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ON FROBENIUS MANIFOLDS FROM GROMOV–WITTEN THEORY OF ORBIFOLD PROJECTIVE LINES WITH r ORBIFOLD POINTS

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Abstract. We prove that the Frobenius structure constructed from the Gromov–Witten theory for an orbifold projective line with at most r orbifold points is uniquely determined by the WDVV equations with certain natural initial conditions.

1. Introduction. The (formal) Frobenius manifold is a certain complex (formal) manifold endowed with the Frobenius algebra structure on its tangent sheaf whose product, unit, non-degenerate bilinear form and grading operator called the Euler vector field satisfy the special properties (for its definition and important properties, see Section 2). This structure was originally discovered by K. Saito in his study of primitive forms and their period mappings on the deformation theory of isolated hypersurface singularities ([13] and references therein) and was rediscovered and formulated by Dubrovin [3] in order to give coordinate–free expression for a solution of the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations considered in two dimensional topological field theories. Namely the Frobenius manifold can be also obtained from the Gromov–Witten theory for manifolds or orbifolds. Here the Gromov–Witten theory for orbifolds by Abramovich–Graber–Vistoli [1] and Chen–Ruan [2] is summarized briefly as follows: Let $\mathcal{X}$ be an orbifold (or a smooth proper Deligne–Mumford stack over $\mathbb{C}$). Then, for non-negative integers $g, n \in \mathbb{Z}_{\geq 0}$ and $\beta \in H_2(X, \mathbb{Z})$, where X is the coarse moduli space of $\mathcal{X}$, the moduli space (stack) $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta)$ of orbifold (twisted) stable maps of genus g with n-marked points of degree β is defined. There exists a virtual fundamental class $\left[\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta)\right]^{\mathrm{vir}}$ and Gromov–Witten invariants of genus g with n-marked points of degree β are defined as usual except for that we have to use the orbifold cohomology group $H^*_{orb}(\mathcal{X}, \mathbb{Q})$:

$$\langle \Delta_1, \ldots, \Delta_n \rangle^{\mathcal{X}}_{g,n,\beta} := \int_{\left[\overline{\mathcal{M}}_{g,n}(\mathcal{X},\beta)\right]^{\mathrm{vir}}} \mathrm{ev}_1^* \Delta_1 \wedge \cdots \wedge \mathrm{ev}_n^* \Delta_n \,, \quad \Delta_1, \ldots, \Delta_n \in H^*_{orb}(\mathcal{X}, \mathbb{Q}) \,,$$

where we denote by $\mathrm{ev}_i^* : H^*_{orb}(\mathcal{X}, \mathbb{Q}) \longrightarrow H^*(\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta), \mathbb{Q})$ the induced homomorphism by the evaluation map. We also consider the generating function (or formal power series)

$$\mathcal{F}^{\mathcal{X}}_g := \sum_{n,\beta} \frac{1}{n!} \langle \mathbf{t}, \ldots, \mathbf{t} \rangle^{\mathcal{X}}_{g,n,\beta} \, q^{\beta} \,, \quad \mathbf{t} = \sum_i t_i \Delta_i$$

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and call it the genus g potential, where $\{\Delta_i\}$ denotes a $\mathbb{Q}$-basis of $H^*_{orb}(\mathcal{X},\mathbb{Q})$. The main result in [1, 2] tells us that the point axiom, the divisor axiom for a class in $H^2(\mathcal{X},\mathbb{Q})$ and the associativity of the quantum product hold similar to the Gromov–Witten theory for a usual manifold (see [1, 2] for details of these axioms). In particular, the associativity of the quantum product implies the WDVV equations and it gives a formal Frobenius manifold M whose structure sheaf $\mathcal{O}_M$, tangent sheaf $\mathcal{T}_M$ and Frobenius potential are defined as the algebra $\Lambda[[H^*_{orb}(\mathcal{X},\mathbb{C})]]$ of formal power series in dual coordinates $\{t_i\}$ of the $\mathbb{Q}$-basis $\{\Delta_i\}$ of $H^*_{orb}(\mathcal{X},\mathbb{Q})$ over the Novikov field Λ (roughly speaking, Λ is the $\mathbb{C}$-algebra of formal Laurent series in $q^{\beta_1},\dots,q^{\beta_\rho}$ where $\beta_1,\dots,\beta_\rho$ are effective 1-cycles which generate the Mori cone of X), $\mathcal{T}_M := H^*_{orb}(\mathcal{X},\mathbb{C})\otimes_{\mathbb{C}}\mathcal{O}_M$ and the genus zero potential $\mathcal{F}_0^{\mathcal{X}}$ respectively.

Let $r\geq 3$ be a positive integer. Let A be a multiplet $(a_1,a_2,\dots,a_r)$ of positive integers such that $2\leq a_1\leq a_2\leq\cdots\leq a_r$ and $\Lambda=(\lambda_1,\dots,\lambda_r)$ a multiplet of pairwise distinct elements of $\mathbb{P}^1(\mathbb{C})$ normalized such that $\lambda_1=\infty$, $\lambda_2=0$ and $\lambda_3=1$. Set $\mu_A=2+\sum_{k=1}^r(a_k-1)$ and $\chi_A:=2+\sum_{k=1}^r(-1+1/a_k)$. We shall consider the orbifold projective line with r-orbifold points at $\lambda_1,\dots,\lambda_r$ whose orders are $a_1,a_2,\dots,a_r$, which is denoted by $\mathbb{P}^1_{A,\Lambda}$ (see Definition 4.2). Here the number μ_A is regarded as the total dimension of the orbifold cohomology group $H^*_{orb}(\mathbb{P}^1_{A,\Lambda},\mathbb{C})$ and the number χ_A is regarded as the orbifold Euler number of $\mathbb{P}^1_{A,\Lambda}$. The main purpose of the present paper is to show that the Frobenius manifold $M^{GW}_{\mathbb{P}^1_{A,\Lambda}}$ constructed from the Gromov–Witten theory for $\mathbb{P}^1_{A,\Lambda}$ can be determined by the WDVV equations with certain natural initial conditions. Then we shall show the following uniqueness theorem which is our main result in the present paper and the natural generalization of the one in our previous paper [6]:

THEOREM (Theorem 3.1). *There exists a unique Frobenius manifold M of rank μ_A and dimension one with flat coordinates $(t_1,t_{1,1},\dots,t_{i,j},\dots,t_{r,a_r-1},t_{\mu_A})$ satisfying the following conditions*:

(i) *The unit vector field e and the Euler vector field E are given by*

$$e=\frac{\partial}{\partial t_1},\quad E=t_1\frac{\partial}{\partial t_1}+\sum_{i=1}^{r}\sum_{j=1}^{a_i-1}\frac{a_i-j}{a_i}t_{i,j}\frac{\partial}{\partial t_{i,j}}+\chi_A\frac{\partial}{\partial t_{\mu_A}}.$$

(ii) *The non-degenerate symmetric bilinear form η on $\mathcal{T}_M$ satisfies*

$$\eta\left(\frac{\partial}{\partial t_1},\frac{\partial}{\partial t_{\mu_A}}\right)=\eta\left(\frac{\partial}{\partial t_{\mu_A}},\frac{\partial}{\partial t_1}\right)=1,$$
$$\eta\left(\frac{\partial}{\partial t_{i_1,j_1}},\frac{\partial}{\partial t_{i_2,j_2}}\right)=\begin{cases}\frac{1}{a_{i_1}} & i_1=i_2 \textit{ and } j_2=a_{i_1}-j_1,\\ 0 & \textit{otherwise}.\end{cases}$$

(iii) *The Frobenius potential $\mathcal{F}_A$ satisfies $E\mathcal{F}_A|_{t_1=0}=2\mathcal{F}_A|_{t_1=0}$,*

$$\mathcal{F}_A|_{t_1=0}\in\mathbb{C}\left[[t_{1,1},\dots,t_{1,a_1-1},\dots,t_{i,j},\dots,t_{r,1},\dots,t_{r,a_r-1},e^{t_{\mu_A}}]\right].$$

(iv) *Assume the condition* (iii). *we have*

$$\mathcal{F}_A|_{t_1=e^{t_{\mu_A}}=0} = \sum_{i=1}^{r} \mathcal{G}_A^{(i)}\,, \quad \mathcal{G}_A^{(i)} \in \mathbb{C}[t_{i,1}, \ldots, t_{i,a_i-1}],\ i = 1, \ldots, r\,.$$

(v) *Assume the condition* (iii). *In the frame* $\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_{1,1}}, \ldots, \frac{\partial}{\partial t_{r,a_r-1}}, \frac{\partial}{\partial t_{\mu_A}}$ *of* $\mathcal{T}_M$, *the product* $\circ$ *can be extended to the limit* $t_1 = t_{1,1} = \cdots = t_{r,a_r-1} = e^{t_{\mu_A}} = 0$. *The* $\mathbb{C}$*-algebra obtained in this limit is isomorphic to*

$$\mathbb{C}[x_1, x_2, \ldots, x_r] \Big/ \left(x_i x_j,\ a_i x_i^{a_i} - a_j x_j^{a_j}\right)_{1\le i\ne j\le r}\,,$$

where $\partial/\partial t_{i,j}$ *are mapped to* x_i^j *for* $i = 1, \ldots, r$, $j = 1, \ldots, a_i - 1$ *and* $\partial/\partial t_{\mu_A}$ *are mapped to* $a_1 x_1^{a_1}$.

(vi) *The term*

$$\left(\prod_{i=1}^{r} t_{i,1}\right) e^{t_{\mu_A}}$$

occurs with the coefficient 1 *in* $\mathcal{F}_A$.

