(t)} R_{g(t)}$,

(iii) contracted Bianchi identity: $\mathrm{tr}\big(\nabla \mathrm{Ric}_{g(t)}\big) = \dfrac{1}{2}\nabla^{g(t)} R_{g(t)}$.

Proof of Proposition 2.8. Let γ be an $\mathcal{L}_0$-minimizing curve from (t', m') to (t'', m''). By Proposition 2.3, Proposition 2.9 and (5), we see

$$\Big\{\frac{\partial L_0^{t',t''}}{\partial t'} + \frac{\partial L_0^{t',t''}}{\partial t''}\Big\}(m', m'')$$

$$= \frac{1}{2}\int_{t'}^{t''} \Big\{ \Delta_{g(t)} R_{g(t)} + 2|\mathrm{Ric}_{g(t)}|^2_{g(t)} - 2\mathrm{Ric}_{g(t)}(\dot\gamma(t), \dot\gamma(t)) \Big\} \mathrm{d}t\,.$$

Next we compute and give an estimate for (a contraction of) the Hessian. For each $i = 1, 2, \dots, d$, we define a system of vector fields $(A_i)_{i=1}^d$ along γ by

$$A_i(t) := /\!\!/^{t',t}_{m',\gamma(t)} u_i' \quad \text{for } t' \le t \le t''$$

and we take a variation $f_i : (-\varepsilon_0, \varepsilon_0) \times [t', t''] \to M$ of γ ($\varepsilon_0 > 0$ being small enough) such that

(a) $f_i(0, \cdot) = \gamma$,

(b) f_i has A_i as its variational field:

$$A_i(0, t) = A_i(t) \quad t' \le t \le t''$$

where $A_i(\varepsilon, t) := \frac{\mathrm{d} f_i}{\mathrm{d}\varepsilon}(\varepsilon, t)$ is the transversal vector field, and

(c) two transversal curves $f_i(\cdot, t')$, $f_i(\cdot, t'') : (-\varepsilon_0, \varepsilon_0) \to M$ are $g(t')$-geodesic and $g(t'')$-geodesic respectively at $\varepsilon = 0$:

$$\nabla^{g(t')}_{A_i(t')} A_i(\cdot, t') = 0 \quad \text{and} \quad \nabla^{g(t'')}_{A_i(t'')} A_i(\cdot, t'') = 0.$$

We further set

$$\ell_i(\varepsilon) := L_0^{t',t''}\big(f_i(\varepsilon, t'), f_i(\varepsilon, t'')\big) \le \mathcal{L}_0(f_i(\varepsilon, \cdot)) =: \widehat{\ell}_i(\varepsilon)\,.$$

It is easy to see that $\ell_i''(0) \le \widehat{\ell}_i''(0)$. Since

$$\nabla^{g(t')\oplus g(t'')}_{A_i(t')\oplus A_i(t'')} A_i(\cdot, t') \oplus A_i(\cdot, t'') = \big\{\nabla^{g(t')}_{A_i(t')} A_i(\cdot, t')\big\} \oplus \big\{\nabla^{g(t'')}_{A_i(t'')} A_i(\cdot, t'')\big\} = 0\,,$$

we can compute the Hessian as

$$\begin{aligned}
&\big[\mathrm{Hess}_{g(t')\oplus g(t'')} L_0^{t',t''}\big]\big(u_i' \oplus /\!\!/^{t',t''}_{m',m''} u_i', u_i' \oplus /\!\!/^{t',t''}_{m',m''} u_i'\big) \\
&\quad = \big[\mathrm{Hess}_{g(t')\oplus g(t'')} L_0^{t',t''}\big]\big(A_i(t') \oplus A_i(t''), A_i(t') \oplus A_i(t'')\big) \\
&\quad = A_i(t') \oplus A_i(t'')\big\{\big(A_i(\cdot, t') \oplus A_i(\cdot, t'')\big) L_0^{t',t''}\big\} \\
&\quad = \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\bigg|_{\varepsilon=0} L_0^{t',t''}\big(f_i(\varepsilon, t'), f_i(\varepsilon, t'')\big) = \ell_i''(0) \le \widehat{\ell}_i''(0) = (\delta_{A_i}\delta_{A_i}\mathcal{L}_0)(\gamma)\,.
\end{aligned}$$

By the second variational formula (Proposition 2.6), we have

$$\begin{aligned}
&(\delta_{A_i}\delta_{A_i}\mathcal{L}_0)(\gamma) \\
&\quad = \mathrm{Ric}_{g(t)}(A_i(t), A_i(t))\big|_{t=t'}^{t=t''} \\
&\qquad + \frac{1}{2}\int_{t'}^{t''} \big\{\mathrm{Hess}_{g(t)} R_{g(t)}(A_i(t), A_i(t)) \\
&\qquad + 2\big\langle \mathcal{R}_{g(t)}(A_i(t), \dot\gamma(t)) A_i(t), \dot\gamma(t)\big\rangle_{g(\tau)} - 2\frac{\mathrm{dRic}_{g(t)}}{\mathrm{d}t}(A_i(t), A_i(t)) \\
&\qquad + 4\big[\big(\nabla^{g(t)}_{A_i(t)}\mathrm{Ric}_{g(t)}\big)(A_i(t), \dot\gamma(t)) - \big(\nabla^{g(t)}_{\dot\gamma(t)}\mathrm{Ric}_{g(t)}\big)(A_i(t), A_i(t))\big]
\end{aligned}$$

$$+2\big|\nabla^{g(t)}_{\dot\gamma(t)}A_i(t)-\mathrm{Ric}_{g(t)}(A_i(t),\cdot)\big|^2_{g(t)}-2|\mathrm{Ric}_{g(t)}(A_i(t),\cdot)|^2_{g(t)}\Big\}\mathrm{d}t\,.$$

Hence, by taking the sum over $i=1,2,\dots,d$ with formulae in Proposition 2.9, we have

$$\begin{aligned}
&\sum_{i=1}^d\big[\mathrm{Hess}_{g(t')\oplus g(t'')}L_0^{t',t''}\big]\big(u_i'\oplus /\!\!/^{t',t''}_{m',m''}u_i',\,u_i'\oplus /\!\!/^{t',t''}_{m',m''}u_i'\big)\\
&\le R_{g(t)}(\gamma(t))\Big|^{t=t''}_{t=t'}\\
&\quad+\frac12\int_{t'}^{t''}\Big\{\Delta_{g(t)}R_{g(t)}-2\mathrm{Ric}_{g(t)}(\dot\gamma(t),\dot\gamma(t))-2\Delta_{g(t)}R_{g(t)}\\
&\qquad\qquad+4\Big[\frac12\nabla^{g(t)}_{\dot\gamma(t)}R_{g(t)}-\nabla^{g(t)}_{\dot\gamma(t)}R_{g(t)}\Big]-2|\mathrm{Ric}_{g(t)}|^2_{g(t)}\Big\}\mathrm{d}t\\
&=\frac12\int_{t'}^{t''}\Big\{2\Delta_{g(t)}R_{g(t)}+4|\mathrm{Ric}_{g(t)}|^2_{g(t)}+2\nabla^{g(t)}_{\dot\gamma(t)}R_{g(t)}\Big\}\mathrm{d}\tau\\
&\quad+\frac12\int_{t'}^{t''}\Big\{-\Delta_{g(t)}R_{g(t)}-2\mathrm{Ric}_{g(t)}(\dot\gamma(t),\dot\gamma(t))-2\nabla^{g(t)}_{\dot\gamma(t)}R_{g(t)}-2|\mathrm{Ric}_{g(t)}|^2_{g(t)}\Big\}\mathrm{d}t\\
&=\frac12\int_{t'}^{t''}\Big\{\Delta_{g(t)}R_{g(t)}+2|\mathrm{Ric}_{g(t)}|^2_{g(t)}-2\mathrm{Ric}_{g(t)}(\dot\gamma(t),\dot\gamma(t))\Big\}\mathrm{d}t\,,
\end{aligned}$$

which is equal to $\Big\{\frac{\partial L_0^{t',t''}}{\partial t'}+\frac{\partial L_0^{t',t''}}{\partial t''}\Big\}(m',m'')$. □

2.2. Some estimates on relatively compact open subsets. We begin with estimates which hold *globally* under the curvature assumption (4). Let $K_->0$ be a constant satisfying $-K_-g(t)\le\mathrm{Ric}_{g(t)}$ for all $t\in[0,T]$. Recall that $\dim M=d$ and our Ricci flow is defined on $[0,T]$. Given any metric g, we denote by ρ_g the corresponding Riemannian distance.

PROPOSITION 2.10.

(i) Comparison of metric $g(t)$ between two different times:

$$g(t)\le\mathrm{e}^{2K_-(t-s)}g(s)\quad\textit{for } 0\le s\le t\le T.$$

(ii) Comparison of distance $\rho_{g(t)}$ between two different times:

$$\rho_{g(t)}(x,y)\le\mathrm{e}^{K_-(t-s)}\rho_{g(s)}(x,y)$$

for any $x,y\in M$ and $0\le s\le t\le T$.

(iii) Lower bound for L_0: *For $0\le t'<t''\le T$, $m',m''\in M$ and a piecewise C^1 curve $\gamma:[t',t'']\to M$ with $\gamma(t')=m'$ and $\gamma(t'')=m''$,*

$$\mathcal{L}_0(\gamma)\ge\frac12\mathrm{e}^{-K_-(t''-t')}\int_{t'}^{t''}|\dot\gamma(t)|^2_{g(t'')}\mathrm{d}t-\frac{dK_-}{2}(t''-t').$$

In particular,

$$L_0^{t',t''}(m',m'')\ge\frac12\mathrm{e}^{-K_-(t''-t')}\ \frac{\rho_{g(t'')}(m',m'')^2}{t''-t'}-\frac{dK_-}{2}(t''-t')\,,$$

$$\inf_{\substack{(t',m';t'',m'')\in([0,T]\times M)^2\\ t'<t''}} L_0^{t',t''}(m',m'') \geq -\frac{dK_-T}{2}.$$

Although the proof is parallel to the corresponding statements in $\mathcal{L}$-geometry (especially (i) and (ii) are irrelevant to $\mathcal{L}$ or $\mathcal{L}_0$), we give it for completeness.

PROOF. (i) Since $\frac{\partial g}{\partial t} = -2\mathrm{Ric}_g \leq 2K_- g$, apply Gronwall's lemma.

(ii) Obviously follows from (i).

(iii) Take a piecewise C^1-curve $\gamma : [t', t''] \to M$ with $\gamma(t') = m'$ and $\gamma(t') = m''$. Then, by the choice of K_- and (ii),

$$\begin{aligned}\mathcal{L}_0(\gamma) &\geq \frac{1}{2}\int_{t'}^{t''} |\dot\gamma(t)|^2_{g(t)}\mathrm{d}t + \frac{dK_-}{2}\\ &\geq \frac{1}{2}\mathrm{e}^{-K_-(t''-t')}\int_{t'}^{t''} |\dot\gamma(t)|^2_{g(t'')}\mathrm{d}t + \frac{dK_-}{2}\\ &\geq \frac{1}{2}\mathrm{e}^{-K_-(t''-t')}\frac{\rho_{g(t'')}(m',m'')^2}{t''-t'} + \frac{dK_-}{2}.\end{aligned}$$

Thus the conclusion holds since γ is arbitrary. □

PROPOSITION 2.11. *Let $m', m'' \in M$, $0 \leq t' < t'' \leq T$ and let $\gamma : [t', t''] \to M$ be an $\mathcal{L}_0$-geodesic joining (t', m') to (t'', m''). For each curve $\eta : [t', t''] \to M$, put*

$$\begin{aligned}K_+(\eta) &:= \inf\left\{K > 0 :\ \mathrm{Ric}_{g(t)} \leq K g(t) \text{ along } \eta(t)\ \right\},\\ C(\eta) &:= \sup_{t\in[t',t'']} |\nabla^{g(t)} R_{g(t)}(\eta(t))|^2_{g(t)}.\end{aligned}$$

Then we have the following:

(i) Upper bound for L_0: *For each $g(t')$-geodesic $c : [t', t''] \to M$ with $c(t') = m'$ and $c(t'') = m''$,*

$$L_0^{t',t''}(m',m'') \leq \frac{\rho_{g(t')}(m',m'')^2}{2(t''-t')^2}\frac{\mathrm{e}^{2K_-(t''-t')}-1}{2K_-} + \frac{dK_+(c)(t''-t')}{2}.$$

(ii) Bound of $\frac{\mathrm{d}}{\mathrm{d}t}|\dot\gamma(t)|^2_{g(t)}$: *For each $t \in [t', t'']$, we have*

$$-\Big(2K_- + \frac{1}{2}\Big)|\dot\gamma(t)|^2_{g(t)} - \frac{C(\gamma)}{2} \leq \frac{\mathrm{d}}{\mathrm{d}t}|\dot\gamma(t)|^2_{g(t)} \leq \Big(2K_+(\gamma) + \frac{1}{2}\Big)|\dot\gamma(t)|^2_{g(t)} + \frac{C(\gamma)}{2}.$$

(iii) Comparison of $|\dot\gamma(t)|^2_{g(t)}$ between different times: *There are constants $c_i > 0$ ($i = 1, 2, 3, 4$) such that, for $t' \leq u' \leq u'' \leq t''$,*

$$\begin{aligned}|\dot\gamma(u'')|^2_{g(u'')} &\leq c_1|\dot\gamma(u')|^2_{g(u')} + c_2,\\ |\dot\gamma(u')|^2_{g(u')} &\leq c_3|\dot\gamma(u'')|^2_{g(u'')} + c_4\end{aligned}$$

where the constants depend only on T, K_-, $K_+(\gamma)$ and $C(\gamma)$.

(iv) Bounding the speed of $\mathcal{L}_0$-minimizing curve at some time by $\mathcal{L}_0$: *If the curve γ is $\mathcal{L}_0$-minimizing, there is $t^* \in (t', t'')$ such that*

$$\frac{1}{2}|\dot\gamma(t^*)|^2_{g(t^*)} \le \frac{L_0^{t',t''}(m', m'')}{t'' - t'} + \frac{dK_-}{2}.$$

PROOF. (i) The proof is similar to Proposition 2.10(iii). By Proposition 2.10(i), we see that

$$L_0^{t',t''}(m', m'') \le \mathcal{L}_0(c) \le \frac{1}{2}\int_{t'}^{t''} \left\{ \mathrm{e}^{2K_-(t-t')}|\dot c(t)|^2_{g(t')} + dK_+(c)\right\}\mathrm{d}t\,.$$

Since $|\dot c(t)|_{g(t')} \equiv \frac{\rho_{g(t')}(m',m'')}{t''-t'}$, the claim holds.

(ii) Using the $\mathcal{L}_0$-geodesic equation,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|\dot\gamma(t)|^2_{g(t)} &= -2\mathrm{Ric}_{g(t)}(\dot\gamma(t), \dot\gamma(t)) + 2\left\langle \nabla^{g(t)}_{\dot\gamma(t)}\dot\gamma(t), \dot\gamma(t)\right\rangle_{g(t)} \\
&= 2\mathrm{Ric}_{g(t)}(\dot\gamma(t), \dot\gamma(t)) + \left\langle \nabla^{g(t)} R_{g(t)}, \dot\gamma(t)\right\rangle_{g(t)} \\
&\le 2\mathrm{Ric}_{g(t)}(\dot\gamma(t), \dot\gamma(t)) + \frac{|\nabla^{g(t)} R_{g(t)}|^2_{g(t)} + |\dot\gamma(t)|^2_{g(t)}}{2} \\
&\le \left(2K_+(\gamma) + \frac{1}{2}\right)|\dot\gamma(t)|^2_{g(t)} + \frac{C(\gamma)}{2}\,.
\end{aligned}$$

The lower bound $\frac{\mathrm{d}}{\mathrm{d}t}|\dot\gamma(t)|^2_{g(t)} \ge -(2K_- + \frac{1}{2})|\dot\gamma(t)|^2_{g(t)} - \frac{C(\gamma)}{2}$ can be obtained similarly.

