

## SPECTRAL ZETA FUNCTIONS OF GRAPHS AND THE RIEMANN ZETA FUNCTION IN THE CRITICAL STRIP

FABIEN FRIEDLI AND ANDERS KARLSSON\*

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**Abstract.** We initiate the study of spectral zeta functions  $\zeta_X$  for finite and infinite graphs  $X$ , instead of the Ihara zeta function, with a perspective towards zeta functions from number theory and connections to hypergeometric functions. The Riemann hypothesis is shown to be equivalent to an approximate functional equation of graph zeta functions. The latter holds at all points where Riemann's zeta function  $\zeta(s)$  is non-zero. This connection arises via a detailed study of the asymptotics of the spectral zeta functions of finite torus graphs in the critical strip and estimates on the real part of the logarithmic derivative of  $\zeta(s)$ . We relate  $\zeta_{\mathbb{Z}}$  to Euler's beta integral and show how to complete it giving the functional equation  $\xi_{\mathbb{Z}}(1-s) = \xi_{\mathbb{Z}}(s)$ . This function appears in the theory of Eisenstein series although presumably with this spectral interpretation unrecognized. In higher dimensions  $d$  we provide a meromorphic continuation of  $\zeta_{\mathbb{Z}^d}(s)$  to the whole plane and identify the poles. From our asymptotics several known special values of  $\zeta(s)$  are derived as well as its non-vanishing on the line  $\operatorname{Re}(s) = 1$ . We determine the spectral zeta functions of regular trees and show it to be equal to a specialization of Appell's hypergeometric function  $F_1$  via an Euler-type integral formula due to Picard.

**Introduction.** In order to study the Laplace eigenvalues  $\lambda_n$  of bounded domains  $D$  in the plane, Carleman employed the function

$$\zeta_D(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s}$$

taking advantage of techniques from the theory of Dirichlet series including Ikehara's Tauberian theorem [Ca34]. This was followed-up in [P39], and developed further in [MP49] for the case of compact Riemannian manifolds. These zeta functions have since played a role in the definitions of determinants of Laplacians and analytic torsion, and they are important in theoretical physics [Ha77, EI12, RV15]. For graphs it has been popular and fruitful to study the Ihara zeta function, which is an analog of the Selberg zeta function in turn modeled on the Euler product of Riemann's zeta function. Serre noted that Ihara's definition made sense for any finite graph and this suggestion was taken up and developed by Sunada, Hashimoto, Hori, Bass and others, see [Su86, Te10].

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The present paper has a three-fold objective. First, we advance the study of spectral zeta functions of graphs, instead of the Ihara zeta function. We do this even for infinite graphs where the spectrum might be continuous. For the most fundamental infinite graphs, this study leads into the theory of hypergeometric function in several variables, such as those of Appell, and gives rise to several questions.

Second, we study the asymptotics of spectral zeta functions for finite torus graphs as they grow to infinity, in a way similar to what is often considered in statistical physics (see for example [DD88]). The study of limiting sequences of graphs is also a subject of significant current mathematical interest, see [Lo12, Ly10, LPS14]. Terms appearing in our asymptotic expansions are zeta functions of lattice graphs and of continuous torus which are Epstein zeta function from number theory. This relies to an important extent on the work of Chinta, Jorgenson, and the second-named author [CJK10], in particular we quote and use without proof several results established in this reference.

Third, we provide a new perspective on some parts of analytic number theory, in two ways. In one way, this comes via replacing partial sums of Dirichlet series by zeta functions of finite graphs. Although the latter looks somewhat more complicated, they have more structure, being a spectral zeta function, and are decidedly easier in some respects. We show the equivalence of the Riemann hypothesis with a conjectural functional equation for graph spectral zeta functions, and this seems substantially different from other known reformulations of this important problem [RH08]. In a second way, the spectral zeta function of the graph  $\mathbb{Z}$  enjoys properties analogous to the Riemann zeta function, notably the relation  $\xi_{\mathbb{Z}}(1-s) = \xi_{\mathbb{Z}}(s)$ , and it appears *incognito* as fudge factor in a few instances in the classical theory, such as in the Fourier development of Eisenstein series.

For us, a spectral zeta function  $\zeta_X$  of a space  $X$  is the Mellin transform of the heat kernel of  $X$  at the origin, removing the trivial eigenvalue if applicable, and divided by a gamma factor (*cf.* [JL12]). Alternatively one can define this function by an integration against the spectral measure.

Consider a sequence of discrete tori  $\mathbb{Z}^d / A_n \mathbb{Z}^d$  indexed by  $n$  and where the matrices  $A_n$  are diagonal with entries  $a_i n$ , and integers  $a_i > 0$ . The matrix  $A$  is the diagonal matrix with entries  $a_i$ . We show the following for any dimension  $d \geq 1$ :

**THEOREM 0.1.** *The following asymptotic expansion as  $n \rightarrow \infty$  is valid for  $Re(s) < d/2 + 1$ , and  $s \neq d/2$ ,*

$$\zeta_{\mathbb{Z}^d / A_n \mathbb{Z}^d}(s) = \zeta_{\mathbb{Z}^d}(s) \det A n^d + \zeta_{\mathbb{R}^d / A \mathbb{Z}^d}(s) n^{2s} + o(n^{2s}).$$

The formula reflects that as  $n$  goes to infinity the finite torus graph can be viewed as converging to  $\mathbb{Z}^d$  on the one hand, and rescaled to the continuous torus  $\mathbb{R}^d / \mathbb{Z}^d$  on the other hand. For  $Re(s) > d/2$  one has

$$(1) \quad \lim_{n \rightarrow \infty} \frac{1}{n^{2s}} \zeta_{\mathbb{Z}^d / A_n \mathbb{Z}^d}(s) = \zeta_{\mathbb{R}^d / A \mathbb{Z}^d}(s),$$

as already shown in [CJK10], see also Section 4 below. One can verify that it is legitimate to differentiate in the asymptotics in Theorem 0.1 and if we then set  $s = 0$ , we recover as

expected the main asymptotic formula in [CJK10] in the case considered. The asymptotics of the determinant of graph Laplacians is a topic of significant interest, see [RV15, Conclusion] for a recent discussion from the point of view of quantum field theory, and see [L02] for related determinants in the context of  $L^2$ -invariants.

We now specialize to the case  $d = 1$ . In particular, the spectral zeta function of the finite cyclic graph  $\mathbb{Z}/n\mathbb{Z}$  (see e.g. [CJK10] for details and Section 1) is

$$\zeta_{\mathbb{Z}/n\mathbb{Z}}(s) = \frac{1}{4^s} \sum_{k=1}^{n-1} \frac{1}{\sin^{2s}(\pi k/n)}.$$

The spectral zeta function of the graph  $\mathbb{Z}$  is

$$\zeta_{\mathbb{Z}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-2t} I_0(2t) t^s \frac{dt}{t},$$

where it converges, which it does for  $0 < \operatorname{Re}(s) < 1/2$ . From this definition it is not immediate that its meromorphic continuation admits a functional equation much analogous to classical zeta functions:

**THEOREM 0.2.** *Let the completed zeta function for  $\mathbb{Z}$  be defined as*

$$\xi_{\mathbb{Z}}(s) = 2^s \cos(\pi s/2) \zeta_{\mathbb{Z}}(s/2).$$

*Then this is an entire function that satisfies for all  $s \in \mathbb{C}$*

$$\xi_{\mathbb{Z}}(s) = \xi_{\mathbb{Z}}(1 - s).$$

This raises the question: Are there other spectral zeta functions of graphs with similar properties?

The function  $\zeta_{\mathbb{Z}}$  actually appears implicitly in classical analytic number theory. Let us exemplify this point. To begin with

$$\zeta_{\mathbb{Z}}(s) = \frac{1}{4^s \sqrt{\pi}} \frac{\Gamma(1/2 - s)}{\Gamma(1 - s)},$$

which is the crucial fact behind the result above. Now, in the main formula of Chowla-Selberg in [SC67] the following term appears:

$$\frac{2^{2s} a^{s-1} \sqrt{\pi}}{\Gamma(s) \Delta^{s-1/2}} \zeta(2s - 1) \Gamma(s - 1/2).$$

Here lurks  $\zeta_{\mathbb{Z}}(1 - s)$ , not only by correctly combining the two gamma factors, but also incorporating the factor  $2^{2s}$  and explaining the appearance of  $\sqrt{\pi}$ . In other words, the term above equals

$$\frac{4\pi a^{s-1}}{\Delta^{s-1/2}} \zeta(2s - 1) \zeta_{\mathbb{Z}}(1 - s).$$

Upon dividing by the Riemann zeta function  $\zeta(s)$ , this term is called scattering matrix (function) in the topic of Fourier expansions of Eisenstein series, see [JL96, p.111], and is complicated or unknown for discrete groups more general than  $SL(2, \mathbb{Z})$ , see [IK04, section

15.4] and [M08]. We believe that the interpretation of such fudge factors as spectral zeta functions is new and may provide some insight into how such factors arise more generally.

The Riemann zeta function is essentially the same as the spectral zeta function of the circle  $\mathbb{R}/\mathbb{Z}$ , more precisely one has

$$(2) \quad \zeta_{\mathbb{R}/\mathbb{Z}}(s) = 2(2\pi)^{-2s} \zeta(2s).$$

Here is a specialization of Theorem 0.1 to  $d = 1$  with explicit functions and some more precision, see also Sidi [Si04]:

**THEOREM 0.3.** *For  $s \neq 1$  with  $\operatorname{Re}(s) < 3$  it holds that*

$$\sum_{k=1}^{n-1} \frac{1}{\sin^s(\pi k/n)} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(1/2 - s/2)}{\Gamma(1 - s/2)} n + 2\pi^{-s} \zeta(s) n^s + o(n^s)$$

as  $n \rightarrow \infty$ . In the critical strip,  $0 < \operatorname{Re}(s) < 1$ , more precise asymptotics can be found, such as

$$\sum_{k=1}^{n-1} \frac{1}{\sin^s(\pi k/n)} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(1/2 - s/2)}{\Gamma(1 - s/2)} n + 2\pi^{-s} \zeta(s) n^s + \frac{s}{3} \pi^{2-s} \zeta(s - 2) n^{s-2} + o(n^{s-2})$$

as  $n \rightarrow \infty$ .