Here we have an important results concerning the condition (iv) in Theorem 3.1. The polynomial $\mathcal{G}_A^{(i)}$ in the condition (iv) can be expressed by the Frobenius potential $\mathcal{F}_{A_i}(t'_1, \mathbf{t}'_3, e^{t_{\mu_{A_i}}})$ of the Frobenius manifold M_{A_i} in Theorem 3.1 in [6] where $A_i = (1, 1, a_i)$ with $a_i \ge 2$ and $(t'_1, \mathbf{t}'_3, t_{\mu_{A_i}}) := (t'_1, t'_{3,1}, \ldots, t'_{3,a_i-1}, t_{\mu_{A_i}})$ is the flat coordinate for M_{A_i}:

PROPOSITION (Proposition 3.18). *For the polynomial* $\mathcal{G}_A^{(i)}$ *in the condition* (iv) *in Theorem* 3.1, *we have*

$$\mathcal{G}_A^{(i)} = \mathcal{F}_{A_i}(0, \mathbf{t}_i, 0)\,,$$

where $\mathbf{t}_i := (t_{i,1}, \ldots, t_{i,a_i-1})$ *is the i-th parts of the flat coordinate in Theorem* 3.1.

As a corollary of Theorem 3.1, the Frobenius structure constructed from the Gromov–Witten theory for $\mathbb{P}^1_{A,\Lambda}$ can be uniquely reconstructed by the conditions in Theorem 3.1:

THEOREM (Theorem 4.5). *The conditions in Theorem 3.1 are satisfied by the Frobenius structure constructed from the Gromov–Witten theory for* $\mathbb{P}^1_{A,\Lambda}$.

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2. Preliminary. In this section, we recall the definition and three elementary properties of the Frobenius manifold [3]. The definition below is taken from Saito-Takahashi [13].

DEFINITION 2.1. Let $M = (M, \mathcal{O}_M)$ be a connected complex manifold or a formal manifold over $\mathbb{C}$ of dimension μ whose holomorphic tangent sheaf and cotangent sheaf are denoted by $\mathcal{T}_M$ and Ω_M^1 respectively. Set a complex number d. A *Frobenius structure of rank μ and dimension d on M* is a tuple $(\eta, \circ, e, E)$, where we denote by η a non-degenerate $\mathcal{O}_M$-symmetric bilinear form on $\mathcal{T}_M$, by $\circ$ an $\mathcal{O}_M$-bilinear product on $\mathcal{T}_M$ of an associative and commutative $\mathcal{O}_M$-algebra structure with the unit e and by E a holomorphic vector field on M called the Euler vector field, satisfying the following axioms:

(i) The product $\circ$ is self-adjoint with respect to η: that is,

$$\eta(\delta \circ \delta', \delta'') = \eta(\delta, \delta' \circ \delta''), \quad \delta, \delta', \delta'' \in \mathcal{T}_M .$$

(ii) The Levi–Civita connection $\nabla : \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{T}_M \to \mathcal{T}_M$ with respect to η is flat: that is,

$$[\nabla_\delta, \nabla_{\delta'}] = \nabla_{[\delta,\delta']}, \quad \delta, \delta' \in \mathcal{T}_M .$$

(iii) The tensor $C : \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{T}_M \to \mathcal{T}_M$ defined by $C_\delta \delta' := \delta \circ \delta'$, $(\delta, \delta' \in \mathcal{T}_M)$ is flat: that is,

$$\nabla C = 0 .$$

(iv) The unit e for the product $\circ$ is a ∇–flat holomorphic vector field: that is,

$$\nabla e = 0 .$$

(v) The non-degenerate bilinear form η and the product $\circ$ are homogeneous of degree $2 - d$ and 1 respectively with respect to the Lie derivative Lie_E of the Euler vector field E: that is,

$$\mathrm{Lie}_E(\eta) = (2 - d)\eta , \quad \mathrm{Lie}_E(\circ) = \circ .$$

We shall expose, without their proof, three basic properties of the Frobenius manifold which are necessary to state Theorem 3.1. Let us consider the space of horizontal sections of the connection ∇:

$$\mathcal{T}_M^f := \{\delta \in \mathcal{T}_M \mid \nabla_{\delta'}\delta = 0 \text{ for all } \delta' \in \mathcal{T}_M\} .$$

Then the axiom (ii) implies that $\mathcal{T}_M^f$ is a local system of rank μ on M:

PROPOSITION 2.2. *At each point of the Frobenius manifold M, there exists a local coordinate $(t_1, \ldots, t_\mu)$, called flat coordinates, such that $e = \partial_1$, $\mathcal{T}_M^f$ is spanned by $\partial_1, \ldots, \partial_\mu$ and $\eta(\partial_i, \partial_j) \in \mathbb{C}$ for all $i, j = 1, \ldots, \mu$ where we denote $\partial/\partial t_i$ by ∂_i.*

The axiom (iii) implies the existence of the Frobenius potential:

PROPOSITION 2.3. *At each point of the Frobenius manifold M, there exists the local holomorphic function $\mathcal{F}$, called Frobenius potential, satisfying*

$$\eta(\partial_i \circ \partial_j, \partial_k) = \eta(\partial_i, \partial_j \circ \partial_k) = \partial_i\partial_j\partial_k\mathcal{F} , \quad i, j, k = 1, \ldots, \mu ,$$

for any system of flat coordinates. In particular, we have

$$\eta_{ij} := \eta(\partial_i, \partial_j) = \partial_1\partial_i\partial_j\mathcal{F} .$$

Furthermore, the associativity of the product $\circ$ implies that the Frobenius potential satisfies the WDVV equations:

PROPOSITION 2.4. *The Frobenius potential $\mathcal{F}$ satisfies the following equations:*

$$\sum_{\sigma,\tau=1}^{\mu} \partial_a\partial_b\partial_\sigma\mathcal{F}\cdot\eta^{\sigma\tau}\cdot\partial_\tau\partial_c\partial_d\mathcal{F} - \sum_{\sigma,\tau=1}^{\mu} \partial_a\partial_c\partial_\sigma\mathcal{F}\cdot\eta^{\sigma\tau}\cdot\partial_\tau\partial_b\partial_d\mathcal{F} = 0\,,$$

where $a, b, c, d \in \{1, \dots, \mu\}$.

3. A Uniqueness Theorem. Let $r \geq 3$ be a positive integer. Let A be a multiplet $(a_1, a_2, \dots, a_r)$ of positive integers such that $2 \leq a_1 \leq a_2 \leq \cdots \leq a_r$ and $\Lambda = (\lambda_1, \dots, \lambda_r)$ a multiplet of pairwise distinct elements of $\mathbb{P}^1(\mathbb{C})$ normalized such that $\lambda_1 = \infty$, $\lambda_2 = 0$ and $\lambda_3 = 1$. Set

$$\mu_A := 2 + \sum_{k=1}^{r}(a_k - 1)\ , \tag{3.1}$$

$$\chi_A := 2 + \sum_{k=1}^{r}\left(\frac{1}{a_k} - 1\right)\ . \tag{3.2}$$

We have the following uniqueness theorem for Frobenius manifolds of rank μ_A and dimension one. The proof of this uniqueness theorem, especially Proposition 3.23, is inspired by Kontsevich–Manin [7] and E. Mann [10]:

THEOREM 3.1. *There exists a unique Frobenius manifold M of rank μ_A and dimension one with flat coordinates $(t_1, t_{1,1}, \dots, t_{i,j}, \dots, t_{r,a_r-1}, t_{\mu_A})$ satisfying the following conditions:*

(i) *The unit vector field e and the Euler vector field E are given by*

$$e = \frac{\partial}{\partial t_1},\ E = t_1\frac{\partial}{\partial t_1} + \sum_{i=1}^{r}\sum_{j=1}^{a_i-1}\frac{a_i - j}{a_i}t_{i,j}\frac{\partial}{\partial t_{i,j}} + \chi_A\frac{\partial}{\partial t_{\mu_A}}\ .$$

(ii) *The non-degenerate symmetric bilinear form η on $\mathcal{T}_M$ satisfies*

$$\eta\left(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_{\mu_A}}\right) = \eta\left(\frac{\partial}{\partial t_{\mu_A}}, \frac{\partial}{\partial t_1}\right) = 1\,,$$

$$\eta\left(\frac{\partial}{\partial t_{i_1,j_1}}, \frac{\partial}{\partial t_{i_2,j_2}}\right) = \begin{cases}\frac{1}{a_{i_1}} & i_1 = i_2 \text{ and } j_2 = a_{i_1} - j_1\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

(iii) *The Frobenius potential $\mathcal{F}$ satisfies $E\mathcal{F}|_{t_1=0} = 2\mathcal{F}|_{t_1=0}$,*

$$\mathcal{F}_A|_{t_1=0} \in \mathbb{C}\left[[t_{1,1}, \dots, t_{1,a_1-1}, \dots, t_{i,j}, \dots, t_{r,1}, \dots, t_{r,a_r-1}, e^{t_{\mu_A}}]\right]\ .$$

(iv) *Assume the condition* (iii). *we have*

$$\mathcal{F}_A|_{t_1=e^{t_{\mu_A}}=0} = \sum_{i=1}^{r} \mathcal{G}_A^{(i)}, \quad \mathcal{G}_A^{(i)} \in \mathbb{C}[t_{i,1}, \ldots, t_{i,a_i-1}],\ i = 1, \ldots, r\,.$$

(v) *Assume the condition* (iii). *In the frame* $\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_{1,1}}, \ldots, \frac{\partial}{\partial t_{r,a_r-1}}, \frac{\partial}{\partial t_{\mu_A}}$ *of* $\mathcal{T}_M$*, the product* $\circ$ *can be extended to the limit* $t_1 = t_{1,1} = \cdots = t_{r,a_r-1} = e^{t_{\mu_A}} = 0$*. The* $\mathbb{C}$*-algebra obtained in this limit is isomorphic to*

$$\mathbb{C}[x_1, x_2, \ldots, x_r] \Big/ \left(x_i x_j,\ a_i x_i^{a_i} - a_j x_j^{a_j}\right)_{1 \le i \ne j \le r},$$

where $\partial/\partial t_{i,j}$ *are mapped to* x_i^j *for* $i = 1, \ldots, r$, $j = 1, \ldots, a_i - 1$ *and* $\partial/\partial t_{\mu_A}$ *are mapped to* $a_1 x_1^{a_1}$.

(vi) *The term*

$$\left(\prod_{i=1}^{r} t_{i,1}\right) e^{t_{\mu_A}}$$

occurs with the coefficient 1 *in* $\mathcal{F}_A$.

