

## WORPITZKY PARTITIONS FOR ROOT SYSTEMS AND CHARACTERISTIC QUASI-POLYNOMIALS

MASAHIKO YOSHINAGA

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**Abstract.** For a given irreducible root system, we introduce a partition of (coweight) lattice points inside the dilated fundamental parallelepiped into those of partially closed simplices. This partition can be considered as a generalization and a lattice points interpretation of the classical formula of Worpitzky.

This partition, and the generalized Eulerian polynomial, recently introduced by Lam and Postnikov, can be used to describe the characteristic (quasi)polynomials of Shi and Linial arrangements. As an application, we prove that the characteristic quasi-polynomial of the Shi arrangement turns out to be a polynomial. We also present several results on the location of zeros of characteristic polynomials, related to a conjecture of Postnikov and Stanley. In particular, we verify the “functional equation” of the characteristic polynomial of the Linial arrangement for any root system, and give partial affirmative results on “Riemann hypothesis” for the root systems of type  $E_6$ ,  $E_7$ ,  $E_8$ , and  $F_4$ .

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## 1. Introduction.

**1.1. Arrangements of hyperplanes.** Let  $\mathcal{A} = \{H_1, \dots, H_n\}$  be an arrangement of affine hyperplanes in a vector space  $V$ . The intersection poset of  $\mathcal{A}$  is the set  $L(\mathcal{A}) = \{\cap S \mid S \subset \mathcal{A}\}$  of intersections of  $\mathcal{A}$ . The intersection poset  $L(\mathcal{A})$  is partially ordered by reverse inclusion, which has a unique minimal element  $\hat{0} = V$ . The arrangement  $\mathcal{A}$  is called essential if the maximal elements of  $L(\mathcal{A})$  are 0-dimensional subspaces. The characteristic polynomial of  $\mathcal{A}$  is defined by

$$\chi(\mathcal{A}, q) = \sum_{X \in L(\mathcal{A})} \mu(X) q^{\dim X},$$

where  $\mu$  is the Möbius function on  $L(\mathcal{A})$ , defined by

$$\mu(X) = \begin{cases} 1, & \text{if } X = \hat{0} \\ -\sum_{Y < X} \mu(Y), & \text{otherwise.} \end{cases}$$

The characteristic polynomial  $\chi(\mathcal{A}, q)$  of  $\mathcal{A}$  is one of the most fundamental invariants. Indeed,  $\chi(\mathcal{A}, q)$  captures combinatorial and topological properties of  $\mathcal{A}$  as follows.

**THEOREM 1.1.** (i) (Zaslavsky [28]). *Suppose that  $V$  is a real vector space. Then the number of connected components of  $V \setminus \cup \mathcal{A}$  is equal to  $(-1)^\ell \chi(\mathcal{A}, -1)$ . If  $\mathcal{A}$  is essential, then the number of bounded connected components of  $V \setminus \cup \mathcal{A}$  is equal to  $(-1)^\ell \chi(\mathcal{A}, 1)$ .*

(ii) (Orlik and Solomon [17]). *Suppose that  $V$  is a complex vector space. Then the Poincaré polynomial of  $V \setminus \cup \mathcal{A}$  is equal to  $(-t)^\ell \chi(\mathcal{A}, -\frac{1}{t})$ .*

**1.2. Main results.** Let  $\Phi$  be a root system of rank  $\ell$ , with exponents  $e_1, \dots, e_\ell$  and Coxeter number  $h$ . Fix a set of positive roots  $\Phi^+ \subset \Phi$ . The structures of truncated affine Weyl arrangements

$$\mathcal{A}_\Phi^{[a,b]} = \{H_{\alpha,k} \mid \alpha \in \Phi^+, a \leq k \leq b\}$$

(see also §2.3 for the notation) have been intensively studied, because of their intriguing combinatorial properties [20, 2, 21]. The characteristic polynomial of the Coxeter arrangement  $\mathcal{A}_\Phi = \mathcal{A}_\Phi^{[0,0]}$  was computed in [7]. Later, this was generalized to the extended Catalan arrangement  $\mathcal{A}_\Phi^{[-k,k]}$  and the extended Shi arrangement  $\mathcal{A}_\Phi^{[1-k,k]}$ . The characteristic polynomial

of these arrangements factor as follows

$$\chi(\mathcal{A}_\Phi^{[-k,k]}, t) = \prod_{i=1}^{\ell} (t - e_i - kh),$$

$$\chi(\mathcal{A}_\Phi^{[1-k,k]}, t) = (t - kh)^\ell,$$

([9, 1, 4, 27]). For other parameters  $a \leq b$ , e.g., the Linal arrangement  $\mathcal{A}_\Phi^{[1,n]}$ , the characteristic polynomial  $\chi(\mathcal{A}_\Phi^{[a,b]}, t)$  does not factor in general. However there are a number of beautiful conjectures concerning  $\chi(\mathcal{A}_\Phi^{[a,b]}, t)$ . Among others, Postnikov and Stanley [19] conjectured that

- (a)  $\chi(\mathcal{A}_\Phi^{[1-k,n+k]}, t) = \chi(\mathcal{A}_\Phi^{[1,n]}, t - kh)$  (“ $h$ -shift reduction”).
- (b)  $\chi(\mathcal{A}_\Phi^{[1,n]}, nh - t) = (-1)^\ell \chi(\mathcal{A}_\Phi^{[1,n]}, t)$  (“Functional equation”).
- (c) All the roots of the polynomial  $\chi(\mathcal{A}_\Phi^{[1,n]}, t)$  have the same real part  $\frac{nh}{2}$  (“Riemann hypothesis”).

Postnikov and Stanley verified these assertions for  $\Phi = A_\ell$  in [19]. Later, Athanasiadis gave proofs for  $\Phi = A_\ell, B_\ell, C_\ell, D_\ell$  in [1, 3].

Recently, Kamiya, Takemura and Terao [15] introduced the notion of the characteristic quasi-polynomial for an arrangement  $\mathcal{A}$  defined over  $\mathbb{Q}$ . The characteristic quasi-polynomial  $\chi_{\text{quasi}}(\mathcal{A}, t)$  is a periodic polynomial (see §2.2 for details) that may be considered as a refinement of the characteristic polynomial  $\chi(\mathcal{A}, t)$ .

Our main result concerns the characteristic quasi-polynomial for  $\mathcal{A}_\Phi^{[a,b]}$ : “ $h$ -shift reduction” and the “functional equation” hold at the level of characteristic quasi-polynomials.

**THEOREM 1.2.** *Let  $\Phi$  be an arbitrary irreducible root system.*

- (i) *The characteristic quasi-polynomial of the extended Shi arrangement is a polynomial,  $\chi_{\text{quasi}}(\mathcal{A}_\Phi^{[1-k,k]}, t) = (t - kh)^\ell$  (Theorem 5.1).*
- (ii) *The characteristic quasi-polynomial satisfies “ $h$ -shift reduction”  $\chi_{\text{quasi}}(\mathcal{A}_\Phi^{[1-k,n+k]}, t) = \chi_{\text{quasi}}(\mathcal{A}_\Phi^{[1,n]}, t - kh)$  (Theorem 5.3). In particular, this holds for the characteristic polynomial (Corollary 5.4).*
- (iii) *The characteristic quasi-polynomial satisfies “Functional equation”  $\chi_{\text{quasi}}(\mathcal{A}_\Phi^{[1,n]}, nh - t) = (-1)^\ell \chi_{\text{quasi}}(\mathcal{A}_\Phi^{[1,n]}, t)$  (Theorem 5.6). In particular, this holds for the characteristic polynomial (Corollary 5.7).*
- (iv) *Suppose  $\Phi \in \{E_6, E_7, E_8, F_4\}$ . Let  $\tilde{n}$  be the period of the Ehrhart quasi-polynomial of the fundamental alcove (see §3.2 and Table 1). If  $n \equiv -1 \pmod{\text{rad}(\tilde{n})}$ , then the “Riemann hypothesis” holds for  $\mathcal{A}_\Phi^{[1,n]}$  (Theorem 5.8).*

**1.3. Outline of the proof.** We follow the strategy adopted in [6, 1, 4, 15] for the computation of  $\chi_{\text{quasi}}(\mathcal{A}_\Phi, q)$ , where  $\mathcal{A}_\Phi := \mathcal{A}_\Phi^{[0,0]}$  is the Coxeter arrangement (see §3.3). The idea is to relate the characteristic quasi-polynomial to the Ehrhart quasi-polynomial  $L_{A^\circ}(q)$  of the fundamental alcove  $A^\circ$ . Consider the associated hyperplane arrangement  $\bar{\mathcal{A}}$  in the quotient  $Z(\Phi)/qZ(\Phi)$ , where  $Z(\Phi)$  is the coweight lattice. Then, by definition,  $\chi_{\text{quasi}}(\mathcal{A}_\Phi, q)$  is

the number of points in the complement of  $\overline{\mathcal{A}}$ , for  $q \gg 0$ . If we define  $P^\diamond = \sum_{k=1}^{\ell} (0, 1] \overline{\omega}_i$ , (where  $\overline{\omega}_i^\vee$  is the basis dual to the simple basis), then there is a bijective correspondence between the points in  $Z(\Phi)/qZ(\Phi)$  and the lattice points in the dilated parallelepiped  $qP^\diamond$ . The parallelepiped  $P^\diamond$  is dissected by the affine Weyl arrangement into open simplices (alcoves). Thus  $\chi_{\text{quasi}}(\mathcal{A}_\Phi, q)$  can be expressed as the sum of Ehrhart quasi-polynomials of these alcoves. Since  $\mathcal{A}_\Phi$  is Weyl group invariant, the above dissection is into simplices of the same size (Figure 1), which yields the simple formula

$$(1) \quad \chi_{\text{quasi}}(\mathcal{A}_\Phi, q) = \frac{f}{|W|} \cdot L_{A^\circ}(q)$$

(see Corollary 3.5, Corollary 3.6 and Proposition 3.7). The formula (1) first appeared explicitly in [6], where it was proved using the classification of root systems. A case free proof was given in [1]. The argument was later extended to the case  $\mathcal{A}_\Phi^{[-m, m]}$  in [4].

