Tohoku Math. J.
69 (2017), 571–583

A POLYNOMIAL DEFINED BY THE $SL(2;\mathbb{C})$-REIDEMEISTER TORSION FOR A HOMOLOGY 3-SPHERE OBTAINED BY A DEHN SURGERY ALONG A $(2P, Q)$-TORUS KNOT

TERUAKI KITANO

(Received June 4, 2015, revised November 12, 2015)

Abstract. Let K be a $(2p, q)$-torus knot. Here p and q are coprime odd positive integers. Let M_n be a 3-manifold obtained by a $1/n$-Dehn surgery along K. We consider a polynomial $\sigma_{(2p,q,n)}(t)$ whose zeros are the inverses of the Reidemeister torsion of M_n for $SL(2;\mathbb{C})$-irreducible representations under some normalization. Johnson gave a formula for the case of the $(2, 3)$-torus knot under another normalization. We generalize this formula for the case of $(2p, q)$-torus knots by using Tchebychev polynomials.

1. Introduction. Reidemeister torsion is a piecewise linear invariant for manifolds. It was originally defined by Reidemeister, Franz and de Rham in the 1930's. In the 1980's Johnson [1] developed a theory of the Reidemeister torsion from the view point of relations to the Casson invariant. He also derived an explicit formula for the Reidemeister torsion of homology 3-spheres obtained by $1/n$-Dehn surgeries along a torus knot for $SL(2;\mathbb{C})$-irreducible representations.

Let K be a $(2p, q)$-torus knot, where p, q are coprime, positive odd integers. Let M_n be a closed 3-manifold obtained by a $1/n$-surgery along K. We consider the Reidemeister torsion $\tau_\rho(M_n)$ of M_n for an irreducible representation $\rho : \pi_1(M_n) \to SL(2;\mathbb{C})$.

Johnson gave a formula for any non-trivial value of $\tau_\rho(M_n)$. Furthermore in the case of the trefoil knot, he proposed to consider the polynomial whose zero set coincides with the set of all non-trivial values $\{\frac{1}{\tau_\rho(M_n)}\}$, which is denoted by $\sigma_{(2,3,n)}(t)$. Under some normalization of $\sigma_{(2,3,n)}(t)$, he gave a 3-term relation among $\sigma_{(2,3,n+1)}(t)$, $\sigma_{(2,3,n)}(t)$ and $\sigma_{(2,3,n-1)}(t)$ by using Tchebychev polynomials.

In this paper we consider one generalization of this polynomial for a $(2p, q)$-torus knot. Main results of this paper are Theorem 4.3 and Proposition 5.1.

Acknowledgements. The author would like to express his gratitude to the referee for reading this article carefully and giving many constructive comments to the author. In particular, the fact in Remark 5.4 was pointed out by the referee.

2. Definition of Reidemeister torsion. First let us describe definitions and properties of the Reidemeister torsion for $SL(2;\mathbb{C})$-representations. See Johnson [1], Kitano [2, 3] and Milnor [5, 6, 7] for details.

2010 *Mathematics Subject Classification.* Primary 57M27.

Key words and phrases. Reidemeister torsion, torus knot, Brieskorn homology 3-sphere, $SL(2;\mathbb{C})$-representation.

This research was partially supported by JSPS KAKENHI 25400101.

Let W be a d-dimensional vector space over $\mathbb{C}$ and let $\mathbf{b} = (b_1, \dots, b_d)$ and $\mathbf{c} = (c_1, \dots, c_d)$ be two bases for W. Setting $b_i = \sum p_{ji} c_j$, we obtain a nonsingular matrix $P = (p_{ij})$ with entries in $\mathbb{C}$. Let $[\mathbf{b}/\mathbf{c}]$ denote the determinant of P.