REMARK 3.2. The conditions in Theorem 3.1 are satisfied by natural ones for the orbifold Gromow–Witten theory of $\mathbb{P}^1_{A,\Lambda}$. The condition (i), (ii) and (v) come from the conditions for a homogeneous basis of the orbifold cohomology group, the orbifold Poincaré pairing and the large radius limit for the orbifold Gromov–Witten theory respectively. The condition (ii) and (v) are essential to obtain coefficients corresponding to genus zero three points degree zero correlators. The condition (iii) comes from the divisor axiom. The condition (iv) and (vi) come from some geometrical meanings of the orbifold Gromov–Witten invariants. Namely, the coefficient of the term in the condition (vi) corresponds to a certain genus zero r-points degree one correlator.

We shall notice different and common points between the present proof of Theorem 3.1 and the one for Theorem 3.1 in [6]. Surprisingly, Theorem 3.1 can be proven by the parallel way to the one in our previous paper [6]. However, for general cases $r \ge 4$, we have to modify the arguments in [6] related to the reconstruction of the coefficients corresponding to the genus zero degree one correlators, e.g., the terms in the WDVV equations whose coefficients give the recursion relations. In particular, the arguments in Proposition 3.9 except for Lemma 3.17 are very natural generalizations of the one for Proposition 3.36 in [6]. In contrast to this, some arguments in [6] can be also applied without major modifications, e.g., the arguments in [6] related to the reconstruction of the coefficients corresponding to the genus zero higher degree correlators. From now on, we will mark with asterisks ($*$) on propositions, lemmas and sublemmas whose proofs are (almost) same with the ones in [6] and mark with daggers (†) on them whose proofs need some modifications. In order to make the proof self-contained, we shall include all details of arguments even if the arguments are common to the ones in our previous paper [6].

We shall use the same notations with the ones in our previous paper [6]. By the condition (iii) in Theorem 3.1, we can expand the non-trivial part of the Frobenius potential $\mathcal{F}_A|_{t_1=0}$ as

$$\mathcal{F}_A|_{t_1=0} = \sum_{\alpha=(\alpha_{1,1},\dots,\alpha_{r,a_r-1})} c(\alpha,m)t^\alpha e^{mt_{\mu_A}}\ , \quad t^\alpha = \prod_{i=1}^{r}\prod_{j=1}^{a_i-1} t_{i,j}^{\alpha_{i,j}}\ .$$

Here we note that, by Proposition 2.3, the terms in $\mathcal{F}_A$ including t_1 are only cubic terms $t_1 t_{i,j} t_{i,a_i-j}$ and their coefficients can be determined by the condition (ii).

Consider a free abelian group $\mathbb{Z}^{\mu_A-2}$ and denote its standard basis by $e_{i,j}$, $i=1,\dots,r$, $j=1,\dots,a_i-1$. The element $\alpha=\sum_{i,j}\alpha_{i,j}e_{i,j}$, $\alpha_{i,j}\in\mathbb{Z}$ of $\mathbb{Z}^{\mu_A-2}$ is called *non-negative* and is denoted by $\alpha\geq 0$ if all $\alpha_{i,j}$ are non-negative integers. We also denote by $c(e_1+e_{i,j}+e_{i,a_i-j},0)$ the coefficient of $t_1t_{i,j}t_{i,a_i-j}$ in the trivial part of the Frobenius potential $\mathcal{F}$. For a non-negative $\alpha\in\mathbb{Z}^{\mu_A-2}$, we set

$$|\alpha| := \sum_{i=1}^{r}\sum_{j=1}^{a_i-1}\alpha_{i,j}\ ,$$

and call it the *length* of α. Define the number $s_{a,b,c}$ for $a,b,c\in\mathbb{Z}$ as follows:

$$s_{a,b,c} = \begin{cases} 1 \ \text{ if } \ a,b,c \text{ are pairwise distinct}\,, \\ 6 \ \text{ if } \ a=b=c\,, \\ 2 \ \text{ otherwise}\,. \end{cases}$$

For $a,b,c,d\in\{1,\dots,\mu_A\}$, denote by $WDVV(a,b,c,d)$ the following equation:

$$\sum_{\sigma,\tau=1}^{\mu_A}\partial_a\partial_b\partial_\sigma\mathcal{F}\cdot\eta^{\sigma\tau}\cdot\partial_\tau\partial_c\partial_d\mathcal{F} - \sum_{\sigma,\tau=1}^{\mu_A}\partial_a\partial_c\partial_\sigma\mathcal{F}\cdot\eta^{\sigma\tau}\cdot\partial_\tau\partial_b\partial_d\mathcal{F} = 0\,,$$

where $(\eta^{\sigma\tau}) := (\eta_{\sigma\tau})^{-1}$.

3.1. Coefficients $c(\alpha,0)$ and $c(\alpha,1)$ can be reconstructed.

PROPOSITION* 3.3. *Coefficients $c(\alpha,0)$ with $|\alpha|=3$ are determined by the condition* (v) *of Theorem* 3.1.

PROOF. Note that $C_{ijk}=\eta(\partial_i\circ\partial_j,\partial_k)$ and the non-degenerate bilinear form η can be extended to the limit $\underline{t}$, $e^t\to 0$. We denote by η' this extended bilinear form. By the condition (v), the relation $x_ix_j=0$ if $i\neq j$ holds in the $\mathbb{C}$-algebra obtained in this limit. Therefore, we have $c(\sum_{k=1}^{3}e_{i_k,j_k},0)\neq 0$ only if $i_1=i_2=i_3$. In particular, we have

$$\begin{aligned} s_{j_1,j_2,j_3}\cdot c\left(\sum_{k=1}^{3}e_{i,j_k},0\right) &= \lim_{\underline{t},e^t\to 0}\partial_{i,j_1}\partial_{i,j_2}\partial_{i,j_3}\mathcal{F}_A = \eta'(x_i^{j_1}\cdot x_i^{j_2},x_i^{j_3}) \\ &= \eta'(1\cdot x_i^{j_1+j_2},x_i^{j_3}) = \lim_{\underline{t},e^t\to 0}\partial_1\partial_{i,j_1+j_2}\partial_{i,j_3}\mathcal{F}_A \end{aligned}$$

by Proposition 2.3 and

$$\lim_{\underline{t}, e^t \to 0} \partial_1 \partial_{i,j_1+j_2} \partial_{i,j_3} \mathcal{F} = \begin{cases} \frac{1}{a_i} & \text{if } \sum_{k=1}^3 j_k = a_i \,, \\ 0 & \text{otherwise} \,. \end{cases}$$

□

PROPOSITION* 3.4. *A coefficient $c(\alpha, 1)$ with $|\alpha| \le r$ is non-zero if and only if $\alpha = \sum_{k=1}^r e_{k,1}$. In particular, we have $c(\sum_{k=1}^r e_{k,1}, 1) = 1$ by the condition* (vi) *of Theorem* 3.1.

PROOF. We shall split the proof into following two cases.

LEMMA* 3.5 (Case 1). *Let $\gamma \in \mathbb{Z}^{\mu_A - 2}$ be a non-negative element satisfying that $|\gamma| = r$ and $\gamma - e_{i,j} \ge 0$ for some i, j. If $a_i \ge 3$ and $j \ge 2$, then we have $c(\gamma, 1) = 0$.*

PROOF. Since $\deg(t^\alpha e^{t_{\mu_A}}) < 2$, we have $c(\alpha, 1) = 0$ if $|\alpha| \le r - 1$. We shall calculate the coefficient of the term $t^{\gamma - e_{i,j}} e^{t_{\mu_A}}$ in $WDVV((i, 1), (i, j-1), \mu_A, \mu_A)$. Then we have

$$s_{1,j-1,a_i-j} \cdot c(e_{i,1} + e_{i,j-1} + e_{i,a_i-j}, 0) \cdot a_i \cdot \gamma_{i,j} \cdot c(\gamma, 1) = 0 \,.$$

Hence we have $c(\gamma, 1) = 0$. □

LEMMA* 3.6 (Case 2). *If a non-negative element $\gamma \in \mathbb{Z}^{\mu_A - 2}$ satisfies that $|\gamma| = r$ and $\gamma = \sum_{k=1}^r \gamma_{k,1} e_{k,1}$ for some $\gamma_{1,1}, \ldots, \gamma_{r,1}$ such that $\prod_{k=1}^r \gamma_{k,1} = 0$, then we have $c(\gamma, 1) = 0$.*

PROOF. Note that $c(\alpha, 0) = 0$ if $|\alpha| = 4$ and $\alpha - e_{i_1,j_1} - e_{i_2,j_2} \ge 0$ for $i_1 \ne i_2$ by the condition (iv) and that $c(\alpha, 1) = 0$ if $|\alpha| \le r - 1$ since $\deg(t^\alpha e^{t_{\mu_A}}) < 2$. Assume that $\gamma_{i,1} = 0$. We shall calculate the coefficient of the term $(\prod_{k \ne i}^r t_{k,1}^{\gamma_{k,1}}) e^{t_{\mu_A}}$ in the WDVV equation $WDVV((i, 1), (i, a_i - 1), \mu_A, \mu_A)$. Then we have

$$c(e_1 + e_{i,a_i-1} + e_{i,1}, 0) \cdot c(\gamma, 1) = 0 \,.$$

Hence we have $c(\gamma, 1) = 0$ and hence Lemma 3.6. □

Therefore we have Proposition 3.4. □

COROLLARY* 3.7. *If $a_i \ge 3$, then we have*

$$c(2e_{i,1} + 2e_{i,a_i-1}, 0) = -\frac{1}{4a_i^2} \,.$$

PROOF. By the condition (iv), we have $c(\gamma, 0) = 0$ if $\gamma - e_{i_1,j_1} - e_{i_2,j_2} \ge 0$ for $i_1 \ne i_2$. We shall calculate the coefficient of the term $(\prod_{k=1}^r t_{k,1}) e^{t_{\mu_A}}$ in $WDVV((i, 1), (i, a_i - 1), \mu_A, \mu_A)$. Then we have

$$c(e_1 + e_{i,1} + e_{i,a_i-1}, 0) \cdot 1 \cdot c\left(\sum_{k=1}^r e_{k,1}, 1\right) + 4 \cdot c(2e_{i,1} + 2e_{i,a_i-1}, 0) \cdot a_i \cdot c\left(\sum_{k=1}^r e_{k,1}, 1\right) = 0 \,.$$