If we apply the same strategy for the case of Shi and Linial arrangements, then  $\chi_{\text{quasi}}(\mathcal{A}_\Phi^{[a, b]}, q)$  can again be expressed as the sum of Ehrhart quasi-polynomials. However the sizes of simplices are no longer uniform (see Figure 4). This difficulty can be overcome by looking at a disjoint partition of  $P^\diamond$  into partially closed alcoves

$$(2) \quad P^\diamond = \bigsqcup_{\xi \in \Xi} A_\xi^\diamond,$$

(see §2.4 for details). Then obviously we have a partition of lattice points

$$(3) \quad qP^\diamond \cap Z(\Phi) = \bigsqcup_{\xi \in \Xi} (qA_\xi^\diamond \cap Z(\Phi)),$$

which we will call a Worpitzky partition. The number of lattice points contained in  $qA_\xi^\diamond$  is expressed as

$$(4) \quad L_{A_\xi^\diamond}(q) = L_{\overline{A^\circ}}(q - \text{asc}(A_\xi^\circ)),$$

(Lemma 4.9), where  $\text{asc}(A_\xi^\circ)$  is a certain integer (Definition 4.1 and (35)). The key result (Theorem 4.7) in the proof of our main results is that the distribution of the quantity  $\text{asc}(A_\xi^\circ)$  is given by the generalized Eulerian polynomial  $R_\Phi(t)$  (Definition 4.4) introduced by Lam and Postnikov [16]. Using the shift operator  $S$  (§2.5), the partition (3) implies the formula,

$$(5) \quad q^\ell = (R_\Phi(S) L_{\overline{A^\circ}})(q).$$

In the case  $\Phi = A_\ell$ , the polynomial  $R_\Phi(t)$  is equal to the classical Eulerian polynomial. Then the above formula (5) is known as the Worpitzky identity [26, 8]. Hence (5) can be considered as a generalization of Worpitzky identity and (3) as its lattice points interpretation.

Using these results, the characteristic quasi-polynomials for Shi and Linial arrangements have expressions similar to the Worpitzky identity (5). We have

$$(6) \quad \begin{aligned} \chi_{\text{quasi}}(\mathcal{A}_\Phi^{[1-k, k]}, q) &= (S^{kh} R_\Phi(S) L_{\overline{A^\circ}})(q) = (q - kh)^\ell, \\ \chi_{\text{quasi}}(\mathcal{A}_\Phi^{[1-k, n+k]}, q) &= (S^{kh} R_\Phi(S^{n+1}) L_{\overline{A^\circ}})(q), \end{aligned}$$

(Theorem 5.1, Theorem 5.3). Using these expressions, the functional equation is obtained from the duality of the generalized Eulerian polynomial

$$(7) \quad t^h \mathbf{R}_\Phi\left(\frac{1}{t}\right) = \mathbf{R}_\Phi(t),$$

(Proposition 4.5).

If  $n \equiv -1 \pmod{\text{rad}(\tilde{n})}$ , then  $1 + n$  is divisible by  $\text{rad}(\tilde{n})$ . Hence  $\gcd(q, \tilde{n}) = 1$  implies  $\gcd(q - k(n + 1), \tilde{n}) = 1$  for  $k \in \mathbb{Z}$ . This enables us to simplify the expression of the characteristic polynomial  $\chi(\mathcal{A}_\Phi^{[1, n]}, t)$ . Using techniques similar to those in Postnikov, Stanley and Athanasiadis [19, 3], we can verify the ‘‘Riemann hypothesis’’ for such parameters  $n$ .

The paper is organized as follows. §2 contains background materials on root systems, characteristic quasi-polynomials and the Eulerian polynomial. The partition of the fundamental parallelepiped, which will play an important role later, is introduced in §2.4 (Definition 2.3). In §3 the relation between the Ehrhart quasi-polynomial of the fundamental alcove and the characteristic quasi-polynomial is discussed. In §4 we first summarize basic properties of the generalized Eulerian polynomial  $\mathbf{R}_\Phi(t)$  introduced by Lam and Postnikov [16]. Then we introduce Worpitzky partitions of the lattice points which provide a Worpitzky-type identity (Theorem 4.8). We also give an explicit example of the Worpitzky partition for  $\Phi = B_2$ . In §5, we obtain formulae for characteristic quasi-polynomials by modifying the Worpitzky-type identity. Using these formulae, we prove our main results.

## 2. Background.

**2.1. Quasi-polynomials with gcd-property.** A function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  is called a *quasi-polynomial* if there exist  $\tilde{n} > 0$  and polynomials  $g_1(t), g_2(t), \dots, g_{\tilde{n}}(t) \in \mathbb{Z}[t]$  such that

$$f(q) = g_r(q), \text{ if } q \equiv r \pmod{\tilde{n}},$$

( $1 \leq r \leq \tilde{n}$ ). The minimal such  $\tilde{n}$  is called the period of the quasi-polynomial  $f$ .

Moreover, the function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  is said to be a *quasi-polynomial with gcd-property* if the polynomial  $g_r(t)$  depends on  $r$  only through  $\gcd(r, \tilde{n})$ . In other words,  $g_{r_1}(t) = g_{r_2}(t)$  if  $\gcd(r_1, \tilde{n}) = \gcd(r_2, \tilde{n})$ .

**2.2. Arrangements and characteristic quasi-polynomials.** Let  $L \simeq \mathbb{Z}^\ell$  be a lattice and  $L^\vee = \text{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$  be the dual lattice. Given  $\alpha_1, \dots, \alpha_n \in L^\vee$  and integers  $k_1, \dots, k_n \in \mathbb{Z}$ , we can associate a hyperplane arrangement  $\mathcal{A} = \{H_1, \dots, H_n\}$  in  $\mathbb{R}^\ell \simeq L \otimes_{\mathbb{A}} \mathbb{R}$ , with  $H_i = \{x \in L \otimes \mathbb{R} \mid \alpha_i(x) = k_i\}$ . For a positive integer  $q > 0$ , define

$$(8) \quad M(\mathcal{A}; q) := \{\bar{x} \in L/qL \mid \forall i, \alpha_i(x) \not\equiv k_i \pmod{q}\}.$$

Kamiya, Takemura and Terao proved the following.

**THEOREM 2.1** ([13, 14]). *There exist  $q_0 > 0$  and a quasi-polynomial  $\chi_{\text{quasi}}(\mathcal{A}, t)$  with gcd-property such that  $\#M(\mathcal{A}, q) = \chi_{\text{quasi}}(\mathcal{A}, q)$  for  $q > q_0$ .*

More precisely, there exists a period  $\tilde{n}$  and a polynomial  $g_d(t) \in \mathbb{Z}[t]$  for each divisor  $d|\tilde{n}$  such that

$$\#M(\mathcal{A}; q) = g_d(q),$$

for  $q > q_0$ , where  $d = \gcd(\tilde{n}, q)$ .

One of the most important invariants of a hyperplane arrangement  $\mathcal{A}$  is the characteristic polynomial  $\chi(\mathcal{A}, t) \in \mathbb{A}[t]$  (see [18] for the definition and basic properties). The characteristic polynomial is one of the polynomials given by Theorem 2.1 (see also [3, Theorem 2.1]), specifically,

$$(9) \quad \chi(\mathcal{A}, t) = g_1(t).$$

**2.3. Root systems.** Let  $V = \mathbb{R}^\ell$  be the Euclidean space with inner product  $(\cdot, \cdot)$ . Let  $\Phi \subset V$  be an irreducible root system with exponents  $e_1, \dots, e_\ell$ , Coxeter number  $h$  and Weyl group  $W$ . For any integer  $k \in \mathbb{Z}$  and  $\alpha \in \Phi^+$ , the affine hyperplane  $H_{\alpha, k}$  is defined by

$$(10) \quad H_{\alpha, k} = \{x \in V \mid (\alpha, x) = k\}.$$

Fix a positive system  $\Phi^+ \subset \Phi$  and the set of simple roots  $\Delta = \{\alpha_1, \dots, \alpha_\ell\} \subset \Phi^+$ . The highest root, denoted by  $\tilde{\alpha} \in \Phi^+$ , can be expressed as a linear combination  $\tilde{\alpha} = \sum_{i=1}^{\ell} c_i \alpha_i$  ( $c_i \in \mathbb{Z}_{>0}$ ). We also set  $\alpha_0 := -\tilde{\alpha}$  and  $c_0 := 1$ . Then we have the linear relation

$$(11) \quad c_0 \alpha_0 + c_1 \alpha_1 + \dots + c_\ell \alpha_\ell = 0.$$

The coweight lattice  $Z(\Phi)$  and the coroot lattice  $\check{Q}(\Phi)$  are defined as follows.

$$Z(\Phi) = \{x \in V \mid (\alpha_i, x) \in \mathbb{Z}, \alpha_i \in \Delta\},$$

$$\check{Q}(\Phi) = \sum_{\alpha \in \Phi} \mathbb{Z} \cdot \frac{2\alpha}{(\alpha, \alpha)}.$$

The coroot lattice  $\check{Q}(\Phi)$  is a finite index subgroup of the coweight lattice  $Z(\Phi)$ . The index  $\# \frac{Z(\Phi)}{\check{Q}(\Phi)} = f$  is called the *index of connection*.

Let  $\varpi_i^\vee \in Z(\Phi)$  be the dual basis to the simple roots  $\alpha_1, \dots, \alpha_\ell$ , that is,  $(\alpha_i, \varpi_j^\vee) = \delta_{ij}$ . Then  $Z(\Phi)$  is a free abelian group generated by  $\varpi_1^\vee, \dots, \varpi_\ell^\vee$ . We also have  $c_i = (\varpi_i^\vee, \tilde{\alpha})$ .