(iii) By (ii), Gronwall's lemma implies

$$\begin{aligned}
&|\dot\gamma(u'')|^2_{g(u'')} \\
&\quad \le \mathrm{e}^{(2K_+(\gamma)+\frac{1}{2})(u''-u')}|\dot\gamma(u')|^2_{g(u')} + \frac{C(\gamma)}{4K_+(\gamma)+1}\left(\mathrm{e}^{(2K_+(\gamma)+\frac{1}{2})(u''-u')} - 1\right).
\end{aligned}$$

The other is obtained similarly.

(iv) By the mean value theorem, we can take $t^* \in (t', t'')$ such that

$$|\dot\gamma(t^*)|^2_{g(t^*)} = \frac{1}{t'' - t'}\int_{t'}^{t''} |\dot\gamma(t)|^2_{g(t)}\mathrm{d}t\,.$$

Since γ is $\mathcal{L}_0$-minimizing, the right hand side is dominated by

$$\frac{L_0^{t',t''}(m', m'')}{t'' - t'} - \frac{1}{2(t'' - t')}\int_{t'}^{t''} R_{g(t)}(\gamma(t))\mathrm{d}t \le \frac{L_0^{t',t''}(m', m'')}{t'' - t'} + \frac{dK_-}{2}\,.$$

□

The following is a starting point of local estimates in this subsection.

LEMMA 2.12. *For each $\delta > 0$ and a relatively compact open subset $M_0 \subset M$, there exists a relatively compact open subset $B \supset M_0$ such that, for each $m', m'' \in M_0$ and $t, t', t'' \in [0, T]$ with $t'' - t' \ge \delta$, all $\mathcal{L}_0$-minimizing curves joining (t', m') and (t'', m'') and all $g(t)$-length-minimizing curves joining m' and m'' are contained in B.*

PROOF. Since M_0 is relatively compact, there is a compact B_0 with $M_0 \subset B_0 \subset M$ which contains all $g(0)$-geodesics joining any pair of points in M_0. Let us define $K_+ = K_+(B_0)$ and $R_1 := R_1(B_0)$ by

$$K_+(B_0) := \inf\left\{K > 0 : \mathrm{Ric}_{g(t)} \le K g(t) \text{ on } B_0\right\},$$
$$R_1(B_0) := \sup_{m',m'' \in M_0} \left\{ \frac{\rho_{g(0)}(m',m'')^2}{2\delta^2} \frac{\mathrm{e}^T - 1}{2K_-} + \frac{dK_+(B_0)T}{2} \right\}.$$

Then, by Proposition 2.11(i) and Proposition 2.10(ii), $L_0^{t',t''}(m',m'') \le R_1$ holds for all $0 \le t' < t'' \le T$ with $t'' - t' \ge \delta$ and $m', m'' \in M_0$. Let $R_2 = R_2(B_0)$ be defined as follows:

$$R_2(B_0) := 2T\mathrm{e}^{K_-T}\left(R_1 + \frac{dK_-T}{2}\right).$$

Take a relatively compact open B with $B_1 \subset B \subset M$ such that $\rho_{g(T)}(M_0, B^c) > \sqrt{R_2/2}$. Then, for any curve $\gamma : [t', t''] \to M$ with $\gamma(t'), \gamma(t'') \in M_0$ and $\gamma([t', t'']) \cap B^c \ne \emptyset$, Proposition 2.10(iii) and Proposition 2.10(i) yields $\mathcal{L}_0(\gamma) > R_1$. Thus B enjoys the claimed property on $\mathcal{L}_0$-minimizing curves. By a similar argument, we can prove the corresponding property for $g(t)$-geodesics by using Proposition 2.10(ii). Thus, the assertion holds by enlarging B if necessary. □

REMARK 2.13. By Proposition 2.11 and Lemma 2.12, we see that for each bounded open set $M_0 \subset M$ and $\delta > 0$, there is positive constants $K_+ = K_+(M_0, \delta) > 0$ and $C = C(M_0, \delta) > 0$ such that

$$\mathrm{Ric}_g \le K_+ g\,, \quad \text{and} \quad |\nabla^{g(t)} R_{g(t)}(m)|^2_{g(t)} < C \quad \text{on } [0,T] \times M_0\,,$$

and further, for each m', $m'' \in M_0$, $t'' - t' \ge \delta$ and each $\mathcal{L}_0$-minimizing curve γ joining (t', m') and (t'', m''), we have

$$L_0^{t',t''}(m',m'') \le \frac{\rho_{g(t')}(m',m'')^2}{2(t''-t')^2} \frac{\mathrm{e}^{2K_-t''} - \mathrm{e}^{2K_-t'}}{2K_-} + \frac{dK_+(t''-t')}{2} \tag{7}$$

and

$$-\left(2K_- + \frac{1}{2}\right)|\dot\gamma(t)|^2_{g(t)} - \frac{C}{2} \le \frac{\mathrm{d}}{\mathrm{d}t}|\dot\gamma(t)|^2_{g(t)} \le \left(2K_+ + \frac{1}{2}\right)|\dot\gamma(t)|^2_{g(t)} + \frac{C}{2}. \tag{8}$$

LEMMA 2.14. *$(t', m'; t'', m'') \mapsto L_0^{t',t''}(m', m'')$ is continuous on $\{(t', m'; t'', m'') \mid 0 \le t' < t'' \le T, m', m'' \in M\}$.*

PROOF. Let $0 \le t'_0 < t''_0 \le T$ and $m'_0, m''_0 \in M$ and take $\varepsilon \in (0,1)$. Take $\delta_0 > 0$ so that $t''_0 - t'_0 \ge 2\delta_0$. Let U' and U'' be $g(0)$-metric balls of radius δ_0 centered at m' and m'' respectively. Let B be as in Lemma 2.12 for $M_0 = U' \cup U''$ and $\delta = \delta_0$. Take $\delta'_1 \in (0, \delta_0/4)$ so that it satisfies the following:

- $(1-\varepsilon)g(s) \le g(t) \le (1+\varepsilon)g(s)$ and $|R_{g(s)} - R_{g(t)}| < \varepsilon$ on B for each $s, t \in [0,T]$ with $|s-t| < 2\delta'_1$,
- $4\delta'_1/(\delta_0 - 2\delta'_1) < \varepsilon$.

Let K_+ and C be as in Remark 2.2, corresponding to M_0 and δ_1'. Take $\delta_1 \in (0, \delta_1')$ so that $dK_+\delta_1/2 < \varepsilon$. Take $\delta_2 > 0$ so that

$$\frac{\delta_2^2(e^{4K_-T} - e^{2K_-(2T-\delta_1)})}{4\delta_1^2 K_-} < \varepsilon$$

holds and take $\delta_3 = \delta_1 \wedge \delta_2$. Let $m', m'' \in M$ with $\rho_{g(0)}(m', m_0') \vee \rho_{g(0)}(m'', m_0'') < \delta_3$ and $t', t'' \in [0, T]$ with $|t' - t_0'| \vee |t'' - t_0''| < \delta_3$. Take a curve $\gamma : [t_0', t_0''] \to M$ from (t_0', m_0') to (t_0'', m_0'') such that $\mathcal{L}_0(\gamma) \le L^{t_0',t_0''}(m_0', m_0'') + \varepsilon$. In addition, we take a curve γ_1 from (t', m') to $(t' + \delta_1, m_0')$, and a curve γ_2 from $(t'' - \delta_1, m_0'')$ to (t'', m''). Let $\alpha : [t' + \delta_1, t'' - \delta_1] \to [t_0', t_0'']$ be a (unique) affine increasing surjection. We define $\tilde{\gamma} : [t', t''] \to M$ as follows:

$$\tilde{\gamma}(t) := \begin{cases} \gamma_1(t) & (t \in [t', t' + \delta_1)), \\ \gamma(\alpha(t)) & (t \in [t' + \delta_1, t'' - \delta_1)), \\ \gamma_2(t) & (t \in [t'' - \delta_1, t'']). \end{cases}$$

Note that, by the choice of δ_1,

$$\alpha'(t) = \frac{t_0'' - t_0'}{t'' - t' - 2\delta_1} \in [1, 1 + \varepsilon], \tag{9}$$

$$\begin{aligned} |\alpha^{-1}(t) - t| &= \left| \frac{t'' - t' - 2\delta_1}{t_0'' - t_0'}(t - t_0') - (t - t' - \delta_1) \right| \\ &\le |t_0' - t' - \delta_1| \vee |t'' - t_0'' - \delta_1| \le 2\delta_1. \end{aligned} \tag{10}$$

Now we turn to the estimate. We begin with the following basic estimate:

$$\begin{aligned} L_0^{t',t''}(m', m'') - L_0^{t_0',t_0''}(m_0', m_0'') \le \mathcal{L}_0(\gamma_1) + \mathcal{L}_0(\gamma_2) \\ + \mathcal{L}_0(\tilde{\gamma}|_{[t'+\delta_1, t''-\delta_1]}) - \mathcal{L}_0(\gamma) + \varepsilon. \end{aligned} \tag{11}$$

By the choice of $\tilde{\gamma}$, we have

$$\begin{aligned} &\mathcal{L}_0(\tilde{\gamma}|_{[t'+\delta_1, t''-\delta_1]}) - \mathcal{L}_0(\gamma) \\ &\quad = \frac{1}{2}\int_{t_0'}^{t_0''} \left\{ \frac{t_0'' - t_0'}{t'' - t' - 2\delta_1} |\dot{\gamma}(t)|^2_{g(\alpha^{-1}(t))} - |\dot{\gamma}(t)|^2_{g(t)} \right\} \mathrm{d}t \\ &\qquad + \frac{1}{2}\int_{t_0'}^{t_0''} \left\{ \frac{t'' - t' - 2\delta_1}{t_0'' - t_0'} R_{g(\alpha^{-1}(t))} - R_{g(t)} \right\} \mathrm{d}t. \end{aligned}$$

Then the choice of δ_1 together with (8), (9) and (10) yields that there is a constant $\hat{C} > 0$ depending only on T, $K_\pm$, d and C such that the right hand side of the last equality is bounded from above by $\hat{C}\varepsilon$. Moreover, by virtue of the choice of δ_1 and δ_2, (7) yields $L_0^{t',t'+\delta_1}(m', m_0') \vee L_0^{t''-\delta_1,t''}(m_0'', m'') \le 2\varepsilon$. Thus, by minimizing the right hand side of (11) over γ_1 and γ_2, the left hand side is bounded from above by $(5 + \hat{C})\varepsilon$. We can give the same upper bound to $L_0^{t_0',t_0''}(m_0', m_0'') - L_0^{t',t''}(m', m'')$ in the same manner and hence the assertion holds. □

We fix a bounded open $M_0 \subset M$ and $\delta > 0$ arbitrarily and denote by $K_+ = K_+(M_0, \delta)$ and $C = C(M_0, \delta)$ the positive constants appeared in Remark 2.13. Let

$$\mathcal{M}_0 := \left\{(t', m'; t'', m'') \in [0, T] \times M_0 \times [0, T] \times M_0 : t' < t''\right\}.$$

For each $(t', m'; t'', m'') \in \mathcal{M}_0$, we denote by $\Gamma_{m',m''}^{t',t''}$ the set of all $\mathcal{L}_0$-minimizing paths joining (t', m') to (t'', m''). We further define

$$\begin{aligned}
\mathcal{M}_\delta &:= \left\{(t', m'; t'', m'') \in \mathcal{M}_0 : t'' - t' > \delta\right\}, \\
\Delta_\delta &:= \left\{(t', t'') \in [0, T] \times [0, T] : t'' - t' > \delta\right\}, \\
\Gamma_\delta &:= \bigcup_{(t',m';t'',m'') \in \overline{\mathcal{M}_\delta}} \Gamma_{m',m''}^{t',t''} \quad \text{and} \\
\iota &: \Gamma_\delta \to \overline{\Delta_\delta} \times C([0, 1] \to M) \\
&\qquad\qquad \text{by} \quad \iota(\gamma) := (t', t'', \widehat{\gamma}) \quad \text{if } \gamma \in \Gamma_{m',m''}^{t',t''}
\end{aligned}$$

where $\widehat{\gamma} \in C([0, 1] \to M)$ is defined by

$$\widehat{\gamma}(u) := \gamma\big(t' + u(t'' - t')\big) \quad \text{for } 0 \le u \le 1.$$

We topologize Γ_δ by the pull-back distance

$$d_{\Gamma_\delta}(\gamma, \eta) := |t' - s'| + |t'' - s''| + \sup_{0 \le u \le 1} \rho_{g(T)}(\widehat{\gamma}(u), \widehat{\eta}(u))$$

for $\gamma \in \Gamma_{m',m''}^{t',t''}$ and $\eta \in \Gamma_{n',n''}^{s',s''}$.

LEMMA 2.15. *There are constant $c_5, c_6 > 0$ such that, for each $0 \le t' < t'' \le T$, m', $m'' \in \overline{M_0}$ and $\gamma \in \Gamma_{m',m''}^{t',t''}$,*

$$\sup_{t' \le t \le t''} |\dot{\gamma}(t)|_{g(t)}^2 \le 2c_5 \left\{ \frac{L_0^{t',t''}(m', m'')}{t'' - t'} + \frac{dK_-}{2} \right\} + c_6.$$

PROOF. For each $0 \le t' < t'' \le T$, $m', m'' \in \overline{M_0}$ and $\gamma \in \Gamma_{m',m''}^{t',t''}$, the claimed bound follows from Proposition 2.11(iii) and (iv) with constants $\max\{c_1, c_3\}$ and $\max\{c_2, c_4\}$ as c_5 and c_6. By virtue of Lemma 2.12 (cf. Remark 2.13), we can choose them to be independent of t', t'', m', m'' and γ. □

PROPOSITION 2.16 (Compactness result). *We have the following*:

(i) Equi-Lipschitz estimate: *For each $0 \le t' < t'' \le T$, m', $m'' \in \overline{M_0}$, $\gamma \in \Gamma_{m',m''}^{t',t''}$ and $t' \le s \le u \le t''$,*

$$\rho_{g(T)}(\gamma(s), \gamma(u)) \le \text{const.}\left(\frac{1}{t'' - t'} + 1\right)|u - s|,$$

where the constant depends on K_-, K_+, C but not on s, u, t', t'', m', m'' and the choice of γ.

(ii) Uniform boundedness: *Γ_δ is uniformly bounded.*

(iii) Closedness: *$\iota(\Gamma_\delta)$ is closed in $\overline{\Delta}_\delta \times C([0,1] \to M)$, where the topology of $\overline{\Delta}_\delta \times C([0,1] \to M)$ is given by the product of the Euclidean one in $\overline{\Delta}_\delta$ and the uniform topology in $C^1([0,1] \to M)$.*

Moreover, Γ_δ is compact.

PROOF. Equi-Lipschitz estimate: For $0 \le s < u \le T$, we have

$$\rho_{g(T)}\big(\gamma(s),\gamma(u)\big)^2 \le \int_s^u |\dot\gamma(t)|^2_{g(T)}\mathrm{d}t \le \mathrm{e}^{2K_-T}\int_s^u |\dot\gamma(t)|^2_{g(t)}\mathrm{d}t,$$

where we have used Proposition 2.10(i) in the last inequality. Then the equi-Lipschitz estimate follows by Lemma 2.15 and the compactness of $\overline{M_0}$.

Uniform boundedness: It is obvious from the Equi-Lipschitz estimate.