We have $c(e_1 + e_{i,1} + e_{i,a_i-1}, 0) = 1/a_i$ and $c(\sum_{k=1}^{r} e_{k,1}, 1) = 1$ by the conditions (ii) and (vi) in Theorem 3.1. Hence we have $c(2e_{i,1} + 2e_{i,a_i-1}, 0) = -1/4a_i^2$. □

COROLLARY* 3.8. *If $a_i = 2$, then we have*

$$c(4e_{i,1}, 0) = -\frac{1}{96}\,.$$

PROOF. By the condition (iv), we have $c(\gamma, 0) = 0$ if $\gamma - e_{i_1,j_1} - e_{i_2,j_2} \geq 0$ for $i_1 \neq i_2$. We shall calculate the coefficient of the term $(\prod_{k=1}^{r} t_{k,1})e^{t_{\mu_A}}$ in $WDVV((i,1),(i,1),\mu_A,\mu_A)$. Then we have

$$2c(e_1 + 2e_{i,1}, 0) \cdot c\left(\sum_{k=1}^{r} e_{k,1}, 1\right) + 24c(2e_{i,1} + 2e_{i,a_i-1}, 0) \cdot 2 \cdot c\left(\sum_{k=1}^{r} e_{k,1}, 1\right) = 0\,.$$

We have $c(e_1 + 2e_{i,1}, 0) = 1/4$ and $c(\sum_{k=1}^{r} e_{i,1}, 1) = 1$ by the conditions (ii) and (vi) in Theorem 3.1. Hence we have $c(4e_{i,1}, 0) = -1/96$. □

PROPOSITION† 3.9. *Assume that $c(\alpha, 0)$ and $c(\alpha', 1)$ are reconstructed if $|\alpha| \leq k + 3$ and $|\alpha'| \leq k + r$ for some $k \in \mathbb{Z}_{\geq 0}$. Then coefficients $c(\gamma, 0)$ with $|\gamma| \leq k + 4$ and $c(\gamma', 1)$ with $|\gamma'| \leq k + r + 1$ are reconstructed from coefficients $c(\alpha, 0)$ with $|\alpha| \leq k + 3$ and $c(\alpha', 1)$ with $|\alpha'| \leq k + r$.*

PROOF. We shall split the proof of Proposition 3.9 into following four steps.

LEMMA† 3.10 (Step 1). *If a non-negative element $\beta \in \mathbb{Z}^{\mu_A-2}$ satisfies that $|\beta| = k+1$, then the coefficient $c(\beta + e_{i,j} + e_{i,j'} + e_{i,a_i-1}, 0)$ for some i, j, j' can be reconstructed from coefficients $c(\alpha, 0)$ with $|\alpha| \leq k + 3$ and $c(\alpha', 1)$ with $|\alpha'| \leq k + r$.*

PROOF. Without loss of generality, we can assume $i = 1$. First we shall show that the coefficient $c(\beta + e_{1,1} + e_{1,j+j'-1} + e_{1,a_1-1}, 0)$ can be reconstructed from coefficients $c(\alpha, 0)$ with $|\alpha| \leq k + 3$ and $c(\alpha', 1)$ with $|\alpha'| \leq k + r$. We have $\deg(t^{\beta} t_{1,j+j'-1}) = 1$. By the condition (iv), there exist $e_{1,l}$, $e_{1,l'}$ such that

- $\beta + e_{1,j+j'-1} - e_{1,l} - e_{1,l'} \geq 0\,,$
- $\deg(t_{1,l}) + \deg(t_{1,l'}) \leq 1\,.$

We put $\beta' := \beta + e_{1,1} + e_{1,j+j'-1} - e_{1,l} - e_{1,l'}$. We shall calculate the coefficient of the term $t^{\beta'}(\prod_{k=2}^{r} t_{k,1})e^{t_{\mu_A}}$ in the WDVV equation $WDVV((1,l),(1,l'),\mu_A,\mu_A)$. Then we have

$$(\beta'_{1,l} + 1)(\beta'_{1,l'} + 1)(\beta'_{1,a_1-1} + 1) \cdot c(\beta + e_{1,1} + e_{1,j+j'-1} + e_{1,a_1-1}, 0) \cdot a_1 \cdot c\left(\sum_{k=1}^{r} e_{k,1}, 1\right)$$
$$+(known\ \ terms) = 0\,.$$

By the condition (vi) in Theorem 3.1, the coefficient $c(\beta + e_{1,1} + e_{1,j+j'-1} + e_{1,a_1-1}, 0)$ can be reconstructed from coefficients $c(\alpha, 0)$ with $|\alpha| \leq k + 3$ and $c(\alpha', 1)$ with $|\alpha'| \leq k + r$.

Next we shall show that the coefficient $c(\beta + (\sum_{k=2}^{r} e_{k,1}) + e_{1,j+j'}, 1)$ can be reconstructed from coefficients $c(\alpha, 0)$ with $|\alpha| \leq k+3$ and $c(\alpha', 1)$ with $|\alpha'| \leq k+r$. We shall calculate the coefficient of the term $t^{\beta}(\prod_{k=4}^{r} t_{k,1})e^{t_{\mu_A}}$ in the WDVV equation $WDVV((1,1),(1,j+j'-1),(2,1),(3,1))$. Then we have

$$
\begin{aligned}
& s_{1,j+j'-1,a_1-j-j'} \cdot c(e_{1,1}+e_{1,j+j'-1}+e_{1,a_1-j-j'},0)\cdot a_1 \cdot \\
& (\beta_{1,j+j'}+1)(\beta_{2,1}+1)(\beta_{3,1}+1)\cdot c(\beta+\left(\sum_{k=2}^{r} e_{k,1}\right)+e_{1,j+j'},1) \\
& +(known\ \ terms) \\
& +(\beta_{1,1}+1)(\beta_{1,j+j'-1}+1)(\beta_{1,a_1-1}+1)\cdot c(\beta+e_{1,1}+e_{1,j+j'-1}+e_{1,a_1-1},0)\cdot a_1 \cdot \\
& c\left(\sum_{k=1}^{r} e_{k,1},1\right)=0\,.
\end{aligned}
$$

By the previous argument and Proposition 3.3, the coefficient $c(\beta+(\sum_{k=2}^{r} e_{k,1})+e_{1,j+j'},1)$ can be reconstructed from $c(\alpha,0)$ with $|\alpha|\le k+3$ and $c(\alpha',1)$ with $|\alpha'|\le k+r$.

Finally we shall show that the coefficient $c(\beta+e_{1,j}+e_{1,j'}+e_{1,a_1-1},0)$ can be reconstructed from coefficients $c(\alpha,0)$ with $|\alpha|\le k+3$ and $c(\alpha',1)$ with $|\alpha'|\le k+r$. We shall calculate the coefficient of the term $t^{\beta}(\prod_{k=2}^{r} t_{k,1})e^{t_{\mu_A}}$ in $WDVV((1,j),(1,j'),\mu_A,\mu_A)$. Then we have

$$
\begin{aligned}
\text{(i)}\ & (\beta_{1,j}+1)(\beta_{1,j'}+1)(\beta_{1,a_1-1}+1)\cdot c(\beta+e_{1,j}+e_{1,j'}+e_{1,a_1-1},0)\cdot a_1\cdot c\left(\sum_{k=1}^{r} e_{k,1},1\right) \\
& +(known\ \ terms) \\
& +s_{j,j'',a_1-j-j'}\cdot c(e_{1,j}+e_{1,j'}+e_{1,a_1-j-j'},0)\cdot \\
& a_1\cdot(\beta_{1,j+j'}+1)\cdot c(\beta+\left(\sum_{k=2}^{r} e_{k,1}\right)+e_{1,j+j'},1)=0 \\
& \text{if}\ \ a_1-j+a_1-j'\ge a_1+1\,,
\end{aligned}
$$

$$
\begin{aligned}
\text{(ii)}\ & (\beta_{1,j}+1)(\beta_{1,j'}+1)(\beta_{1,a_1-1}+1)\cdot c(\beta+e_{1,j}+e_{1,j'}+e_{1,a_1-1},0)\cdot a_1\cdot c\left(\sum_{k=1}^{r} e_{k,1},1\right) \\
& +(known\ \ terms) \\
& +c(e_{1,j}+e_{1,j'}+e_1,0)\cdot 1\cdot c(\beta+\left(\sum_{k=2}^{r} e_{k,1}\right),1)=0 \\
& \text{if}\ \ a_1-j+a_1-j'=a_1\,,
\end{aligned}
$$

$$
\begin{aligned}
\text{(iii)}\ & (\beta_{1,j}+1)(\beta_{1,j'}+1)(\beta_{1,a_1-1}+1)\cdot c(\beta+e_{1,j}+e_{1,j'}+e_{1,a_1-1},0)\cdot a_1\cdot c\left(\sum_{k=1}^{r} e_{k,1},1\right) \\
& +(known\ \ terms)=0 \\
& \text{if}\ \ a_1-j+a_1-j'<a_1\,.
\end{aligned}
$$

By the second argument and Proposition 3.3, the coefficient $c(\beta+e_{1,j}+e_{1,j'}+e_{1,a_1-1},0)$ can be reconstructed from coefficients $c(\alpha,0)$ with $|\alpha|\le k+3$ and $c(\alpha',1)$ with $|\alpha'|\le k+r$. □

LEMMA† 3.11 (Step 2). *For a non-negative $\gamma\in\mathbb{Z}^{\mu_A-2}$ with $|\gamma|=k+r+1$, a coefficient $c(\gamma,1)$ can be reconstructed from coefficients $c(\alpha,0)$ with $|\alpha|\le k+3$ and $c(\alpha',1)$ with $|\alpha'|\le k+r$.*

PROOF. We shall split the proof of Lemma 3.11 into following three cases.