A connected component of  $V \setminus \bigcup_{\substack{\alpha \in \Phi^+ \\ k \in \mathbb{Z}}} H_{\alpha, k}$  is called an *alcove*. Let us define the fundamental alcove  $A^\circ$  by

$$\begin{aligned} A^\circ &= \left\{ x \in V \mid \begin{array}{l} (\alpha_i, x) > 0, \quad (1 \leq i \leq \ell) \\ (\tilde{\alpha}, x) < 1 \end{array} \right\} \\ &= \left\{ x \in V \mid \begin{array}{l} (\alpha_i, x) > 0, \quad (1 \leq i \leq \ell) \\ (\alpha_0, x) > -1 \end{array} \right\}. \end{aligned}$$

The closure  $\overline{A^\circ} = \{x \in V \mid (\alpha_i, x) \geq 0 \ (1 \leq i \leq \ell), (\tilde{\alpha}, x) \leq 1\}$  is the convex hull of  $0, \frac{\varpi_1^\vee}{c_1}, \dots, \frac{\varpi_\ell^\vee}{c_\ell} \in V$ . The closed alcove  $\overline{A^\circ}$  is a simplex. The supporting hyperplanes of

facets of  $\overline{A^\circ}$  are  $H_{\alpha_1,0}, \dots, H_{\alpha_\ell,0}, H_{\tilde{\alpha},1}$ . We note that  $\overline{A^\circ}$  is a fundamental domain of the affine Weyl group  $W_{\text{aff}} = W \ltimes \check{Q}(\Phi)$ .

Let  $P^\diamond$  denote the fundamental domain of the coweight lattice  $Z(\Phi)$  defined by

$$(12) \quad \begin{aligned} P^\diamond &= \sum_{i=1}^{\ell} (0, 1] \varpi_i^\vee \\ &= \{x \in V \mid 0 < (\alpha_i, x) \leq 1, i = 1, \dots, \ell\}. \end{aligned}$$

Here we summarize without proofs some useful facts on root systems [12].

- PROPOSITION 2.2. (i)  $c_0 + c_1 + \dots + c_\ell = h$ .  
 (ii)  $\frac{|W|}{f} = \frac{\text{vol}(P^\diamond)}{\text{vol}(A^\circ)} = l! \cdot c_1 \cdot c_2 \cdot \dots \cdot c_\ell$ .  
 (iii)  $|\Phi^+| = \frac{\ell h}{2}$ .

**2.4. Partition of the fundamental parallelepiped.** Let us consider the set of alcoves contained in  $P^\diamond$ , denoted by  $\{A_\xi^\circ \mid \xi \in \Xi\}$ , where  $\Xi$  is a finite set with  $|\Xi| = \frac{|W|}{f}$  (by Proposition 2.2 (ii)). In other words,

$$(13) \quad P^\diamond \setminus \bigcup_{\alpha \in \Phi^+, k \in \mathbb{Z}} H_{\alpha,k} = \bigsqcup_{\xi \in \Xi} A_\xi^\circ.$$

Each  $A_\xi^\circ$  can be written uniquely as

$$(14) \quad A_\xi^\circ = \left\{ x \in V \mid \begin{array}{ll} (\alpha, x) > k_\alpha & \text{for } \alpha \in I \\ (\beta, x) < k_\beta & \text{for } \beta \in J \end{array} \right\},$$

for some positive roots  $I, J \subset \Phi^+$  with  $|I \sqcup J| = \ell + 1$ , and  $k_\alpha, k_\beta \in \mathbb{Z}$  ( $\alpha \in I, \beta \in J$ ). By definition, the facets of  $\overline{A_\xi^\circ}$  are supported by the hyperplanes  $H_{\alpha,k_\alpha}$  ( $\alpha \in I$ ) and  $H_{\beta,k_\beta}$  ( $\beta \in J$ ).

DEFINITION 2.3. With notation as above, let us define the partially closed alcove  $A_\xi^\diamond$  by

$$(15) \quad A_\xi^\diamond := \left\{ x \in V \mid \begin{array}{ll} (\alpha, x) > k_\alpha & \text{for } \alpha \in I \\ (\beta, x) \leq k_\beta & \text{for } \beta \in J \end{array} \right\}.$$

Obviously, the interior of  $A_\xi^\diamond$  is  $A_\xi^\circ$ . Although  $A_\xi^\diamond$  is not a closure of  $A_\xi^\circ$ ,  $A_\xi^\diamond$  may be considered as the partial closure of  $A_\xi^\circ$ .

PROPOSITION 2.4. Let  $\rho = \sum_{i=1}^{\ell} \varpi_i^\vee$ . Then  $x \in A_\xi^\diamond$  if and only if for sufficiently small  $0 < \varepsilon \ll 1$ ,  $x - \varepsilon \cdot \rho \in A_\xi^\circ$ , (that is, there exists  $\varepsilon_0 > 0$  such that if  $0 < \varepsilon < \varepsilon_0$ , then  $x - \varepsilon \cdot \rho \in A_\xi^\circ$ ).

PROOF. Straightforward. □

From Proposition 2.4, we have a partition of  $P^\diamond$ .

PROPOSITION 2.5.

$$(16) \quad P^\diamond = \bigsqcup_{\xi \in \Xi} A_\xi^\diamond.$$

PROOF. It is enough to show that each  $x \in P^\diamond$  is contained in the unique  $A_\xi^\diamond$ . Let  $x \in P^\diamond$ . Then for sufficiently small  $\varepsilon > 0$ ,  $(\alpha, x - \varepsilon \cdot \rho) \notin \mathbb{Z}$  for all  $\alpha \in \Phi^+$ , hence  $x - \varepsilon \cdot \rho$  is contained in the unique alcove  $A_\xi^\circ$ . By Proposition 2.4,  $x$  is contained in the corresponding  $A_\xi^\diamond$ .  $\square$

**2.5. Shift operator and ‘‘Riemann hypothesis’’.** Let  $a, b \in \mathbb{Z}$  be integers with  $a \leq b$ . Let us denote by  $\mathcal{A}_\Phi^{[a,b]}$  the hyperplane arrangement

$$\mathcal{A}_\Phi^{[a,b]} = \{H_{\alpha,k} \mid \alpha \in \Phi^+, k \in \mathbb{Z}, a \leq k \leq b\}.$$

By Proposition 2.2 (iii), we have  $|\mathcal{A}_\Phi^{[a,b]}| = \frac{\ell \cdot h \cdot (b-a+1)}{2}$ . For special cases, the characteristic polynomial  $\chi(\mathcal{A}_\Phi^{[a,b]}, t)$  factors.

THEOREM 2.6. (i) If  $k \geq 0$ , then  $\chi(\mathcal{A}_\Phi^{[-k,k]}, t) = \prod_{i=1}^\ell (t - e_i - kh)$ .  
(ii) If  $k \geq 1$ , then  $\chi(\mathcal{A}_\Phi^{[1-k,k]}, t) = (t - kh)^\ell$ .

The above result had been conjectured by Edelman and Reiner [9]. Theorem 2.6 (i) was proved in [4] by using lattice point counting techniques, which will be developed further in this paper. Theorem 2.6 (ii) was proved in [27] by use of the theory of free arrangements ([9, 18, 24, 25]).

For an interval  $[a, b] \neq [-k, k], [1-k, k]$ , the characteristic polynomial  $\chi(\mathcal{A}_\Phi^{[a,b]}, t)$  does not factor in general. Postnikov and Stanley pose the following ‘‘Riemann hypothesis’’.

CONJECTURE 2.7 ([19, Conjecture 9.14]). Let  $a, b \in \mathbb{Z}$  with  $a \leq 1 \leq b$ . Suppose  $a + b \geq 1$ . Then every root  $t \in \mathbb{C}$  of the equation  $\chi(\mathcal{A}_\Phi^{[a,b]}, t) = 0$  satisfies  $\operatorname{Re} t = \frac{h(b-a+1)}{2}$ .

Conjecture 2.7 has been proved by Stanley, Postnikov and Athanasiadis in [3, 19] for  $\Phi \in \{A_\ell, B_\ell, C_\ell, D_\ell, G_2\}$ . We recall their results.

Let  $f : \mathbb{N} \rightarrow \mathbb{R}$  be a partial function, that is, a function defined on a subset of  $\mathbb{N}$ . Define the action of the *shift operator*  $S$  by

$$(Sf)(t) = f(t - 1).$$

More generally, for a polynomial  $P(S) = \sum_k a_k S^k$  in  $S$ , the action is defined by

$$(P(S)f)(t) = \sum_k a_k f(t - k).$$

PROPOSITION 2.8. Let  $g(S) \in \mathbb{R}[S]$  and  $f(t) \in \mathbb{R}[t]$ . Suppose  $\deg f = n$ . Then  $g(S)f = 0$  if and only if  $(1 - S)^{n+1} | g(S)$ .

PROOF. First note that since  $(1 - S)f(t) = f(t) - f(t - 1)$  is the difference operator,  $\deg((1 - S)f) = n - 1$ . Suppose  $(1 - S)^{n+1} | g(S)$ . Then by induction, it is easily seen that  $(1 - S)^{n+1} f = 0$ . Hence  $g(S)f = 0$ .



Conversely, suppose  $g(S)f = 0$ . Consider the Taylor expansion of  $g(S)$  at  $S = 1$ . Set  $g(S) = b_0 + b_1(S-1) + b_2(S-1)^2 + \cdots + b_n(S-1)^n + (S-1)^{n+1}\tilde{g}(S)$ . Since  $(S-1)^{n+1}f = 0$ , we have

$$(17) \quad (b_0 + b_1(S-1) + b_2(S-1)^2 + \cdots + b_n(S-1)^n)f = 0.$$

Set  $f(t) = r_0t^n + r_1t^{n-1} + \cdots + r_n$  with  $r_0 \neq 0$ . The coefficient of the term of degree  $n$  in (17) is  $b_0r_0$ . Hence  $b_0 = 0$ . Similarly,  $b_1 = \cdots = b_n = 0$ , and we have  $g(S) = (S-1)^{n+1}\tilde{g}(S)$ .  $\square$

The shift operator can be used to express characteristic polynomials.

THEOREM 2.9 ([3, 19]).

- (1) Let  $n \geq 1$  and  $k \geq 0$ . For the cases  $\Phi \in \{A_\ell, B_\ell, C_\ell, D_\ell\}$ ,  $\chi(\mathcal{A}_\Phi^{[1-k, n+k]}, t) = \chi(\mathcal{A}_\Phi^{[1, n]}, t - kh)$ .
- (2) Let  $n \geq 1$ . Then the characteristic polynomial  $\chi(\mathcal{A}_\Phi^{[1, n]}, t)$  has the following expression.