Closedness: Define $\mathscr{L}_0 : \overline{\Delta}_\delta \times C^1([0,1] \to M) \to (-\infty, +\infty]$ by

$$\mathscr{L}_0(t',t'',c) := \frac{1}{2}\int_{t'}^{t''} \big\{|\dot{\widetilde{c}}(t)|^2_{g(t)} + R_{g(t)}(\widetilde{c}(t))\big\}\mathrm{d}t$$

where $\widetilde{c}(t) := c\big(\frac{t-t'}{t''-t'}\big)$ for $t' \le t \le t''$. Then $\mathscr{L}_0$ is lower semicontinuous (see e.g., [19, Appendix 1, pp. 198–201]). By using $\mathscr{L}_0$, $\iota(\Gamma_\delta)$ is expressed as follows:

$$\iota(\Gamma_\delta) = \left\{(t',t'',c) : \begin{array}{l} c(0), c(1) \in \overline{M},\, t''-t' \ge \delta \\ \text{and } \mathscr{L}_0(t',t'',c) \le L_0^{t',t''}(c(0),c(1)) \end{array}\right\}.$$

This expression yields that $\iota(\Gamma_\delta)$ is closed in $\overline{\Delta}_\delta \times C([0,1] \to M)$.

Finally, by combining them with Ascoli's compactness theorem, we conclude that Γ_δ is compact. □

2.3. The $\mathcal{L}_0$-cut locus. Set

$$\mathcal{M} := \big\{(t',m';t'',m'') \in [0,T]\times M \times [0,T]\times M : t' < t''\big\}.$$

DEFINITION 2.17 ($\mathcal{L}_0$-exponential map and $\mathcal{L}_0$-cut locus).

(i) $\mathcal{L}_0\exp_{m'}^{t',t''} : T_{m'}M \to M$ is defined by

$$\mathcal{L}_0\exp_{m'}^{t',t''}(v) := \gamma(t'')$$

where the curve γ is the solution to the $\mathcal{L}_0$-geodesic equation

$$\begin{cases} \nabla^{g(t)}_{\dot\gamma(t)}\dot\gamma(t) - \dfrac{1}{2}\nabla^{g(t)}R_{g(t)} - 2\mathrm{Ric}_{g(t)}(\dot\gamma(t),\cdot) = 0\,, \\ \gamma(t') = m'\,, \quad \dot\gamma(t') = v \in T_{m'}M\,. \end{cases}$$

See the Remark 2.18(i) below.

(ii) $\mathcal{L}_0\mathrm{cut}_{m'}^{t',t''}$ is the set of all points $m'' \in M$ such that there is more than one $\mathcal{L}_0$-minimizing curve $\gamma : [t',t''] \to M$ with $\gamma(t') = m'$ and $\gamma(t'') = m''$ or there is a $v \in T_{m'}M$ such that $m'' = \mathcal{L}_0\exp_{m'}^{t',t''}(v)$ and v is a critical point of $\mathcal{L}_0\exp_{m'}^{t',t''}$. We also use the notation $\mathcal{L}_0\mathrm{cut}^{t',t''} := \{(m',m'') : m'' \in \mathcal{L}_0\mathrm{cut}_{m'}^{t',t''}\}$.

(iii) $\mathcal{L}_0\mathrm{cut} := \left\{(t', m'; t'', m'') \in \mathcal{M} : m'' \in \mathcal{L}_0\mathrm{cut}_{m'}^{t',t''}\right\}$.

REMARK 2.18. (i) Since our Ricci flow is assumed to be complete, we see that $\mathcal{L}_0 \exp_{m'}^{t',t''}$ is well-defined as follows. Given initial data $\gamma(t') = m'$ and $\dot\gamma(t') = v$, let $I \subset [t', T]$ be the maximal interval, on which the $\mathcal{L}_0$-geodesic equation for γ can be solved (Recall that our Ricci flow is defined on $[0, T]$). Since the $\mathcal{L}_0$-geodesic equation is of the normal form, the standard theory of ODE shows that I is open in $[t', T]$. On the other hand, the Gronwall lemma, applied to the first inequality in Proposition 2.11(iii) (take $u' = t'$ and $u'' = t$), gives an upper bound for $\sup_{t\in I} |\dot\gamma(t)|_{g(t)}$. This implies that $\{\gamma(t) : t \in I\}$ is a bounded set. From the completeness of the metric, $\gamma(t)$ converges as $t \to \sup I$, which turns to show that I is closed in $[t', T]$. Since $[t', T]$ is connected and $I \neq \emptyset$, we must have $I = [t', T]$. As v is arbitrary, we conclude that $\mathcal{L}_0 \exp_{m'}^{t',t''}$ is defined on the whole of $T_{m'}M$.

(ii) For any $\mathcal{L}_0$-geodesic $\gamma : [t', t''] \to M$, if $\dot\gamma(t') \neq 0$, then $\dot\gamma(t'') \neq 0$ must hold. This can be seen as follows: Let $\eta(t) := \gamma(t'' + t' - t)$ and $\widetilde{g}(t) := g(t'' + t' - t)$ for $t \in [t', t'']$. Then η satisfies the differential equation

$$\nabla^{\widetilde{g}(t)}_{\dot\eta(t)}\dot\eta(t) - \frac{1}{2}\nabla^{\widetilde{g}(t)} R_{\widetilde{g}(t)} + 2\mathrm{Ric}_{\widetilde{g}(t)}(\dot\eta(t), \cdot) = 0$$

with $\eta(t') = \gamma(t'')$ and $\dot\eta(t') = -\dot\gamma(t'')$. If $\dot\gamma(t'') = 0$, then $\nabla^{\widetilde{g}(t')} R_{\widetilde{g}(t')}(\eta(t')) = \nabla^{g(t'')} R_{g(t'')}$ $(\gamma(t'')) = 0$. Therefore $\eta^*(t) \equiv \eta(t')$ also satisfies the same differential equation as η with the same initial condition. By uniqueness, $\eta^*(t) \equiv \eta(t)$ must hold and hence $\dot\gamma(t') = -\dot\eta(t'') = 0$. Thus the assertion holds.

PROPOSITION 2.19.

(i) *For each $t' < t''$ and $m', m'' \in M$, there is a smooth path $\gamma : [t', t''] \to M$ joining m' to m'' such that γ has the minimal $\mathcal{L}_0$-length among all such paths* (*see Lemma* 7.27 *in* [6]).

(ii) *For any $m' \in M$ and $t' < t''$, $\mathcal{L}_0\mathrm{cut}_{m'}^{t',t''}$ is closed and of measure zero* (*see Lemma* 7.99 *in* [6] *and Lemma* 5 *in* [20]).

(iii) *The $\mathcal{L}_0$*cut *is closed.*

(iv) *The function $L_0^{t',t''}(m', m'')$ is smooth on $\mathcal{M}\backslash\mathcal{L}_0$*cut.

REMARK 2.20. By Proposition 2.1, any $\mathcal{L}_0$-minimizing curve must be $\mathcal{L}_0$-geodesic and then Proposition 2.19(i) shows that $\exp_{m'}^{t',t''}$ is surjective. Additionally, by (iv), the statements in Propositions 2.2, 2.3 and 2.8 hold outside $\mathcal{L}_0$cut.

Since they can be shown by the same arguments as the usual Riemannian geometry or $\mathcal{L}$-geometry, we omit the proof of (i) and (iv). The proof of (iii) is along the same line of Lemma 5 in [20].

PROOF OF PROPOSITION 2.19 (ii). Since the closedness follows from (iii), we prove that $\mathcal{L}_0\mathrm{cut}_{m'}^{t',t''}$ is of measure zero. First we decompose $\mathcal{L}_0\mathrm{cut}_{m'}^{t',t''} = \mathcal{C} \cup \mathcal{C}'$ where $\mathcal{C}$ is the set

of all critical values in $\mathcal{L}_0\mathrm{cut}_{m'}^{t',t''}$ of $\mathcal{L}_0 \exp_{m'}^{t',t''}$ and

$$\mathcal{C}' := \left\{ m'' \in \mathcal{L}_0\mathrm{cut}_{m'}^{t',t''} : \begin{array}{l} m'' \text{ is a regular value of } \mathcal{L}_0 \exp_{m'}^{t',t''} \text{ and} \\ \text{there is more than one } \mathcal{L}_0\text{-minimizing} \\ \text{curve joining } (t', m') \text{ and } (t'', m''). \end{array} \right\}.$$

By Sard's theorem, $\mathcal{C}$ has measure zero and hence we need only to prove so is also $\mathcal{C}'$. For this, consider the map $\phi : T_{m'}M \times T_{m'}M \to \mathbb{R}$ defined by

$$\phi(v, w) := \mathcal{L}_0(\gamma_v) - \mathcal{L}_0(\gamma_w)$$

where for each $v \in T_{m'}M$, the curve $\gamma_v : [t', t''] \to M$ is given by

$$\gamma_v(t) := \mathcal{L}_0 \exp_{m'}^{t',t}(v), \quad t' \le t \le t''.$$

Then by the first variational formula for $\mathcal{L}_0$ (Proposition 2.1), we have

$$(\mathrm{d}\phi)_{(v,w)} = \left(\left\langle \dot\gamma_v(t''), (\mathrm{d}\mathcal{L}_0 \exp_{m'}^{t',t''})_v(\cdot) \right\rangle_{g(t'')}, -\left\langle \dot\gamma_w(t''), (\mathrm{d}\mathcal{L}_0 \exp_{m'}^{t',t''})_w(\cdot) \right\rangle_{g(t'')} \right).$$

Therefore by Remark 2.18(ii), the implicit function theorem tells us that

$$N := \left\{ (v, w) \in T_{m'}M \times T_{m'}M : \begin{array}{l} \phi(v, w) = 0, v \ne w; \\ \text{both } v \text{ and } w \text{ are regular points of } \mathcal{L}_0 \exp_{m'}^{t',t''} \end{array} \right\}$$

is a $(2d-1)$-dimensional submanifold of $T_{m'}M \times T_{m'}M$.

Take a countable cover $\{(U_i, \psi_i)\}_i$ of M which consists of local coordinate neighborhoods and consider the map

$$\xi_i : N_i := N \cap \left[\left(\mathcal{L}_0 \exp_{m'}^{t',t''} \right)^{-1}(U_i) \right]^2 \to \mathbb{R}^d$$

defined by

$$\xi_i(v, w) := \psi_i\left(\mathcal{L}_0 \exp_{m'}^{t',t''}(v) \right) - \psi_i\left(\mathcal{L}_0 \exp_{m'}^{t',t''}(w) \right).$$

Under the identification $T_{m''}M \cong T_{\psi_i(m'')}\mathbb{R}^d$ for $m'' \in U_i$, we have

$$(\mathrm{d}\xi_i)_{(v,w)} = \left((\mathrm{d}\mathcal{L}_0 \exp_{m'}^{t',t''})_v, -(\mathrm{d}\mathcal{L}_0 \exp_{m'}^{t',t''})_w \right).$$

Again by the implicit function theorem, we see that

$$N_i' := \left\{ (v, w) \in N_i : \begin{array}{l} \text{both } (\mathrm{d}\mathcal{L}_0 \exp_{m'}^{t',t''})_v \text{ and } (\mathrm{d}\mathcal{L}_0 \exp_{m'}^{t',t''})_w \text{ are} \\ \text{non-singular and } \mathcal{L}_0 \exp_{m'}^{t',t''}(v) = \mathcal{L}_0 \exp_{m'}^{t',t''}(w) \end{array} \right\}$$

is a $(d-1)$-dimensional submanifold in $T_{m'}M \times T_{m'}M$.

Now, letting

$$\eta_i : N_i' \ni (v, w) \mapsto \mathcal{L}_0 \exp_{m'}^{t',t''}(v) \in M,$$

$\mathcal{C}'$ is included in the countable union of hypersurfaces $\eta_i(N_i')$ of M. Therefore it has measure zero. □

PROOF OF PROPOSITION 2.19 (iii). Assume that $\mathcal{L}_0$cut is not closed. Then we can take a sequence $(t_i', m_i'; t_i'', m_i'') \in \mathcal{L}_0\mathrm{cut}$, $i = 1, 2, \dots$ which converges to some point $(t_0', m_0'; t_0'', m_0'') \notin \mathcal{L}_0\mathrm{cut}$. We denote $\Delta_0 := \{(t', t'') \in [0, T]^2 : t' < t''\}$.

Then the map

$$\varphi : \Delta_0 \times TM \to ([0, T] \times M)^2$$

defined by

$$\varphi\big(t', t'', (m', v')\big) := \big(t', m'; t'', \mathcal{L}_0 \exp_{m'}^{t', t''}(v)\big)$$

is non-singular at $\big(t_0', t_0'', (m_0', v_0')\big)$ where $v_0' := \dot\gamma(t_0'')$ and γ is the unique $\mathcal{L}_0$-minimizing curve joining (t_0', m_0') and (t_0'', m_0''). Hence by the inverse function theorem, we can take an open neighborhood U of $\big(t_0', t_0'', (m_0', v_0')\big)$ and an open neighborhood V of $(t_0', m_0'; t_0'', m_0'')$ such that $\varphi|_U : U \to V$ is diffeomorphic. Define $v_i' \in T_{m_i'}M$ by $\big(t_i', t_i'', (m_i', v_i')\big) = \varphi|_U^{-1}(t_i', m_i'; t_i'', m_i'')$. We may assume that $(t_i', m_i'; t_i'', m_i'') \in V$ for all i with neglecting finite numbers of points if necessary.

Note that, for each $\big(t', t'', (m', v')\big) \in U$, v' is not a critical point of $\mathcal{L}_0 \exp_{m'}^{t', t''} : T_{m'}M \to M$. Therefore we can choose $w_i' \in T_{m_i'}M$ so that $[t_i', t_i''] \ni t \mapsto \mathcal{L}_0 \exp_{m_0'}^{t', t}(w_i')$ is an $\mathcal{L}_0$-minimizing geodesic joining (t_i', m_i') to (t_i'', m_i'') but $v_i' \neq w_i'$. Taking a subsequence if necessary, we may assume that $\big(t_i', t_i'', (m_i', w_i')\big) \overset{i\to\infty}{\to} \big(t_0', t_0'', (m_0', w_0')\big)$ for some $w_0' \in T_{m_0'}M$. Since $\big(t_i', t_i'', (m_i', w_i')\big) \notin U$ for all i and $\varphi|_U : U \to V$ is diffeomorphic, we must have $w_0' \neq v_0'$.