SUBLEMMA† 3.12 (Step 2–Case 1). *If a non-negative element $\gamma\in\mathbb{Z}^{\mu_A-2}$ satisfies that $|\gamma|=k+r+1$ and $\gamma-e_{i,j}\ge 0$ for some i,j such that $j\ge 2$, then the coefficient $c(\gamma,1)$ can be reconstructed from coefficients $c(\alpha,0)$ with $|\alpha|\le k+3$ and $c(\alpha',1)$ with $|\alpha'|\le k+r$.*

PROOF. Put $\gamma':=\gamma-e_{i.j}-(\sum_{k\neq i}^{r}e_{k,1})+e_{i,1}+e_{i,j-1}+e_{i,a_i-1}$. We shall calculate the coefficient of the term $t^{\gamma-e_{i,j}}e^{t_{\mu_A}}$ in $WDVV((i,1),(i,j-1),\mu_A,\mu_A)$. Then we have

$$\begin{aligned}&s_{1,j-1,a_i-j}\cdot c(e_{i,1}+e_{i,j-1}+e_{i,a_i-j},0)\cdot a_i\cdot\gamma_{i_1,j_1}\cdot c(\gamma,1)\\&+(known\ \ terms)\\&+\gamma'_{i,1}\gamma'_{i,j-1}\gamma'_{i,a_i-1}\cdot c(\gamma',0)\cdot a_i\cdot c\left(\sum_{k=1}^{r}e_{k,1},1\right)=0\,.\end{aligned}$$

By Lemma 3.10, the coefficient $c(\gamma',0)$ can be reconstructed from $c(\alpha,0)$ and $c(\alpha',1)$ with $|\alpha|\le k+3$ and $|\alpha'|\le k+r$. Hence the coefficient $c(\gamma,1)$ can be reconstructed from coefficients $c(\alpha,0)$ with $|\alpha|\le k+3$ and $c(\alpha',1)$ with $|\alpha'|\le k+r$. □

SUBLEMMA* 3.13 (Step 2–Case 2). *If a non-negative element $\gamma\in\mathbb{Z}^{\mu_A-2}$ satisfies that $|\gamma|=k+r+1$ and $\gamma=\sum_{k=1}^{r}\gamma_{k,1}e_{k,1}$ for some $\gamma_{1,1},\dots,\gamma_{r,1}$ such that $\prod_{k=1}^{r}\gamma_{k,1}\neq 0$, then we have $c(\gamma,1)=0$.*

PROOF. By counting the degree of the term $t^{\gamma}e^{t_{\mu_A}}$, we have

$$\deg(t^{\gamma}e^{t_{\mu_A}})>\deg\left(\left(\prod_{k=1}^{r}t_{k,1}\right)e^{t_{\mu_A}}\right)=2\,.$$

Then we have $c(\gamma,1)=0$. □

SUBLEMMA† 3.14 (Step2–Case 3). *If a non-negative element $\gamma\in\mathbb{Z}^{\mu_A-2}$ satisfies that $|\gamma|=k+r+1$ and $\gamma=\sum_{k=1}^{r}\gamma_{k,1}e_{k,1}$ for some $\gamma_{1,1},\dots,\gamma_{r,1}$ such that $\prod_{k=1}^{r}\gamma_{k,1}=0$, then the coefficient $c(\gamma,1)$ can be reconstructed from coefficients $c(\alpha,0)$ with $|\alpha|\le k+3$ and $c(\alpha',1)$ with $|\alpha'|\le k+r$.*

PROOF. Assume that $\gamma_{i,1}=0$ and put $\gamma':=\gamma-(\sum_{k\neq i}^{r}e_{k,1})+e_{i,1}+e_{i,a_i-1}+e_{i,a_i-1}$. We shall calculate the coefficient of the term $(\prod_{k\neq i}^{r}t_{k,1}^{\gamma_{k,1}})e^{t_{\mu_A}}$ in the WDVV equation $WDVV((i,1),(i,a_i-1),\mu_A,\mu_A)$. Then we have

$$c(e_{i,1}+e_{1,a_i-1}+e_1,0)\cdot c(\gamma,1)+(known\ \ terms)$$

$$+\gamma'_{i,1}\gamma'_{i,a_i-1}\gamma'_{i,a_i-1}\cdot c(\gamma',0)\cdot a_i\cdot c\left(\sum_{k=1}^{r} e_{k,1},\,1\right)=0\,.$$

We have $\gamma' - e_{k,1} \geq 0$ for some $k \neq i$. Then we have $c(\gamma', 0) = 0$ by the condition (iv). Hence the coefficient $c(\gamma, 1)$ can be reconstructed from coefficients $c(\alpha, 0)$ with $|\alpha| \leq k + 3$ and $c(\alpha', 1)$ with $|\alpha'| \leq k + r$. Therefore a coefficient $c(\gamma, 1)$ can be reconstructed from coefficients $c(\alpha, 0)$ with $|\alpha| \leq k + 3$ and $c(\alpha', 1)$ with $|\alpha'| \leq k + r$. □

Then we have Lemma 3.11. □

LEMMA† 3.15. *If $a_i \geq 3$, then we have*

$$c\left(e_{i,j+1}+e_{i,a_i-j}+\sum_{k\neq i}^{r} e_{k,1},\,1\right)=\begin{cases}\frac{1}{a_i} & \text{if } a_i-j\neq j+1\,,\\ \frac{1}{2a_i} & \text{if } a_i-j=j+1\,.\end{cases}$$

PROOF. We shall calculate the coefficient of the term $(\prod_{k\neq i}^{r} t_{k,1})e^{t_{\mu_A}}$ in the WDVV equation $WDVV((i, a_i - j - 1), (i, 1), (i, j + 1), \mu_A)$. Then we have

$$\begin{aligned}
&\text{(i) } \frac{1}{a_i}\cdot a_i\cdot c\left(e_{i,j+1}+e_{i,a_i-j}+\sum_{k\neq i}^{r} e_{k,1},\,1\right)\\
&\qquad -1\cdot 1\cdot c(e_1+e_{i,a_i-j-1}+e_{i,j+1},0)=0\\
&\qquad \text{if } a_i-j\neq j+1,\ a_i-j-1\neq j+1\,,
\end{aligned}$$

$$\begin{aligned}
&\text{(ii) } \frac{1}{a_i}\cdot a_i\cdot c\left(e_{i,j+1}+e_{i,a_i-j}+\sum_{k\neq i_1}^{r} e_{k,1},\,1\right)\\
&\qquad -1\cdot 1\cdot 2\cdot c(e_1+e_{i,a_i-j-1}+e_{i,j+1},0)=0\\
&\qquad \text{if } a_i-j\neq j+1,\ a_i-j-1= j+1\,,\\
&\text{(iii) } \frac{1}{a_i}\cdot a_i\cdot 2\cdot c\left(e_{i,j+1}+e_{i,a_i-j}+\sum_{k\neq i_1}^{r} e_{k,1},\,1\right)\\
&\qquad -1\cdot 1\cdot c(e_1+e_{i,a_i-j-1}+e_{i,j+1},0)=0\\
&\qquad \text{if } a_i-j= j+1\,.
\end{aligned}$$

Hence we have Lemma 3.15. □

LEMMA† 3.16 (Step 3). *If a non-negative element $\gamma \in \mathbb{Z}^{\mu_A-2}$ satisfies that $|\gamma| = k+4$ and $\gamma - e_{i,1} \geq 0$ for some i, then the coefficient $c(\gamma, 0)$ can be reconstructed from coefficients $c(\alpha, 0)$ with $|\alpha| \leq k + 3$ and $c(\alpha', 1)$ with $|\alpha'| \leq k + r$.*

PROOF. We will show this claim by the induction on the degree of parameter $t_{i,j}$. By Lemma 3.10, the coefficient $c(\beta+e_{i,j}+e_{i,j'}+e_{i,a_i-1}, 0)$ with $|\beta| = k+1$ can be reconstructed from coefficients $c(\alpha, 0)$ with $|\alpha| \leq k+3$ and $c(\alpha', 1)$ with $|\alpha'| \leq k+r$. Assume that $c(\gamma', 0)$ with $|\gamma'| = k + 4$ is known if $\gamma' - e_{i,1} - e_{i,n} \geq 0$, $n \geq l$.

We shall show that a coefficient $c(\gamma, 0)$ can be reconstructed from coefficients $c(\alpha, 0)$ with $|\alpha| \leq k+3$ and $c(\alpha', 1)$ with $|\alpha'| \leq k+r$ if $|\gamma| = k+4$ and $\gamma - e_{i,1} - e_{i,l-1} \geq 0$. We have $\deg(t^{\gamma - e_{i,1} - e_{i,l-1}}) = l/a_i$. By the condition (iv), there exist $e_{i,m}, e_{i,m'}$ such that

- $\gamma - e_{i,1} - e_{i,l-1} - e_{i,m} - e_{i,m'} \geq 0$,
- $\deg(t_{i,m}) + \deg(t_{i,m'}) \leq l/a_i$.