For example, with  $s = 0$  the sum on the left equals  $n - 1$ , and the asymptotic formula hence confirms the well-known values  $\Gamma(1/2) = \sqrt{\pi}$  and  $\zeta(0) = -1/2$ . On the line  $\operatorname{Re}(s) = 1$ , the asymptotics is critical in the sense that the two first terms on the right balance each other in size as a power of  $n$ . As a consequence, for all  $t \neq 0$  we have that  $\zeta(1 + it) \neq 0$  if and only if

$$\frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{\sin^{1+it}(\pi k/n)}$$

diverges as  $n \rightarrow \infty$ . The latter sum does indeed diverge. We do not have a direct proof of this at the moment, but it does follow from a theorem of Wintner [W47] since the improper integral  $\int \sin^{-1-it}(x) dx$  diverges at  $x = 0$ . So we have that the Riemann zeta function has no zeros on the line  $\operatorname{Re}(s) = 1$ , which is the crucial input in the standard proof of the prime number theorem. It should however be said that Wintner's theorem is known to already be intimately related to the prime number theorem via works of Hardy-Littlewood.

As suggested to us by Jay Jorgenson, one may differentiate the formula in Theorem 0.1 for  $d = 1$ , as can be verified via the formulas in Section 4, and get a criterion for multiple zeros:

**COROLLARY 0.4.** *Let*

$$c(s) = \frac{1}{2\sqrt{\pi}} \frac{\Gamma(1/2 - s/2)}{\Gamma(1 - s/2)} \left( \frac{\Gamma'(1/2 - s/2)}{\Gamma(1/2 - s/2)} - \frac{\Gamma'(1 - s/2)}{\Gamma(1 - s/2)} \right)$$

and

$$S(s, n) = c(s)n - \sum_{k=1}^{n-1} \frac{\log(\sin(\pi k/n))}{\sin^s(\pi k/n)}.$$

Then  $\zeta$  has a multiple zero at  $s$ ,  $0 < \text{Re}(s) < 1$  if and only if  $S(s, n) \rightarrow 0$  as  $n \rightarrow \infty$ , and otherwise  $S(s, n) \rightarrow \infty$  as  $n \rightarrow \infty$ .

It is believed that all Riemann zeta zeros are simple.

Similarly to the above discussion about the prime number theorem, the Riemann hypothesis has a formulation in terms of the behaviour of the sum of sines (here we can refer to [So98] for comparison). It turns out that with some further investigation there is, what we think, a more intriguing formulation of the Riemann hypothesis. This is in terms of functional equations and provides perhaps some further heuristic evidence for its validity. Let

$$h_n(s) = (4\pi)^{s/2} \Gamma(s/2) n^{-s} (\zeta_{\mathbb{Z}/n\mathbb{Z}}(s/2) - n\zeta_{\mathbb{Z}}(s/2)).$$

CONJECTURE. Let  $s \in \mathbb{C}$  with  $0 < \text{Re}(s) < 1$ . Then

$$\lim_{n \rightarrow \infty} \left| \frac{h_n(1-s)}{h_n(s)} \right| = 1.$$

This is an asymptotic or approximative functional equation, and it is true almost everywhere as follows from the asymptotics above, see Corollary 0.5 below. Although we came to this via graph zeta functions, it is important to emphasize, as a referee pointed out, that the asymptotics in one dimension hold with the same proofs for more general sums, instead of the inverse sine sums coming from cyclic graphs. More precisely, let  $f$  be an analytic function being real and positive on the open interval  $(0, 1)$ , satisfying  $f(z) = f(1-z)$  for any  $z \in \mathbb{C}$ , with  $f(0) = 0$ ,  $f'(0) > 0$ ,  $f''(0) = 0$  and  $f^{(3)}(0) \neq 0$ .

Now let for  $0 < \text{Re}(s) < 1$

$$h_n[f](s) = f'(0)^s \pi^{-s/2} \Gamma(s/2) n^{-s} \left[ \sum_{j=1}^{n-1} \frac{1}{f(j/n)^s} - n \int_0^1 \frac{dx}{f(x)^s} \right].$$

As in Section 5 applying [Si04] one gets

$$h_n[f](s) = 2\xi(s) - \frac{f^{(3)}(0)}{f'(0)\pi^2} \alpha(s)n^{-2} + o(n^{-2})$$

as  $n \rightarrow \infty$  and where  $\alpha$  is the function appearing in Section 8. So one may formulate the same conjecture above, and Corollary 0.5 and Theorem 0.6 below hold for  $h_n[f]$ . This being said, we still feel that there might be something special with the sum of reciprocal sines in this context since, as discussed above,  $\zeta_{\mathbb{Z}}(s/2)$  has a functional equation of the desired type,  $s \longleftrightarrow 1-s$ , and also  $\zeta_{\mathbb{Z}/n\mathbb{Z}}(s/2)$  in an asymptotic sense, see Section 7. These functional relations do not depend on the Riemann zeta function and may not hold for the corresponding sum and integral defined by  $f$  as above. Independently of number theory, we are interested in functional equations for graph zeta functions, which ultimately may also reflect similar relations for spectral zeta functions of manifolds. See also our concluding remarks below.

COROLLARY 0.5. *The conjecture holds in the critical strip wherever  $\zeta(s) \neq 0$ .*

So the question is whether it also holds at the Riemann zeros. Here is the relation to the Riemann hypothesis:

THEOREM 0.6. *The conjecture is equivalent to the Riemann hypothesis.*

Section 8 is devoted to the proof of this statement. This relies in particular on properties of the logarithmic derivative of  $\zeta$ , in the proof of Lemma 8.4, and the Riemann functional equation.

**Some concluding remarks.** Why do we think that the study of sums like

$$\sum_{k=1}^{n-1} \frac{1}{\sin^s(\pi k/n)}$$

could in some ways be better than the standard Dirichlet series  $\sum_1^n k^{-s}$ , or some other sum of similar type for that matter? For example, it has been pointed out to us that we could also derive version of Theorems 0.3 and 0.6 for more general functions, as described above, for example replacing sine with  $x - 2x^3 + x^4 = x(1 - x)(1 + x - x^2)$ . In this case the function corresponding to our  $\zeta_{\mathbb{Z}}(s)$ , say in the definition of  $h_n$ , would be

$$\int_0^1 \frac{1}{x^{2s}(1-x)^{2s}(1+x-x^2)^{2s}} dx,$$

which is a less standard function.

Let us now address this legitimate question with several answers that reinforce each other:

1. The graph zeta functions are defined in a parallel way to the definition of Riemann's zeta. Functions arising in this way may have greater chance to have more symmetries and structure, for example, keep in mind the remarkable relation

$$\xi_{\mathbb{Z}}(1-s) = \xi_{\mathbb{Z}}(s),$$

which is far from being just an abstract generality. Furthermore, it appears in the theory of Eisenstein series as observed above in a way that is difficult to deny, and in our opinion, unwise to dismiss. On the other hand, it is not clear whether the integral above satisfies a functional equation. In Section 7, we obtain an asymptotic functional relation of the desired type for the completed finite  $1/\sin$  sums:

$$\lim_{n \rightarrow \infty} \frac{1}{n} (\xi_{\mathbb{Z}/n\mathbb{Z}}(1-s) - \xi_{\mathbb{Z}/n\mathbb{Z}}(s)) = 0$$

in the critical strip. We do not see a similar relation for, say

$$\sum_{k=1}^{n-1} \frac{1}{((k/n)(1-k/n)(1+k/n-k^2/n^2))^{2s}}.$$

Relations when  $s \longleftrightarrow 1 - s$  is at the heart of the matter for our reformulation of the Riemann Hypothesis.

2. Symmetric functions of graph eigenvalues often have combinatorial interpretations as counting something (starting with Kirchhoff's matrix tree theorem), see our Section 6.2 for a small illustration. This also motivates further study of spectral zeta functions for graphs. In particular, the analogous functions for manifolds play a role in various branches of mathematical physics. In this connection, Theorem 0.1 is of definite interest, see the comments after this theorem.
3. It is also noteworthy to recall that for  $s = 2m$ , the even positive integers, our finite sums admit a closed form expression as a polynomial in  $n$ , for example (which can be shown combinatorially in line with the previous point),

$$\sum_{k=1}^{n-1} \frac{1}{\sin^2(\pi k/n)} = \frac{1}{3}n^2 - \frac{1}{3},$$

while  $\sum_1^n k^{-2m}$  does not admit such a formula. The sine series evaluation implies, in view of (1) and (2) above, Euler's formulas for  $\zeta(2m)$ , for example  $\zeta(2) = \pi^2/6$ . See Section 6 for more about how our asymptotical relations imply known special values, and also references to contexts where the finite  $1/\sin$  sums are studied.

*Higher dimensions.* For  $d > 1$  the torus zeta functions are Epstein zeta functions also appearing in number theory. Some of these are known not to satisfy the Riemann hypothesis, the statement that all non-trivial zeros lie on one vertical line (see [RH08] and [PT34]). It seems interesting to understand this difference between  $d = 1$  and certain higher dimensional cases from our perspective. Theorem 0.1 gives precise asymptotics in higher dimensions, but to get even further terms in the expansion, as in Theorem 0.3, there are some complications, especially when trying to assemble a nice expression, like  $\zeta(s - 2)$  as in Theorem 0.3. Therefore this is left for future study.

*Generalized Riemann Hypothesis (GRH).* In a forthcoming sequel about Dirichlet  $L$ -functions [F15], by the first-named author, it similarly emerges that the GRH is essentially equivalent to an expected asymptotic functional equation of the corresponding graph  $L$ -function. More precisely, spectral  $L$ -functions for graphs (different from those considered in [H92] and [STe00]) are introduced, and in the case of  $\mathbb{Z}/n\mathbb{Z}$ , the  $L$ -functions completed with suitable fudge factors, and denoted  $\Lambda_n(s, \chi)$ , satisfy

$$\lim_{n \rightarrow \infty} \left| \frac{\Lambda_n(s, \chi)}{\Lambda_n(1 - s, \bar{\chi})} \right| = 1,$$

for  $0 < \operatorname{Re}(s) < 1$  and  $\operatorname{Im}(s) \geq 8$ , if and only if the GRH holds (for  $s$  in the same range) for Dirichlet's  $L$ -function  $L(s, \chi)$ .