Consequently, the curves

$$t \mapsto \mathcal{L}_0 \exp_{m_0'}^{t_0', t}(v_0') \quad \text{and} \quad t \mapsto \mathcal{L}_0 \exp_{m_0'}^{t_0', t}(w_0')$$

must be distinct $\mathcal{L}_0$-minimizing curves joining (t_0', m_0') and (t_0'', m_0''), which contradicts to $(t_0', m_0'; t_0'', m_0'') \notin \mathcal{L}_0\mathrm{cut}$. Hence $\mathcal{L}_0$cut is closed. □

We introduce the notion of $\mathcal{L}_0$-Jacobi fields. For a smooth curve $\gamma : [t', t''] \to M$, a C^2-vector field V along γ and $t \in [t', t'']$, we define a linear form $\mathscr{J}_t(V)$ as follows:

$$\begin{aligned}\mathscr{J}_t(V) := &-\nabla^{g(t)}_{\dot\gamma(t)}\nabla^{g(t)}_{\dot\gamma(t)}V(t) + \frac{1}{2}\mathrm{Hess}_{g(t)}R_{g(t)}(V(t), \cdot) \\ &-\mathcal{R}_{g(t)}(V(t), \dot\gamma(t))\dot\gamma(t) + 2\big(\nabla^{g(t)}_{V(t)}\mathrm{Ric}_{g(t)}\big)(\dot\gamma(t), \cdot) \\ &+2\mathrm{Ric}_{g(t)}\big(\nabla^{g(t)}_{V(t)}\dot\gamma(t), \cdot\big)\,.\end{aligned} \tag{12}$$

We call a vector field J along γ an $\mathcal{L}_0$-*Jacobi field* if $\mathscr{J}_t(J) = 0$ for each $t \in [t', t'']$. Note that some computation yields the following relation between $\mathscr{J}_t$ and $\mathcal{L}_0 I_\gamma$:

$$\mathcal{L}_0 I_\gamma(V, W) = \big\langle \nabla^{g(t)}_{\dot\gamma(t)}V(t), W(t)\big\rangle_{g(t)}\Big|_{t=t'}^{t=t''} + \int_{t'}^{t''} \langle \mathscr{J}_t(V), W(t)\rangle_{g(t)}\mathrm{d}t\,. \tag{13}$$

LEMMA 2.21. *Let $\gamma : [t', t''] \to M$ be an $\mathcal{L}_0$-geodesic. Then $\dot\gamma(t')$ is a critical point of $\mathcal{L}_0 \exp^{t',t''}_{\gamma(t')}$ if and only if there is an $\mathcal{L}_0$-Jacobi field J along γ with $J(t') = 0$, $J(t'') = 0$ and $J \not\equiv 0$.*

PROOF. Let $\dot\gamma(t')$ be a critical point of $\mathcal{L}_0 \exp^{t',t''}_{\gamma(t')}$. It means that there is a non-zero vector $V \in T_{\dot\gamma(t')}(T_{\gamma(t')}M)$ satisfying $(\mathrm{d}\mathcal{L}_0 \exp^{t',t''}_{\gamma(t')})_{\dot\gamma(t')}(V) = 0$. We consider a variation $f(u,t) : (-\varepsilon, \varepsilon) \times [t', t''] \to M$ of γ given by

$$f(u,t) := \mathcal{L}_0 \exp^{t',t}_{\gamma(t')} \left(\dot\gamma(t') + u(t-t')V\right). \tag{14}$$

Then the vector field J along γ given by $J(t) := \frac{\mathrm{d}}{\mathrm{d}u}\Big|_{u=0} f(u,t)$ is an $\mathcal{L}_0$-Jacobi field. Indeed, we can verify it by applying $\nabla^{g(t)}_{(\partial/\partial u)|_{u=0} f(u,t)}$ to the $\mathcal{L}_0$-geodesic equation $\mathscr{G}_t(f(u,\cdot)) = 0$ for $f(u,\cdot)$. Then $J(t') = 0$ by definition and $J(t'') = 0$ by the choice of V. In addition, $\nabla^{g(t')}_{\dot\gamma(t')} J(t') = V$ holds (Here we are identifying V with a vector in $T_{\gamma(t')}M$). Since $V \neq 0$, we have $J \not\equiv 0$.

Conversely, suppose that there is an $\mathcal{L}_0$-Jacobi field J along γ with $J(t') = 0$, $J(t'') \neq 0$ and $J \not\equiv 0$. Then $V := \nabla^{g(t')}_{\dot\gamma(t')} J \neq 0$. Indeed, if $V = 0$, then $J \equiv 0$ must hold by the uniqueness of the solution to the second order linear differential equation $\mathscr{J}_t(J) = 0$. By identifying V with a vector in $T_{\dot\gamma(t')}(T_{\gamma(t')}M)$, we consider a variation f given by (14). Then again by the uniqueness of the solution to $\mathscr{J}_t(J) = 0$, the variational vector field corresponding to f coincides with J. Then $J(t'') = 0$ means $(\mathrm{d}\mathcal{L}_0 \exp^{t',t''}_{\gamma(t')})_{\dot\gamma(t')}(V) = 0$ and hence the conclusion holds. □

PROPOSITION 2.22. *For each $(t', m'; t'', m'') \in \mathcal{M}$, the following two conditions are equivalent:*

(i) $(t', m'; t'', m'') \notin \mathcal{L}_0\mathrm{cut}$.

(ii) *Every $\mathcal{L}_0$-minimizing curve γ joining (t', m') and (t'', m'') extends to an $\mathcal{L}_0$-minimizing curve beyond (t'', m''), that is, there exist an $\varepsilon > 0$ and a curve $\widetilde\gamma : [t', t'' + \varepsilon] \to M$ such that $\widetilde\gamma|_{[t',t'']} = \gamma$ and $\widetilde\gamma$ is an $\mathcal{L}_0$-minimizing curve joining its endpoints.*

PROOF. Assume that (i) holds. If γ is an $\mathcal{L}_0$-minimizing curve joining (t', m') and (t'', m''), then γ is unique among such curves and satisfies the $\mathcal{L}_0$-geodesic equation. Since the map $T_{m'}M \ni w \mapsto \mathcal{L}_0 \exp^{t',t''}_{m'}(w) \in M$ is regular at $\dot\gamma(t')$, there exists an open neighborhood U of $\dot\gamma(t')$ such that the map is diffeomorphic on U to its image.

We extend γ forward in time, to $\widetilde\gamma : [t', t'' + \varepsilon_0] \to M$ by solving the same $\mathcal{L}_0$-geodesic equation ($\varepsilon_0 > 0$ being small enough). Since $\mathcal{L}_0\mathrm{cut}$ is closed (Proposition 2.19(iii)), we can assume that $\widetilde\gamma[t'', t'' + \varepsilon_0] \cap \mathcal{L}_0\mathrm{cut}^{t',t''+\varepsilon_0}_{\widetilde\gamma(t')} = \emptyset$ for sufficiently small $\varepsilon_0 > 0$. For each $\varepsilon \in (0, \varepsilon_0]$, we define $w_\varepsilon \in T_{\gamma(t')}M$ to be the tangent vector $\dot\gamma_\varepsilon(t')$ of the unique $\mathcal{L}_0$-minimizing curve γ_ε joining $(t', \gamma(t'))$ and $(t'' + \varepsilon, \gamma(t'' + \varepsilon))$. By Proposition 2.19(iv) and Proposition 2.3, L_0 is smooth around $(t', m'; t'' + \varepsilon, \gamma(t'' + \varepsilon))$ and

$$|w_\varepsilon|^2_{g(t')} = 2\frac{\partial L_0^{t',t''+\varepsilon}}{\partial t'}\big(m', \gamma(t'' + \varepsilon)\big) + R_{g(t')}(m'),$$

so that $\{w_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ is a bounded set. Therefore, taking a subsequence, we can assume that w_ε converges to a vector $w \in T_{\dot\gamma(t')}M$ as $\varepsilon \downarrow 0$.

Then we show that $w_\varepsilon \in U$ for a sufficiently small $\varepsilon > 0$. Since $\gamma_\varepsilon(t)$ smoothly depends on w_ε and $\mathcal{L}_0$ is lower semi-continuous (cf. the proof of Proposition 2.16), by using Lemma 2.14, we have

$$\begin{aligned}\mathcal{L}_0(\gamma) &= L_0^{t',t''}\big(\gamma(t'),\gamma(t'')\big)\\ &= \lim_{\varepsilon\to 0} L_0^{t',t''}\big(\gamma_\varepsilon(t'),\gamma_\varepsilon(t''+\varepsilon)\big)\\ &= \lim_{\varepsilon\to 0}\mathcal{L}_0(\gamma_\varepsilon) \ge \mathcal{L}_0(\gamma_w) \ge \mathcal{L}_0(\gamma)\,,\end{aligned}$$

where $\gamma_w : [t', t''] \to M$ is the $\mathcal{L}_0$-geodesic of initial conditions $\gamma_w(t') = m'$ and $\dot\gamma_w(t') = w$. It clearly holds that $\gamma_w(t'') = m''$. Therefore, γ_w is also an $\mathcal{L}_0$-minimizing curve joining (t', m') and (t'', m''). By the uniqueness, we have $\gamma = \gamma_w$, so that $\dot\gamma(t') = w$. Thus $w_\varepsilon \in U$ for a sufficiently small $\varepsilon > 0$.

Now, for a sufficiently small $\varepsilon > 0$, we have $\dot\gamma(t')$, $w_\varepsilon \in U$ and

$$\mathcal{L}_0 \exp_{\gamma(t')}^{t',t''+\varepsilon}(\dot\gamma(t')) = \widetilde\gamma(t''+\varepsilon) = \gamma_w(t''+\varepsilon) = \mathcal{L}_0 \exp_{\gamma(t')}^{t',t''+\varepsilon}(w_\varepsilon)\,,$$

which implies $\dot\gamma(t') = w_\varepsilon$, so that $\widetilde\gamma = \gamma_\varepsilon$. Hence the curve $\widetilde\gamma$ is uniquely $\mathcal{L}_0$-minimizing.

Conversely, assume that (ii) holds. Suppose that γ and η are two $\mathcal{L}_0$-minimizing curves joining (t', m') and (t'', m''). By the assumption (ii), γ has an $\mathcal{L}_0$-minimizing extension $\widetilde\gamma : [t', t''+\varepsilon] \to M$. Then the piecewise smooth curve c defined by

$$c(t) := \begin{cases}\eta(t) & \text{if } t \in [t', t'']\,,\\ \widetilde\gamma(t) & \text{if } t \in [t'', t''+\varepsilon]\,,\end{cases}$$

which is the concatenation of η and $\widetilde\gamma|_{[t',t''+\varepsilon]}$, must be also $\mathcal{L}_0$-minimizing. A standard variational argument shows that the curve $c(t)$ becomes C^1 at $t = t''$ and must be $\mathcal{L}_0$-geodesic (so that c is found to be smooth). By the uniqueness result (with respect to the initial conditions at time $t''+\varepsilon$) of the ODE theory, we must have $\widetilde\gamma = c$, in particular, we have $\gamma = \eta$. Therefore, the $\mathcal{L}_0$-minimizing geodesic joining (t', m') and (t'', m'') must be unique.

Next, we show that $\mathcal{L}_0 \exp_{m'}^{t',t''}$ is not singular at $\dot\gamma(t') \in T_{m'}M$. Suppose on the contrary, that $\dot\gamma(t')$ is a critical point of $\mathcal{L}_0 \exp_{m'}^{t',t''}$. Then there is an $\mathcal{L}_0$-Jacobi field along γ with $J(t') = 0$, $J(t'') = 0$ and $J \not\equiv 0$ by Lemma 2.21. Note that $\nabla^{g(t'')}_{\gamma(t'')} J(t'') \ne 0$ holds (cf. the proof of Lemma 2.21). We take an $\mathcal{L}_0$-minimizing extension $\widetilde\gamma : [t', t''+\varepsilon] \to M$ of γ and extend J to a piecewise smooth vector field on $[t', t''+\varepsilon]$ (which we denote again by J) by requiring that $J|_{[t'',t''+\varepsilon]} \equiv 0$. We further take a vector field V along $\widetilde\gamma$ with $V(t') = 0$, $V(t'') = -\nabla^{g(t'')}_{\dot\gamma(t'')} J(t''-)$ and $V(t''+\varepsilon) = 0$, and then consider any *proper* variation $g(u, t) : (-\delta, \delta) \times [t', t''] \to M$ of $\widetilde\gamma$, with the variational vector field $W = J + aV$ ($a > 0$). Since W vanishes at the endpoints of $\widetilde\gamma$, such a proper variation exists. By a piecewise use of the second variational formula (Proposition 2.4) together with (13) and the symmetry of $\mathcal{L}_0 I_\gamma$,

we have

$$(\delta_W\delta_W\mathcal{L}_0)(\widetilde{\gamma}) = -2a|\nabla^{g(t'')}_{\dot{\gamma}(t'')}J(t''-)|^2_{g(t'')} + a^2(\delta_V\delta_V\mathcal{L}_0)(\widetilde{\gamma})$$

which is negative for sufficiently small $a > 0$. On the other hand, since $\widetilde{\gamma}$ is $\mathcal{L}_0$-minimizing and g is proper, it must hold that

$$\mathcal{L}_0(\widetilde{\gamma}) = \mathcal{L}_0(g(0,\cdot)) \le \mathcal{L}_0(g(u,\cdot))\,, \quad \text{for each } u \in (-\delta,\delta)$$

which implies $(\delta_W\delta_W\mathcal{L}_0)(\widetilde{\gamma}) = (\mathrm{d}^2/\mathrm{d}u^2)\big|_{u=0}\mathcal{L}_0(g(u,\cdot)) \ge 0$. This is a contradiction. Therefore $\dot{\gamma}(t')$ is not a critical point of $\mathcal{L}_0\exp^{t',t''}_{m'}$.

Hence we have proved (ii)⇒(i). □

Finally we study relations in the time-reversal and $\mathcal{L}_0$-cut locus. Let $\widetilde{g}(\tau) = g(T-\tau)$. Then $\widetilde{g}(\tau)$ evolves under the backward Ricci flow

$$\frac{\partial}{\partial\tau}\widetilde{g}(\tau) = 2\mathrm{Ric}_{\widetilde{g}(\tau)}\,.$$

For $\tau' < \tau''$, we consider the corresponding functional

$$\widetilde{\mathcal{L}}_0(c) := \frac{1}{2}\int_{\tau'}^{\tau''}\big\{|\dot{c}(\tau)|^2_{\widetilde{g}(\tau)} + R_{\widetilde{g}(\tau)}(c(\tau))\big\}\mathrm{d}\tau\,,$$

where $c : [\tau',\tau''] \to M$ is any curve. For $x' \in M$ we define the $\widetilde{\mathcal{L}}_0$-exponential map $\widetilde{\mathcal{L}}_0\exp^{\tau',\tau''}_{x'} : T_{x'}M \to M$ by

$$\widetilde{\mathcal{L}}_0\exp^{\tau',\tau''}_{x'}(w) := \eta(\tau'')$$

where η is the solution to the $\widetilde{\mathcal{L}}_0$-geodesic equation

$$\text{(15)}\qquad \begin{cases} \nabla^{\widetilde{g}(\tau)}_{\dot{\eta}(\tau)}\dot{\eta}(\tau) - \dfrac{1}{2}\nabla^{\widetilde{g}(\tau)}R_{\widetilde{g}(\tau)} + 2\mathrm{Ric}_{\widetilde{g}(\tau)}(\dot{\eta}(\tau),\cdot) = 0\,, \\ \eta(\tau') = x', \quad \dot{\eta}(\tau') = w\,. \end{cases}$$

One can see that this is actually the Euler-Lagrange equation for $\widetilde{\mathcal{L}}_0$. Take $\tau',\tau'',t',t'' \in [0,T]$ with $\tau' < \tau''$, $\tau' = T - t''$ and $t'' - t' = \tau'' - \tau'$. For a curve $\eta : [\tau',\tau''] \to M$, we define $\gamma : [t',t''] \to M$ by $\gamma(t'+s) = \eta(\tau''-s)$ for $0 \le s \le t''-t' = \tau''-\tau'$. We call γ the time-reversal of η. By definition, $\widetilde{\mathcal{L}}_0(\eta) = \mathcal{L}_0(\gamma)$ holds. In addition, by comparing (15) with (5), we can easily show that γ is an $\mathcal{L}_0$-geodesic if and only if η is an $\widetilde{\mathcal{L}}_0$-geodesic.