Suppose

$$C_* : 0 \to C_k \overset{\partial_k}{\to} C_{k-1} \overset{\partial_{k-1}}{\to} \cdots \overset{\partial_2}{\to} C_1 \overset{\partial_1}{\to} C_0 \to 0$$

is an acyclic chain complex of finite dimensional vector spaces over $\mathbb{C}$. We assume that a preferred basis $\mathbf{c}_i$ for C_i is given for each i. Choose some basis $\mathbf{b}_i$ for $B_i = \mathrm{Im}(\partial_{i+1})$ and take a lift of it in C_{i+1}, which we denote by $\tilde{\mathbf{b}}_i$. Since $B_i = Z_i = \mathrm{Ker}\partial_i$, the basis $\mathbf{b}_i$ can serve as a basis for Z_i. Furthermore since the sequence

$$0 \to Z_i \to C_i \overset{\partial_i}{\to} B_{i-1} \to 0$$

is exact, the vectors $(\mathbf{b}_i, \tilde{\mathbf{b}}_{i-1})$ form a basis for C_i. Here $\tilde{\mathbf{b}}_{i-1}$ is a lift of $\mathbf{b}_{i-1}$ in C_i. It is easily shown that $[\mathbf{b}_i, \tilde{\mathbf{b}}_{i-1}/\mathbf{c}_i]$ does not depend on the choice of a lift $\tilde{\mathbf{b}}_{i-1}$. Hence we can simply denote it by $[\mathbf{b}_i, \mathbf{b}_{i-1}/\mathbf{c}_i]$.

Definition 2.1. The torsion of the chain complex C_* is given by the alternating product

$$\prod_{i=0}^{k} [\mathbf{b}_i, \mathbf{b}_{i-1}/\mathbf{c}_i]^{(-1)^{i+1}}$$

and we denote it by $\tau(C_*)$.

Remark 2.2. It is easy to see that $\tau(C_*)$ does not depend on the choices of the bases $\{\mathbf{b}_0, \dots, \mathbf{b}_k\}$.

Now we apply this torsion invariant of chain complexes to the following geometric situations. Let X be a finite CW-complex and $\tilde{X}$ a universal covering of X. The fundamental group $\pi_1 X$ acts on $\tilde{X}$ from the right-hand side as deck transformations. Then the chain complex $C_*(\tilde{X}; \mathbb{Z})$ has the structure of a chain complex of free $\mathbb{Z}[\pi_1 X]$-modules.

Let $\rho : \pi_1 X \to SL(2; \mathbb{C})$ be a representation. We denote the 2-dimensional vector space $\mathbb{C}^2$ by V. Using the representation ρ, V admits the structure of a $\mathbb{Z}[\pi_1 X]$-module and then we denote it by V_ρ. Define the chain complex $C_*(X; V_\rho)$ by $C_*(\tilde{X}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1 X]} V_\rho$ and choose a preferred basis

$$(\tilde{u}_1 \otimes \mathbf{e}_1, \tilde{u}_1 \otimes \mathbf{e}_2, \dots, \tilde{u}_d \otimes \mathbf{e}_1, \tilde{u}_d \otimes \mathbf{e}_2)$$

of $C_i(X; V_\rho)$ where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a canonical basis of $V = \mathbb{C}^2$, $\{u_1, \dots, u_d\}$ are the i-cells giving a basis of $C_i(X; \mathbb{Z})$ and $\{\tilde{u}_1, \dots, \tilde{u}_d\}$ are lifts of them in $C_i(\tilde{X}; \mathbb{Z})$.

Now we suppose that $C_*(X; V_\rho)$ is acyclic, namely all homology groups $H_*(X; V_\rho)$ are vanishing. In this case we call ρ an acyclic representation.

Definition 2.3. Let $\rho : \pi_1(X) \to SL(2; \mathbb{C})$ be an acyclic representation. Then the Reidemeister torsion $\tau_\rho(X) \in \mathbb{C} \setminus \{0\}$ is defined by the torsion $\tau(C_*(X; V_\rho))$ of $C_*(X; V_\rho)$.

REMARK 2.4.

(1) We define $\tau_\rho(X) = 0$ for a non-acyclic representation ρ.

(2) The definition of $\tau_\rho(X)$ depends on several choices. However it is well known that the Reidemeister torsion is a piecewise linear invariant for X with ρ.

Now let M be a closed orientable 3-manifold with an acyclic representation $\rho : \pi_1(M) \to SL(2;\mathbb{C})$. Here we take a torus decomposition of $M = A \cup_{T^2} B$. For simplicity, we write the same symbol ρ for restricted representations to images of $\pi_1(A), \pi_1(B), \pi_1(T^2)$ in $\pi_1(M)$ by inclusions. By this decomposition, we have the following formula.