Note that $\deg(t_{i,m}) + \deg(t_{i,l}) < 1$ and a coefficient $c(\alpha, 1)$ with $|\alpha| = k+r+1$ can be reconstructed by Lemma 3.11. We put $\beta := \gamma - e_{i,l-1} - e_{i,m} - e_{i,m'}$. We shall calculate the coefficient of the term $t^{\beta}(\prod_{k \neq i}^{r} t_{k,1}) e^{t_{\mu_A}}$ in $WDVV((i,m),(i,m'),(i_1,l),\mu_A)$. Then we have

$$\gamma_{i,m}\gamma_{i,m'}\gamma_{i,l-1} \cdot c(\gamma, 0) \cdot a_i \cdot c\left(e_{i,a_i+1-l} + e_{i,l} + \sum_{k \neq i}^{r} e_{k,1}, 1\right) + (known\ \ terms) = 0.$$

By Lemma 3.15, the coefficient $c(\gamma, 0)$ can be reconstructed from coefficients $c(\alpha, 0)$ with $|\alpha| \leq k+3$ and $c(\alpha', 1)$ with $|\alpha'| \leq k+r$ if $\gamma - e_{i,1} \geq 0$. □

LEMMA* 3.17 (Step 4). *A coefficient $c(\gamma, 0)$ with $|\gamma| = k+4$ can be reconstructed from $c(\alpha, 0)$ with $|\alpha| \leq k+3$ and $c(\alpha', 1)$ with $|\alpha'| \leq k+r$.*

PROOF. We will show this claim by the induction on the degree of parameter $t_{i,j}$. By Lemma 3.16, a coefficient $c(\gamma, 0)$ can be reconstructed from $c(\alpha, 0)$ with $|\alpha| \leq k+3$ and $c(\alpha', 1)$ with $|\alpha'| \leq k+r$ if $\gamma - e_{i,1} \geq 0$. Assume that $c(\gamma', 0)$ with $|\gamma'| = k+4$ is known if $\gamma' - e_{i,n} \geq 0$ for $n \leq l$. We shall show a coefficient $c(\gamma, 0)$ can be reconstructed from coefficients $c(\alpha, 0)$ with $|\alpha| \leq k+3$ and $c(\alpha', 1)$ with $|\alpha'| \leq k+r$ if $\gamma - e_{i,l+1} \geq 0$. We shall calculate the coefficient of the term $t^{\gamma - e_{i,j} - e_{i,j'} - e_{i,l+1}}$ in $WDVV((i,1),(i,l),(i,j),(i,j'))$. Then we have

$$s_{1,l,a_i-1-l} \cdot c(e_{i,1} + e_{i,l} + e_{i,a_i-1-l}, 0) \cdot a_i \cdot \gamma_{i,j}\gamma_{i,j'}\gamma_{i,l+1} \cdot c(\gamma, 0) + (known\ \ terms) = 0.$$

Hence a coefficient $c(\gamma, 0)$ can be reconstructed from coefficients $c(\alpha, 0)$ with $|\alpha| \leq k+3$ and $c(\alpha', 1)$ with $|\alpha'| \leq k+r$. □

Therefore we have Proposition 3.9 □

By Proposition 3.3, Proposition 3.4 and Proposition 3.9, coefficients $c(\gamma, 0)$ and $c(\gamma, 1)$ can be reconstructed from $c(\beta, 0)$ with $|\beta| = 3$.

Let $\mathcal{F}_{A_i}(t'_1, \mathbf{t}'_3, e^{t_{\mu_{A_i}}})$ be the Frobenius potential for the Frobenius manifold M_{A_i} in Theorem 3.1 in [6] where a multiplet of positive integers A_i is $(1, 1, a_i)$ such that $a_i \geq 2$ and we denote by $(t'_1, \mathbf{t}'_3, t_{\mu_{A_i}}) := (t'_1, t'_{3,1}, \ldots, t'_{3,a_i-1}, t_{\mu_{A_i}})$ the flat coordinate for the Frobenius manifold M_{A_i}. Inspired by Proposition 3.3, Corollary 3.7, Corollary 3.8 and Lemma 3.15, we have the following Proposition 3.18.

PROPOSITION 3.18. *For the polynomial $\mathcal{G}_A^{(i)}$ in the condition* (iv) *in Theorem* 3.1*, we have*

$$\mathcal{G}_A^{(i)} = \mathcal{F}_{A_i}(0, \mathbf{t}_i, 0)$$

where $\mathbf{t}_i := (t_{i,1}, \dots, t_{i,a_i-1})$ *is the i-th parts of the flat coordinate in Theorem* 3.1.

PROOF. We can expand the Frobenius potential $\mathcal{F}_{A_i}(0, \mathbf{t}'_3, e^{t_{\mu_{A_i}}})$ uniquely as follows:

$$\mathcal{F}_{A_i}(0, \mathbf{t}'_3, e^{t_{\mu_{A_i}}}) = \sum_{\alpha=(\alpha_{i,1},\dots,\alpha_{i,a_i-1})} c'(\alpha, m) t'^{\alpha} e^{mt_{\mu_{A_i}}}, \quad t'^{\alpha} = \prod_{j=1}^{a_i-1} t'^{\alpha_{i,j}}_{3,j}.$$

The coefficients $c'(\alpha, m)$ are uniquely determined by Theorem 3.1 in [6]. We already proved $c(\alpha, 0) = c'(\alpha, 0)$ if $|\alpha| = 3$ and $c(\alpha' + \sum_{k\neq i}^{r} e_{k,1}, 1) = c'(\alpha', 1)$ if $|\alpha| = 1$ in Proposition 3.3 and Proposition 3.4. We shall show the proposition by the induction concerning the length and split the proof into the following four steps.

LEMMA 3.19 (Step 1). *Assume that* $c(\alpha, 0) = c'(\alpha, 0)$ *and* $c(\alpha' + \sum_{k\neq i}^{r} e_{k,1}, 1) = c'(\alpha', 1)$ *if* $|\alpha| \leq k + 2$ *and* $|\alpha'| \leq k$ *for some* $k \in \mathbb{N}$. *Then we have* $c(\beta + \sum_{k\neq i}^{r} e_{k,1}, 1) = c'(\beta, 1)$ *if* $|\beta| = k + 1$.

PROOF. If $\beta - e_{i,1} \geq 0$, then we have $c(\beta + \sum_{k\neq i}^{r} e_{k,1}, 1) = c'(\beta, 1) = 0$ since both $\deg(t^{\beta} \prod_{k\neq i} t_{k,1} e^{t_{\mu_A}})$ and $\deg(t^{\beta} e^{t_{\mu_{A_i}}})$ are greater than 2. Hence we have $\beta - e_{i,j} \geq 0$ for some $j \geq 2$. The coefficient of the term $t^{\beta - e_{i,j}} (\prod_{k\neq i} t_{k,1}) e^{t_{\mu_A}}$ in the WDVV equation $WDVV((i, 1), (i, j-1), \mu_A, \mu_A)$ for $\mathcal{F}_A$ gives the same recursion relation with the one provided by the coefficient of the term $t^{\beta - e_{i,j}} e^{t_{\mu_{A_i}}}$ in $WDVV((i, 1), (i, j-1), \mu_{A_i}, \mu_{A_i})$ for $\mathcal{F}_{A_i}$ by the assumption and elementary calculation. □

LEMMA 3.20 (Step 2). *Assume that* $c(\alpha, 0) = c'(\alpha, 0)$ *and* $c(\alpha' + \sum_{k\neq i}^{r} e_{k,1}, 1) = c'(\alpha', 1)$ *if* $|\alpha| \leq k + 2$ *and* $|\alpha'| \leq k$ *for some* $k \in \mathbb{N}$. *Then we have* $c(\beta + e_{i,j} + e_{i,j'} + e_{i,a_i-1}, 0) = c'(\beta + e_{i,j} + e_{i,j'} + e_{i,a_i-1}, 0)$ *for some* i, j, j' *if* $|\beta| = k$.

PROOF. The coefficient of the term $t^{\beta} (\prod_{k=2}^{r} t_{k,1}) e^{t_{\mu_A}}$ in $WDVV((1, j), (1, j'), \mu_A, \mu_A)$ for $\mathcal{F}_A$ gives the same recursion relation with the one provided by the coefficient of the term $t^{\beta} e^{t_{\mu_A}}$ in the WDVV equation $WDVV((1, j), (1, j'), \mu_{A_i}, \mu_{A_i})$ for $\mathcal{F}_{A_i}$ by the assumption and Lemma 3.19. □

LEMMA 3.21 (Step 3). *Assume that* $c(\alpha, 0) = c'(\alpha, 0)$ *and* $c(\alpha' + \sum_{k\neq i}^{r} e_{k,1}, 1) = c'(\alpha', 1)$ *if* $|\alpha| \leq k + 2$ *and* $|\alpha'| \leq k$ *for some* $k \in \mathbb{N}$. *Then we have* $c(\gamma, 0) = c'(\gamma, 0)$ *if* $|\gamma| = k + 3$ *and* $\gamma - e_{i,1} \geq 0$.

PROOF. We will show this claim by the induction on the degree of parameter $t_{i,j}$. By Lemma 3.20, we have $c(\beta + e_{i,j} + e_{i,j'} + e_{i,a_i-1}, 0) = c'(\beta + e_{i,j} + e_{i,j'} + e_{i,a_i-1}, 0)$ if $|\beta| = k$. Assume that $c(\gamma', 0) = c'(\gamma', 0)$ if $|\gamma'| = k + 3$ and $\gamma' - e_{i,1} - e_{i,n} \geq 0$ for $n \geq l$. Then we shall show that $c(\gamma, 0) = c'(\gamma, 0)$ for $|\gamma'| = k + 3$ and $\gamma' - e_{i,1} - e_{i,l-1} \geq 0$. We have $\deg(t^{\gamma - e_{i,1} - e_{i,l-1}}) = l/a_i$. By the condition (iv), there exist $e_{i,m}, e_{i,m'}$ such that

- $\gamma - e_{i,1} - e_{i,l-1} - e_{i,m} - e_{i,m'} \geq 0$,
- $\deg(t_{i,m}) + \deg(t_{i,m'}) \leq l/a_i$.

Note that $\deg(t_{i,m}) + \deg(t_{i,l}) < 1$. We put $\beta := \gamma - e_{i,l-1} - e_{i,m} - e_{i,m'}$. Then the coefficient of the term $t^{\beta} (\prod_{k\neq i} t_{k,1}) e^{t_{\mu_A}}$ in the WDVV equation $WDVV((i, m), (i, m'), l, \mu_A)$ for $\mathcal{F}_A$

gives the same recursion relation with the one provided by the coefficient of the term $t^{\beta}e^{t_{\mu_{A_i}}}$ in the WDVV equation $WDVV((i,m),(i,m'),l,\mu_{A_i})$ for $\mathcal{F}_{A_i}$ by the assumption and Lemma 3.19. □

LEMMA 3.22 (Step 4). *Assume that $c(\alpha,0)=c'(\alpha,0)$ and $c(\alpha'+\sum_{k\neq i}^{r}e_{k,1},1)=c'(\alpha',1)$ if $|\alpha|\le k+2$ and $|\alpha'|\le k$ for some $k\in\mathbb{N}$. Then we have $c(\gamma,0)=c'(\gamma,0)$ with $|\gamma|=k+3$.*

PROOF. We will show this claim by the induction on the degree of parameter $t_{i,j}$. By Lemma 3.21, we have $c(\gamma,0)=c'(\gamma,0)$ if $\gamma-e_{i,1}\ge 0$. Assume that $c(\gamma',0)=c'(\gamma',0)$ with $|\gamma'|=k+3$ and $\gamma'-e_{i,n}\ge 0$ for $n\le l$. We shall show that $c(\gamma,0)=c'(\gamma,0)$ with $|\gamma|\le k+3$ and $\gamma-e_{i,l+1}\ge 0$. The coefficient of the term $t^{\gamma-e_{i,j}-e_{i,j'}-e_{i,l+1}}$ in the WDVV equation $WDVV((i,1),(i,l),(i,j),(i,j'))$ for $\mathcal{F}_A$ gives the same recursion relation with the one provided by the coefficient of the term $t^{\gamma-e_{i,j}-e_{i,j'}-e_{i,l+1}}$ in the WDVV equation $WDVV((i,1),(i,l),(i,j),(i,j'))$ for $\mathcal{F}_{A_i}$ by the assumption. □

Therefore we have Proposition 3.18. □

3.2. Coefficients $c(\alpha,m)$ can be reconstructed. In Subsection 3.1, we showed that $c(\alpha,0)$ and $c(\alpha,1)$ can be reconstructed from $c(\beta,0)$ with $|\beta|=3$. We define the total order $\prec$ on $\mathbb{Z}^2_{\ge 0}$ as follows:

- $(|\alpha|,m)\prec(|\beta|,n)$ if $m<n$.
- $(|\alpha|,m)\prec(|\beta|,m)$ if $|\alpha|<|\beta|$.