(i) For  $\Phi = A_\ell$ ,

$$(18) \quad \chi(\mathcal{A}_{A_\ell}^{[1, n]}, t) = \left( \frac{1 + S + S^2 + \cdots + S^n}{1 + n} \right)^{\ell+1} t^\ell.$$

(ii) For  $\Phi = B_\ell$  or  $C_\ell$ ,

$$(19) \quad \chi(\mathcal{A}_\Phi^{[1, n]}, t) = \begin{cases} \frac{4S(1+S^2+S^4+\cdots+S^{2n})^{\ell-1}(1+S^2+S^4+\cdots+S^{n-1})^2}{(1+n)^{\ell+1}} t^\ell, & \text{if } n \text{ odd,} \\ \frac{(1+S^2+S^4+\cdots+S^{2n})^{\ell-1}(1+S^1+S^2+\cdots+S^n)^2}{(1+n)^{\ell+1}} t^\ell, & \text{if } n \text{ even.} \end{cases}$$

(iii) For  $\Phi = D_\ell$ ,

$$(20) \quad \chi(\mathcal{A}_{D_\ell}^{[1, n]}, t) = \begin{cases} \frac{8S(1+S^2)(1+S^2+S^4+\cdots+S^{2n})^{\ell-3}(1+S^2+S^4+\cdots+S^{n-1})^4}{(1+n)^{\ell+1}} t^\ell, & \text{if } n \text{ odd,} \\ \frac{(1+S^2+S^4+\cdots+S^{2n})^{\ell-3}(1+S^1+S^2+\cdots+S^n)^4}{(1+n)^{\ell+1}} t^\ell, & \text{if } n \text{ even.} \end{cases}$$

Owing to the next result, the above expressions implies Conjecture 2.7 for  $\mathcal{A}_\Phi^{[a, b]}$  with  $\Phi = A_\ell, B_\ell, C_\ell$ , or  $D_\ell$  and  $a + b \geq 2$ .

LEMMA 2.10 ([19, Lemma 9.13]). Let  $f(t) \in \mathbb{C}[t]$ . Suppose all the roots of the equation  $f(t) = 0$  have real part equal to  $a$ . Let  $g(S) \in \mathbb{C}[S]$  be a polynomial such that every root of the equation  $g(z) = 0$  satisfies  $|z| = 1$ . Then all roots of the equation  $(g(S)f)(t) = 0$  have real part equal to  $a + \frac{\deg g}{2}$ .

REMARK 2.11. (1) The ‘‘Riemann hypothesis’’ for the special case  $a + b = 1$  is a consequence of Theorem 2.6 (ii).

(2) Conjecture 2.7 implies the ‘‘functional equation’’ ([19, (9.12)])

$$(21) \quad \chi(\mathcal{A}_\Phi^{[a, b]}, h(b-a+1) - t) = (-1)^\ell \chi(\mathcal{A}_\Phi^{[a, b]}, t),$$

for  $a \leq 1 \leq b$  satisfying  $a + b \geq 1$ . The relation (21) for characteristic quasi-polynomials will be proved later (Theorem 5.6 and Corollary 5.7).

- (3) The “functional equation” (21) is also valid for the case  $[a, b] = [-k, k]$ , owing to the duality of exponents  $e_i + e_{\ell-i+1} = h$ .

**2.6. Eulerian polynomial.** The Eulerian polynomial was originally introduced by Euler for the purpose of describing the special value of the zeta function  $\zeta(n)$  at negative integers  $n < 0$  [11]. Currently, it plays an important role in enumerative combinatorics [22].

DEFINITION 2.12. For a permutation  $\sigma \in \mathfrak{S}_n$ , define

$$\begin{aligned} a(\sigma) &= \#\{i \mid 1 \leq i \leq n-1, \sigma(i) < \sigma(i+1)\}, \\ d(\sigma) &= \#\{i \mid 1 \leq i \leq n-1, \sigma(i) > \sigma(i+1)\}. \end{aligned}$$

Then

$$A(n, k) = \#\{\sigma \in \mathfrak{S}_n \mid a(\sigma) = k-1\},$$

( $1 \leq k \leq n-1$ ) is called the *Eulerian number* and the generating polynomial

$$A_n(t) = \sum_{k=1}^n A(n, k)t^k = \sum_{\sigma \in \mathfrak{S}_n} t^{1+a(\sigma)}$$

is called the *Eulerian polynomial*. It is easily seen that  $A(n, k) = A(n, n-k+1)$ . It follows immediately that  $t^{n+1} A_n(\frac{1}{t}) = A_n(t)$ .

The next formula is one of the classical results concerning Eulerian numbers.

THEOREM 2.13 (Worpitzky [26], see also [8]).

$$(22) \quad t^n = \sum_{k=1}^n A(n, k) \binom{t+k-1}{n}.$$

REMARK 2.14. Using the shift operator  $S$  (in §2.5), the Worpitzky identity (22) can be reformulated as

$$(23) \quad t^n = A_n(S) \frac{(t+n)(t+n-1) \cdots (t+1)}{n!}.$$

In §4.2 we will give another proof of (23). The polynomial  $\frac{(t+n)(t+n-1) \cdots (t+1)}{n!}$  is the Ehrhart polynomial of the fundamental alcove for the root system of type  $A_n$ . If we replace it with the Ehrhart quasi-polynomial of the fundamental alcove then we obtain similar formulae for root systems. (See Theorem 4.8 and Remark 4.10.)

### 3. Ehrhart quasi-polynomial for the fundamental alcove.

**3.1. Ehrhart quasi-polynomial.** A convex polytope  $\mathcal{P}$  is a convex hull of finitely many points in  $\mathbb{R}^n$ . A polytope  $\mathcal{P} \subset \mathbb{R}^n$  is said to be integral (resp. rational) if all vertices of  $\mathcal{P}$  are contained in  $\mathbb{Z}^n$  (resp.  $\mathbb{Q}^n$ ). We denote by  $\mathcal{P}^\circ$  the relative interior of  $\mathcal{P}$ .

Let  $\mathcal{P}$  be a rational polytope. For a positive integer  $q \in \mathbb{Z}_{>0}$ , define

$$(24) \quad L_{\mathcal{P}}(q) = \#\{q\mathcal{P} \cap \mathbb{Z}^n\}.$$

Similarly, define  $L_{\mathcal{P}^\circ}(q) = \#(q\mathcal{P}^\circ \cap \mathbb{Z}^n)$ . These functions are known to be quasi-polynomials ([5, Theorem 3.23]). (Moreover, the minimal period of the quasi-polynomial divides the least common multiple of the denominators of vertex coordinates.) Thus the value  $L_{\mathcal{P}}(q)$  makes sense for negative  $q$  and is related to  $L_{\mathcal{P}^\circ}(q)$  by the following reciprocity property

$$(25) \quad L_{\mathcal{P}}(-q) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(q),$$

for  $q > 0$ .

**3.2. Ehrhart quasi-polynomial for  $\overline{A^\circ}$ .** Let  $\overline{A^\circ}$  be the closed fundamental alcove of type  $\Phi$  (§2.3). Suter computes the Ehrhart quasi-polynomial  $L_{\overline{A^\circ}}(q)$  (with respect to the coweight lattice  $Z(\Phi)$ ) in [23] (see also [6, 1, 4, 10, 15]). See Example 3.2 for (some of) the explicit formulae. Several useful conclusions may be summarized as follows.

THEOREM 3.1 (Suter [23]).

- (i) *The Ehrhart quasi-polynomial  $L_{\overline{A^\circ}}(q)$  has the gcd-property.*
- (ii) *The leading coefficient of  $L_{\overline{A^\circ}}(q)$  is  $\frac{f}{|W|}$ .*
- (iii) *The minimal period  $\tilde{n}$  is as given in the table (Table 1).*
- (iv) *If  $q$  is relatively prime to the period  $\tilde{n}$ , then*

$$L_{\overline{A^\circ}}(q) = \frac{f}{|W|} (q + e_1)(q + e_2) \cdots (q + e_\ell).$$

- (v)  *$\text{rad}(\tilde{n})|h$ , where  $\text{rad}(\tilde{n}) = \prod_{p:\text{prime}, p|\tilde{n}} p$  is the radical of  $\tilde{n}$ .*

$\Phi$	$e_1, \dots, e_\ell$	$c_1, \dots, c_\ell$	$h$	$f$	$ W $	$\tilde{n}$	$\text{rad}(\tilde{n})$
$A_\ell$	$1, 2, \dots, \ell$	$1, 1, \dots, 1$	$\ell + 1$	$\ell + 1$	$(\ell + 1)!$	1	1
$B_\ell, C_\ell$	$1, 3, 5, \dots, 2\ell - 1$	$1, 2, 2, \dots, 2$	$2\ell$	2	$2^\ell \cdot \ell!$	2	2
$D_\ell$	$1, 3, 5, \dots, 2\ell - 3, \ell - 1$	$1, 1, 1, 2, \dots, 2$	$2\ell - 2$	4	$2^{\ell-1} \cdot \ell!$	2	2
$E_6$	$1, 4, 5, 7, 8, 11$	$1, 1, 2, 2, 2, 3$	12	3	$2^7 \cdot 3^4 \cdot 5$	6	6
$E_7$	$1, 5, 7, 9, 11, 13, 17$	$1, 2, 2, 2, 3, 3, 4$	18	2	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	12	6
$E_8$	$1, 7, 11, 13, 17, 19, 23, 29$	$2, 2, 3, 3, 4, 4, 5, 6$	30	1	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	60	30
$F_4$	$1, 5, 7, 11$	$2, 2, 3, 4$	12	1	$2^7 \cdot 3^2$	12	6
$G_2$	$1, 5$	$2, 3$	6	1	$2^2 \cdot 3$	6	6

TABLE 1. Table of root systems.