*Zeta functions of graphs.* As recalled in the beginning, the more standard zeta function of a graph is the one going back to a paper by Ihara. Ihara zeta functions for infinite graphs appear in a few places, three recent papers are [D14, CJK15, LPS14], which contain further generalizations and where references to papers by Grigorchuk-Zuk and Guido, Isola, and

Lapidus on this topic can be found. A two variable extension of the Ihara zeta function was introduced by Bartholdi [B99] developed out of a formula in [G78]. Zeta functions more closely related to the ones considered in the present paper, are the spectral zeta functions of fractals in works by Teplyaev, Lapidus and van Frankenhuijsen.

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**1. Spectral zeta functions.** At least since Carleman [Ca34] one forms a spectral zeta function

$$\sum_j \frac{1}{\lambda_j^s}$$

over the set of non-zero Laplace eigenvalues, convergent for  $s$  in some right half-plane. For a finite graph the elementary symmetric functions in the eigenvalues admit a combinatorial interpretation starting with Kirchhoff, see *e.g.* [CL96] for a more recent discussion. For infinite graphs or manifolds one does at least not a priori have such symmetric functions (since the spectrum may be continuous or the eigenvalues are infinite in number). This is one reason for defining spectral zeta functions, since these are symmetric, and via transforms one can get the analytic continued interpretations of the elementary symmetric functions, such as the (restricted) determinant. As has been recognized at least for the determinant, the combinatorial interpretation persists in a certain sense, see [Ly10].

As often is the case, since Riemann, in order to define its meromorphic continuation one writes the zeta function as the Mellin transform of the associated theta series, or trace of the heat kernel. For this reason and in view of that some spaces have no eigenvalues but continuous spectrum, a case important to us in this paper, we suggest (as advocated by Jorgenson-Lang, see for example [JL12]) to start from the heat kernel to define spectral zeta functions. Recall that the Mellin transform of a function  $f(t)$  is

$$\mathbf{M}f(s) = \int_0^\infty f(t)t^s \frac{dt}{t}.$$

For example when  $f(t) = e^{-t}$ , the transform is  $\Gamma(s)$ .

More precisely, for a finite or compact space  $X$  we can sum over  $x_0$  of the unique bounded fundamental solution  $K_X(t, x_0, x_0)$  of the heat equation (see for example [JL12, CJK10] for more background on this), which gives the heat trace  $Tr(K_X)$ , typically on the form  $\sum e^{-\lambda t}$ , and define

$$\zeta_X(s) = \frac{1}{\Gamma(s)} \int_0^\infty (Tr(K_X) - 1)t^s \frac{dt}{t}.$$

When the spectrum is discrete this formula gives back Carleman's definition above. For a non-compact space with a heat kernel independent of the point  $x_0$ , for example a Cayley graph of an infinite, finitely generated group, it makes sense to take Mellin transform



of  $K_X(t, x_0, x_0)$  without the trace. Moreover since zero is no longer an eigenvalue for the Laplacian acting on  $L^2(X)$  we should no longer subtract 1, so the definition in this case is

$$\zeta_X(s) = \frac{1}{\Gamma(s)} \int_0^\infty K_X(t, x_0, x_0) t^s \frac{dt}{t}.$$

Let us also note that in the graph setting as shown in [CJK15], it holds that if we start with the heat kernel one may via instead a Laplace transform obtain the Ihara zeta function and the fundamental determinant formula.

An alternative, equivalent, definition is given by the spectral measure  $d\mu = d\mu_{x_0, x_0}$ , see [MW89],

$$\zeta_X(s) = \int \lambda^{-s} d\mu(\lambda).$$

Here and in the next two sections we provide some examples:

EXAMPLE 1.1. For a finite torus graph defined as in the introduction we have by calculating the eigenvalues (see for example [CJK10])

$$\zeta_{\mathbb{Z}^d / A\mathbb{Z}^d}(s) = \frac{1}{2^{2s}} \sum_k \frac{1}{(\sin^2(\pi k_1/a_1) + \dots + \sin^2(\pi k_d/a_d))^s},$$

where the sum runs over all  $0 \leq k_i \leq a_i - 1$  except for all  $k_i$ s being zero.

EXAMPLE 1.2. For real tori we have again by calculating the eigenvalues (see [CJK12]) as is well known

$$\zeta_{\mathbb{R}^d / A\mathbb{Z}^d}(s) = \frac{1}{(2\pi)^{2s}} \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{\|A^*k\|^{2s}},$$

where  $A^* = (A^{-1})^t$ .

In the following sections we will discuss the zeta function of some infinite graphs, namely the standard lattice graphs  $\mathbb{Z}^d$ . Before that let us mention yet another example, that we again do not think one finds in the literature.

EXAMPLE 1.3. The  $(q + 1)$ -regular tree  $T_{q+1}$  with  $q \geq 2$  is a fundamental infinite graph ( $q = 1$  corresponds to  $\mathbb{Z}$  treated in the next section). Also here the spectral measure is well-known, our reference is [MW89]. Thus

$$\begin{aligned} \zeta_{T_{q+1}}(s) &= \int_{-2\sqrt{q}}^{2\sqrt{q}} \frac{1}{(q+1-\lambda)^s} \frac{(q+1)}{2\pi} \frac{\sqrt{4q-\lambda^2}}{((q+1)^2-\lambda^2)} d\lambda \\ &= \frac{q+1}{2\pi} \int_{-2\sqrt{q}}^{2\sqrt{q}} \frac{1}{(q+1-\lambda)^{s+1}} \frac{\sqrt{4q-\lambda^2}}{(q+1+\lambda)} d\lambda. \end{aligned}$$

We change variable  $u = 2\sqrt{q} - \lambda$ . So

$$\begin{aligned} \zeta_{T_{q+1}}(s) &= \frac{q+1}{2\pi} \int_0^{4\sqrt{q}} \frac{1}{(q+1-2\sqrt{q}+u)^{s+1}} \frac{\sqrt{4\sqrt{q}u-u^2}}{(q+1+2\sqrt{q}-u)} du \\ &= \frac{q+1}{2\pi} \int_0^{4\sqrt{q}} \frac{u^{1/2}}{(q+1-2\sqrt{q}+u)^{s+1}} \frac{\sqrt{4\sqrt{q}-u}}{(q+1+2\sqrt{q}-u)} du. \end{aligned}$$

We change again:  $u = 4\sqrt{q}t$ , so

$$\begin{aligned} \zeta_{T_{q+1}}(s) &= \frac{q+1}{2\pi} \int_0^1 \frac{(4\sqrt{q})^{1/2} t^{1/2}}{(q+1-2\sqrt{q}+4\sqrt{q}t)^{s+1}} \frac{\sqrt{4\sqrt{q}-4\sqrt{q}t}}{(q+1+2\sqrt{q}-4\sqrt{q}t)} 4\sqrt{q} dt \\ &= \frac{d}{2\pi} \frac{16q}{(q+1-2\sqrt{q})^{s+1}(q+1+2\sqrt{q})} \int_0^1 \frac{t^{1/2}\sqrt{1-t}}{(1-ut)^{s+1}(1-vt)} dt, \end{aligned}$$

where  $u = -4\sqrt{q}/(q+1-2\sqrt{q})$  and  $v = 4\sqrt{q}/(q+1+2\sqrt{q})$ . This is an Euler-type integral that Picard considered in [Pi1881] and which lead him to Appell’s hypergeometric function  $F_1$ ,

$$\zeta_{T_{q+1}}(s) = \frac{q+1}{2\pi} \frac{16q}{(q+1-2\sqrt{q})^{s+1}(q+1+2\sqrt{q})} \frac{\Gamma(3/2)\Gamma(3/2)}{\Gamma(3)} F_1(3/2, s+1, 1, 3; u, v).$$

Simplifying this somewhat we have proved:

**THEOREM 1.4.** *For  $q > 1$ , the spectral zeta function of the  $(q+1)$ -regular tree is*

$$\zeta_{T_{q+1}}(s) = \frac{q(q+1)}{(q-1)^2(\sqrt{q}-1)^{2s}} F_1(3/2, s+1, 1, 3; u, v),$$

with  $u = -4\sqrt{q}/(\sqrt{q}-1)^2$  and  $v = 4\sqrt{q}/(\sqrt{q}+1)^2$ , and where  $F_1$  is one of Appell’s hypergeometric functions.

The topic of functional relations between hypergeometric functions is a very classical one. In spite of the many known formulas, we were not able to derive a functional equation for  $\zeta_{T_{q+1}}$  with  $s \longleftrightarrow 1-s$ .

**2. The spectral zeta function of the graph  $\mathbb{Z}$ .** The heat kernel of  $\mathbb{Z}$  is  $e^{-2t} I_x(2t)$  where  $I_x$  is a Bessel function (see [CJK10] and its references). Therefore

$$\zeta_{\mathbb{Z}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-2t} I_0(2t) t^s \frac{dt}{t},$$

which converges for  $0 < Re(s) < 1/2$ . It is not so clear why this function should have a meromorphic continuation and functional equation very similar to Riemann’s zeta.

PROPOSITION 2.1. For  $0 < \operatorname{Re}(s) < 1/2$  it holds that

$$\zeta_{\mathbb{Z}}(s) = \frac{1}{4^s \sqrt{\pi}} \frac{\Gamma(1/2 - s)}{\Gamma(1 - s)} = \frac{1}{4^s \pi} B(1/2, 1/2 - s),$$

where  $B$  denotes Euler's beta function. This formula provides the meromorphic continuation of  $\zeta_{\mathbb{Z}}(s)$ .