PROPOSITION 2.23. *Let $\tau',\tau'',t',t'' \in [0,T]$ with $\tau' < \tau''$, $\tau' = T - t''$ and $t''-t' = \tau''-\tau'$.*

(i) *There is more than one $\mathcal{L}_0$-minimizing curve joining (t',x) and (t'',y) iff there is more than one $\widetilde{\mathcal{L}}_0$-minimizing curve joining (τ',y) and (τ'',x).*

(ii) *Let $\eta : [\tau',\tau''] \to M$ be an $\widetilde{\mathcal{L}}_0$-minimizing curve and γ its time-reversal. Then the vector $\dot{\gamma}(t')$ is a critical point of $\mathcal{L}_0\exp^{t',t''}_{\gamma(t')}$ if and only if $\dot{\eta}(\tau')$ is a critical point of $\widetilde{\mathcal{L}}_0\exp^{\tau',\tau''}_{\eta(\tau')}$.*

(iii) *Define* $\widetilde{\mathcal{L}}_0\mathrm{cut}^{\tau',\tau''}_{x'}$ *as the set of all points* $x'' \in M$ *such that there is more than one* $\widetilde{\mathcal{L}}_0$*-minimizing curve joining* (τ', x') *and* (τ'', x'') *or there is a* $w \in T_{x'}M$ *such that* $x'' = \widetilde{\mathcal{L}}_0 \exp^{\tau',\tau''}_{x'}(w)$ *and* w *is a critical point of* $\widetilde{\mathcal{L}}_0 \exp^{\tau',\tau''}_{x'}$*. We also define*

$$\widetilde{\mathcal{L}}_0\mathrm{cut}^{\tau',\tau''} := \left\{(x', x'') : x'' \in \widetilde{\mathcal{L}}_0\mathrm{cut}^{\tau',\tau''}_{x'}\right\}.$$

Then the map $M \times M \ni (x, y) \mapsto (y, x) \in M \times M$ *gives an isomorphism between* $\mathcal{L}_0\mathrm{cut}^{t',t''}$ *and* $\widetilde{\mathcal{L}}_0\mathrm{cut}^{\tau',\tau''}$.

(iv) $(x', x'') \notin \widetilde{\mathcal{L}}_0\mathrm{cut}^{\tau',\tau''}$ *if and only if* $\widetilde{\mathcal{L}}_0$*-minimizing curve joining* (τ', x') *and* (τ'', x'') *can be extended beyond* (τ'', x'') *with keeping its minimality.*

PROOF. The claim (i) is obvious by remarks just before Proposition 2.23. For (ii), we introduce the notion of $\widetilde{\mathcal{L}}_0$-Jacobi field: For $\eta : [\tau', \tau''] \to M$, $\widetilde{J}$ is an $\widetilde{\mathcal{L}}_0$-Jacobi field if $\widetilde{\mathscr{J}_\tau}(V) = 0$, where a linear form $\widetilde{\mathscr{J}_\tau}$ is defined by replacing $\gamma(t)$, $\dot\gamma(t)$ and $g(t)$ in (12) with $\eta(\tau)$, $\dot\eta(\tau)$ and $\widetilde{g}(\tau)$ respectively and changing the sign of all terms involving the Ricci curvature. Then the criticality of $\widetilde{\mathcal{L}}_0$-exponential map is also characterized by $\widetilde{\mathcal{L}}_0$-Jacobi fields as in Lemma 2.21 by the same argument. Moreover, a vector field $\widetilde{J}$ along η is an $\widetilde{\mathcal{L}}_0$-Jacobi field if and only if a vector field J along the time-reversal of η given by $J(t'+s) := \widetilde{J}(\tau''-s)$ is an $\mathcal{L}_0$-Jacobi field. Then the conclusion easily follows by combining these observations with Lemma 2.21. The assertion (iii) follows immediately from (i) and (ii). The assertion (iv) can be proved in the same way as Proposition 2.22. □

3. Construction of a coupled Brownian motions in the absence of $\mathcal{L}_0$-cut locus. In this section, we will show Theorem 1.1 under the assumption that $\mathcal{L}_0\mathrm{cut}$ is empty where $\mathcal{L}_0\mathrm{cut}$ is defined in Subsection 2.3.

Under $\mathcal{L}_0\mathrm{cut} = \emptyset$, $L_0^{t',t''}(m', m'')$ is a smooth function of $(t', m'; t'', m'')$ (Theorem 2.19(iv)), and for each $m', m'' \in M$ and $t' < t''$, we can take a unique minimizer for $\mathcal{L}_0$ joining (t', m') to (t'', m''). Therefore, for each pair of points, the space-time parallel transport as a map between their tangent spaces is uniquely determined. In the following, we construct a coupled Brownian motions by space-time parallel transport, introduced in Definition 2.7.

Let $e_1, e_2, \ldots, e_d \in \mathbb{R}^d$ be the canonical basis of $\mathbb{R}^d$. We denote by $\pi : \mathscr{F}(M) \to M$ the frame bundle over M and by $\pi : \mathscr{O}^{g(t)}(M) \to M$ the orthonormal frame bundle with respect to the metric $g(t)$. Note that $\mathscr{O}^{g(t)}(M)$ varies along our Ricci flow but not for $\mathscr{F}(M)$. We canonically have the map defined by

$$\mathscr{F}(M)_m \times \mathbb{R}^d \ni (u, x) \mapsto u.x := u(x) \in T_mM,$$

where $\mathscr{F}(M)_m$ stands for the fiber at m of $\mathscr{F}(M)$.

Recall our notations on time parameters $\tau'(s)$, $\tau''(s)$ given in Section 1: We sometimes drop s and simply write τ' or τ''. Let $(U_s, V_s)_{0\le s\le S} = (U_s(u'), V_s(v''))_{0\le s\le S}$ be the solution

to the SDE

$$\begin{cases} \mathrm{d}U_s = \sqrt{2}H_i^{g(\tau')}(U_s)\circ \mathrm{d}W_s^i - \displaystyle\sum_{\alpha,\beta=1}^{d} \frac{\partial g}{\partial t}(\tau')\Big(U_s.e_\alpha, U_s.e_\beta\Big)\mathcal{V}^{\alpha,\beta}(U_s)\mathrm{d}s\,, \\ \mathrm{d}V_s = \sqrt{2}H_i^{g(\tau'')}(V_s)\circ \mathrm{d}B_s^i - \displaystyle\sum_{\alpha,\beta=1}^{d} \frac{\partial g}{\partial t}(\tau'')\Big(V_s.e_\alpha, V_s.e_\beta\Big)\mathcal{V}^{\alpha,\beta}(V_s)\mathrm{d}s\,, \\ \mathrm{d}B_s = V_s^{-1}/\!\!/_{\pi U_s,\pi V_s}^{\tau',\tau''} U_s \mathrm{d}W_s\,, \end{cases}$$

starting from $(U_0, V_0) = (u', v'') \in \mathscr{O}^{g(t_1')}(M)_{m'} \times \mathscr{O}^{g(t_1'')}(M)_{m''}$, where $(H_i^{g(t)})_{i=1}^d$ is the *system of canonical horizontal vector fields* on $\mathscr{F}(M)$ associated with $g(t)$, and $(\mathcal{V}^{\alpha,\beta})_{\alpha,\beta=1}^d$ is the *system of canonical vertical vector fields*, each of which is defined by

$$\mathcal{V}^{\alpha,\beta} f(u) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} f(u\circ \mathrm{e}^{\varepsilon e_\alpha\otimes e_\beta})\,, \quad f\in C^\infty(\mathscr{F}(M))\,.$$

It is known that the random variable (U_s, V_s) takes its value in $\mathscr{O}^{g(\tau'(s))}(M)\times\mathscr{O}^{g(\tau''(s))}(M)$ for every $0\le s\le S$ (see [2, Proposition 1.1]). We put $(X_s, Y_s) := (\pi U_s, \pi V_s)$.

The next statement is an Itô formula for the process $(X, Y) = (X_s, Y_s)_{0\le s\le S}$. We omit the proof because it is straightforward.

PROPOSITION 3.1. *For any smooth function $f(s, m', m'')$ on $[0,\infty)\times M\times M$, we have*

$$\begin{aligned} &\mathrm{d}f(s, X_s, Y_s) \\ &\quad = \Big\{\frac{\partial f}{\partial s}(s, X_s, Y_s) + \sum_{i=1}^{d}\Big[\mathrm{Hess}_{g(\tau')\oplus g(\tau'')} f(s,\cdot,\cdot)\Big]\Big(U_s.e_i\oplus V_s.e_i^*, U_s.e_i\oplus V_s.e_i^*\Big)\Big\}\mathrm{d}s \\ &\qquad + [U_s.e_i\oplus V_s.e_i^*] f(s,\cdot,\cdot)\bullet \mathrm{d}W_s^i, \end{aligned}$$

where $e_i^* := V_s^{-1}/\!\!/_{X_s,Y_s}^{\tau',\tau''} U_s.e_i \in \mathbb{R}^d$ for each $i = 1, 2, \ldots, d$ and $\bullet\mathrm{d}W_s^i$ means the Itô integral.

COROLLARY 3.2. *Suppose $\mathcal{L}_0\mathrm{cut} = \emptyset$. Then we have the following:*

$$\begin{aligned} &\mathrm{d}L_0^{\tau'(s),\tau''(s)}(X_s, Y_s) \\ &= -\Big\{\frac{\partial L_0^{t',t''}}{\partial t'} + \frac{\partial L_0^{t',t''}}{\partial t''}\Big\}\bigg|_{(t',t'')=(\tau'(s),\tau''(s))}(X_s, Y_s)\mathrm{d}s \\ &\quad + \sum_{i=1}^{d}\big[\mathrm{Hess}_{g(\tau')\oplus g(\tau'')} L_0^{\tau',\tau''}\big]\big(U_s.e_i\oplus V_s.e_i^*, U_s.e_i\oplus V_s.e_i^*\big)\mathrm{d}s \\ &\quad + [U_s.e_i\oplus V_s.e_i^*] L_0^{\tau',\tau''}\bullet \mathrm{d}W_s^i\,. \end{aligned}$$

Since $\mathcal{L}_0\mathrm{cut} = \emptyset$, by using the result in Proposition 2.8, the stochastic process $L_0^{\tau'(s),\tau''(s)}(X_s, Y_s)$ is a semi-martingale whose bounded variation part is non-positive. Therefore we can conclude that $s\mapsto L_0^{\tau'(s),\tau''(s)}(X_s, Y_s)$ is a supermartingale after we prove the

integrability. For proving the integrability, we consider a family of functions $\varphi_n \in C^2(\mathbb{R})$ with $\varphi_n(x) \uparrow x$ as $n \to \infty$ for each $x \in \mathbb{R}$. Suppose in addition that φ_n is nondecreasing, concave and bounded from above. Since L_0 is bounded from below, we can easily show that $\varphi_n(L_0^{\tau'(s),\tau''(s)}(X_s, Y_s))$ is a supermartingale. Then the integrability will be ensured by the monotone convergence theorem and the fact $L_0^{\tau'(0),\tau''(0)}(X_0, Y_0)$ is deterministic. The proof of the integrability in the next section will go along the same idea but we will show it together with the rigorous proof of Theorem 1.1.

4. Supermartingale property of a coupled Brownian motion in the presence of $\mathcal{L}_0$-cut locus.

4.1. Coupling via approximation by geodesic random walks. For the construction of a suitable coupling of Brownian motions in the presence of $\mathcal{L}_0$-cut locus, we use the approximation by geodesic random walks to avoid the technical difficulty coming from the singularity of L_0. Indeed L_0 is smooth outside $\mathcal{L}_0$cut (Proposition 2.19(iv)) but not on $\mathcal{L}_0$cut. To carry out this procedure, we will rely on some basic properties of $\mathcal{L}_0$cut summarized in Subsection 2.3.

We fix a measurable section (or selection)

$$\gamma : (t', m'; t'', m'') \mapsto \gamma_{m',m''}^{t',t''}$$

of minimal $\mathcal{L}_0$-geodesics, where, the measurability is with respect to the Borel σ-field generated by the uniform topology on the path space. Since $\mathcal{L}_0$-minimizing curves with fixed endpoints form a compact set (Proposition 2.16), the existence of such a section is ensured by a measurable selection theorem (e.g. [3, Theorem 6.9.6] and Proposition 2.16). We further fix a measurable section $\sigma(t, \cdot) : M \to \mathscr{O}^{g(t)}(M)$ of $g(t)$-orthonormal frame bundle for each $t \in [t_0', t_1']$.

To construct a geodesic random walk, we prepare an $\mathbb{R}^d$-valued i.i.d. sequence $(\lambda_n)_{n=1}^{\infty}$, each of which is uniformly distributed on the unit ball in $\mathbb{R}^d$ centered at the origin. Since it will be needed when working with these λ's, we shall summarize necessary formulae as follows. We omit the proof because it is obvious or an easy consequence of the divergence theorem.

LEMMA 4.1. *Let V be an n-dimensional real Euclidean space. Let $\ell : V \to \mathbb{R}$ be a linear function and $\alpha : V \times V \to \mathbb{R}$ be a symmetric 2-form on V. Let B^n is the unit ball in V centered at origin. Then*

(i) $\displaystyle\int_{B^n} \ell(x)\mathrm{d}x = 0$ *and* $\displaystyle\int_{B^n} \alpha(x, x)\mathrm{d}x = \frac{\mathrm{vol}(B^n)}{n+2}\mathrm{tr}\,\alpha$,
where in the last equality, we have naturally regarded α as the linear homomorphism $V \to V^* \cong V$.

Suppose further that we are given another n-dimensional real vector space W, a linear function $f : V \oplus W \to \mathbb{R}$, a symmetric 2-form β on $V \oplus W$ and a linear homomorphism $A : V \to W$. Then

(ii) $\displaystyle\int_{B^n} f(x \oplus Ax)\mathrm{d}x = 0$ *and*

(iii) $\displaystyle\int_{B^n} \beta\big(x \oplus Ax, x \oplus Ax\big)\mathrm{d}x = \frac{\mathrm{vol}(B^n)}{n+2}\sum_{i=1}^{n}\beta\big(e_i \oplus Ae_i, e_i \oplus Ae_i\big)$
where $(e_i)_{i=1}^n$ *is an any orthonormal basis of* V.

Now for each m', $m'' \in M$ and $\varepsilon > 0$, let us construct a coupling $(X_s^\varepsilon, Y_s^\varepsilon)_{0\le s\le S}$ of geodesic random walks starting from (m', m'') by

$$\begin{cases} X_s^\varepsilon := \exp_{X_0^\varepsilon}^{g(\tau'(0))}\left(\dfrac{\sqrt{2}s}{\varepsilon}\sigma(\tau'(0), X_0^\varepsilon).\sqrt{d+2}\lambda_1\right), \\ Y_s^\varepsilon := \exp_{Y_0^\varepsilon}^{g(\tau''(0))}\left(\dfrac{\sqrt{2}s}{\varepsilon}/\!\!/_{X_0^\varepsilon, Y_0^\varepsilon}^{\tau'(0),\tau''(0)}\sigma(\tau'(0), X_0^\varepsilon).\sqrt{d+2}\lambda_1\right), \end{cases} \quad \text{for } 0 \le s \le s_1,$$

$$\vdots$$

$$\begin{cases} X_s^\varepsilon := \exp_{X_{s_n}^\varepsilon}^{g(\tau'(s_n))}\left(\dfrac{\sqrt{2}s - \sqrt{2}s_n}{\varepsilon}\sigma(\tau'(s_n), X_{s_n}^\varepsilon).\sqrt{d+2}\lambda_{n+1}\right), \\ Y_s^\varepsilon := \exp_{Y_{s_n}^\varepsilon}^{g(\tau''(s_n))}\left(\dfrac{\sqrt{2}s - \sqrt{2}s_n}{\varepsilon}/\!\!/_{X_{s_n}^\varepsilon, Y_{s_n}^\varepsilon}^{\tau'(s_n),\tau''(s_n)}\sigma(\tau'(s_n), X_{s_n}^\varepsilon).\sqrt{d+2}\lambda_{n+1}\right) \end{cases} \quad \text{for } s_n \le s \le s_{n+1}$$

$$\vdots$$

where $X_0^\varepsilon := m'$, $Y_0^\varepsilon := m''$ and $s_n := (n\varepsilon^2) \wedge S$ for each $n = 0, 1, 2, \ldots$. From Lemma 4.1, we see that the factor $\sqrt{d+2}$ is the normalization constant in the sense of that $(d+2)\mathbf{E}[\lambda_n \otimes \lambda_n] = \mathrm{id}_{\mathbb{R}^d}$.