PROPOSITION 2.5. *Let $\rho : \pi_1(M) \to SL(2;\mathbb{C})$ a representation. Assume all homogy groups $H_*(T^2; V_\rho) = 0$. Then all homology groups $H_*(M; V_\rho) = 0$ if and only if both of all homology groups $H_*(A; V_\rho) = H_*(B; V_\rho) = 0$. In this case, it holds*

$$\tau_\rho(M) = \tau_\rho(A)\tau_\rho(B)\,.$$

3. Johnson's theory. We apply the above proposition to a 3-manifold obtained by Dehn-surgery along a knot. Now let $K \subset S^3$ be a $(2p, q)$-torus knot with coprime odd integers p, q. Further let $N(K)$ be an open tubular neighborhood of K and $E(K)$ the knot exterior $S^3 \setminus N(K)$. We denote its closure of $N(K)$ by $\overline{N}$ which is homeomorphic to $S^1 \times D^2$. Now we write M_n to a closed orientable 3-manifold obtained by a $1/n$-surgery along K. Naturally there exists a torus decomposition $M_n = E(K) \cup \overline{N}$ of M_n.

REMARK 3.1. This manifold M_n is diffeomorphic to a Brieskorn homology 3-sphere $\Sigma(2p, q, N)$ where $N = |2pqn + 1|$.

Here the fundamental group of $E(K)$ has a presentation as follows.

$$\pi_1(E(K)) = \pi_1(S^3 \setminus K) = \langle x, y \mid x^{2p} = y^q \rangle\,.$$

Furthermore the fundamental group $\pi_1(M_n)$ admits the presentation as follows;

$$\pi_1(M_n) = \langle x, y \mid x^{2p} = y^q, ml^n = 1 \rangle$$

where $m = x^{-r}y^s$ ($r, s \in \mathbb{Z}$, $2ps - qr = 1$) is a meridian of K and $l = x^{-2p}m^{2pq} = y^{-q}m^{2pq}$ is similarly a longitude.

Let $\rho : \pi_1(E(K)) = \pi_1(S^3 \setminus K) \to SL(2;\mathbb{C})$ a representation. It is easy to see a given representation ρ can be extended to $\pi_1(M_n) \to SL(2;\mathbb{C})$ as a representation if and only if $\rho(ml^n) = E$. Here E is the identity matrix in $SL(2;\mathbb{C})$. In this case by applying Proposition 2.5,

$$\tau_\rho(M_n) = \tau_\rho(E(K))\tau_\rho(\overline{N})$$

for any acyclic representation $\rho : \pi_1(M_n) \to SL(2;\mathbb{C})$.

Now we consider only irreducible representations of $\pi_1(M_n)$, which is extended from the one on $\pi_1(E(K))$. It is seen that the set of the conjugacy classes of the $SL(2;\mathbb{C})$-irreducible representations is finite. Any conjugacy class can be represented by $\rho_{(a,b,k)}$ for some (a, b, k) such that

(1) $0 < a < 2p, 0 < b < q, a \equiv b \bmod 2$,
(2) $0 < k < N = |2pqn + 1|, k \equiv na \bmod 2$,
(3) $\mathrm{tr}(\rho_{(a,b,k)}(x)) = 2\cos\frac{a\pi}{2p}$,
(4) $\mathrm{tr}(\rho_{(a,b,k)}(y)) = 2\cos\frac{b\pi}{q}$,
(5) $\mathrm{tr}(\rho_{(a,b,k)}(m)) = 2\cos\frac{k\pi}{N}$.

Johnson computed $\tau_{\rho_{(a,b,k)}}(M_n)$ as follows.

THEOREM 3.2 (Johnson).

(1) *A representation $\rho_{(a,b,k)}$ is acylic if and only if $a \equiv b \equiv 1, k \equiv n$ mod 2.*
(2) *For any acyclic representation $\rho_{(a,b,k)}$ with $a \equiv b \equiv 1$, $k \equiv n$ mod 2, then*

$$\tau_{\rho_{(a,b,k)}}(M_n) = \frac{1}{2\left(1-\cos\frac{a\pi}{2p}\right)\left(1-\cos\frac{b\pi}{q}\right)\left(1+\cos\frac{2pqk\pi}{N}\right)}.$$

REMARK 3.3.