PROOF. By formula 11.4.13 in [AS64], we have

$$\mathbf{M}(e^{-t} I_x(t))(s) = \frac{\Gamma(s + x) \Gamma(1/2 - s)}{2^s \pi^{1/2} \Gamma(1 + x - s)},$$

valid for  $\operatorname{Re}(s) < 1/2$  and  $\operatorname{Re}(s + x) > 0$ . This implies the first formula. Finally, using that  $\Gamma(1/2) = \sqrt{\pi}$  and the definition of the beta function the proposition is established.  $\square$

We proceed to determine a functional equation for this zeta function. Recall that

$$\Gamma(z) \Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}.$$

Therefore

$$\begin{aligned} 2^s \sqrt{\pi} \zeta_{\mathbb{Z}}(s/2) &= \frac{\Gamma(1 - (1/2 + s/2))}{\Gamma(1 - s/2)} = \frac{\sin(\pi s/2) \Gamma(s/2)}{\pi} \frac{\pi}{\sin(\pi(s + 1)/2) \Gamma(1/2 + s/2)} \\ &= \tan(\pi s/2) \frac{\Gamma(1/2 - (1 - s)/2)}{\Gamma(1 - (1 - s)/2)} = 2^{1-s} \sqrt{\pi} \tan(\pi s/2) \zeta_{\mathbb{Z}}((1 - s)/2). \end{aligned}$$

Hence in analogy with Riemann's case we have

$$\zeta_{\mathbb{Z}}(s/2) = 2^{1-2s} \tan(\pi s/2) \zeta_{\mathbb{Z}}((1 - s)/2).$$

(The passage from  $s$  to  $s/2$  is also the same.) If we define the completed zeta to be

$$\xi_{\mathbb{Z}}(s) = 2^s \cos(\pi s/2) \zeta_{\mathbb{Z}}(s/2),$$

then one verifies that the above functional equation can be written in the familiar more symmetric form

$$\xi_{\mathbb{Z}}(s) = \xi_{\mathbb{Z}}(1 - s)$$

for all  $s \in \mathbb{C}$ . Moreover, note that this is an entire function since the simple poles coming from  $\Gamma$  are cancelled by the cosine zeros and it takes real values on the critical line. We call  $\xi_{\mathbb{Z}}$  the *entire completion* of  $\zeta_{\mathbb{Z}}$ .

Let us determine some special values. In view of that for integers  $n \geq 0$ ,

$$\Gamma(1/2 + n) = \frac{(2n)!}{4^n n!} \sqrt{\pi}$$

and  $\Gamma(1 + n) = n!$ , we have for  $s = -n$ ,

$$\zeta_{\mathbb{Z}}(-n) = \frac{1}{4^{-n} \sqrt{\pi}} \frac{\Gamma(1/2 + n)}{\Gamma(1 + n)} = \frac{(2n)!}{n! n!} = \binom{2n}{n}.$$

This number equals the number of paths of length  $2n$  from the origin to itself in  $\mathbb{Z}$ .

Furthermore, in a similar way for  $n \geq 1$ ,

$$\zeta_{\mathbb{Z}}(-n + 1/2) = \frac{1}{4^{-n}\sqrt{\pi}} \frac{\Gamma(n)}{\Gamma(1/2 + n)} = \frac{4^{2n} n!n!}{2\pi n (2n)!} = \frac{4^{2n}}{2\pi n \binom{2n}{n}}.$$

It is well-known that the gamma function is a meromorphic function in the whole complex plane with simple poles at the negative integers and no zeros. Note that if we pass from  $s$  to  $s/2$  we have that  $\zeta_{\mathbb{Z}}(s/2)$  has simple poles at the positive odd integers, and the special values determined above appear at the even negative numbers.

We may thus summarize:

**THEOREM 2.2.** *The spectral zeta function  $\zeta_{\mathbb{Z}}(s)$  can be extended to a meromorphic function on  $\mathbb{C}$  satisfying*

$$\zeta_{\mathbb{Z}}(s) = \frac{1}{4^s\sqrt{\pi}} \frac{\Gamma(1/2 - s)}{\Gamma(1 - s)}.$$

*It has zeros for  $s = n, n = 1, 2, 3, \dots$ , and simple poles for  $s = 1/2 + n, n = 0, 1, 2, \dots$ . Moreover, its completion  $\xi_{\mathbb{Z}}$ , which is entire, admits the functional equation*

$$\xi_{\mathbb{Z}}(s) = \xi_{\mathbb{Z}}(1 - s).$$

*Finally we have the special values*

$$\zeta_{\mathbb{Z}}(-n) = \binom{2n}{n} \text{ and } \zeta_{\mathbb{Z}}(-n + 1/2) = \frac{4^{2n}}{2\pi n \binom{2n}{n}},$$

*where  $n \geq 0$  is an integer.*

**3. The spectral zeta function of the lattice graphs  $\mathbb{Z}^d$ .** The heat kernel on  $\mathbb{Z}^d$  is the product of heat kernels on  $\mathbb{Z}$  and this gives that

$$\zeta_{\mathbb{Z}^d}(s) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-2dt} I_0(2t)^d t^s \frac{dt}{t},$$

which converges for  $0 < \text{Re}(s) < d/2$ . For  $d = 2$  taking instead the equivalent definition with the spectral measure, the spectral zeta function is a variant of the Selberg integral with two variables.

The integrals like

$$\int_0^\infty e^{-zt} I_0(2t)^d t^s \frac{dt}{t},$$

and more general ones, have been studied by Saxena in [Sa66], see also the discussion in [SK85, sect. 9.4]. For  $\text{Re}(z) > 2d$  and  $\text{Re}(s) > 0$  one has

$$\int_0^\infty e^{-zt} I_0(2t)^d t^s \frac{dt}{t} = \frac{2^{s-1}}{\sqrt{\pi}} z^{-s+1/2} \Gamma\left(\frac{s+1}{2}\right) F_C^{(d)}(s/2, (s+1)/2; 1, 1, \dots, 1; 4/z^2, 4/z^2, \dots, 4/z^2),$$

where  $F_C^{(d)}$  is one of the Lauricella hypergeometric functions in  $d$  variables [Ex76]. The condition  $Re(z) > 2d$  can presumably be relaxed by the principle of analytic continuation giving up the multiple series definition of  $F_C^{(d)}$ . This point is discussed in [SE79]. Formally we would then have that

$$\zeta_{\mathbb{Z}^d}(s) = \frac{d^{-s+1/2} \Gamma((s+1)/2)}{\sqrt{2\pi} \Gamma(s)} F_C^{(d)}(s/2, (s+1)/2; 1, 1, \dots, 1; 1/d^2, 1/d^2, \dots, 1/d^2),$$

which is rather suggestive as far as functional relations go. It is however not clear at present time that for  $d > 1$  there is a relation as nice as the functional equation in the case  $d = 1$ . Related to this, it is remarked in [Ex76, p. 49] that no integral representation of Euler type has been found for  $F_C$ . We note that if one instead of the heat kernel start with the spectral measure in defining  $\zeta_{\mathbb{Z}^d}(s)$ , we do get such an integral representation, at least for special parameters. This aspect is left for future investigation.

We will now provide an independent and direct meromorphic continuation of these functions. To do this, we take advantage of the heat kernel definition of the zeta function. Fix a dimension  $d \geq 1$ . Recall that on the one hand there are explicit positive non-zero coefficients  $a_n$  such that

$$e^{-2dt} I_0(2t)^d = \sum_{n \geq 0} a_n t^n$$

which converges for every positive  $t$ , and on the other hand we similarly have an expansion at infinity,

$$e^{-2dt} I_0(2t)^d = \sum_{n=0}^{N-1} b_n t^{-n-d/2} + O(t^{-N-d/2})$$

as  $t \rightarrow \infty$  for any integer  $N > 0$ .

Therefore we write

$$\begin{aligned} \int_0^\infty e^{-2dt} I_0(2t)^d t^{s-1} dt &= \int_0^1 \sum_{n=0}^{N-1} a_n t^n t^{s-1} dt + \int_0^1 \sum_{n \geq N} a_n t^n t^{s-1} dt \\ &+ \int_1^\infty \left( e^{-2dt} I_0(2t)^d - \sum_{n=0}^{N-1} b_n t^{-n-d/2} \right) t^{s-1} dt + \int_1^\infty \sum_{n=0}^{N-1} b_n t^{-n-d/2} t^{s-1} dt \\ &= \sum_{n=0}^{N-1} \frac{a_n}{s+n} - \sum_{n=0}^{N-1} \frac{b_n}{s-(n+d/2)} + \int_0^1 O(t^N) t^{s-1} dt + \int_1^\infty O(t^{-N-d/2}) t^{s-1} dt. \end{aligned}$$

This last expression defines a meromorphic function in the region  $-N < Re(s) < N + d/2$ , with simple poles at  $s = -n$  and  $s = n + d/2$ .

The spectral zeta function  $\zeta_{\mathbb{Z}^d}(s)$  is the above integral divided by  $\Gamma(s)$ . In view of that the entire function  $1/\Gamma(s)$  has zeros at the non-positive integers, this will cancel the simple poles at  $s = -n$ . Since we can take  $N$  as large as we want we obtain in this way the meromorphic

continuation of  $\zeta_{\mathbb{Z}^d}(s)$ . Moreover, thanks to that the coefficients  $b_n$  are non-zero we have established:

**PROPOSITION 3.1.** *The function  $\zeta_{\mathbb{Z}^d}(s)$  admits a meromorphic continuation to the whole complex plane with simple poles at the points  $s = n + d/2$  with  $n \geq 0$ .*

It is natural to wonder whether this function also for  $d > 1$  can be completed like in the case  $d = 1$  giving an entire function with functional relation  $\xi_{\mathbb{Z}^d}(1 - s) = \xi_{\mathbb{Z}^d}(s)$ . Indeed, more generally we find the question interesting for which graph, finite or infinite, the zeta functions have a functional relation in some way analogous to the classical type of functional equations.

Finally we point out a non-trivial special value that we derive in a later section:

$$\zeta_{\mathbb{Z}^d}(0) = 1 .$$

**4. Asymptotics of the zeta functions of torus graphs.** We consider a sequence of torus graphs  $\mathbb{Z}^d / A_n \mathbb{Z}^d$  indexed by  $n$  and where the matrices  $A_n$  are diagonal with entries  $a_i n$ , with integers  $a_i > 0$ . (A more general setting could be considered (cf. [CJK12]) but it will not be important to us in the present context.) We denote by  $\zeta_n$  the corresponding zeta function defined as in the previous section. We let the matrix  $A$  be the diagonal matrix with entries  $a_i$ . In this section we take advantage of the theory developed in [CJK10] without recalling the proofs which would take numerous pages.