We shall give a remark here. From the definition, the random curves $s \mapsto X_s^\varepsilon$ and $s \mapsto Y_s^\varepsilon$ are clearly piecewise smooth and then we see that $X_{s_n}^\varepsilon$ and $Y_{s_n}^\varepsilon$ are $\sigma\big(\lambda_k : 1 \le k \le n\big)$-measurable and $\dot{X}_{s_n+}^\varepsilon$ and $\dot{Y}_{s_n+}^\varepsilon$ are $\sigma\big(\lambda_k : 1 \le k \le n+1\big)$-measurable but not $\sigma\big(\lambda_k : 1 \le k \le n\big)$-measurable.

As shown in [16], each of $X^\varepsilon = (X_s^\varepsilon)_{0\le s\le S}$ and $Y^\varepsilon = (Y_s^\varepsilon)_{0\le s\le S}$ converges in law to $g(\tau'(s))$-Brownian motion starting from m' and $g(\tau''(s))$-Brownian motion starting from m'' respectively. As a result, the collection $\{(X^\varepsilon, Y^\varepsilon)\}_{\varepsilon>0}$ of couplings forms a tight family and hence we can find a convergent subsequence of $(X^\varepsilon, Y^\varepsilon)_{\varepsilon>0}$. We fix such a subsequence and denote the subsequence by the same notation $(X^\varepsilon, Y^\varepsilon)_{\varepsilon>0}$ for simplicity. We denote the limit by $(X, Y) = (X_s, Y_s)_{0\le s\le S}$.

4.2. Supermartingale property. Let $\mathcal{G}_0$ be the trivial σ-field and $\mathcal{G}_n := \sigma\big(\lambda_k : 1 \le k \le n\big)$ for each $n = 1, 2, \ldots$.

PROPOSITION 4.2. *Set* $\Lambda_s^\varepsilon := L_0^{\tau'(s),\tau''(s)}(X_s^\varepsilon, Y_s^\varepsilon)$. *For each relatively compact open set* M_0 *in* M *including* $X_0^\varepsilon = m'$ *and* $Y_0^\varepsilon = m''$, *there is* $\varepsilon_0 > 0$ *which depends only on* M_0 *such that the following holds: For each* $0 < \varepsilon \le \varepsilon_0$, *there are a family* $(Q_n^\varepsilon)_{n=1}^\infty$ *of random*

variables and a deterministic constant $\delta(\varepsilon)$ such that

$$\Lambda^{\varepsilon}_{s_{n+1}} \le \Lambda^{\varepsilon}_{s_n} + \varepsilon\zeta^{\varepsilon}_{n+1} + \varepsilon^2\Sigma^{\varepsilon}_{n+1} + Q^{\varepsilon}_{n+1} \tag{16}$$

with the estimate

$$\sum_{n:\, s_n < \sigma_{M_0}(X^{\varepsilon}, Y^{\varepsilon})\wedge S} Q^{\varepsilon}_n \le \delta(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0\,, \tag{17}$$

where ζ^{ε}_n and Σ^{ε}_n are $\mathcal{G}_n$-measurable and integrable random variables such that

$$\mathbf{E}[\,\zeta^{\varepsilon}_n \,|\, \mathcal{G}_{n-1}\,] = 0 \quad \text{and} \quad \overline{\Sigma}^{\varepsilon}_n := \mathbf{E}[\,\Sigma^{\varepsilon}_n \,|\, \mathcal{G}_{n-1}\,] \le 0 \tag{18}$$

for each $n \ge 1$, and

$$\sigma_{M_0}(w', w'') := \inf\left\{s > 0 : (w'(s), w''(s)) \notin M_0 \times M_0\right\}$$

for each $(w', w'') \in C([0, S] \to M \times M)$.

REMARK 4.3. Intuitively, it is clear that the difference inequality (16) comes from the Taylor expansion with respect to ε. Therefore, when $(\tau'(s_n), X^{\varepsilon}_{s_n}; \tau''(s_n), Y^{\varepsilon}_{s_n}) \notin \mathcal{L}_0\mathrm{cut}$,

$$\begin{aligned}
\zeta^{\varepsilon}_{n+1} &= \left[\varepsilon\dot{X}^{\varepsilon}_{s_n+} \oplus \varepsilon\dot{Y}^{\varepsilon}_{s_n+}\right] L_0^{\tau'(s_n),\tau''(s_n)} \\
\Sigma^{\varepsilon}_{n+1} &= -\left\{\frac{\partial L_0^{t',t''}}{\partial t'} + \frac{\partial L_0^{t',t''}}{\partial t''}\right\}\Bigg|_{(t',t'')=(\tau'(s_n),\tau''(s_n))} (X^{\varepsilon}_{s_n}, Y^{\varepsilon}_{s_n}) \\
&\quad + \frac{1}{2}\left[\mathrm{Hess}_{g(\tau'(s_n))\oplus g(\tau''(s_n))} L_0^{\tau'(s_n),\tau''(s_n)}\right]\left(\varepsilon\dot{X}^{\varepsilon}_{s_n+} \oplus \varepsilon\dot{Y}^{\varepsilon}_{s_n+}, \varepsilon\dot{X}^{\varepsilon}_{s_n+} \oplus \varepsilon\dot{Y}^{\varepsilon}_{s_n+}\right).
\end{aligned}$$

However, we must avoid using this expression since $\mathcal{L}_0\mathrm{cut} \neq \emptyset$.

PROOF. If $\mathcal{L}_0\mathrm{cut} = \emptyset$, the Taylor expansion with respect to the parameter ε easily yields the desired estimate. Thus we will modify the argument by taking the presence of $\mathcal{L}_0\mathrm{cut}$ into account. We begin with determining Q^{ε}_{n+1} which enjoys (16) and we verify (17) after that. Let us define $K \subset ([0, S] \times M)^3$ by

$$K := \left\{ (u_i, m_i)_{i=1}^3 \;\middle|\; \begin{array}{l} (u_1, m_1), (u_3, m_3) \in [0, S] \times \overline{M_0}, \\ u_3 - u_1 \ge t_1'' - t_1', \\ u_2 = (u_1 + u_3)/2, \\ L_0^{u_1,u_3}(m_1, m_3) \\ \quad = L_0^{u_1,u_2}(m_1, m_2) + L_0^{u_2,u_3}(m_2, m_3) \end{array} \right\}.$$

By Proposition 2.10 (iii), there are constants $c_1, c_2 > 0$ such that, for each $(u_i, m_i)_{i=1}^3 \in K$,

$$\begin{aligned}
&\rho_{g(t_1')}(m_1, m_2) + \rho_{g(t_1'')}(m_2, m_3) \\
&\quad \le c_1(L_0^{u_1,u_2}(m_1, m_2) + L_0^{u_2,u_3}(m_2, m_3)) + c_2 \\
&\quad = c_1 L_0^{u_1,u_3}(m_1, m_3) + c_2\,.
\end{aligned}$$

Since the last quantity does not depend on m_2 and continuously depends on $(m_1, t_1; m_3, t_3)$ which moves on a compact set, there is a bounded set D, which is possibly larger than M_0, such that $(u_2, m_2) \in D$ holds for any $(u_i, m_i)_{i=1}^3 \in K$. Therefore K is compact.

Take $(u_i, m_i)_{i=1}^3 \in K$. Then (u_2, m_2) must be on a minimal $\mathcal{L}_0$-geodesic joining (u_1, m_1) and (u_3, m_3). We denote it by γ. Since $u_1 \neq u_3$ by the definition of K, both $\gamma|_{[u_1,u_2]}$ and $\gamma|_{[u_2,u_3]}$ are not constant as a space-time curve (It means that $(t, \gamma(t))$ is not constant both on $[u_1, u_2]$ and on $[u_2, u_3]$). Hence we can extend $\gamma|_{[u_1,u_2]}$ and $\gamma|_{[u_2,u_3]}$ with keeping their minimalities. It implies $(u_i, m_i; u_{i+1}, m_{i+1}) \notin \mathcal{L}_0\mathrm{cut}$ for $i = 1, 2$ by Proposition 2.22 and Proposition 2.23(iii)(iv). For $j = 1, 2$, let p_j be a projection from K given by $p_j((u_i, m_i)_{i=1}^3) = (u_j, m_j; u_{j+1}, m_{j+1})$. Then $(p_j(K))_{j=1,2}$ are compact and away from $\mathcal{L}_0\mathrm{cut}$. Let us define a compact set $K_1^\varepsilon \subset ([0, S] \times M)^2$ by

$$A := \left\{ (u', m', v'; u'', m'', v'') \;\middle|\; \begin{array}{l} (u', m'; u'', m'') \in p_1(K) \cup p_2(K), \\ v' \in T_{m'}M,\ v'' \in T_{m''}M, \\ \|v'\|_{g(u')} = \|v''\|_{g(u'')} \\ \qquad\qquad = \sqrt{2(d+2)} \end{array} \right\},$$

$$K_1^\varepsilon := \left\{ \left(u' + a, \exp_{m'}^{g(\tau'(u'))}(\varepsilon a v'); u'' + a, \exp_{m''}^{g(\tau''(u''))}(\varepsilon a v'')\right) \;\middle|\; (u', m', v'; u'', m'', v'') \in A, a \in [0, 1] \right\}.$$

Since $\mathcal{L}_0\mathrm{cut}$ is closed by Proposition 2.19(iii), there is $\varepsilon_0 > 0$ such that $K_1^\varepsilon \cap \mathcal{L}_0\mathrm{cut} = \emptyset$ when $\varepsilon \leq \varepsilon_0$. Note that the map

$$(t', m'; t'', m'') \mapsto \left(\frac{t' + t''}{2}, \gamma_{m',m''}^{t',t''}\left(\frac{t' + t''}{2}\right)\right)$$

is measurable.

Let $\varepsilon < \varepsilon_0$. For simplicity of notations, we denote the "midpoint" of $(\tau'(s_n), X_{s_n}^\varepsilon)$ and $(\tau''(s_n), Y_{s_n}^\varepsilon)$, and the associated variational vector by the following:

$$\begin{aligned} \hat{\tau} &:= \frac{\tau'(s_n) + \tau''(s_n)}{2}, \\ \hat{X} &:= \gamma_{X_{s_n}^\varepsilon, Y_{s_n}^\varepsilon}^{\tau'(s_n), \tau''(s_n)}(\hat{\tau}), \\ \hat{V} &:= /\!\!/_{X_{s_n}^\varepsilon, \hat{X}}^{\tau'(s_n), \hat{\tau}} \dot{X}_{s_n+}^\varepsilon. \end{aligned}$$

Then we can easily verify the following:

$$\begin{aligned} &\left(\tau'(s_n), X_{s_n}^\varepsilon; \hat{\tau}, \hat{X}; \tau''(s_n), Y_{s_n}^\varepsilon\right) \in K, \\ &\left(\tau'(s_n), X_{s_n}^\varepsilon, \varepsilon \dot{X}_{s_n+}^\varepsilon; \hat{\tau}, \hat{X}, \varepsilon \hat{V}\right) \in K_1^\varepsilon, \\ &\left(\hat{\tau}, \hat{X}, \varepsilon \hat{V}; \tau''(s_n), Y_{s_n}^\varepsilon, \varepsilon \dot{Y}_{s_n+}^\varepsilon\right) \in K_1^\varepsilon. \end{aligned}$$

Since $K_1^\varepsilon \cap \mathcal{L}_0\mathrm{cut} = \emptyset$, the Taylor expansion yields

$$\begin{aligned} (19) \qquad & L_0^{\tau'(s_{n+1}), \hat{\tau} + \varepsilon^2}(X_{s_{n+1}}^\varepsilon, \exp_{\hat{X}}^{g(\hat{\tau})}(\varepsilon \hat{V})) \\ & \leq L_0^{\tau'(s_n), \hat{\tau}}(X_{s_n}^\varepsilon, \hat{X}) + \varepsilon\left[\varepsilon \dot{X}_{s_n+}^\varepsilon \oplus \varepsilon \hat{V}\right] L_0^{\tau'(s_n), \hat{\tau}} \end{aligned}$$

$$
\begin{aligned}
&- \varepsilon^2 \left[\frac{\partial L_0^{t',t''}}{\partial t'} + \frac{\partial L_0^{t',t''}}{\partial t''} \right] \Big|_{(t',t'')=(\tau'(s_n),\hat{\tau})} (X^{\varepsilon}_{s_n}, \hat{X}) \\
&+ \frac{\varepsilon^2}{2} \left[\mathrm{Hess}_{g(\tau'(s_n))\oplus g(\hat{\tau})} L_0^{\tau'(s_n),\hat{\tau}} \right] \left(\varepsilon \dot{X}^{\varepsilon}_{s_n+} \oplus \varepsilon \hat{V}, \varepsilon \dot{X}^{\varepsilon}_{s_n+} \oplus \varepsilon \hat{V} \right) \\
&+ o(\varepsilon^2)
\end{aligned}
$$

and

$$
\begin{aligned}
(20) \qquad & L_0^{\hat{\tau}+\varepsilon^2,\tau''(s_{n+1})}(\exp_{\hat{X}}^{g(\hat{\tau})}(\varepsilon \hat{V}), Y^{\varepsilon}_{s_{n+1}}) \\
&\le L_0^{\hat{\tau},\tau''(s_n)}(\hat{X}, Y^{\varepsilon}_{s_n}) + \varepsilon \left[\varepsilon \hat{V} \oplus \varepsilon \dot{Y}^{\varepsilon}_{s_n+} \right] L_0^{\hat{\tau},\tau''(s_n)} \\
&- \varepsilon^2 \left[\frac{\partial L_0^{t',t''}}{\partial t'} + \frac{\partial L_0^{t',t''}}{\partial t''} \right] \Big|_{(t',t'')=(\hat{\tau},\tau''(s_n))} (\hat{X}, Y^{\varepsilon}_{s_n}) \\
&+ \frac{\varepsilon^2}{2} \left[\mathrm{Hess}_{g(\hat{\tau})\oplus g(\tau''(s_n))} L_0^{\hat{\tau},\tau''(s_n)} \right] \left(\varepsilon \hat{V} \oplus \varepsilon \dot{Y}^{\varepsilon}_{s_n+}, \varepsilon \hat{V} \oplus \varepsilon \dot{Y}^{\varepsilon}_{s_n+} \right) \\
&+ o(\varepsilon^2) .
\end{aligned}
$$