- In fact Johnson proved this theorem for any torus knot, not only for a $(2p, q)$-torus knot.
- This Johnson's result was generalized for any Seifert fiber manifold in [2]. Please see [2] as a reference.
- In general, it is not true that the set of $\{\tau_\rho(M_n)\}$ is finite. There exists a manifold whose Reidemeister torsion can be variable continuously. Please see [3].

Here assume $K = T(2, 3)$ is the trefoil knot. By considering the set of non-trivial values of $\tau_\rho(M_n)$ for any irreducible representation $\rho : \pi_1(M_n) \to SL(2; \mathbb{C})$, Johnson defined the polynomial $\bar\sigma_{(2,3,n)}(t)$ of one variable t whose zeros are the set of $\left\{\frac{1}{\frac{1}{2}\tau_\rho(M_n)}\right\}$, which is well defined up to multiplications of nonzero constants.

THEOREM 3.4 (Johnson). *Under normalization by $\bar\sigma_{(2,3,n)}(0) = (-1)^n$, there exists the 3-term relation such that*

$$\bar\sigma_{(2,3,n+1)}(t) = (t^3 - 6t^2 + 9t - 2)\bar\sigma_{(2,3,n)}(t) - \bar\sigma_{(2,3,n-1)}(t).$$

REMARK 3.5. The polynomial $t^3 - 6t^2 + 9t - 2$ is given by $2T_6\left(\frac{1}{2}\sqrt{t}\right)$. Here $T_6(x)$ is the sixth Tchebychev polynomial.

Recall the n-th Tchebychev polynomial $T_n(x)$ of the first kind can be defined by expressing $\cos n\theta$ as a polynomial in $\cos\theta$. We give a summary of these polynomials.

PROPOSITION 3.6. *The Tchebychev polynomials have following properties.*

(1) $T_0(x) = 1, T_1(x) = x$.
(2) $T_{-n}(x) = T_n(x)$.
(3) $T_n(1) = 1, T_n(-1) = (-1)^n$.
(4) $T_n(0) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ (-1)^{\frac{n}{2}} & \text{if } n \text{ is even.} \end{cases}$

(5) $T_{n+1}(x) = 2xT_n - T_{n-1}(x)$.
(6) *The degree of* $T_n(x)$ *is* n.
(7) $2T_m(x)T_n(x) = T_{m+n}(x) + T_{m-n}(x)$.

He we put a short list of $T_n(x)$.

- $T_0(x) = 1$,
- $T_1(x) = x$,
- $T_2(x) = 2x^2 - 1$,
- $T_3(x) = 4x^3 - 3x$,
- $T_4(x) = 8x^4 - 8x^2 + 1$,
- $T_5(x) = 16x^5 - 20x^3 + 5x$,
- $T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$.

EXAMPLE 3.7. Put $p = 1, q = 3$ and $n = -1$. Then $N = |2 \cdot 3 \cdot (-1) + 1| = 5$ and $M_{-1} = \Sigma(2, 3, 5)$. In this case, it is easy to see that $a = b = 1$ and $k = 1, 3$. By the above formula, we obtain

$$\begin{aligned}
\tau_{\rho_{(1,1,k)}}(M_{-1}) &= \frac{1}{2\left(1 - \cos\frac{\pi}{2}\right)\left(1 - \cos\frac{\pi}{3}\right)\left(1 + \cos\frac{6k\pi}{5}\right)} \\
&= \frac{1}{2(1-0)(1-\frac{1}{2})(1+\cos\frac{6k\pi}{5})} \\
&= \frac{1}{1+\cos\frac{6k\pi}{5}} \\
&= 3 \pm \sqrt{5}\,.
\end{aligned}$$

Hence we have two non-trivial values of $\frac{1}{\frac{1}{2}\tau_\rho(M_{-1})}$ as

$$\begin{aligned}
\frac{1}{\frac{1}{2}\tau_\rho(M_{-1})} &= \frac{1}{\frac{3\pm\sqrt{5}}{2}} \\
&= \frac{2}{3\pm\sqrt{5}} \\
&= \frac{3\pm\sqrt{5}}{2}\,.
\end{aligned}$$