Following [CJK10] we have

$$\theta_n(t) := \sum_m e^{-\lambda_m t} = \det(A_n) \sum_{k \in \mathbb{Z}^d} \prod_{1 \leq j \leq d} e^{-2t I_{a_j n k_j}(2t)},$$

where  $\lambda_m$  denotes the Laplace eigenvalues. From the left hand side it is clear that this function is entire. Let

$$\theta_A(t) = \sum_{\lambda} e^{-\lambda t},$$

where the sum is over the eigenvalues of the torus  $\mathbb{R}^d / A \mathbb{Z}^d$ . The meromorphic continuation of the corresponding spectral zeta function is, as is well-known (see e.g. [CJK10]),

$$\begin{aligned} \zeta_{\mathbb{R}^d / A \mathbb{Z}^d}(s) &= \frac{1}{\Gamma(s)} \int_1^\infty (\theta_A(t) - 1) t^s \frac{dt}{t} + \frac{1}{\Gamma(s)} \int_0^1 (\theta_A(t) - \det A (4\pi t)^{-d/2}) t^s \frac{dt}{t} \\ &+ \frac{(4\pi)^{-d/2} \det A}{\Gamma(s)(s - d/2)} - \frac{1}{s\Gamma(s)}. \end{aligned}$$

Recall the asymptotics for the I-Bessel functions:

$$I_n(x) = \frac{e^x}{\sqrt{2\pi x}} \left( 1 - \frac{4n^2 - 1}{8x} + O(x^{-2}) \right)$$

as  $x \rightarrow \infty$ .

For  $0 < \operatorname{Re}(s) < d/2$  we may write

$$\Gamma(s)\zeta_n(s) = \int_0^\infty (\theta_n(t) - 1)t^s \frac{dt}{t} = n^{2s} \int_0^\infty (\theta_n(n^2t) - 1)t^s \frac{dt}{t}.$$

We decompose the integral on the right and let  $n \rightarrow \infty$ , the first piece being

$$S_1(n) := \int_1^\infty (\theta_n(n^2t) - 1)t^s \frac{dt}{t} \rightarrow \int_1^\infty (\theta_A(t) - 1)t^s \frac{dt}{t}$$

for every  $s \in \mathbb{C}$  as  $n \rightarrow \infty$ . The convergence is proved in [CJK10]. The second piece is for  $\operatorname{Re}(s) > -n$ ,

$$S_2(n) := \int_0^1 (\theta_n(n^2t) - \det A_n e^{-2dn^2t} I_0(2n^2t)^d)t^s \frac{dt}{t} \rightarrow \int_0^1 (\theta_A(t) - \det A (4\pi t)^{-d/2})t^s \frac{dt}{t},$$

as  $n \rightarrow \infty$  which is proved in [CJK10].

What remains is now the third piece

$$S_3(n) := \int_0^1 (\det A_n e^{-2dn^2t} I_0(2n^2t)^d - 1)t^s \frac{dt}{t} = n^{-2s} \int_0^{n^2} (\det A_n e^{-2dt} I_0(2t)^d - 1)t^s \frac{dt}{t}.$$

This we write as follows

$$S_3(n) = \left( \det A_n \int_0^\infty e^{-2dt} I_0(2t)^d t^s \frac{dt}{t} - \det A_n \int_{n^2}^\infty e^{-2dt} I_0(2t)^d t^s \frac{dt}{t} - \int_0^{n^2} t^s \frac{dt}{t} \right) n^{-2s}.$$

The first integral is the spectral zeta of  $\mathbb{Z}^d$  times  $\Gamma(s)$  and the last integral is

$$\int_0^{n^2} t^s \frac{dt}{t} = \frac{n^{2s}}{s}.$$

We continue with the middle integral here:

$$\int_{n^2}^\infty e^{-2dt} I_0(2t)^d t^s \frac{dt}{t} = \int_{n^2}^\infty (e^{-2dt} I_0(2t)^d - (4\pi t)^{-d/2})t^s \frac{dt}{t} + \int_{n^2}^\infty (4\pi t)^{-d/2} t^s \frac{dt}{t},$$

hence

$$\int_{n^2}^\infty e^{-2dt} I_0(2t)^d t^s \frac{dt}{t} = \int_{n^2}^\infty (e^{-2dt} I_0(2t)^d - (4\pi t)^{-d/2})t^s \frac{dt}{t} - (4\pi)^{-d/2} \frac{n^{2s-d}}{s-d/2}.$$

We denote

$$S_{rest}(n) = \int_{n^2}^\infty (e^{-2dt} I_0(2t)^d - (4\pi t)^{-d/2})t^s \frac{dt}{t},$$

which is a convergent integral for  $\operatorname{Re}(s) < d/2 + 1$  in view of the asymptotics for  $I_0(t)$ . Notice also that for fixed  $s$  with  $\operatorname{Re}(s) < d/2 + 1$  the integral is of order  $n^{2s-2-d}$  as  $n \rightarrow \infty$ .

Taken all together we have

$$\begin{aligned} n^{-2s}\zeta_n(s) &= \frac{1}{\Gamma(s)}S_1(n) + \frac{1}{\Gamma(s)}S_2(n) - \frac{1}{s\Gamma(s)} + (4\pi)^{-d/2} \frac{\det A}{\Gamma(s)(s-d/2)} \\ &+ n^{d-2s} \det A \zeta_{\mathbb{Z}^d}(s) - n^{d-2s} \frac{\det A}{\Gamma(s)} S_{rest}(n). \end{aligned}$$

This is valid for all  $s$  in the intersection of where  $\zeta_{\mathbb{Z}^d}(s)$  is defined,  $-n < \operatorname{Re}(s) < d/2 + 1$ , and  $s \neq d/2$ . As remarked above coming from [CJK10] as  $n \rightarrow \infty$  the first four terms combines to give  $\zeta_{\mathbb{R}^d/A\mathbb{Z}^d}(s)$ . This means that we have in particular proved Theorem 0.1.

**5. The one dimensional case.** We now specialize to  $d = 1$  and  $A_n = n$ . In this case recall that

$$\zeta_n(s) = \zeta_{\mathbb{Z}/n\mathbb{Z}}(s) = \frac{1}{4^s} \sum_{k=1}^{n-1} \frac{1}{\sin(\pi k/n)^{2s}}$$

and

$$\zeta_{\mathbb{R}/\mathbb{Z}}(s) = 2(2\pi)^{-2s} \zeta(2s),$$

where  $\zeta$  is the Riemann zeta function. Moreover,

$$\zeta_{\mathbb{Z}}(s) = \frac{1}{4^s \sqrt{\pi}} \frac{\Gamma(1/2 - s)}{\Gamma(1 - s)}.$$

In view of the previous section the first part of Theorem 0.3 is established. Let us remark that this can also be viewed as a special case of Gauss-Chebyshev quadrature but with a more precise error term.

With more work one can also find the next term in the asymptotic expansion in the critical strip. This can be achieved with some more detailed analysis, in particular of Proposition 4.7 in [CJK10] and an application of Poisson summation. For the purpose of the present discussion we only need to look at the more precise asymptotics in the critical strip and here for  $d = 1$  there is an alternative approach available by using a non-standard version of the Euler-Maclaurin formula established in [Si04]. The asymptotics is:

$$\sum_{k=1}^{n-1} \frac{1}{\sin(\pi k/n)^s} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(1/2 - s/2)}{\Gamma(1 - s/2)} n + 2\pi^{-s} \zeta(s) n^s + \frac{s}{3} \pi^{2-s} \zeta(s - 2) n^{s-2} + o(n^{s-2})$$

where  $0 < \operatorname{Re}(s) < 1$  as  $n \rightarrow \infty$ . This is the second statement in Theorem 0.3.

EXAMPLE 5.1. Although we did not verify this asymptotics outside of the critical strip, it may nevertheless be convincing to specialize to  $s = 2$ , we then would have

$$\frac{1}{3}n^2 - \frac{1}{3} = \frac{1}{\sqrt{\pi}} 0 \cdot n + 2\pi^{-2} \zeta(2) n^2 + \frac{2}{3} \zeta(0) + o(1),$$

which confirms the values  $\zeta(0) = -1/2$  and  $\zeta(2) = \pi^2/6$ . As remarked in the introduction, from [CJK10], the value of  $\zeta(2)$  can also be derived via

$$\frac{2}{\pi^2} \zeta(2) = \lim_{n \rightarrow \infty} \frac{1}{n^2} \left( \frac{1}{3}n^2 - \frac{1}{3} \right).$$



**6. Special values.**

**6.1. The case of  $s = 0$ .** Setting  $s = 0$  in Theorem 0.1 we clearly have

$$\det A n^d - 1 = \zeta_{\mathbb{Z}^d}(0) \det A n^d + \zeta_{\mathbb{R}^d / A\mathbb{Z}^d}(0) + o(1),$$

which implies that  $\zeta_{\mathbb{Z}^d}(0) = 1$ , and that  $\zeta_{\mathbb{R}^d / A\mathbb{Z}^d}(0) = -1$ , which is a known special value of Epstein zeta functions.

**6.2. The case of  $s$  being negative integers.** Let us now recall some known results about the sums:

$$\sum_{k=1}^{n-1} \frac{1}{\sin(\pi k/n)^s}$$

for special  $s$ . We begin with a simple calculation (see for example [BH10, Lemma 3.5]) namely that for integers  $0 < m < n$

$$\sum_{k=1}^{n-1} \sin^{2m}(\pi k/n) = \frac{n}{4^m} \binom{2m}{m}.$$

In view of the asymptotics in Theorem 0.3 this immediately imply that  $\zeta(-2m) = 0$ , the so-called trivial zeros of Riemann’s zeta function. It also verifies with the special values of  $\zeta_{\mathbb{Z}}$  stated in Section 2. There is a probabilistic interpretation for this: when the number of steps  $m$  is smaller than  $n$ , the random walker cannot tell the difference between the graphs  $\mathbb{Z}$  and  $\mathbb{Z}/n\mathbb{Z}$ .