EXAMPLE 3.2. (1)  $\Phi = A_\ell$ . The closed fundamental alcove  $\overline{A^\circ}$  is the convex hull of  $0, \varpi_1^\vee, \dots, \varpi_\ell^\vee$ , which is an integral simplex. Hence the period is  $\tilde{n} = 1$ . Moreover,

$$(26) \quad L_{\overline{A^\circ}}(t) = \frac{(t + 1)(t + 2) \cdots (t + \ell)}{\ell!}.$$

- (2)  $\Phi = B_\ell$  or  $C_\ell$ . The closed fundamental alcove  $\overline{A^\circ}$  is the convex hull of  $0, \frac{\varpi_1^\vee}{2}, \varpi_2^\vee, \dots, \varpi_\ell^\vee$ . The period is  $\tilde{n} = 2$ .

$$L_{\overline{A^\circ}}(t) = \begin{cases} \frac{(t+1)(t+3)\cdots(t+2\ell-1)}{2^{\ell-1}\cdot\ell!}, & \text{if } t \text{ is odd} \\ \frac{(t+\ell)\prod_{i=1}^{\ell-1}(t+2i)}{2^{\ell-1}\cdot\ell!}, & \text{if } t \text{ is even.} \end{cases}$$

- (3)  $\Phi = D_\ell$ . The period is  $\tilde{n} = 2$ .

$$L_{\overline{A^\circ}}(t) = \begin{cases} \frac{(t+\ell-1)\prod_{i=1}^{\ell-1}(t+2i-1)}{2^{\ell-3}\cdot\ell!}, & \text{if } t \text{ is odd} \\ \frac{(t^2+2(\ell-1)t+\frac{\ell(\ell-1)}{2})\cdot\prod_{i=1}^{\ell-2}(t+2i)}{2^{\ell-3}\cdot\ell!}, & \text{if } t \text{ is even.} \end{cases}$$

- (4)  $\Phi = E_6$ . The period is  $\tilde{n} = 6$ .

$$L_{\overline{A^\circ}}(t) = \begin{cases} \frac{(t+1)(t+4)(t+5)(t+7)(t+8)(t+11)}{2^3\cdot 3\cdot 6!}, & \text{if } t \equiv 1, 5 \pmod{6} \\ \frac{(t+3)(t+9)(t^4+24t^3+195t^2+612t+480)}{2^3\cdot 3\cdot 6!}, & \text{if } t \equiv 3 \pmod{6} \\ \frac{(t+2)(t+4)(t+8)(t+10)(t^2+12t+26)}{2^3\cdot 3\cdot 6!}, & \text{if } t \equiv 2, 4 \pmod{6} \\ \frac{(t+6)^2(t^4+24t^3+186t^2+504t+480)}{2^3\cdot 3\cdot 6!}, & \text{if } t \equiv 0 \pmod{6}. \end{cases}$$

Let  $\Phi$  be an arbitrary root system. For a positive integer  $q \in \mathbb{Z}_{>0}$ , the simplex  $q\overline{A^\circ}$  has  $\ell + 1$  facets, which will be denoted by

$$\begin{aligned} F_0 &= \overline{A^\circ} \cap H_{\tilde{\alpha}, q}, \\ F_1 &= \overline{A^\circ} \cap H_{\alpha_1, 0}, \\ F_2 &= \overline{A^\circ} \cap H_{\alpha_2, 0}, \\ &\vdots \\ F_\ell &= \overline{A^\circ} \cap H_{\alpha_\ell, 0}. \end{aligned}$$

We shall count the lattice points after removing a facet.

LEMMA 3.3. *Let  $0 \leq i \leq \ell$ . Suppose  $q \gg 0$  (indeed  $q > c_i$  is sufficient). Then,*

$$(27) \quad \#((q\overline{A^\circ} \cap Z(\Phi)) \setminus F_i) = L_{\overline{A^\circ}}(q - c_i).$$

PROOF. First we consider the case  $i = 0$ . Let

$$x \in (q\overline{A^\circ} \cap Z(\Phi)) \setminus F_0.$$

Then  $(\tilde{\alpha}, x) < q$ . Since  $(\tilde{\alpha}, x)$  is an integer, we have  $(\tilde{\alpha}, x) \leq q - 1$ . Therefore,

$$(q\overline{A^\circ} \cap Z(\Phi)) \setminus F_0 = (q - 1)\overline{A^\circ} \cap Z(\Phi).$$

Since  $c_0 = 1$ , the number of lattice points is  $L_{\overline{A^\circ}}(q - c_0)$ .

Next we consider the case  $i = 1$ . By an argument similar to that in the case  $i = 0$ ,  $(q\overline{A^\circ} \cap Z(\Phi)) \setminus F_1$  is described as

$$(q\overline{A^\circ} \cap Z(\Phi)) \setminus F_1 = \{x \in Z(\Phi) \mid (\alpha_1, x) \geq 1, (\alpha_2, x) \geq 0, \dots, (\alpha_\ell, x) \geq 0, (\tilde{\alpha}, x) \leq q\}.$$

Since  $(\tilde{\alpha}, \varpi_1^\vee) = c_1$ , the map  $x \mapsto x + \varpi_1^\vee$  induces a bijection between

$$(q - c_1)\overline{A^\circ} \cap Z(\Phi) \xrightarrow{\cong} (q\overline{A^\circ} \cap Z(\Phi)) \setminus F_1.$$

Hence  $\#((q\overline{A^\circ} \cap Z(\Phi)) \setminus F_1) = L_{\overline{A^\circ}}(q - c_1)$ . This completes the proof for  $i = 1$ . The proof for  $i \geq 2$  is similar.  $\square$

Applying Lemma 3.3 repeatedly, we obtain the following.

**COROLLARY 3.4.** *Let  $\{i_1, \dots, i_k\} \subset \{0, 1, 2, \dots, \ell\}$ . Suppose  $q \gg 0$  (indeed  $q > c_{i_1} + \dots + c_{i_k}$  is sufficient). Then*

$$\#((q\overline{A^\circ} \cap Z(\Phi)) \setminus (F_{i_1} \cup \dots \cup F_{i_k})) = L_{\overline{A^\circ}}(q - c_{i_1} - \dots - c_{i_k}).$$

**COROLLARY 3.5** ([23, 4]). *Let  $q \in \mathbb{Z}$ . Then*

$$(28) \quad L_{A^\circ}(q) = L_{\overline{A^\circ}}(q - h).$$

**PROOF.** Since both sides of (28) are quasi-polynomials, it is sufficient to check the equality for  $q \gg 0$ .

$$(qA^\circ) \cap Z(\Phi) = (q\overline{A^\circ} \cap Z(\Phi)) \setminus \bigcup_{i=0}^{\ell} F_i.$$

Hence (28) follows from Corollary 3.4 and the equality  $\sum_{i=0}^{\ell} c_i = h$ .  $\square$

Finally, combining Corollary 3.5 and the reciprocity of Ehrhart quasi-polynomials (25), we obtain the following duality of  $L_{\overline{A^\circ}}(q)$ .

**COROLLARY 3.6.** *If  $q \in \mathbb{Z}$ , then*

$$L_{\overline{A^\circ}}(q - h) = (-1)^\ell L_{\overline{A^\circ}}(-q).$$

**3.3. Characteristic quasi-polynomial.** In this section, we recall the relation between the Ehrhart quasi-polynomial of  $\overline{A^\circ}$  and the characteristic quasi-polynomial of the Weyl arrangement  $\mathcal{A}_\Phi$ , following [6, 1, 4, 15] (which will be refined later). Recall the definition (12) of the fundamental parallelepiped  $P^\diamond = \{x \in V \mid 0 < (\alpha_i, x) \leq 1, i = 1, \dots, \ell\}$ . Let  $q > 0$ . Let us consider the projection

$$(29) \quad \pi : Z(\Phi) \longrightarrow Z(\Phi)/qZ(\Phi).$$

The restriction of  $\pi$  to  $qP^\diamond$  induces a bijection

$$(30) \quad \pi|_{qP^\diamond \cap Z(\Phi)} : qP^\diamond \cap Z(\Phi) \xrightarrow{\cong} Z(\Phi)/qZ(\Phi).$$

To compute the characteristic quasi-polynomial, let us define  $X_q$  by

$$(31) \quad X_q = Z(\Phi) \setminus \bigcup_{\substack{\alpha \in \Phi^+ \\ k \in \mathbb{Z}}} H_{\alpha, kq}.$$

Then the projection  $\pi$  induces a bijection between  $qP^\diamond \cap X_q$  and  $M(\mathcal{A}_\Phi; q)$ .

The set  $qP^\diamond \cap X_q$  is a disjoint union of  $qA_\xi^\circ \cap Z(\Phi)$ , ( $\xi \in \Xi$ ). Therefore, by using the reciprocity (25), we have

$$\begin{aligned} |P^\diamond \cap X_q| &= \frac{|W|}{f} L_{A^\circ}(q) \\ &= \frac{|W|}{f} \cdot (-1)^\ell L_{\overline{A^\circ}}(-q). \end{aligned}$$

(The case  $\Phi = B_2$ ,  $q = 6$  is described in Figure 1. See Example 4.11 for the notation.) Thus we have the following.

PROPOSITION 3.7 ([15]). *The characteristic quasi-polynomial of  $\mathcal{A}_\Phi$  is*

$$\chi_{\text{quasi}}(\mathcal{A}_\Phi, q) = \frac{|W|}{f} \cdot (-1)^\ell L_{\overline{A^\circ}}(-q).$$

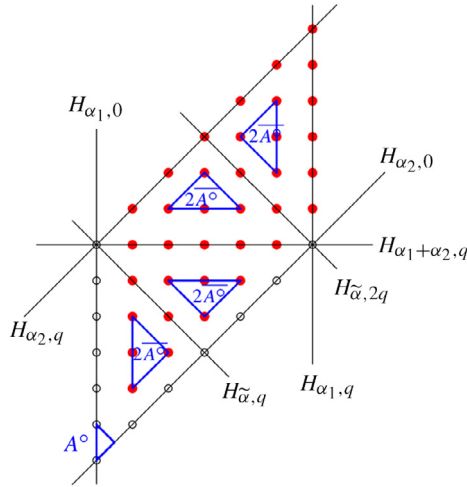


FIGURE 1.  $\chi_{\text{quasi}}(\mathcal{A}_\Phi, q) = 4L_{\overline{A^\circ}}(q-4)$ , ( $\Phi = B_2$ ,  $q = 6$ ).

We also have the duality of characteristic quasi-polynomial of the Weyl arrangement.

COROLLARY 3.8.  $\chi_{\text{quasi}}(\mathcal{A}_\Phi, q) = (-1)^\ell \cdot \chi_{\text{quasi}}(\mathcal{A}_\Phi, h-q)$ .