We shall prove that $c(\alpha,m)$ can be reconstructed from $c(\beta,0)$ with $|\beta|=3$ by the induction on the well order $\prec$ on $\mathbb{Z}^2_{\ge 0}$.

PROPOSITION* 3.23. *A coefficient $c(\gamma,m)$ with $m\ge 2$ can be reconstructed from coefficients $c(\beta,0)$ with $|\beta|=3$.*

PROOF. Assume that $c(\alpha,n)$ can be reconstructed from coefficients $c(\beta,0)$ with $|\beta|=3$ if $(|\alpha|,n)\prec(0,m-1)$. First, we shall show that $c(0,m)$ can be reconstructed. This coefficient must be zero for the case $\chi_A\le 0$. For the case that $\chi_A>0$, we have the following Lemma 3.24:

LEMMA* 3.24. *A coefficient $c(0,m)$ can be reconstructed from coefficients $c(\alpha,n)$ with $(|\alpha|,n)\prec(0,m)$.*

PROOF. We shall calculate the coefficient of $e^{mt_{\mu_A}}$ in *WDVV*$((i,1),(i,a_i-1),\mu_A,\mu_A)$. Then we have

$$c(e_{i,1}+e_{i,a_i-1}+e_1,0)\cdot 1\cdot m^3\cdot c(0,m)+(known\ \ terms)=0\,.$$

Therefore, the cofficient $c(0,m)$ can be reconstructed from coefficients $c(\alpha,n)$ satisfying $(|\alpha|,n)\prec(0,m)$. □

Next, we shall split the second step of the induction into following three cases.

LEMMA* 3.25 (Case 1). *If a non-negative element* $\gamma \in \mathbb{Z}^{\mu_A-2}$ *satisfies that* $|\gamma| = k+1$ *and* $\gamma - e_{i,j} \geq 0$ *for some* j *such that* $j \geq 2$, *then the coefficient* $c(\gamma, m)$ *can be reconstructed from coefficients* $c(\alpha, n)$ *with* $(|\alpha|, n) \prec (k+1, m)$.

PROOF. We shall calculate the coefficient of the term $t^{\gamma - e_{i,j}} e^{mt_{\mu_A}}$ in the WDVV equation $WDVV((i,1),(i,j-1),\mu_A,\mu_A)$. Then we have

$$s_{1,j-1,a_i-j} \cdot c(e_{i,1} + e_{i,j-1} + e_{i,a_i-j}, 0) \cdot a_i \cdot m^2 \cdot \gamma_{i,j} \cdot c(\gamma, m) + (known\ \ terms) = 0\,.$$

Therefore, the coefficient $c(\gamma, m)$ can be reconstructed from coefficients $c(\alpha, n)$ satisfying $(|\alpha|, n) \prec (k+1, m)$. □

LEMMA* 3.26 (Case 2). *If a non-negative element* $\gamma \in \mathbb{Z}^{\mu_A-2}$ *satisfies that* $|\gamma| = k+1$ *and* $\gamma = \sum_{k=1}^{r} \gamma_{k,1} e_{k,1}$ *for some* $\gamma_{1,1}, \dots, \gamma_{r,1}$ *such that* $\prod_{k=1}^{r} \gamma_{k,1} \neq 0$, *then the coefficient* $c(\gamma, m)$ *can be reconstructed from coefficients* $c(\alpha, n)$ *with* $(|\alpha|, n) \prec (k+1, m)$.

PROOF. We shall calculate the coefficient of the term $(\prod_{k=1}^{r} t_{k,1}^{\gamma_{k,1}}) e^{mt_{\mu_A}}$ in the WDVV equation $WDVV((i,1),(i,a_i-1),\mu_A,\mu_A)$. Then we have

$$\begin{aligned}
&\text{(i) } \{c(e_{i,1} + e_{i,a_i-1} + e_1, 0) \cdot m^3 + 4 \cdot c(2e_{i,1} + 2e_{i,a_i-1}, 0) \cdot a_i \cdot m^2 \cdot \gamma_{i,1}\} \cdot c(\gamma, m) \\
&\qquad\qquad\qquad\qquad + (known\ \ terms) = 0 \\
&\qquad\qquad\qquad\qquad \text{if } a_i \geq 3\,, \\
&\text{(ii) } \{2c(2e_{i,1} + e_1, 0) \cdot m^3 + 24 \cdot c(4e_{i,1}, 0) \cdot 2 \cdot m^2 \cdot \gamma_{i,1}\} \cdot c(\gamma, m) \\
&\qquad\qquad\qquad\qquad + (known\ \ terms) = 0 \\
&\qquad\qquad\qquad\qquad \text{if } a_i = 2\,.
\end{aligned}$$

If $\gamma_{i,1} \neq m$ for some i, the coefficient $c(\gamma, m)$ can be reconstructed from $c(\alpha, n)$ with $(\alpha, n) \prec (k+1, m)$. If $\gamma_{i,1} = m$ for all i, we have $\deg((\prod_{k=1}^{r} t_{k,1}^{\gamma_{k,1}}) e^{mt_{\mu_A}}) = 2m \geq 2$ and hence $c(\gamma, m) = 0$ except for the case $m = 1$. □

LEMMA* 3.27 (Case 3). *If a non-negative element* $\gamma \in \mathbb{Z}^{\mu_A-2}$ *satisfies that* $|\gamma| = k+1$ *and* $\gamma = \sum_{k=1}^{r} \gamma_{k,1} e_{k,1}$ *for some* $\gamma_{1,1}, \dots, \gamma_{r,1}$ *such that* $\prod_{k=1}^{r} \gamma_{k,1} = 0$, *then the coefficient* $c(\gamma, m)$ *can be reconstructed from coefficients* $c(\alpha, n)$ *with* $(|\alpha|, n) \prec (k+1, m)$.

PROOF. Assume that $\gamma_{i,1} = 0$. We shall calculate the coefficient of the term $(\prod_{k=1}^{r} t_{k,1}^{\gamma_{k,1}}) e^{mt_{\mu_A}}$ in the WDVV equation $WDVV((i,1),(i,a_i-1),\mu_A,\mu_A)$. Then we have

$$c(e_1 + e_{i,a_i-1} + e_{i,1}, 0) \cdot m^3 \cdot c(\gamma, m) + (known\ \ terms) = 0.$$

Therefore, the coefficient $c(\gamma, m)$ can be reconstructed from coefficients $c(\alpha, n)$ satisfying $(|\alpha|, n) \prec (k+1, m)$. □

Hence, we have Proposition 3.23. □

We finish the proof of Theorem 3.1.

4. The Gromov–Witten Theory for Orbifold Projective Lines. Let $r \geq 3$ be a positive integer. Let $A = (a_1, \dots, a_r)$ be a multiplet of positive integers and $\Lambda = (\lambda_1, \dots, \lambda_r)$

a multiplet of pairwise distinct elements of $\mathbb{P}^1(\mathbb{C})$ normalized such that $\lambda_1 = \infty$, $\lambda_2 = 0$ and $\lambda_3 = 1$.

Following Geigle–Lenzing (cf. Section 1.1 in [5]), we shall introduce an orbifold projective line. First, we prepare some notations.

DEFINITION 4.1. Let r, A and Λ be as above.

(i) Define a ring $R_{A,\Lambda}$ by

$$R_{A,\Lambda} := \mathbb{C}[X_1, \dots, X_r] \big/ I_\Lambda \ , \tag{4.1a}$$

where I_Λ is an ideal generated by $r-2$ homogeneous polynomials

$$X_i^{a_i} - X_2^{a_2} + \lambda_i X_1^{a_1}, \quad i = 3, \dots, r\,. \tag{4.1b}$$

(ii) Denote by L_A an abelian group generated by r-letters $\vec{X}_i$, $i = 1, \dots, r$ defined as the quotient

$$L_A := \bigoplus_{i=1}^{r} \mathbb{Z}\vec{X}_i \big/ M_A \ , \tag{4.2a}$$

where M_A is the subgroup generated by the elements

$$a_i\vec{X}_i - a_j\vec{X}_j, \quad 1 \le i < j \le r\,. \tag{4.2b}$$

We then consider the following quotient stack:

DEFINITION 4.2. Let r, A and Λ be as above. Define a stack $\mathbb{P}^1_{A,\Lambda}$ by

$$\mathbb{P}^1_{A,\Lambda} := \left[\left(\mathrm{Spec}(R_{A,\Lambda})\backslash\{0\}\right)/\mathrm{Spec}(\mathbb{C}L_A)\right]\,, \tag{4.3}$$

which is called the *orbifold projective line* of type (A, Λ).