Conversely, for  $s$  being an odd negative integer our asymptotic formula gives information about the sine sum which is somewhat more complicated in this case, as the fact that  $\zeta$  does not vanish implies. For low exponent  $m$  one can find formulas in [GR07], the simplest one being

$$\sum_{k=1}^{n-1} \sin(k\pi/n) = \cot(\pi/2n).$$

**6.3. The case of  $s$  being even positive integers.** In view of the elementary equality

$$\frac{1}{\sin^2 x} = 1 + \cot^2 x,$$

one sees that for positive integers  $a$ ,

$$\sum_{k=1}^{n-1} \frac{1}{\sin^{2a}(\pi k/n)}$$

can be expressed in terms of higher Dedekind sums considered by Zagier [Z73]. There is also a literature more specialized on this type of finite sums which can be evaluated with a closed form expression already mentioned in the introduction (see [CM99, BY02]):

$$\sum_{k=1}^{n-1} \frac{1}{\sin^{2a}(\pi k/n)} = -\frac{1}{2} \sum_{m=0}^{2a} \frac{(-4)^a}{n^m} \binom{2a+1}{m+1}$$

$$\times \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} \frac{m+1-2k}{m+1} \binom{a+kn+(m-1)/2}{2a+m}$$

These sums apparently arose in physics in Dowker’s work and in mathematical work of Verlinde (see [CS12]). The first order asymptotics is known to be

$$\sum_{k=1}^{n-1} \sin^{-2m}(\pi k/n) \sim (-1)^{m+1} (2n)^{2m} \frac{B_{2m}}{(2m)!},$$

where  $m$  is a positive integer, see for example [BY02, CS12] and their references. As explained in the introduction these evaluations together with the asymptotics formulated in the introduction re-proves Euler’s celebrated calculations of  $\zeta(2m)$ .

At  $s = 1$ , the point where our asymptotic expansion does not apply because of the pole of  $\zeta$ , one has (see [He77, p. 460] attributed to J. Waldvogel)

$$\zeta_{\mathbb{Z}/n\mathbb{Z}}(1) = \frac{2n}{\pi} (\log(2n/\pi) - \gamma) + O(1),$$

where as usual  $\gamma$  is Euler’s constant.

**6.4. Further special values.** Recall the values  $\Gamma(1/2) = \sqrt{\pi}$ ,  $\Gamma'(1) = -\gamma$  and  $\Gamma'(1/2) = -\gamma\sqrt{\pi} - \log 4$ , or in the logarithmic derivative, the psi-function,  $\psi(1) = -\gamma$  and  $\psi(1/2) = -\gamma - 2 \log 2$ . We differentiate  $\zeta_{\mathbb{Z}}(s)$  which gives

$$\zeta'_{\mathbb{Z}}(s) = \zeta_{\mathbb{Z}}(s) (-2 \log 2 - \psi(1/2 - s) + \psi(1 - s)).$$

Setting  $s = 0$  and inserting the special values mentioned we see that

$$\zeta'_{\mathbb{Z}}(0) = 0.$$

This value has the interpretation of being the tree entropy of  $\mathbb{Z}$ , which is the exponential growth rate of spanning trees of subgraphs converging to  $\mathbb{Z}$ , see e.g. [DD88, Ly10, CJK10], studied via the Fuglede-Kadison determinant of the Laplacian. This has also a role in the theory of operator algebras, but in any case it is not evaluated in this way in the literature. Of course one could in our way compute other special values of  $\zeta'_{\mathbb{Z}}$ . For example, at positive integers and half-integers this function has zeros and poles, respectively, and at negative integers we have for integers  $n > 0$  the following:

PROPOSITION 6.1. *It holds that*

$$\zeta'_{\mathbb{Z}}(-n) = \binom{2n}{n} \left( 1 + \frac{1}{2} + \dots + \frac{1}{n} - 2 \left( 1 + \frac{1}{3} + \dots + \frac{1}{2n-1} \right) \right)$$

and

$$\zeta'_{\mathbb{Z}}(-n+1/2) = \frac{4^{2n}}{2\pi n \binom{2n}{n}} \left( -4 \log 4 - 1 - \frac{1}{2} - \frac{1}{3} - \dots - \frac{1}{n-1} + 2 \left( 1 + \frac{1}{3} + \dots + \frac{1}{2n-1} \right) \right).$$

We remark that this section concerned mostly  $d = 1$ , we have not investigated the case of higher dimensions.

**6.5. No real zero in the critical strip.** The non-vanishing of number theoretic zeta functions on the real line in the critical strip is of importance, see e.g. [SC67]. We outline one possible strategy for this problem in general from our asymptotics. We treat here only Riemann’s zeta function for illustration, in this case there are however other more elementary arguments available.

Already the beginning of this section indicates that certain Epstein zeta functions have a tendency to be negative on the real line in the critical strip. It is as if the number of terms in the finite graph zeta is not enough to account for the limit graph zeta function, leaving the relevant Epstein zeta function negative.

The function  $\sin(\pi x)^{-s}$  for  $0 < s < 1$  is positive, convex and symmetric around  $x = 1/2$ . The graph zeta function in question,  $\zeta_{\mathbb{Z}}(s)$  is via a change of variables

$$\int_0^1 \sin^{-s}(\pi x) dx .$$

If we compare this with the sum, using the symmetry, we have for odd  $n$

$$2\left(\frac{1}{n} \sum_{k=1}^{(n-1)/2} \sin^{-s}(\pi k/n) - \int_0^{1/2} \sin^{-s}(\pi x) dx\right) = \frac{2\zeta(s)}{\pi^s} n^{s-1} + o(n^{s-1}) .$$

If we interpret the sum as the Riemann sum of the integral (with not enough terms) the integral can be thought of as always lying above the rectangles. Ignoring all but one rectangle then gives

$$\begin{aligned} & \frac{1}{n} \sum_{k=1}^{(n-1)/2} \sin^{-s}(\pi k/n) - \int_0^{1/2} \sin^{-s}(\pi x) dx \\ & < \frac{1}{n} \sin^{-s}(\pi/n) - \int_0^{1/n} \sin^{-s}(\pi x) dx = \frac{1}{n} \frac{n^s}{\pi^2} - \frac{1}{\pi^s} \int_0^{1/n} x^{-s} dx + o(n^{s-1}) \\ & = \frac{n^{s-1}}{\pi^s} \left(1 - \frac{1}{1-s}\right) + o(n^{s-1}) . \end{aligned}$$

This shows by letting  $n$  go to infinity that

$$\zeta(s) \leq -\frac{s}{1-s} < 0 ,$$

which is consistent with numerics, for example,  $\zeta(1/2) = -1.460 \dots < -1$ .

As with several other aspects of this paper, we leave higher dimensions to future study.

**7. Approximative functional equations.** It is natural to wonder about to what extent  $\zeta_{\mathbb{Z}/n\mathbb{Z}}$  has a functional equation. In view of our asymptotics and the, in this context crucial, relation  $\xi_{\mathbb{Z}}(s) = \xi_{\mathbb{Z}}(1-s)$ , one could expect at least an asymptotic version. Indeed, we start by completing the finite torus zeta functions as  $\xi_{\mathbb{Z}/n\mathbb{Z}}(s) := 2^s \cos(\pi s/2) \zeta_{\mathbb{Z}/n\mathbb{Z}}(s/2)$ , and multiply the asymptotics at  $s$  in the critical strip with the corresponding fudge factors, and do the similar thing for the corresponding formula at  $1-s$ . After that, we subtract the two

expressions, the one at  $s$  with the one at  $1 - s$ , and obtain after further calculations, notably using  $\xi_{\mathbb{Z}}(s) = \xi_{\mathbb{Z}}(1 - s)$ :

$$\begin{aligned} \xi_{\mathbb{Z}/n\mathbb{Z}}(s) - \xi_{\mathbb{Z}/n\mathbb{Z}}(1 - s) &= X(s)n^s - X(1 - s)n^{1-s} \\ &= -\frac{s}{6}X(s - 2)n^{s-2} + \frac{1-s}{6}X((1 - s) - 2)n^{(1-s)-2} + o(n^a), \end{aligned}$$

where  $a = \max \{Re(s) - 2, -1 - Re(s)\}$  and  $X(s) = 2\pi^{-s} \cos(\pi s/2)\zeta(s)$ . Thus:

**COROLLARY 7.1.** *The Riemann zeta function has a zero at  $s$  in the critical strip iff*

$$\lim_{n \rightarrow \infty} (\xi_{\mathbb{Z}/n\mathbb{Z}}(1 - s) - \xi_{\mathbb{Z}/n\mathbb{Z}}(s)) = 0$$

as  $n \rightarrow \infty$ , unless  $s = 1/2$ . In any case, for all  $s$  in the critical strip

$$\lim_{n \rightarrow \infty} \frac{1}{n} (\xi_{\mathbb{Z}/n\mathbb{Z}}(1 - s) - \xi_{\mathbb{Z}/n\mathbb{Z}}(s)) = 0.$$

As is well known there is a very useful approximative functional equation for  $\zeta(s)$ , sometimes called the Riemann-Siegel formula, which states that

$$\zeta(s) = \sum_{k=1}^n \frac{1}{k^s} + \pi^{s-1/2} \frac{\Gamma((1-s)/2)}{\Gamma(s/2)} \sum_{k=1}^m \frac{1}{k^{1-s}} + R_{n,m}(s),$$

where  $R_{m,n}$  is the error term. Notice that the two partial Dirichlet series here have the same sign, which is a different feature from the formulas above. A question here is what functional equations prevail in higher dimension  $d$ .

**8. The Riemann hypothesis.** From the asymptotics given in the theorems above there is a straightforward reformulation of the Riemann hypothesis in terms of the asymptotical behaviour of

$$\sum_{k=1}^{n-1} \frac{1}{\sin(\pi k/n)^s}$$

as  $n \rightarrow \infty$  as a function of  $s$ . It turns out however, that there is a more unexpected, nontrivial, and, what we think, more interesting equivalence with the Riemann hypothesis.