PROOF. Suppose  $q \gg 0$ . Using Corollary 3.6 and Proposition 3.7, we have

$$\begin{aligned} \chi_{\text{quasi}}(\mathcal{A}_\Phi, q) &= \frac{|W|}{f} \cdot (-1)^\ell \cdot L_{A^\circ}(-q) \\ &= \frac{|W|}{f} \cdot L_{A^\circ}(q - h) \\ &= (-1)^\ell \chi_{\text{quasi}}(\mathcal{A}_\Phi; h - q). \end{aligned}$$

□

#### 4. Generalized Eulerian polynomial.

**4.1. Definition and basic property.** Using the linear relation (11) in §2.3, we define the function  $\text{asc}, \text{dsc} : W \rightarrow \mathbb{Z}$ .

DEFINITION 4.1. Let  $w \in W$ . Then  $\text{asc}(w)$  and  $\text{dsc}(w) \in \mathbb{Z}$  are defined by

$$\begin{aligned} \text{asc}(w) &= \sum_{\substack{0 \leq i \leq \ell \\ w(\alpha_i) > 0}} c_i, \\ \text{dsc}(w) &= \sum_{\substack{0 \leq i \leq \ell \\ w(\alpha_i) < 0}} c_i. \end{aligned}$$

REMARK 4.2. Note that  $\text{dsc}(w)$  in this paper is equal to  $\text{cdes}(w)$  in [16].

Let  $w_0 \in W$  be the longest element. Since  $w_0$  exchanges positive and negative roots, we have

$$(32) \quad \begin{aligned} \text{asc}(w_0 w) &= \text{dsc}(w) = h - \text{asc}(w), \\ \text{dsc}(w_0 w) &= \text{asc}(w) = h - \text{dsc}(w). \end{aligned}$$

LEMMA 4.3. (1) Let  $w \in W$ . Suppose that  $w$  induces a permutation on  $\{\alpha_0, \alpha_1, \dots, \alpha_\ell\}$ . If  $w(\alpha_i) = \alpha_{p_i}$ , then  $c_i = c_{p_i}$ .

(2) Let  $w_1, w_2 \in W$ . If there exists  $\gamma \in V$  (actually  $\gamma \in \check{Q}(\Phi)$ ) such that  $w_2 A^\circ = w_1 A^\circ + \gamma$ , then  $\text{asc}(w_1) = \text{asc}(w_2)$ .

PROOF. (1) Applying  $w$  to the linear relation (11), we have

$$(33) \quad \sum_{i=0}^{\ell} c_i w(\alpha_i) = \sum_{i=0}^{\ell} c_i \alpha_{p_i} = 0.$$

Note that any  $\ell$  distinct members of  $\{\alpha_0, \alpha_1, \dots, \alpha_\ell\}$  are linearly independent. Therefore, the space of linear relations has dimension 1. Both (11) and (33) are linear relations with positive coefficients normalized in such a way that the minimal coefficient is equal to 1 ( $c_0 = 1$ ). Hence (11) and (33) are identical, which yields  $c_i = c_{p_i}$ .

(2) Suppose  $w_2A^\circ = w_1A^\circ + \gamma$ . Each side is

$$w_1A^\circ + \gamma = \left\{ x \in V \mid \begin{array}{l} (w_1\alpha_0, x) > (w_1\alpha_0, \gamma) - 1, \\ (w_1\alpha_i, x) > (w_1\alpha_i, \gamma), \quad i = 1, \dots, \ell \end{array} \right\},$$

$$w_2A^\circ = \left\{ x \in V \mid \begin{array}{l} (w_2\alpha_0, x) > -1, \\ (w_2\alpha_i, x) > 0, \quad i = 1, \dots, \ell \end{array} \right\}.$$

Since the supporting hyperplanes should coincide, we have

$$\{w_1\alpha_0, w_1\alpha_1, \dots, w_1\alpha_\ell\} = \{w_2\alpha_0, w_2\alpha_1, \dots, w_2\alpha_\ell\}.$$

Thus (a modified version of) (1) enables us to deduce  $\text{asc}(w_1) = \text{asc}(w_2)$ .  $\square$

DEFINITION 4.4. The generalized Eulerian polynomial  $R_\Phi(t)$  is defined by

$$R_\Phi(t) = \frac{1}{f} \sum_{w \in W} t^{\text{asc}(w)}.$$

The following proposition gives some basic properties of  $R_\Phi(t)$ . We omit the proof, since they are immediate consequences of Theorem 4.6 by Lam and Postnikov. (Direct proofs are also easy. In particular, the duality (2) is immediately deduced from (32).)

PROPOSITION 4.5. (1)  $\deg R_\Phi(t) = h - 1$ .

(2) (Duality)  $t^h \cdot R_\Phi(\frac{1}{t}) = R_\Phi(t)$ .

(3)  $R_\Phi(t) \in \mathbb{Z}[t]$ .

(4)  $R_{A_\ell}(t) = A_\ell(t)$ .

The polynomial  $R_\Phi(t)$  was introduced by Lam and Postnikov in [16]. They proved that  $R_\Phi(t)$  can be expressed in terms of cyclotomic polynomials and the classical Eulerian polynomial.

THEOREM 4.6 ([16, Theorem 10.1]). Let  $\Phi$  be a root system of rank  $\ell$ . Then

$$(34) \quad R_\Phi(t) = [c_1]_t \cdot [c_2]_t \cdots [c_\ell]_t \cdot A_\ell(t),$$

where  $[c]_t = \frac{t^c - 1}{t - 1}$ .

Let  $A' \subset V \setminus \bigcup_{\alpha \in \Phi^+, k \in \mathbb{Z}} H_{\alpha, k}$  be an arbitrary alcove. We can write  $A' = w(A^\circ) + \gamma$  for some  $w \in W$  and  $\gamma \in \mathcal{Q}(\Phi)$ . Then let us define

$$(35) \quad \text{asc}(A') := \text{asc}(w),$$

which is indeed well-defined because of the translational invariance (Lemma 4.3 (2)). Thus we can extend  $\text{asc}$  as a function on the set of all alcoves. Using this extension, we have another expression for  $R_\Phi(t)$ .

THEOREM 4.7.

$$(36) \quad R_\Phi(t) = \sum_{A' \subset P^\diamond} t^{\text{asc}(A')} = \sum_{\xi \in \Xi} t^{\text{asc}(A_\xi^\circ)}.$$



PROOF. For any  $w \in W$ , there exists a unique  $\gamma \in \check{Q}(\Phi)$  such that  $w(A^\circ) + \gamma \subset P^\diamond$ . This induces a map  $\varphi : W \longrightarrow \{A_\xi^\circ \mid \xi \in \Xi\}$ . The map is surjective and  $\#\varphi^{-1}(A_\xi^\circ) = f$  holds for any alcove  $A_\xi^\circ \subset P^\diamond$  (see [12, page 99]). The assertion follows from the definition of  $R_\Phi(t)$ .  $\square$

**4.2. Worpitzky partition.** From the definition  $P^\diamond = \sum_{i=1}^\ell (0, 1] \varpi_i^\vee$ ,

$$(37) \quad qP^\diamond \cap Z(\Phi) = \{t_1 \varpi_1^\vee + \cdots + t_\ell \varpi_\ell^\vee \mid t_i \in \mathbb{Z}, 0 < t_i \leq q\}.$$

Hence we have

$$(38) \quad L_{P^\diamond}(q) = \#(qP^\diamond \cap Z(\Phi)) = q^\ell.$$

The partition (16)  $P^\diamond = \bigsqcup_{\xi \in \Xi} A_\xi^\diamond$  in Proposition 2.5 induces a partition of lattice points,

$$(39) \quad qP^\diamond \cap Z(\Phi) = \bigsqcup_{\xi \in \Xi} qA_\xi^\circ \cap Z(\Phi),$$

which we shall call the *Worpitzky partition*.

**THEOREM 4.8.** *Suppose  $q \gg 0$  (indeed  $q > h$  is sufficient). Then*

$$(40) \quad q^\ell = (R_\Phi(S) L_{\overline{A^\circ}}(q)).$$

Before the proof of this theorem, we will analyze the case of a single alcove.

**LEMMA 4.9.** *Suppose  $q \gg 0$  (indeed  $q > h$  is sufficient). Then*

$$(41) \quad \#(qA_\xi^\diamond \cap Z(\Phi)) = L_{\overline{A^\circ}}(q - \text{asc}(A_\xi^\circ)).$$

PROOF. In the notation of §2.3 (see (15)),  $qA_\xi^\diamond$  is expressed as

$$(42) \quad qA_\xi^\diamond = \left\{ x \in V \mid \begin{array}{l} (\alpha, x) > qk_\alpha \quad \text{for } \alpha \in I \\ (\beta, x) \leq qk_\beta \quad \text{for } \beta \in J \end{array} \right\}.$$

Hence we have

$$(43) \quad qA_\xi^\diamond \cap Z(\Phi) = \left\{ x \in Z(\Phi) \mid \begin{array}{l} (\alpha, x) \geq qk_\alpha + 1, \quad \text{for } \alpha \in I \\ (\beta, x) \leq qk_\beta, \quad \text{for } \beta \in J \end{array} \right\}.$$

From Corollary 3.4 and the definition (35) of  $\text{asc}(A_\xi^\circ)$ , we have the equality (41).  $\square$

We now turn to the proof of Theorem 4.8. Using the partition (39) and Lemma 4.9, we have

$$(44) \quad \begin{aligned} q^\ell &= \#(qP^\diamond \cap Z(\Phi)) \\ &= \sum_{\xi \in \Xi} \#(qA_\xi^\diamond \cap Z(\Phi)) \\ &= \sum_{\xi \in \Xi} L_{\overline{A^\circ}}(q - \text{asc}(A_\xi^\circ)). \end{aligned}$$

Then applying Theorem 4.7 and the shift operator, the right-hand side can be written as  $(R_\Phi(S) L_{\overline{A^\circ}}(q))$ , which completes the proof.

REMARK 4.10. As we noted in Proposition 4.5 (4), if  $\Phi = A_\ell$  then the Eulerian polynomial is equal to the classical one. Furthermore, the Ehrhart polynomial is explicitly known (26). Theorem 4.8 gives the classical Worpitzky identity (23).