Note that the two remainder terms $o(\varepsilon^2)$ appeared in the last equalities consist of higher derivatives of L_0 on $K_1^{\varepsilon_0}$. We denote the sum of these two remainder terms by Q^{ε}_{n+1}. Since $K_1^{\varepsilon_0}$ is compact, Q^{ε}_{n+1} is controlled uniformly in the position of $(X^{\varepsilon}_{s_n}, Y^{\varepsilon}_{s_n})$ and n as long as $s_n < \sigma_M(X^{\varepsilon}, Y^{\varepsilon})$. It means that there is a constant $\tilde{\delta}(\varepsilon)$ being independent of $(X^{\varepsilon}_{s_n}, Y^{\varepsilon}_{s_n})$ and n such that $Q^{\varepsilon}_{n+1} \le \tilde{\delta}(\varepsilon)$ and $\tilde{\delta}(\varepsilon)/\varepsilon^2 \to 0$. The triangular inequality for L_0 and the definition of $(\hat{\tau}, \hat{X})$ together with (19) and (20) yield

$$
\begin{aligned}
& L_0^{\tau'(s_{n+1}),\tau''(s_{n+1})}(X^{\varepsilon}_{s_{n+1}}, Y^{\varepsilon}_{s_{n+1}}) - L_0^{\tau'(s_n),\tau''(s_n)}(X^{\varepsilon}_{s_n}, Y^{\varepsilon}_{s_n}) \\
&\quad \le \left(L_0^{\tau'(s_{n+1}),\hat{\tau}+\varepsilon^2}(X^{\varepsilon}_{s_{n+1}}, \exp_{\hat{X}}^{g(\hat{\tau})}(\varepsilon \hat{V})) - L_0^{\tau'(s_n),\hat{\tau}}(X^{\varepsilon}_{s_n}, \hat{X}) \right) \\
&\qquad + \left(L_0^{\hat{\tau}+\varepsilon^2,\tau''(s_{n+1})}(\exp_{\hat{X}}^{g(\hat{\tau})}(\varepsilon \hat{V}), Y^{\varepsilon}_{s_{n+1}}) - L_0^{\hat{\tau},\tau''(s_n)}(\hat{X}, Y^{\varepsilon}_{s_n}) \right) \\
&\quad = \varepsilon \zeta^{\varepsilon}_{n+1} + \varepsilon^2 \Sigma^{\varepsilon}_{n+1} + Q^{\varepsilon}_{n+1},
\end{aligned}
$$

where random variables $\zeta^{\varepsilon}_{n+1}$ and $\Sigma^{\varepsilon}_{n+1}$ are defined by

$$
\zeta^{\varepsilon}_{n+1} := [\varepsilon \dot{X}^{\varepsilon}_{s_n+} \oplus \varepsilon \hat{V}] L_0^{\tau'(s_n),\hat{\tau}} + [\varepsilon \hat{V} \oplus \varepsilon \dot{Y}^{\varepsilon}_{s_n+}] L_0^{\hat{\tau},\tau''(s_n)}
$$

and

$$
\begin{aligned}
\Sigma^{\varepsilon}_{n+1} := & - \left\{ \frac{\partial L_0^{t',t''}}{\partial t'} + \frac{\partial L_0^{t',t''}}{\partial t''} \right\} \Big|_{(t',t'')=(\tau'(s_n),\hat{\tau})} (X^{\varepsilon}_{s_n}, \hat{X}) \\
& + \frac{1}{2} \left[\mathrm{Hess}_{g(\tau'(s_n))\oplus g(\hat{\tau})} L_0^{\tau'(s_n),\hat{\tau}} \right] \left(\varepsilon \dot{X}^{\varepsilon}_{s_n+} \oplus \varepsilon \hat{V}, \varepsilon \dot{X}^{\varepsilon}_{s_n+} \oplus \varepsilon \hat{V} \right) \\
& - \left\{ \frac{\partial L_0^{t',t''}}{\partial t'} + \frac{\partial L_0^{t',t''}}{\partial t''} \right\} \Big|_{(t',t'')=(\hat{\tau},\tau''(s_n))} (\hat{X}, Y^{\varepsilon}_{s_n})
\end{aligned}
$$

$$+\frac{1}{2}\big[\mathrm{Hess}_{g(\hat{\tau})\oplus g(\tau''(s_n))}L_0^{\hat{\tau},\tau'(s_n)}\big]\big(\varepsilon\hat{V}\oplus\varepsilon\dot{Y}^{\varepsilon}_{s_n+},\varepsilon\hat{V}\oplus\varepsilon\dot{Y}^{\varepsilon}_{s_n+}\big).$$

Thus (16) holds. Since

$$\#\{k\in\mathbb{N}: s_k<\sigma_{M_0}(X^{\varepsilon},Y^{\varepsilon})\wedge S\}\le S\varepsilon^{-2},$$

we have

$$\sum_{n:\ s_n<\sigma_{M_0}(X^{\varepsilon},Y^{\varepsilon})\wedge S}\mathcal{Q}^{\varepsilon}_{n+1}\le S\varepsilon^{-2}\tilde{\delta}(\varepsilon)\to 0$$

as $\varepsilon\to 0$. It asserts (17).

Finally, we prove the required properties for ζ^{ε}_n and Σ^{ε}_n. The measurability are obvious. The integrabilities hold because $X^{\varepsilon}_{s_n}$, $\hat{X}$ and $Y^{\varepsilon}_{s_n}$ lie on a bounded domain in M. Finally (18) follows from Lemma 4.1 and Proposition 2.8 since λ_n is independent of $\mathcal{G}_{n-1}$. □

Let $(\mathcal{F}_s)_{0\le s\le S}$ be the filtration defined by

$$\mathcal{F}_s:=\sigma\big((X_u,Y_u):0\le u\le s\big),\quad 0\le s\le S$$

and set $(\mathcal{F}^{\varepsilon}_{s_n})_{n=0}^{\infty}$ by $\mathcal{F}^{\varepsilon}_0:=$ the trivial σ-field, and

$$\mathcal{F}^{\varepsilon}_{s_n}:=\sigma\big((X^{\varepsilon}_{s_k},Y^{\varepsilon}_{s_k}):k=1,2,\dots,n\big)$$

for each $n=1,2,\dots$.

THEOREM 4.4. *Set $\Lambda_s:=L_0^{\tau'(s),\tau''(s)}(X_s,Y_s)$. Then Λ_s is integrable for each $s\in[0,S]$ and for each $u\le s$ we have*

$$\mathbf{E}[\Lambda_s|\mathcal{F}_u]\le\Lambda_u,$$

that is, $\Lambda=(\Lambda_s)_{0\le s\le S}$ is an $(\mathcal{F}_s)_{0\le s\le S}$-supermartingale.

PROOF. Take $\eta>0$ arbitrarily and choose a bounded open $M_0\subset M$ sufficiently large so that $\mathbf{P}[\sigma_{M_0}(X,Y)\le s]\le\eta$. We first claim that for each $\beta>0$ there exists $C=C(\beta)>0$ being independent of M_0 and η such that

$$\mathbf{E}\big[(\beta\wedge\Lambda_{s\wedge\sigma_{M_0}(X,Y)}-\beta\wedge\Lambda_{u\wedge\sigma_{M_0}(X,Y)})F\big]\le C\eta \tag{21}$$

for each nonnegative bounded $\mathcal{F}_u$-measurable random variable F. For proving (21), we may assume that F is of the form $F=f(Z_{u_1},\dots,Z_{u_n})$ where f is a nonnegative and bounded continuous function on $(M^2)^n$, $Z_u=(X_u,Y_u)$ and $0\le u_1<\cdots<u_n\le u$.

By Proposition 2.10 (iii), we have

$$\begin{aligned}&\mathbf{E}\big[(\beta\wedge\Lambda_{s\wedge\sigma_{M_0}(X,Y)}-\beta\wedge\Lambda_{u\wedge\sigma_{M_0}(X,Y)})F\big]\\&\quad\le\mathbf{E}\big[(\beta\wedge\Lambda_s-\beta\wedge\Lambda_u)F:\sigma_{M_0}(X,Y)>s\big]+C_1\eta\end{aligned}$$

for some constant $C_1>0$ which is independent of M_0 but may depend on β (Indeed, one can take $C_1=\beta+(dK_-T)/2$). Let $[u]_{\varepsilon}:=\sup\{\varepsilon^2 n:n\in\mathbb{N},\ \varepsilon^2 n\le u\}$ and $Z^{\varepsilon}:=(X^{\varepsilon},Y^{\varepsilon})$. Since $\{\mathbf{w}\mid\sigma_{M_0}(\mathbf{w})>s\}$ is open,

$$\mathbf{E}\big[(\beta\wedge\Lambda_s-\beta\wedge\Lambda_u)F:\sigma_{M_0}(X,Y)>s\big]$$

$$\leq \liminf_{\varepsilon\to 0} \mathbf{E}\big[(\beta \wedge \Lambda^\varepsilon_s - \beta \wedge \Lambda^\varepsilon_u) f(Z^\varepsilon_{u_1}, \dots, Z^\varepsilon_{u_n}) : \sigma_{M_0}(X^\varepsilon, Y^\varepsilon) > s\big]$$
$$= \liminf_{\varepsilon\to 0} \mathbf{E}\big[(\beta \wedge \Lambda^\varepsilon_{[s]_\varepsilon} - \beta \wedge \Lambda^\varepsilon_{[u]_\varepsilon}) f(Z^\varepsilon_{[u_1]_\varepsilon}, \dots, Z^\varepsilon_{[u_n]_\varepsilon}) : \sigma_{M_0}(X^\varepsilon, Y^\varepsilon) > s\big],$$

where the last equality follows from the continuity of $L_0^{\tau',\tau''}$ and f. Then

$$\mathbf{E}\big[(\beta \wedge \Lambda^\varepsilon_{[s]_\varepsilon} - \beta \wedge \Lambda^\varepsilon_{[u]_\varepsilon}) f(Z^\varepsilon_{[u_1]_\varepsilon}, \dots, Z^\varepsilon_{[u_n]_\varepsilon}) : \sigma_{M_0}(X^\varepsilon, Y^\varepsilon) > s\big]$$
$$\leq \mathbf{E}\big[(\beta \wedge \Lambda^\varepsilon_{[s]_\varepsilon \wedge [\sigma_{M_0}(X^\varepsilon,Y^\varepsilon)]_\varepsilon} - \beta \wedge \Lambda^\varepsilon_{[u]_\varepsilon \wedge [\sigma_{M_0}(X^\varepsilon,Y^\varepsilon)]_\varepsilon}) f(Z^\varepsilon_{[u_1]_\varepsilon}, \dots, Z^\varepsilon_{[u_n]_\varepsilon})\big]$$
$$+ C_1 \mathbf{P}\big[\sigma_{M_0}(X^\varepsilon, Y^\varepsilon) \leq s\big].$$

For the second term of the right hand side,

$$\limsup_{\varepsilon\to 0} \mathbf{P}[\sigma_{M_0}(X^\varepsilon, Y^\varepsilon) \leq s] \leq \eta$$

holds since $\{\mathbf{w} \mid \sigma_{M_0}(\mathbf{w}) \leq s\}$ is closed. Let us estimate the first term. Since Proposition 4.2 ensures $\overline{\Sigma}^\varepsilon_i \leq 0$ for each $i = 1, 2, \dots$, Proposition 4.2 together with the conditional Jensen inequality yields

$$\mathbf{E}\big[(\beta \wedge \Lambda^\varepsilon_{[s]_\varepsilon \wedge [\sigma_{M_0}(X^\varepsilon,Y^\varepsilon)]_\varepsilon}) f(Z^\varepsilon_{[u_1]_\varepsilon}, \dots, Z^\varepsilon_{[u_n]_\varepsilon})\big]$$
$$\leq \mathbf{E}\Bigg[\beta \wedge \Bigg\{\mathbf{E}\Bigg[\sum_{i=[u]_\varepsilon \wedge [\sigma_{M_0}(X^\varepsilon,Y^\varepsilon)]_\varepsilon + 1}^{[s]_\varepsilon \wedge [\sigma_{M_0}(X^\varepsilon,Y^\varepsilon)]_\varepsilon} \varepsilon\zeta^\varepsilon_i + \varepsilon^2(\Sigma^\varepsilon_i - \overline{\Sigma}^\varepsilon_i) + Q^\varepsilon_i \;\Big|\mathcal{F}^\varepsilon_{[u]_\varepsilon}\Bigg]$$
$$+ \Lambda^\varepsilon_{[u]_\varepsilon \wedge [\sigma_{M_0}(X^\varepsilon,Y^\varepsilon)]_\varepsilon}\Bigg\} f(Z^\varepsilon_{[u_1]_\varepsilon}, \dots, Z^\varepsilon_{[u_n]_\varepsilon})\Bigg].$$

Since $[\sigma_{M_0}(X^\varepsilon, Y^\varepsilon)]_\varepsilon$ is an $(\mathcal{F}^\varepsilon_{s_n})_n$-stopping time, we can apply the optional sampling theorem to conclude that the terms involving ζ^ε_i and $\Sigma^\varepsilon_i - \bar{\Sigma}^\varepsilon_i$ vanish. Again by Proposition 4.2, we have

$$\mathbf{E}\left[\sum_{i=[u]_\varepsilon \wedge [\sigma_{M_0}(X^\varepsilon,Y^\varepsilon)]_\varepsilon + 1}^{[s]_\varepsilon \wedge [\sigma_{M_0}(X^\varepsilon,Y^\varepsilon)]_\varepsilon} Q^\varepsilon_i \;\Big|\mathcal{F}^\varepsilon_{[u]_\varepsilon}\right] \leq \delta(\varepsilon) \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

Therefore, by combining all these estimates, we obtain the claim (21) with $C = 2C_1$.

In (21), letting $M_0 \uparrow M$ with the dominated convergence theorem and letting $\eta \downarrow 0$ yield

$$\mathbf{E}\big[(\beta \wedge \Lambda_s)F\big] \leq \mathbf{E}\big[(\beta \wedge \Lambda_u)F\big].$$

Since Λ_s is bounded from below by Proposition 2.10 (iii), the monotone convergence theorem yields the conclusion by $\beta \uparrow \infty$. Indeed, we obtain the integrability of Λ_s by applying this argument with $u = 0$. □

Now we are in turn to complete the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. In Theorem 4.4, we have proved the existence of a coupling (X, Y) of $g(\tau'(s))$- and $g(\tau''(s))$-Brownian motions with deterministic initial data such that $s \mapsto L_0^{\tau'(s),\tau''(s)}(X_s, Y_s)$ is a supermartingale. Thus it suffices to show that we can choose the family of laws of couplings as a measurable function of initial data.

To complete it, we shall employ a measurable selection theorem. Note that the space of all Borel probability measures on the path space $C([0, S] \to M \times M)$ equipped with the weak topology is a Polish space.