To show this we begin from the second asymptotical formula in Theorem 0.3:

$$\sum_{k=1}^{n-1} \frac{1}{\sin(\pi k/n)^s} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(1/2 - s/2)}{\Gamma(1 - s/2)} n + 2\pi^{-s} \zeta(s)n^s + \frac{s}{3} \pi^{2-s} \zeta(s - 2)n^{s-2} + o(n^{s-2})$$

for  $0 < Re(s) < 1$  as  $n \rightarrow \infty$ .

Let

$$\begin{aligned} h_n(s) &= (4\pi)^{s/2} \Gamma(s/2)n^{-s} (\zeta_{\mathbb{Z}/n\mathbb{Z}}(s/2) - n\zeta_{\mathbb{Z}}(s/2)) \\ &= \pi^{s/2} \Gamma(s/2)n^{-s} \left( \sum_{k=1}^{n-1} \frac{1}{\sin(\pi k/n)^s} - \frac{1}{\sqrt{\pi}} \frac{\Gamma(1/2 - s/2)}{\Gamma(1 - s/2)} n \right). \end{aligned}$$

Using the completed Riemann zeta function  $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$  the above asymptotics can be restated as

$$h_n(s) = 2\xi(s) + \alpha(s)n^{-2} + o(n^{-2}),$$

where  $\alpha(s) := \frac{s}{3}\pi^{2-s/2}\Gamma(s/2)\zeta(s-2)$ .

From this asymptotics and in view of  $\xi(1-s) = \xi(s)$  we conclude immediately:

PROPOSITION 8.1. *Let  $s \in \mathbb{C}$  with  $0 < \operatorname{Re}(s) < 1$  and  $\zeta(s) \neq 0$ . Then  $h_n(1-s) \sim h_n(s)$  in the sense that*

$$\lim_{n \rightarrow \infty} \frac{h_n(1-s)}{h_n(s)} = 1.$$

We now conjecture that a weakened version of this asymptotic functional relation is valid even at zeta zeros:

CONJECTURE. *Let  $s \in \mathbb{C}$  with  $0 < \operatorname{Re}(s) < 1$ . Then*

$$\lim_{n \rightarrow \infty} \left| \frac{h_n(1-s)}{h_n(s)} \right| = 1.$$

From now on we will prove that this is equivalent to the Riemann hypothesis:

THEOREM. *The conjecture above is equivalent to the statement that all non-trivial zeros of  $\zeta$  have real part  $1/2$ .*

We begin the proof with a simple observation:

LEMMA 8.2. *Suppose  $\zeta(s) = 0$ . Then the asymptotic relation*

$$\lim_{n \rightarrow \infty} \left| \frac{h_n(1-s)}{h_n(s)} \right| = 1$$

*is equivalent to  $|\alpha(1-s)| = |\alpha(s)|$ .*

Next we have:

LEMMA 8.3. *The equation  $|\alpha(1-s)| = |\alpha(s)|$  holds for all  $s$  on the critical line  $\operatorname{Re}(s) = 1/2$ .*

PROOF. Recall that

$$\alpha(s) = \frac{s}{3}\pi^{2-s/2}\Gamma(s/2)\zeta(s-2).$$

Since  $\zeta(\bar{s}) = \overline{\zeta(s)}$  and  $\Gamma(\bar{s}) = \overline{\Gamma(s)}$ , we have that  $\alpha(\bar{s}) = \overline{\alpha(s)}$ . Therefore if  $s = 1/2 + it$ , then

$$\alpha(1-s) = \alpha(1-1/2-it) = \alpha(\overline{1/2+it}) = \overline{\alpha(s)},$$

which implies the lemma. □

Note that using  $\xi((1-s)-2) = \xi(s+2)$  and Euler’s reflection formula  $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$  we have

$$\left| \frac{\alpha(1-s)}{\alpha(s)} \right| = \left| \frac{\frac{(s-1)(s+1)}{6} \pi \pi^{-(s+2)/2} \Gamma((s+2)/2) \zeta(s+2)}{\frac{s(s-2)}{6} \pi \pi^{-(s-2)/2} \Gamma((s-2)/2) \zeta(s-2)} \right| = \left| \frac{\zeta(s+2)(s-1)(s+1)}{\zeta(s-2)4\pi^2} \right|.$$

As a consequence  $|\alpha(1-s)| = |\alpha(s)|$  is equivalent to

$$\left| \frac{\zeta(s+2)}{\zeta(s-2)} \right| = \frac{4\pi^2}{|s^2-1|}.$$

We will study the right and left hand sides as functions of  $\sigma$ , in the interval  $0 < \sigma < 1$ , with  $s = \sigma + it$  and  $t > 0$  fixed. In view of that

$$\frac{1}{|s^2-1|^2} = \frac{1}{\sigma^4 + 2\sigma^2 + (t^2-1)^2},$$

we see that the right hand side is strictly decreasing in  $\sigma$ . On the other hand we have the following:

LEMMA 8.4. *Let  $s = \sigma + it$ , with  $t$  fixed such that  $|t| > 26$ . Then the function*

$$\left| \frac{\zeta(s+2)}{\zeta(s-2)} \right|$$

*is strictly increasing in  $0 < \sigma < 1$ .*

PROOF. As remarked in [MSZ14], for a holomorphic function  $f$ , a simple calculation, using the Cauchy-Riemann equation, leads to

$$Re(f'(s)/f(s)) = \frac{1}{|f(s)|} \frac{\partial |f(s)|}{\partial \sigma},$$

in any domain where  $f(z) \neq 0$ . This implies that for  $|f|$  to be increasing in  $\sigma$  we should show that the real part of its logarithmic derivative is positive.

We begin with one of the two terms in the logarithmic derivative of  $\zeta(s+2)/\zeta(s-2)$ :

$$Re(\zeta'(s+2)/\zeta(s+2)) = -Re\left(\sum_{n \geq 1} \Lambda(n)n^{-s-2}\right) = -\sum_{n \geq 1} \Lambda(n)n^{-\sigma-2} \cos(t \log n),$$

where  $\Lambda(n)$  is the von Mangoldt function. So

$$\left| Re(\zeta'(s+2)/\zeta(s+2)) \right| \leq \sum_{n \geq 1} \Lambda(n)n^{-2} = -\frac{\zeta'(2)}{\zeta(2)} = \gamma + \log(2\pi) - 12 \log A < 0.57,$$

by known numerics. We are therefore left to show that the other term

$$Re(-\zeta'(s-2)/\zeta(s-2)) \geq 0.57.$$

On the one hand, following the literature, see [L99, SD10, MSZ14], from the Mittag-Leffler expansion we have

$$\frac{\tilde{\xi}'(s)}{\tilde{\xi}(s)} = \sum_{\rho} \frac{1}{s - \rho}$$

where the sum is taken over the zeros which all lie in the critical strip. (The function  $\tilde{\xi}(s)$  is defined by  $\tilde{\xi}(s) = (s-1)\Gamma(1+s/2)\pi^{-s/2}\zeta(s)$ .) This implies by a simple termwise calculation ([MSZ14]) that since  $s-2$  is to the left of the critical strip, we have  $Re(\tilde{\xi}'(s-2)/\tilde{\xi}(s-2)) < 0$  in the interval  $0 < \sigma < 1$ . On the other hand

$$0 > Re(\tilde{\xi}'(s-2)/\tilde{\xi}(s-2)) = Re(1/(s-3)) + \frac{1}{2}Re(\psi(s/2)) - \frac{1}{2}\log \pi + Re(\zeta'(s-2)/\zeta(s-2)),$$

where  $\psi$  is the logarithmic derivative of the gamma function. We estimate

$$Re(1/(s-3)) = \frac{\sigma-3}{(\sigma-3)^2+t^2} > \frac{-3}{4+t^2} > -\frac{3}{4+144} > -0.03$$

and  $-\log \pi > -1.2$ . Hence

$$Re(-\zeta'(s-2)/\zeta(s-2)) > -0.7 + Re(\psi(s/2))/2.$$

The last thing to do is to estimate the psi-function. Following [MSZ14], we have using Stirling's formula for  $\psi$ ,

$$Re(\psi(s)) = \log |s| - \frac{\sigma}{2|s|^2} + Re(R(s)),$$

where  $|R(s)| \leq \sqrt{2}/(6|s|^2)$ . This is valid for any  $s = \sigma + it$  in the critical strip. We observe that

$$-\frac{\sigma}{2|s|^2} \geq -\frac{1}{2t^2}$$

so

$$Re(\psi(s/2)) \geq \log \frac{|t|}{2} - \frac{2}{t^2} - \frac{2\sqrt{2}}{3t^2} \geq 2.56$$

if  $|t| \geq 26$ . This completes the proof. □

Note that by numerics one can see that the lemma does not hold for small  $t$ . The lemma implies that the left and right hand sides can be equal only once for a fixed  $t$ , and this occurs at  $Re(s) = 1/2$  as shown above. We summarize this in the following statement which concerns just the Riemann zeta function:

**PROPOSITION 8.5.** *For  $s \in \mathbb{C}$  with  $0 < Re(s) < 1$ , with  $|Im(s)| > 26$ , the equality  $|\alpha(1-s)| = |\alpha(s)|$  holds if and only if  $Re(s) = 1/2$ .*

Therefore, since it is known that the Riemann zeta zeros in the critical strip having imaginary part less than 26 in absolute value all lie on the critical line and in view of Lemma 8.2, the equivalence between the graph zeta functional equation and the Riemann hypothesis is established.