EXAMPLE 4.11. Let  $\Phi = B_2$ . Set the simple roots  $\alpha_1, \alpha_2$  as in Figure 2. Then  $\tilde{\alpha} = \varpi_1$ . Since  $f = 2$  and  $|W| = 8$ ,  $P^\diamond$  contains 4 alcoves, say  $\{A_\xi^\circ \mid \xi \in \Xi\} = \{A_{\xi_1}^\circ, A_{\xi_2}^\circ, A_{\xi_3}^\circ, A_{\xi_4}^\circ\}$  with the fundamental alcove  $A_{\xi_1}^\circ = A^\circ$ . Figure 3 is the Worpitzky partition of  $qP^\diamond \cap Z(B_2)$  for  $q = 6$ . The dots in Figure 3 are the set  $6P^\diamond \cap Z(B_2)$ , which is decomposed into a disjoint sum of simplices of sizes 3, 4, 4, and 5. The Eulerian polynomial is  $\mathbf{R}_{B_2}(t) = t + 2t^2 + t^3$ . Hence

$$\begin{aligned} 6^2 &= L_{\overline{A^\circ}}(5) + 2L_{\overline{A^\circ}}(4) + L_{\overline{A^\circ}}(3) \\ &= ((S + 2S^2 + S^3) L_{\overline{A^\circ}})(6) \\ &= (\mathbf{R}_{B_2}(S) L_{\overline{A^\circ}})(6). \end{aligned}$$

We can apply the above techniques to the Shi and Linal arrangements. The number of lattice points in  $qP^\diamond$  minus corresponding hyperplanes are expressed in terms of the generalized Eulerian polynomial and the Ehrhart quasi-polynomial. (See the next section for details.)

EXAMPLE 4.12. Figure 4 shows the lattice points of  $(qP^\diamond \cap Z(B_2)) \setminus \bigcup_{\alpha, k} (H_{\alpha, kq} \cup H_{\alpha, kq+1})$  and  $(qP^\diamond \cap Z(B_2)) \setminus \bigcup_{\alpha, k} H_{\alpha, kq+1}$  with  $q = 10$ , which correspond to the Shi and Linal arrangements, respectively. In both cases, the decomposition can be described by using the shift operator, the Eulerian polynomial and the Ehrhart quasi-polynomial.

$$\begin{aligned} L_{\overline{A^\circ}}(5) + 2L_{\overline{A^\circ}}(4) + L_{\overline{A^\circ}}(3) &= ((S^5 + 2S^6 + S^7) L_{\overline{A^\circ}})(10) \\ &= (S^4 \mathbf{R}_{B_2}(S) L_{\overline{A^\circ}})(10). \end{aligned}$$

$$\begin{aligned} L_{\overline{A^\circ}}(8) + 2L_{\overline{A^\circ}}(6) + L_{\overline{A^\circ}}(4) &= ((S^2 + 2S^4 + S^6) L_{\overline{A^\circ}})(10) \\ &= (\mathbf{R}_{B_2}(S^2) L_{\overline{A^\circ}})(10). \end{aligned}$$

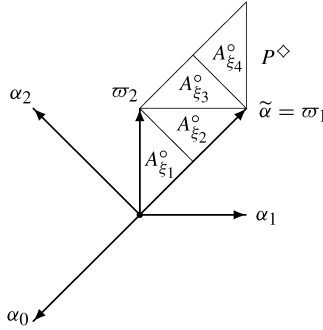


FIGURE 2. Root system of type  $B_2$ .

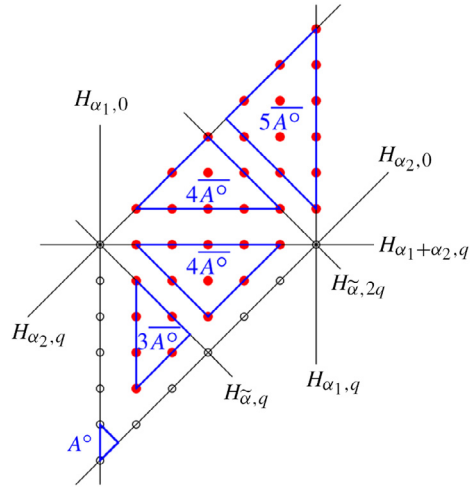


FIGURE 3. Worpitzky partition for  $\Phi = B_2$  with  $q = 6$ .

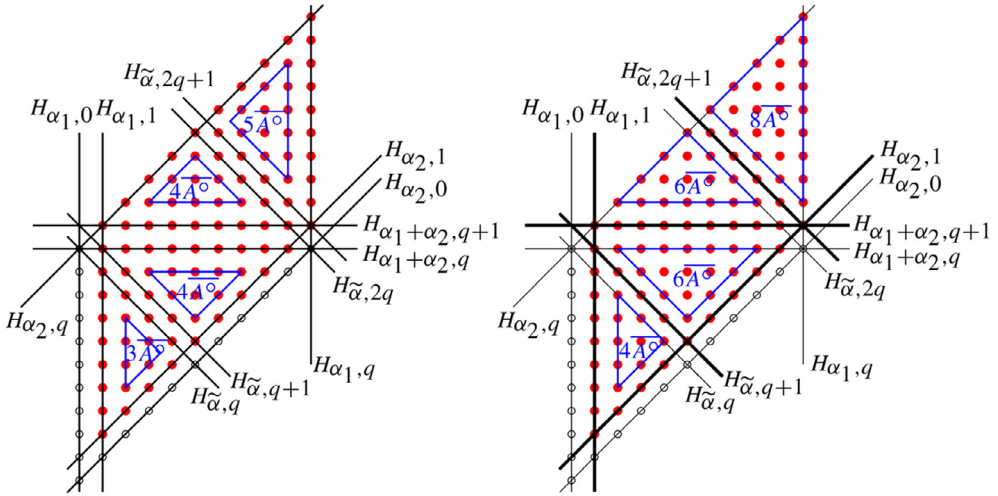


FIGURE 4. Shi and Linial arrangements ( $\Phi = B_2, q = 10$ ).

These computations will be generalized to all the root systems in the next section.

**5. Shi and Linial arrangements.** We will apply the Worpitzky partition from the previous section to the computation of characteristic quasi-polynomials for the Shi and Linial arrangements.

### 5.1. Shi arrangements.

**THEOREM 5.1.** *Let  $k \in \mathbb{Z}_{>0}$ . The characteristic quasi-polynomial  $\chi_{\text{quasi}}(\mathcal{A}_{\Phi}^{[1-k, k]}, t)$  of the extended Shi arrangement  $\mathcal{A}_{\Phi}^{[1-k, k]}$  is equal to the polynomial  $(t - kh)^{\ell}$ .*

**PROOF.** Suppose  $q \gg 0$  (indeed  $q > (k + 1)h$  is sufficient). Set

$$(45) \quad X_q := Z(\Phi) \setminus \bigcup_{\substack{\alpha \in \Phi^+, \\ i, m \in \mathbb{Z}, \\ 1-k \leq i \leq k}} H_{\alpha, mq+i}.$$

We have to compute (cf. §3.3),

$$(46) \quad \chi_{\text{quasi}}(\mathcal{A}_{\Phi}^{[1-k, k]}, q) = \#(qP^{\diamond} \cap X_q).$$

Consider the Worpitzky partition  $qP^{\diamond} \cap Z(\Phi) = \bigsqcup_{\xi \in \Xi} (qA_{\xi}^{\diamond} \cap Z(\Phi))$ . We have

$$(47) \quad \chi_{\text{quasi}}(\mathcal{A}_{\Phi}^{[1-k, k]}, q) = \sum_{\xi \in \Xi} \#(qA_{\xi}^{\diamond} \cap X_q).$$

In the notation of Definition 2.3, we have

$$(48) \quad qA_{\xi}^{\diamond} \cap X_q = \left\{ x \in Z(\Phi) \mid \begin{array}{ll} (\alpha, x) \geq qk_{\alpha} + k + 1 & \text{for } \alpha \in I \\ (\beta, x) \leq qk_{\beta} - k & \text{for } \beta \in J \end{array} \right\}.$$

Hence by Corollary 3.4 and Lemma 4.9,

$$(49) \quad \#(qA_{\xi}^{\diamond} \cap X_q) = L_{\overline{A^{\circ}}}(q - kh - \text{asc}(A_{\xi}^{\circ})).$$

Then, applying Theorem 4.7,

$$(50) \quad \chi_{\text{quasi}}(\mathcal{A}_{\Phi}^{[1-k, k]}, q) = (R_{\Phi}(S) L_{\overline{A^{\circ}}})(q - kh).$$

By Theorem 4.8, the right-hand side is equal to  $(q - kh)^{\ell}$ .  $\square$

By considering the case that  $q$  is relatively prime to  $\tilde{n}$ , we can conclude that the characteristic polynomial is

$$\chi(\mathcal{A}_{\Phi}^{[1-k, k]}, t) = (t - kh)^{\ell}.$$

This gives an alternative proof of Theorem 2.6 (ii).

**5.2. Linal arrangements.** In this section, we express the characteristic quasi-polynomial for the Linal arrangement  $\mathcal{A}_{\Phi}^{[1, 1+n]}$  (with  $n \geq 1$ ) and its extension  $\mathcal{A}_{\Phi}^{[1-k, 1+n+k]}$  (with  $n \geq 1, k \geq 0$ ) in terms of generalized Eulerian polynomials and Ehrhart quasi-polynomials.

**THEOREM 5.2.** *Let  $n \geq 1$ . The characteristic quasi-polynomial of the Linal arrangement  $\mathcal{A}_{\Phi}^{[1, n]}$  is*

$$(51) \quad \chi_{\text{quasi}}(\mathcal{A}_{\Phi}^{[1, n]}, q) = (R_{\Phi}(S^{n+1}) L_{\overline{A^{\circ}}})(q).$$

PROOF. Suppose  $q \gg 0$  (indeed  $q > (n+1)h$  is sufficient). Set

$$(52) \quad X_q := Z(\Phi) \setminus \bigcup_{\substack{\alpha \in \Phi^+, \\ i, m \in \mathbb{Z}, \\ 1 \leq i \leq n}} H_{\alpha, mq+i}.$$

In view of the bijection (30), we have to compute (cf. §3.3)

$$(53) \quad \chi_{\text{quasi}}(\mathcal{A}_{\Phi}^{[1, n]}, q) = q P^{\diamond} \cap X_q.$$

By the Worpitzky partition  $q P^{\diamond} \cap Z(\Phi) = \bigsqcup_{\xi \in \Xi} (q A_{\xi}^{\diamond} \cap Z(\Phi))$ , we have

$$(54) \quad \chi_{\text{quasi}}(\mathcal{A}_{\Phi}^{[1, n]}, q) = \sum_{\xi \in \Xi} \#(q A_{\xi}^{\diamond} \cap X_q).$$

In the notation of Definition 2.3, we have

$$(55) \quad q A_{\xi}^{\diamond} \cap X_q = \left\{ x \in Z(\Phi) \mid \begin{array}{ll} (\alpha, x) \geq qk_{\alpha} + n + 1 & \text{for } \alpha \in I \\ (\beta, x) \leq qk_{\beta} & \text{for } \beta \in J \end{array} \right\}.$$

Hence by Corollary 3.4 and Lemma 4.9,

$$(56) \quad \#(q A_{\xi}^{\diamond} \cap X_q) = L_{\overline{A^{\circ}}}(q - (n+1) \text{asc}(A_{\xi}^{\circ})).$$

Then, applying Theorem 4.7, we obtain

$$(57) \quad \chi_{\text{quasi}}(\mathcal{A}_{\Phi}^{[1, n]}, q) = (\mathbf{R}_{\Phi}(S^{n+1}) L_{\overline{A^{\circ}}})(q).$$

□

Moreover, by an argument similar to that in the proof of Theorem 5.1, we have the following.