An orbifold projective line of type (A, Λ) is a Deligne–Mumford stack whose coarse moduli space is a smooth projective line $\mathbb{P}^1$. The orbifold cohomology group of $\mathbb{P}^1_{A,\Lambda}$ is, as a vector space, just the singular cohomology group of the inertia orbifold:

$$\mathcal{I}\mathbb{P}^1_{A,\Lambda} = \mathbb{P}^1_{A,\Lambda} \bigsqcup \bigsqcup_{1\le i\le r} \left(\bigsqcup_{j=1}^{a_i-1} (B(\mathbb{Z}/a_i\mathbb{Z}))_j \right)$$

where $(B(\mathbb{Z}/a_i\mathbb{Z}))_j := B(\mathbb{Z}/a_i\mathbb{Z})$. The age associated to the component $\mathbb{P}^1_{A,\Lambda}$ is 0 and the age associated to $(B(\mathbb{Z}/a_i\mathbb{Z}))_j$ is j/a_i. The orbifold Poincaré pairing is given by twisting the usual Poincaré pairing:

$$\int_{\mathbb{P}^1_{A,\Lambda}} \alpha \cup_{orb} \beta := \int_{\mathcal{I}\mathbb{P}^1_{A,\Lambda}} \alpha \cup I\beta\,,$$

where I is the involution defined in [1, 2]. Then we have the following:

LEMMA* 4.3. *We can choose a* $\mathbb{Q}$*-basis* $1 = \Delta_1, \Delta_{1,1}, \dots, \Delta_{i,j}, \dots, \Delta_{r,a_r-1}, \Delta_{\mu_A}$ *of the orbifold cohomology group* $H^*_{orb}(\mathbb{P}^1_{A,\Lambda}, \mathbb{Q})$ *such that*

$$H^0_{orb}(\mathbb{P}^1_{A,\Lambda}, \mathbb{Q}) \simeq \mathbb{Q}\Delta_1, \quad \Delta_{i,j} \in H^{2\frac{j}{a_i}}_{orb}(\mathbb{P}^1_{A,\Lambda}, \mathbb{Q}), \quad H^2_{orb}(\mathbb{P}^1_{A,\Lambda}, \mathbb{Q}) \simeq \mathbb{Q}\Delta_{\mu_A}$$

and

$$\int_{\mathbb{P}^1_{A,\Lambda}} \Delta_1 \cup_{orb} \Delta_{\mu_A} = 1\,, \quad \int_{\mathbb{P}^1_{A,\Lambda}} \Delta_{i,j} \cup_{orb} \Delta_{k,l} = \begin{cases} \frac{1}{a_i} & \text{if } k = i,\ l = a_i - j \\ 0 & \text{otherwise}\,. \end{cases}$$

PROOF. The decomposition of $H^*_{orb}(\mathbb{P}^1_{A,\Lambda}, \mathbb{C})$ follows from the decomposition of the inertia orbifold $\mathcal{I}\mathbb{P}^1_{A,\Lambda}$. The latter assertion immediately follows from the definition of the orbifold Poincaré pairing. □

Denote by $t_1, t_{1,1}, \dots, t_{i,j}, \dots, t_{r,a_r-1}, t_{\mu_A}$ the dual coordinates of the $\mathbb{Q}$-basis $\Delta_1, \Delta_{1,1}, \dots, \Delta_{i,j}, \dots, \Delta_{r,a_r-1}, \Delta_{\mu_A}$ of $H^*_{orb}(\mathbb{P}^1_{A,\Lambda}, \mathbb{Q})$ in Lemma 4.3. Consider a formal manifold M whose structure sheaf $\mathcal{O}_M$ and tangent sheaf $\mathcal{T}_M$ are given by

$$\text{(4.4)} \quad \mathcal{O}_M := \mathbb{C}((e^{t_{\mu_A}}))[[t_1, t_{1,1}, \dots, t_{i,j}, \dots, t_{r,a_r-1}]], \quad \mathcal{T}_M := H^*_{orb}(\mathbb{P}^1_{A,\Lambda}, \mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_M\,,$$

where $\mathbb{C}((e^{t_{\mu_A}}))$ denotes the $\mathbb{C}$-algebra of formal Laurent series in $e^{t_{\mu_A}}$.

The Gromov–Witten theory for orbifolds developed by Abramovich–Graber–Vistoli [1] and Chen–Ruan [2] gives us the following proposition. Note here that, by using the divisor axiom, it turns out that third derivatives of the genus zero Gromov–Witten potential $\mathcal{F}_0^{\mathbb{P}^1_{A,\Lambda}}$ are elements of $\mathbb{C}[[t_{1,1}, \dots, t_{i,j}, \dots, t_{r,a_r-1}, q^{[\mathbb{P}^1]}e^{t_{\mu_A}}]]$ and hence they can be considered as elements of $\mathcal{O}_M$ by formally setting $q^{[\mathbb{P}^1]} = 1$.

PROPOSITION 4.4 ([1, 2]). *There exists a structure of a formal Frobenius manifold of rank* μ_A *and dimension one on* M *whose non-degenerate symmetric* $\mathcal{O}_M$*-bilinear form* η *on* $\mathcal{T}_M$ *is given by the orbifold Poincaré pairing.*

PROOF. See Theorem 6.2.1 of [1] and Theorem 3.4.3 of [2]. □

The following theorem is the main result in this section:

THEOREM 4.5. *The conditions in Theorem 3.1 are satisfied by the Frobenius structure constructed from the Gromov–Witten theory for* $\mathbb{P}^1_{A,\Lambda}$.

We shall check the conditions in Theorem 3.1 one by one.

4.1. Condition (i). It follows from Lemma 4.3 that the unit vector field $e \in \mathcal{T}_M$ and the Euler vector field $E \in \mathcal{T}_M$ are given as

$$e = \frac{\partial}{\partial t_1}\,, \quad E = t_1\frac{\partial}{\partial t_1} + \sum_{i=1}^{r}\sum_{j=1}^{a_i-1} \frac{a_i - j}{a_i} t_{i,j}\frac{\partial}{\partial t_{i,j}} + \chi_A \frac{\partial}{\partial t_{\mu_A}}\,,$$

which is the condition (i).

4.2. Condition (ii). It is obvious from Lemma 4.3.

4.3. Condition (iii). The condition (iii) follows from the divisor axiom and the definition of the genus zero potential $\mathcal{F}_0^{\mathbb{P}^1_{A,\Lambda}}$.

4.4. Condition (iv). The condition (iv) is satisfied since the image of degree zero orbifold map with marked points on orbifold points on the source must be one of orbifold points on the target $\mathbb{P}^1_{A,\Lambda}$.

4.5. Condition (v). The orbifold cup product is the specialization of the quantum product at $t_1 = t_{1,1} = \cdots = t_{r,a_r-1} = e^{t_{\mu_A}} = 0$. Therefore, it turns out that the orbifold cup product can be determined by the degree zero three point Gromov–Witten invariants.

LEMMA* 4.6. *There is a $\mathbb{C}$-algebra isomorphism between the orbifold cohomology ring $H^*_{orb}(\mathbb{P}^1_{A,\Lambda}, \mathbb{C})$ and $\mathbb{C}[x_1, x_2, \ldots, x_r]\Big/\big(x_i x_j,\ a_i x_i^{a_i} - a_j x_j^{a_j}\big)_{1\le i\ne j\le r}$, where $\partial/\partial t_{i,j}$ are mapped to x_i^j for $i = 1, \ldots, r$, $j = 1, \ldots, a_i - 1$ and $\partial/\partial t_{\mu_A}$ are mapped to $a_1 x_1^{a_1}$.*

PROOF. Under the same notation in Lemma 4.3, the orbifold cup product is given as follows:

$$\Delta_\alpha \cup_{orb} \Delta_\beta = \sum_\delta \big\langle \Delta_\alpha, \Delta_\beta, \Delta_\gamma \big\rangle_{0,3,0}^{\mathbb{P}^1_{A,\Lambda}} \eta^{\gamma\delta} \Delta_\delta \,,$$

where we set $\eta^{\gamma\delta}$ as follows:

$$(\eta^{\gamma\delta}) = \Big(\int_{\mathbb{P}^1_{A,\Lambda}} \Delta_\gamma \cup_{orb} \Delta_\delta \Big)^{-1} .$$

By the previous argument in Subsection 4.4, we have

$$\Delta_{i_1,j_1} \cup_{orb} \Delta_{i_2,j_2} = 0 \text{ if } i_1 \ne i_2 \,.$$

By the formula

$$\begin{aligned} \int_{\mathbb{P}^1_{A,\Lambda}} \Delta_{i_1,j_1} \cup_{orb} \Delta_{i_1,j_1'} \cup_{orb} \Delta_{i_1,j_1''} &= \frac{1}{|\mathbb{Z}/a_{i_1}\mathbb{Z}|} \int_{pt} ev_1^*(\Delta_{i_1,j_1}) \cup ev_2^*(\Delta_{i_1,j_1'}) \cup ev_3^*(\Delta_{i_1,j_1''}) \\ &= \begin{cases} \frac{1}{a_{i_1}} & \text{if } j_1 + j_1' + j_1'' = a_{i_1} \,, \\ 0 & \text{otherwise} \,, \end{cases} \end{aligned}$$

we have

$$\Delta_{i_1,j_1} \cup_{orb} \Delta_{i_1,j_1'} = \Delta_{i_1,j_1+j_1'} \text{ if } j_1 + j_1' \le a_{i_1} - 1 \,,$$

and hence

$$\Delta_{i_1,1}^p := \underbrace{\Delta_{i_1,1} \cup_{orb} \cdots \cup_{orb} \Delta_{i_1,1}}_{a_{i_1}\ times} = \frac{1}{a_{i_1}} \Delta_{\mu_A}$$

Therefore we have Lemma 4.6. $\square$

LEMMA* 4.7. *The term*

$$\left(\prod_{i=1}^{r} t_{i,1}\right) e^{t_{\mu_A}}$$

occurs with the coefficient 1 *in the* $\mathcal{F}_0^{\mathbb{P}^1_{A,\Lambda}}$.

PROOF. This lemma follows from the fact that the Gromov–Witten invariant counts the number of orbifold maps from $\mathbb{P}^1_{A,\Lambda}$ to $\mathbb{P}^1_{A,\Lambda}$ of degree 1 fixing r marked (orbifold) points, which is exactly the identity map. □

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