We define $\mathscr{K}$ as the set of all laws of a coupling (X, Y) of $g(\tau'(s))$-Brownian motion $X = (X_s)_{0\le s\le S}$ and $g(\tau''(s))$-Brownian motion $Y = (Y_s)_{0\le s\le S}$ such that

(a) (X_0, Y_0) is deterministic and

(b) for $0 \le u \le s$, it holds that $L_0^{\tau'(s),\tau''(s)}(X_s, Y_s)$ is integrable and

$$\mathbf{E}[L_0^{\tau'(s),\tau''(s)}(X_s, Y_s)|\mathcal{F}_u] \le L_0^{\tau'(u),\tau''(u)}(X_u, Y_u) \quad \text{a.s.}$$

We denote the probability or the expectation with respect to $Q \in \mathcal{K}$ by $\mathbf{P}^Q$ or $\mathbf{E}^Q$ respectively. For each $m', m'' \in M$, let us define $\mathcal{K}_{m',m''} \subset \mathcal{K}$ by

$$\mathcal{K}_{m',m''} := \{Q \in \mathcal{K} \mid \mathbf{P}^Q[(X_0, Y_0) = (m', m'')] = 1\}.$$

By [3, Theorem 6.9.6], the claim holds once we show that $\mathcal{K}_{m',m''}$ is compact for each m', m''. Since the marginal distributions of elements in $\mathcal{K}_{m',m''}$ is fixed, the Prokhorov theorem yields that $\mathcal{K}_{m',m''}$ is relatively compact. To show that $\mathcal{K}_{m',m''}$ is closed, take a sequence $Q_n \in \mathcal{K}_{m',m''}$ which converges to Q. The following argument is similar to the one in the proof of Theorem 4.4. First, $\mathbf{P}^Q[(X_0, Y_0) = (m', m'')] = 1$ obviously holds and hence Q verifies the condition (a). Second, for each $k \in \mathbb{N}$, $f_i \in C_b(M^2)$ $(i = 1, \dots, k)$, $0 \le u_1 \le u_2 \le \cdots \le u_k \le s \le S$ and $R > 0$,

$$\begin{aligned}
&\mathbf{E}^{Q_n}\left[L_0^{\tau'(s),\tau''(s)}(X_s, Y_s) \wedge R \;\middle|\; \prod_{i=1}^k f_i(X_{u_i}, Y_{u_i})\right] \\
&\quad \le \mathbf{E}^{Q_n}\left[L_0^{\tau'(u_k),\tau''(u_k)}(X_{u_k}, Y_{u_k}) \wedge R \;\middle|\; \prod_{i=1}^k f_i(X_{u_i}, Y_{u_i})\right].
\end{aligned}$$

Thus, by tending $n \to \infty$ and $R \to \infty$ after it, we obtain

$$\begin{aligned}
&\mathbf{E}^{Q}\left[L_0^{\tau'(s),\tau''(s)}(X_s, Y_s) \;\middle|\; \prod_{i=1}^k f_i(X_{u_i}, Y_{u_i})\right] \\
&\quad \le \mathbf{E}^{Q}\left[L_0^{\tau'(u_k),\tau''(u_k)}(X_{u_k}, Y_{u_k}) \;\middle|\; \prod_{i=1}^k f_i(X_{u_i}, Y_{u_i})\right].
\end{aligned}$$

In particular, by applying the same argument for $k = 1$, $f_1 \equiv 1$ and $u_1 = 0$, we obtain that $L_0^{\tau'(s),\tau''(s)}(X_s, Y_s)$ is integrable with respect to Q. Thus Q verifies the condition (b) and hence $Q \in \mathcal{K}_{m',m''}$. It means $\mathcal{K}_{m',m''}$ is closed and the proof is completed. □

PROOF OF COROLLARY 1.2. Since Proposition 2.10 (iii) and Lemma 2.14 ensure that $L_0^{t_1',t_1''}$ is continuous and bounded from below, there is a minimizer $\pi \in \Pi\big(c'(t_1'), c''(t_1'')\big)$ for

$$C_{0,\varphi}^{t_1',t_1''}\big(c'(t_1'), c''(t_1'')\big) = C_{0,\varphi}^{\tau'(0),\tau''(0)}\big(c'(\tau'(0)), c''(\tau''(0))\big)\,.$$

Let $\mathbf{P}_{(m',m'')}$ be the law of the coupling (X, Y) with $(X_0, Y_0) = (m', m'')$ obtained in Theorem 1.1. This is a probability measure on $\mathscr{W}(M) \times \mathscr{W}(M)$, where $\mathscr{W}(M)$ is the space of continuous paths in M defined on $[0, S]$. Let $\mathbf{P}$ be the probability measure on $\mathscr{W}(M)\times\mathscr{W}(M)$ given by

$$\mathbf{P}(\mathrm{d}w', \mathrm{d}w'') := \int_{M\times M} \pi(\mathrm{d}m', \mathrm{d}m'')\mathbf{P}_{(m',m'')}(\mathrm{d}w', \mathrm{d}w'')\,.$$

Note that $\mathbf{P}$ is well-defined by virtue of the measurability result in Theorem 1.1. Under $\mathbf{P}$, the canonical process $(w'(s), w''(s))$ is a coupling of $g(\tau'(s))$-Brownian motion and $g(\tau''(s))$-Brownian motion with the initial distribution π such that $L_0^{\tau'(s),\tau''(s)}(w'(s), w''(s))$ is a supermartingale. In particular, the law of $(w'(s), w''(s))$ gives a coupling of $c'(\tau'(s))$ and $c''(\tau''(s))$ for each $s \in [0, S]$. Since $\varphi\big(L_0^{\tau'(s),\tau''(s)}(w'(s), w''(s))\big)$ is still a supermartingale under $\mathbf{P}$, we have

$$\begin{aligned}
&C_{0,\varphi}^{\tau'(s),\tau''(s)}\big(c'(\tau'(s)), c''(\tau''(s))\big)\\
&\quad\le \int_{M\times M} \mathbf{E}_{(m',m'')}\big[\varphi\big(L_0^{\tau'(s),\tau''(s)}(w'(s), w''(s))\big)\big]\pi(\mathrm{d}m', \mathrm{d}m'')\\
&\quad\le \int_{M\times M} \mathbf{E}_{(m',m'')}\big[\varphi\big(L_0^{\tau'(0),\tau''(0)}(w'(0), w''(0))\big)\big]\pi(\mathrm{d}m', \mathrm{d}m'')\\
&\quad= \int_{M\times M} \varphi\big(L_0^{\tau'(0),\tau''(0)}(m', m'')\big)\pi(\mathrm{d}m', \mathrm{d}m'')\\
&\quad= C_{0,\varphi}^{\tau'(0),\tau''(0)}\big(c'(\tau'(0)), c''(\tau''(0))\big)\,,
\end{aligned}$$

where $\mathbf{E}_{(m',m'')}$ stands for the expectation with respect to $\mathbf{P}_{(m',m'')}$. Hence $C_{0,\varphi}^{t_1'-s,t_1''-s}\big(c'(t_1' - s), c''(t_1'' - s)\big)$ is non-increasing in s because we can repeat the same argument even if we replace the initial time 0 with any $s' \in [0, s]$. □

REMARK 4.5. The proof of the integrability of the $\mathcal{L}$-distance between the coupling of Brownian motions by space-time parallel transport in [15] seems to be incorrect. It is worth mentioning that we can recover the same integrability as in Theorem 4.4 even in that case by the same argument.

More precisely, the argument in [15, Lemma 6] seems to require some modification. In the proof, they claimed an inequality analogous to Doob's L^p-martingale inequality for a positive supermartingale. However it is not true in general. Indeed there is a counterexample to Doob's inequality when $p = 1$ for a positive martingale M_n (See [8, Example 5.4.2]). Since the $x \mapsto x^{1/p}$ is nonincreasing and concave on $[0, \infty)$ for $p \ge 1$, $M_n^{1/p}$ gives a counterexample to Doob's L^p-inequality to positive supermartingales.

On the other hand, if we further assume the stronger restriction on the Ricci curvature, we can recover a similar integrability as stated in [15, Lemma 6]:

PROPOSITION 4.6. *Suppose* $\sup_{0\le t\le T}|\mathrm{Ric}|_{g(t)} < \infty$. *Fix arbitrary two points* $m', m'' \in M$. *For any coupling* $(X_s, Y_s)_{0\le s\le S}$ *of* $g(\tau'(s))$*-Brownian motion* $X = (X_s)_{0\le s\le S}$ *with* $X_0 = m'$ *and* $g(\tau''(s))$*-Brownian motion* $Y = (Y_s)_{0\le s\le S}$ *with* $Y_0 = m''$, $\sup_{0\le s\le S} L_0^{\tau'(s),\tau''(s)}(X_s, Y_s)$ *is integrable.*

PROOF. Let $o \in M$ be a fixed reference point of M. By Proposition 2.10(iii) and Proposition 2.11(i), we have estimates

$$\begin{aligned}\text{const.} &\le L_0^{\tau'(s),\tau''(s)}(X_s, Y_s)\\ &\le \text{const.}\Big\{\rho_{g(\tau'(s))}(X_s, o)^2 + \rho_{g(\tau''(s))}(o, Y_s)^2\Big\} + \text{const.},\end{aligned}$$

where the constants depend on $\sup_{0\le t\le T}|\mathrm{Ric}|_{g(t)}$, T, t_1' and t_1'' but not on s. Therefore, for our statement, it is sufficient to prove the integrability of $\sup_{0\le s\le T}\rho_{g(\tau'(s))}(X_s, o)^2$. By [14, Theorem 2], we see that

$$\mathrm{d}\rho(s, X_s) = \left\{\frac{1}{2}\Delta_{g(\tau'(s))} - \frac{\partial}{\partial s}\right\}\rho(s, X_s)\mathrm{d}s + (U_s.e_i)\rho(s, X_s)\mathrm{d}W_s^i - \mathrm{d}L_s,$$

denoting $\rho(s, x) := \rho_{g(\tau'(s))}(o, x)$, where L_s is a nondecreasing continuous process which increases only when X_s belongs to the cut locus of o with respect to $g(\tau'(s))$. Note that the set of $s \in [0, S]$ where X_s is in $g(\tau'(s))$-cut locus has null Lebesgue measure and hence other quantities are also well-defined. Therefore we have

$$\begin{aligned}\mathrm{d}\rho(s, X_s)^2 = \left\{\rho(s, X_s)\left(\Delta_{g(\tau'(s))} - 2\frac{\partial}{\partial s}\right)\rho(s, X_s)\mathrm{d}s + \sum_{i=1}^{d}|(U_s.e_i)\rho(s, X_s)|^2\right\}\mathrm{d}s\\ -2\rho(s, X_s)\mathrm{d}L_s + 2\rho(s, X_s)\mathrm{d}\beta_s\end{aligned}$$

where $\mathrm{d}\beta_s = (U_s.e_i)\rho(s, X_s)\mathrm{d}W_s^i$ is a one-dimensional Brownian motion. Furthermore, Proposition 2 in [14] shows that

$$\begin{aligned}&\rho(s, X_s)\left\{\frac{1}{2}\Delta_{g(\tau'(s))} - \frac{\partial}{\partial s}\right\}\rho(s, X_s)\\ &\quad\le \frac{d-1}{2}\rho(s, X_s)\Big\{k_1\coth(k_1\cdot\rho(s, X_s)\wedge r_1) + k_1^2\cdot\rho(s, X_s)\wedge r_1\Big\}\end{aligned}$$

for some positive constants k_1 and r_1 from which we find that there is some constant $c > 0$ such that

$$\rho(s, X_s)\left\{\frac{1}{2}\Delta_{g(\tau'(s))} + \frac{\partial}{\partial s}\right\}\rho(s, X_s) \le c\rho(s, X_s)^2 \quad \text{for each } s \ge 0 \text{ a.s.}$$

We also find that $\sum_{i=1}^{d} |(U_s.e_i)\rho(s, X_s)|^2 = 1$. So we shall apply a comparison argument between ρ's SDE and

$$\begin{cases} \mathrm{d}Z_s = \sqrt{4Z_s \vee 0}\,\mathrm{d}\beta_s + (cZ_s + 1)\mathrm{d}s\,, \\ Z_0 = \rho(0, X_0) = \rho_{g(t_1')}(m', o)\,. \end{cases}$$

It is well-known that there is a global unique strong solution $Z = (Z_s)_{s\geq 0}$ to the above SDE and this satisfies $Z_s \geq 0$ for all $s \geq 0$ a.s. By the comparison theorem (e.g., see Theorem 1.1, Chapter VI in [10]), we see that $\rho(s, X_s) \leq Z_s$ for each $s \geq 0$ a.s. Since $\sup_{0\leq u\leq s} Z_u^p$ is integrable for any $p \geq 1$ (e.g., see Lemma 2.1, Chapter V in [10]), the conclusion follows. □

REFERENCES

[1] M. ARNAUDON, K. A. COULIBALY AND A. THALMAIER, Horizontal Diffusion in C^1 Path Space, Séminaire de Probabilités XLIII, 73–94, Lecture Notes in Math., 2006, Springer, Berlin, 2011.

[2] M. ARNAUDON, K. A. COULIBALY AND A. THALMAIER, Brownian motion with respect to a metric depending on time: definition, existence and applications to Ricci flow, C. R. Math. Acad. Sci. Paris 346 (2008), no. 13–14, 773–778.

[3] V. BOGACHEV, Measure theory. Vol. I, II, Springer-Verlag, Berlin, 2007.

[4] B. CHEN AND X. ZHU, Uniqueness of the Ricci flow on complete noncompact manifolds, J. Differential Geom. 74 (2006), no. 1, 119–154.

[5] L. CHENG, The radial part of Brownian motion with respect to $\mathcal{L}$-distance under Ricci flow, arXiv:1211.3626v2.

[6] B. CHOW, S.-C. CHU, D. GLICKENSTEIN, C. GUENTHER, J. ISENBERG, T. IVEY, D. KNOPF, P. LU, F. LUO AND L. NI, The Ricci flow: techniques and applications. Part I. Geometric aspects, Mathematical Surveys and Monographs, 135. American Mathematical Society, Providence, RI, 2007.

[7] M. CRANSTON, Gradient estimates on manifolds using coupling, J. Funct. Anal. 99 (1991), no. 1, 110–124.

[8] R. DURRETT, Probability: theory and examples. Fourth edition, Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, Cambridge, 2010.

[9] R. S. HAMILTON, Three-manifolds with positive Ricci curvature, J. Differential Geom. 17 (1982), no. 2, 255–306.

[10] N. IKEDA AND S. WATANABE, Stochastic differential equations and diffusion processes, Second edition. North-Holland Mathematical Library, 24. North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, 1989.

[11] W. S. KENDALL, Nonnegative Ricci curvature and the Brownian coupling property, Stochastics 19 (1986), no. 1–2, 111–129.

[12] W. S. KENDALL, From stochastic parallel transport to harmonic maps, New directions in Dirichlet forms, 49–115, Amer. Math. Soc., Providence, RI; International Press, Cambridge, MA, 1998.

[13] B. KLEINER AND J. LOTT, Notes on Perelman's papers, Geom. Topol., 12 (2008), no. 5, 2587–2855.

[14] K. KUWADA AND R. PHILIPOWSKI, Non-explosion of diffusion processes on manifolds with time-dependent metric, Math. Z. 268 (2011), no. 3-4, 979–991.

[15] K. KUWADA AND R. PHILIPOWSKI, Coupling of Brownian motions and Perelman's $\mathcal{L}$-functional, J. Funct. Anal. 260 (2011), 2742–2766.

[16] K. KUWADA, Convergence of time-inhomogeneous geodesic random walks and its application to coupling methods, Ann. Probab. 40 (2012), no. 5, 1945–1979.

[17] T. LINDVALL AND L. C. G. ROGERS, Coupling of multidimensional diffusions by reflection, Ann. Probab. 14 (1986), no. 3, 860–872.

[18] J. LOTT, Optimal transport and Perelman's reduced volume, Calc. Var. Partial Differential Equations 36 (2009), no. 1, 49–84.

[19] J. N. MATHER, Action minimizing invariant measures for positive definite Lagrangian systems, Math. Z. 207 (1991), no. 2, 169–207.

[20] R. J. MCCANN AND P. TOPPING, Ricci flow, entropy and optimal transportation, Amer. J. Math. 132 (2010), no. 3, 711–730.

[21] G. PEREL'MAN, The entropy formula for the Ricci flow and its geometric applications, arXiv preprint math/0211159, 2002.

[22] M.-K. VON RENESSE AND K.-T. STURM, Transport inequalities, gradient estimates, entropy and Ricci curvature, Comm. Pure. Appl. Math. 58 (2005), no. 7, 923–940.

[23] W.-X. SHI, Deforming the metric on complete Riemannian manifolds, J. Differential Geom. 30 (1989), no. 1, 223–301.

[24] P. TOPPING, Lectures on the Ricci flow, London Mathematical Society Lecture Note Series, 325. Cambridge University Press, Cambridge, 2006.

[25] P. TOPPING, $\mathscr{L}$-optimal transportation for Ricci flow, J. Reine Angew. Math. 636 (2009), 93–122.

[26] C. VILLANI, Optimal transport. Old and new, Grundlehren der Mathematischen Wissenschaften, 338. Springer-Verlag, Berlin, 2009.

[27] F.-Y. WANG, Functional inequalities, Markov semigroups, and spectral theory, Science Press, Beijing, China, 2005.

DEPARTMENT OF APPLIED MATHEMATICS
FACULTY OF SCIENCES
FUKUOKA UNIVERSITY
FUKUOKA 813–0180
JAPAN

E-mail address: fmamaba@fukuoka-u.ac.jp

MATHEMATICAL INSTITUTE
TOHOKU UNIVERSITY
SENDAI 980–8578
JAPAN

E-mail address: kuwada@m.tohoku.ac.jp