**THEOREM 5.3.** *Let  $n \geq 1$  and  $k \geq 0$ . The characteristic quasi-polynomial of the Linial arrangement  $\mathcal{A}_{\Phi}^{[1-k, n+k]}$  is*

$$(58) \quad \chi_{\text{quasi}}(\mathcal{A}_{\Phi}^{[1-k, n+k]}, q) = \chi_{\text{quasi}}(\mathcal{A}_{\Phi}^{[1, n]}, q - kh).$$

Recall that by Theorem 2.1,  $\chi_{\text{quasi}}(\mathcal{A}_{\Phi}^{[1-k, n+k]}, q)$  is a quasi-polynomial with gcd-property. Furthermore, the Coxeter number  $h$  is divisible by the radical  $\text{rad}(\tilde{n})$  of the period  $\tilde{n}$  (Theorem 3.1 (v)). Hence if  $q$  is relatively prime to the period  $\tilde{n}$ , then  $q - kh$  is also relatively prime to  $\tilde{n}$ . Hence  $\#M(\mathcal{A}_{\Phi}^{[1, n]}, q)$  and  $\#M(\mathcal{A}_{\Phi}^{[1, n]}, q - kh)$  are computed by using the same polynomial, the characteristic polynomial. Thus we obtain the following.

**COROLLARY 5.4.**

$$(59) \quad \chi(\mathcal{A}_{\Phi}^{[1-k, n+k]}, t) = \chi(\mathcal{A}_{\Phi}^{[1, n]}, t - kh).$$

Now we have obtained two expressions of  $\chi(\mathcal{A}_\Phi^{[1,n]}, t)$  for  $\Phi = A_\ell$ . The comparison of these two expressions yields a useful congruence relation concerning the classical Eulerian polynomial  $A_\ell(t)$ . Let  $\Phi = A_\ell$ . Set  $g(t) = \frac{(t+1)(t+2)\cdots(t+\ell)}{\ell!}$ . Then Theorem 5.2 asserts that

$$(60) \quad \chi(\mathcal{A}_\Phi^{[1,n]}, t) = A_\ell(S^{n+1})g(t).$$

On the other hand, by formula (18) and the Worpitzky identity (23), we have another expression

$$(61) \quad \chi(\mathcal{A}_\Phi^{[1,n]}, t) = \left( \frac{1 + S + S^2 + \cdots + S^n}{1 + n} \right)^{\ell+1} A_\ell(S)g(t).$$

By comparing the two formulae (60) and (61) and using Proposition 2.8, we have the following congruence relation.

PROPOSITION 5.5. *Let  $\ell \geq 1, m \geq 2$ . Then*

$$(62) \quad A_\ell(S^m) \equiv \left( \frac{1 + S + S^2 + \cdots + S^{m-1}}{m} \right)^{\ell+1} A_\ell(S) \pmod{(S-1)^{\ell+1}}.$$

**5.3. The functional equation.** Next we prove the functional equation at the level of characteristic quasi-polynomials. The duality of the generalized Eulerian polynomial plays a crucial role in the proof.

THEOREM 5.6.

$$(63) \quad \chi_{\text{quasi}}(\mathcal{A}_\Phi^{[1,n]}, nh - t) = (-1)^\ell \chi_{\text{quasi}}(\mathcal{A}_\Phi^{[1,n]}, t).$$

PROOF. Let  $q \in \mathbb{Z}$ . We set  $R_\Phi(t) = \sum_{i=1}^{h-1} a_i t^i$ . Using Corollary 3.6,

$$\begin{aligned} \chi_{\text{quasi}}(\mathcal{A}_\Phi^{[1,n]}, nh - q) &= R_\Phi(S^{n+1}) L_{\overline{A^\circ}}(nh - q) \\ &= \sum_{i=1}^{h-1} a_i L_{\overline{A^\circ}}(nh - q - (n+1)i) \\ &= (-1)^\ell \sum_{i=1}^{h-1} a_i L_{\overline{A^\circ}}(q + (n+1)i - nh - h) \\ &= (-1)^\ell \sum_{i=1}^{h-1} a_i L_{\overline{A^\circ}}(q - (n+1)(h-i)). \end{aligned}$$

By applying the duality of  $a_i = a_{h-i}$  (Proposition 4.5 (2)), the right-hand side is equal to

$$\begin{aligned} (-1)^\ell \sum_{i=1}^{h-1} a_i L_{\overline{A^\circ}}(q - (n+1)i) &= (-1)^\ell R_\Phi(S^{n+1}) L_{\overline{A^\circ}}(q) \\ &= (-1)^\ell \chi_{\text{quasi}}(\mathcal{A}_\Phi^{[1,n]}, q). \end{aligned}$$

□

Recall that if  $q$  is relatively prime to  $\tilde{n}$ , then  $mh - q$  is also relatively prime to  $\tilde{n}$  (Theorem 3.1 (v)). By combining Theorem 2.6, Theorem 5.3, and Theorem 5.6, we can formulate the functional equation.

**COROLLARY 5.7.** *Let  $a \leq 1 \leq b$ . Then*

$$\chi(\mathcal{A}_\Phi^{[a,b]}, (b-a+1)h-t) = (-1)^\ell \chi(\mathcal{A}_\Phi^{[a,b]}, t).$$

**5.4. Partial results on the ‘‘Riemann hypothesis’’.** We will prove the ‘‘Riemann hypothesis’’ for several cases in  $\Phi = E_6, E_7, E_8$  and  $F_4$ . Recall that

$$\text{rad}(\tilde{n}) = \begin{cases} 6, & \text{for } \Phi = E_6, E_7, F_4 \\ 30, & \text{for } \Phi = E_8. \end{cases}$$

**THEOREM 5.8.** *Let  $\Phi$  be either  $E_6, E_7, E_8$  or  $F_4$ . Suppose*

$$n \equiv -1 \pmod{\text{rad}(\tilde{n})}.$$

*Then each root of the equation  $\chi(\mathcal{A}_\Phi^{[1,n]}, t) = 0$  satisfies  $\text{Re} = \frac{nh}{2}$ .*

**PROOF.** We give the proof only for the case  $\Phi = E_6$ . The proof for the other cases are similar. Let  $n = 6m - 1$  ( $m \in \mathbb{Z}$ ). By Theorem 5.2  $\chi_{\text{quasi}}(\mathcal{A}_\Phi^{[1,6m-1]}, q) = \mathbf{R}_\Phi(S^{6m}) \mathbf{L}_{\overline{A^\circ}}(q)$  for  $q \gg 0$ . Set  $g(t) = \frac{(t+1)(t+4)(t+5)(t+7)(t+8)(t+11)}{2^3 \cdot 3 \cdot 6!}$  and recall Example 3.2 that if  $q$  is prime to  $\text{rad}(\tilde{n}) = 6$  then

$$\mathbf{L}_{\overline{A^\circ}}(q) = g(q).$$

In this case  $q - 6k$  is also relatively prime to 6. Hence

$$\chi_{\text{quasi}}(\mathcal{A}_\Phi^{[1,6m-1]}, q) = \mathbf{R}_\Phi(S^{6m})g(q).$$

Thus we have a formula for the characteristic polynomial.

$$\chi(\mathcal{A}_\Phi^{[1,6m-1]}, t) = \mathbf{R}_\Phi(S^{6m})g(t).$$

Set  $c(t) = [2]_t^3 \cdot [3]_t$ . Using the formula proved by Lam and Postnikov (Theorem 4.6),  $\mathbf{R}_{E_6}(t) = c(t) \cdot \mathbf{A}_6(t)$ . Hence

$$\chi(\mathcal{A}_\Phi^{[1,6m-1]}, t) = c(S^{6m}) \mathbf{A}_6(S^{6m})g(t).$$

Now we employ Proposition 5.5; replacing  $S$  by  $S^6$ , we have

$$\mathbf{A}_6(S^{6m}) \equiv \left( \frac{1 + S^6 + S^{12} + \dots + S^{6(m-1)}}{m} \right)^7 \mathbf{A}_6(S^6) \pmod{(S^6 - 1)^7}.$$

Therefore, using Proposition 2.8,  $\chi(\mathcal{A}_\Phi^{[1,6m-1]}, t)$  can be written as

$$c(S^{6m}) \left( \frac{1 + S^6 + S^{12} + \dots + S^{6(m-1)}}{m} \right)^7 \mathbf{A}_6(S^6)g(t).$$

The first two factors are clearly cyclotomic polynomials in  $S$ . In view of Lemma 2.10, it is sufficient to check  $A_6(S^6)g(t)$  satisfies the Riemann hypothesis. The explicit computation of  $A_6(S^6)g(t)$  (up to constant factor) gives

$$29288834 - 8855550t + 1159185t^2 - 84600t^3 + 3660t^4 - 90t^5 + t^6.$$

We can check by explicit computation that the six complex roots of this polynomial have common real part 15.  $\square$

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DEPARTMENT OF MATHEMATICS  
HOKKAIDO UNIVERSITY  
KITA 10, NISHI 8, KITA-KU  
SAPPORO 060–0810  
JAPAN

*E-mail address:* yoshinaga@math.sci.hokudai.ac.jp