## REFERENCES

- [AS64] MILTON ABRAMOWITZ AND IRENE A. STEGUN, Handbook of mathematical functions with formulas, graphs, and mathematical tables. National Bureau of Standards Applied Mathematics Series, 55 For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 1964 xiv+1046 pp.
- [RH08] The Riemann hypothesis. A resource for the aficionado and virtuoso alike. Edited by Peter Borwein, Stephen Choi, Brendan Rooney and Andrea Weirathmueller. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York, 2008. xiv+533 pp.
- [B99] LAURENT BARTHOLDI, Counting paths in graphs, Enseign. Math. 45 (1999), 83–131.
- [BH10] MATTHIAS BECK AND MARY HALLORAN, Finite trigonometric character sums via discrete Fourier analysis, Int. J. Number Theory 6 (2010), no. 1, 51–67.
- [BY02] BRUCE C. BERNDT AND BOON PIN YEAP, Explicit evaluations and reciprocity theorems for finite trigonometric sums, Adv. in Appl. Math. 29 (2002), no. 3, 358–385.
- [Ca34] TORSTEN CARLEMAN, Propriétés asymptotiques des fonctions fondamentales des membranes vibrantes, Comptes rendus du VIIIe Congrès des Math. Scand. Stockholm, 1934, pp. 34–44.
- [CJK10] GAUTAM CHINTA, JAY JORGENSON AND ANDERS KARLSSON, Zeta functions, heat kernels, and spectral asymptotics on degenerating families of discrete tori, Nagoya Math. J. 198 (2010), 121–172.
- [CJK12] GAUTAM CHINTA, JAY JORGENSON AND ANDERS KARLSSON, Complexity and heights of tori, In: Dynamical systems and group actions, 89–98, Contemp. Math., 567, Amer. Math. Soc., Providence, RI, 2012.
- [CJK15] GAUTAM CHINTA, JAY JORGENSON AND ANDERS KARLSSON, Heat kernels on regular graphs and generalized Ihara zeta function formulas, Monatsh. Math. 178 (2015), no. 2, 171–190.
- [CM99] WENCHANG CHU AND ALBERTO MARINI, Partial fractions and trigonometric identities, Adv. in Appl. Math. 23 (1999), no. 2, 115–175.
- [CL96] F. R. K. CHUNG AND ROBERT LANGLANDS, A combinatorial Laplacian with vertex weights, J. Combin. Theory Ser. A 75 (1996), no. 2, 316–327.
- [CS12] DJURDJE CVIJOVIĆ AND H. M. SRIVASTAVA, Closed-form summations of Dowker’s and related trigonometric sums, J. Phys. A 45 (2012), no. 37, 374015, 10 pp.
- [D14] A. DIETMAR, Ihara Zeta functions of infinite weighted graphs, SIAM J. Discrete Math. 29 (2015), no.4, 2100–2116.
- [DD88] BERTRAND DUPLANTIER AND FRANÇOIS DAVID, Exact partition functions and correlation functions of multiple Hamiltonian walks on the Manhattan lattice, J. Statist. Phys. 51 (1988), no. 3-4, 327–434.
- [EI12] E. ELIZALDE, Ten Physical Applications of Spectral Zeta Functions, Second edition. Lecture Notes in Physics, 855. Springer, Heidelberg, 2012. xiv+227 pp.
- [Ex76] HAROLD EXTON, Multiple hypergeometric functions and applications. Foreword by L. J. Slater. Mathematics & its Applications. Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], New York-London-Sydney, 1976. 312 pp.
- [F15] FABIEN FRIEDLI, A functional relation for  $L$ -functions of graphs equivalent to the Riemann hypothesis for Dirichlet  $L$ -functions, J. Number Theory 169 (2016), 342–352.
- [GR07] I. S. GRADSHTEYN AND I. M. RYZHIK, Table of integrals, series, and products. Translated from the Russian. Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger. Seventh edition. Elsevier/Academic Press, Amsterdam, 2007. xlviii+1171 pp.
- [G78] ROSTISLAV I. GRIGORCHUK, Banach invariant means on homogeneous spaces and random walks, (Russian), Ph.D. thesis, Moscow State University, 1978.
- [H92] KI-ICHIRO HASHIMOTO, Artin type  $L$ -functions and the density theorem for prime cycles on finite graphs, Internat. J. Math. 3 (1992), no. 6, 809–826.
- [Ha77] S. W. HAWKING, Zeta function regularization of path integrals in curved spacetime, Comm. Math. Phys.



- 55 (1977), no. 2, 133–148.
- [He77] P. HENRICI, *Applied and Computational Complex Analysis*, Vol. 2, John Wiley & Sons, New York, 1977.
- [IK04] HENRYK IWANIEC AND EMMANUEL KOWALSKI, *Analytic number theory*. American Mathematical Society Colloquium Publications, 53. American Mathematical Society, Providence, RI, 2004. xii+615 pp.
- [JL96] JAY JORGENSON AND SERGE LANG, Extension of analytic number theory and the theory of regularized harmonic series from Dirichlet series to Bessel series, *Math. Ann.* 306 (1996), no. 1, 75–124.
- [JL12] JAY JORGENSON AND SERGE LANG, The heat kernel, theta inversion and zetas on  $\Gamma \backslash G/K$ . In: *Number theory, analysis and geometry*, 273–306, Springer, New York, 2012.
- [L99] JEFFREY C. LAGARIAS, On a positivity property of the Riemann  $\xi$ -function, *Acta Arith.* 89 (1999), no. 3, 217–234.
- [LPS14] D. LENZ, F. POGORZELSKI AND M. SCHMIDT, The Ihara Zeta function for infinite graphs, arXiv:1408.3522
- [Lo12] LASZLO LOVASZ, *Large networks and graph limits*. American Mathematical Society Colloquium Publications, 60. American Mathematical Society, Providence, RI, 2012. xiv+475 pp.
- [L02] WOLFGANG LÜCK,  $L^2$ -invariants: theory and applications to geometry and  $K$ -theory. *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*, 44. Springer-Verlag, Berlin, 2002. xvi+595 pp.
- [Ly10] RUSSELL LYONS, Identities and inequalities for tree entropy, *Combin. Probab. Comput.* 19 (2010), no. 2, 303–313.
- [MSZ14] YU. MATIYASEVICH, F. SAIDAK AND P. ZVENGROWSKI, Horizontal monotonicity of the modulus of the zeta function, L-functions, and related functions, *Acta Arith.* 166 (2014), no. 2, 189–200.
- [MP49] S. MINAKSHISUNDARAM AND Å. PLEIJEL, Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds, *Canadian J. Math.* 1 (1949), 242–256.
- [MW89] BOJAN MOHAR AND WOLFGANG WOESS, A survey on spectra of infinite graphs, *Bull. London Math. Soc.* 21 (1989), no. 3, 209–234.
- [M08] WERNER MÜLLER, A spectral interpretation of the zeros of the constant term of certain Eisenstein series, *J. Reine Angew. Math.* 620 (2008), 67–84.
- [Pi1881] ÉMILE PICARD, Sur une extension aux fonctions de deux variables du problème de Riemann relatif aux fonctions hypergéométriques, *Ann. Sci. École Norm. Sup. (2)* 10 (1881), 305–322.
- [P39] ÅKE PLEIJEL, Sur les propriétés asymptotiques des fonctions et valeurs propres des plaques vibrantes, *C. R. Acad. Sci. Paris* 209 (1939), 717–718.
- [PT34] H. S. A. POTTER AND E. C. TITCHMARSH, The Zeros of Epstein's Zeta-Functions, *Proc. London Math. Soc.* S2-39 no. 1 (1934), 372–384.
- [RV15] NICOLAI RESHETIKHIN AND BORIS VERTMAN, Combinatorial quantum field theory and gluing formula for determinants, *Lett. Math. Phys.* 105 (2015), no. 3, 309–340.
- [Sa66] R. K. SAXENA, Integrals involving products of Bessel functions. II, *Monatsh. Math.* 70 (1966), 161–163.
- [SC67] ATLE SELBERG AND S. CHOWLA, On Epstein's zeta-function, *J. Reine Angew. Math.* 227 (1967), 86–110.
- [Si04] AVRAM SIDI, Euler-Maclaurin expansions for integrals with endpoint singularities: a new perspective, *Numer. Math.* 98 (2004), no. 2, 371–387.
- [So98] JONATHAN SONDOW, The Riemann hypothesis, simple zeros and the asymptotic convergence degree of improper Riemann sums, *Proc. Amer. Math. Soc.* 126 (1998), 1311–1314.
- [SD10] JONATHAN SONDOW AND CRISTIAN DUMITRESCU, A monotonicity property of Riemann's xi function and a reformulation of the Riemann hypothesis, *Period. Math. Hungar.* 60 (2010), no. 1, 37–40.
- [SE79] H. M. SRIVASTAVA, AND HAROLD EXTON, A generalization of the Weber-Schafheitlin integral, *J. Reine Angew. Math.* 309 (1979), 1–6.
- [SK85] H. M. SRIVASTAVA AND PER W. KARLSSON, *Multiple Gaussian hypergeometric series*. Ellis Horwood

- Series: Mathematics and its Applications. Ellis Horwood Ltd., Chichester; Halsted Press, New York, 1985. 425 pp.
- [STe00] H. M. STARK AND A. A. TERRAS, Zeta functions of finite graphs and coverings. II, Adv. Math. 154 (2000), no. 1, 132–195.
- [Su86] TOSHIKAZU SUNADA,  $L$ -functions in geometry and some applications. In: Curvature and topology of Riemannian manifolds (Katata, 1985), 266–284, Lecture Notes in Math., 1201, Springer, Berlin, 1986.
- [Te10] AUDREY TERRAS, Zeta Functions of Graphs: A Stroll through the Garden. Cambridge Studies in Advanced Mathematics 128. Cambridge University Press, 2010.
- [W47] AUREL WINTNER, The sum formula of Euler-Maclaurin and the inversions of Fourier and Möbius, Amer. J. Math. 69 (1947), 685–708.
- [Z73] DON ZAGIER, Higher dimensional Dedekind sums, Math. Ann. 202 (1973), 149–172.

SECTION DE MATHÉMATIQUES  
UNIVERSITÉ DE GENÈVE  
2–4 RUE DU LIÈVRE, CASE POSTALE 64  
1211 GENÈVE 4  
SWITZERLAND

*E-mail address:* fabien.friedli@unige.ch

SECTION DE MATHÉMATIQUES  
UNIVERSITÉ DE GENÈVE  
2–4 RUE DU LIÈVRE, CASE POSTALE 64  
1211 GENÈVE 4  
SWITZERLAND  
AND  
MATEMATISKA INSTITUTIONEN  
UPPSALA UNIVERSITET  
BOX 256  
751 05 UPPSALA  
SWEDEN

*E-mail address:* anders.karlsson@unige.ch