Therefore we have

$$\left(t - \left(\frac{3-\sqrt{5}}{2}\right)\right)\left(t - \left(\frac{3+\sqrt{5}}{2}\right)\right) = t^2 - 3t + 1\,.$$

Under Johnson's normalization $\bar{\sigma}_{(2,3,-1)}(0) = -1$,

$$\bar{\sigma}_{(2,3,-1)}(t) = -t^2 + 3t - 1\,.$$

Next put $n = 1$. In this case

$$\tau_{\rho_{(1,1,k)}}(M_1) = \frac{1}{2\left(1-\cos\frac{\pi}{2}\right)\left(1-\cos\frac{\pi}{3}\right)\left(1+\cos\frac{6k\pi}{7}\right)} = \frac{1}{\left(1+\cos\frac{6k\pi}{7}\right)}.$$

We can see as

$$\begin{aligned}&\left(t-2\left(1+\cos\frac{6\pi}{7}\right)\right)\left(t-2\left(1+\cos\frac{6\cdot 3\pi}{7}\right)\right)\left(t-2\left(1+\cos\frac{6\cdot 5\pi}{7}\right)\right)\\ =&t^3-5t^2+6t-1\\ =&\bar{\sigma}_{(2,3,1)}(t).\end{aligned}$$

On the other hand, by using Johnson's formula

$$\begin{aligned}(t^3-6t^2+9t-2)\bar{\sigma}_{(2,3,0)}(t)-\bar{\sigma}_{(2,3,-1)}(t) =&(t^3-6t^2+9t-2)\cdot 1-(-t^2+3t-1)\\ =&t^3-5t^2+6t-1,\end{aligned}$$

we obtain the same polynomial.

4. Main theorem. From this section, we consider the generalization for a $(2p, q)$-torus knot. Here p, q are coprime odd integers. In this section we give a formula of the torsion polynomial $\sigma_{(2p,q,n)}(t)$ for $M_n = \Sigma(2p, q, N)$ obtained by a $1/n$-Dehn surgery along K. Although Johnson considered the inverses of the half of $\tau_\rho(M_n)$, we simply treat torsion polynomials as follows.

DEFINITION 4.1. A one variable polynomial $\sigma_{(2p,q,n)}(t)$ is called the torsion polynomial of M_n if the zero set coincides with the set of all non-trivial values $\left\{\frac{1}{\tau_\rho(M_n)}\right\}$ and it satisfies the following normalization condition as $\sigma_{(2p,q,n)}(0) = (-1)^{\frac{np(q-1)}{2}}$.

REMARK 4.2.
If $n = 0$, then clearly $M_n = S^3$ with the trivial fundamental group. Hence we define the torsion polynomial to be trivial.

From here assume $n \neq 0$. Recall Johnson's formula

$$\frac{1}{\tau_{\rho_{(a,b,k)}}(M_n)} = 2\left(1-\cos\frac{a\pi}{2p}\right)\left(1-\cos\frac{b\pi}{q}\right)\left(1+\cos\frac{2pqk\pi}{N}\right)$$

where $0 < a < 2p$, $0 < b < q$, $a \equiv b \equiv 1 \bmod 2$, $k \equiv n \bmod 2$. Here we put

$$C_{(2p,q,a,b)} = \left(1-\cos\frac{a\pi}{2p}\right)\left(1-\cos\frac{b\pi}{q}\right)$$

and we have

$$\frac{1}{\tau_{\rho_{(a,b,k)}}(M_n)} = 4C_{(2p,q,a,b)}\cdot\frac{1}{2}\left(1+\cos\frac{2pqk\pi}{N}\right).$$

Main result is the following.

THEOREM 4.3. *The torsion polynomial of M_n is given by*

$$\sigma_{(2p,q,n)}(t) = \prod_{(a,b)} Y_{(n,a,b)}(t)$$

where

$$Y_{(n,a,b)}(t) = \begin{cases} 2C_{(2p,q,a,b)} \dfrac{T_{N+1}\left(\frac{\sqrt{t}}{2\sqrt{C_{(2p,q,a,b)}}}\right) - T_{N-1}\left(\frac{\sqrt{t}}{2\sqrt{C_{(2p,q,a,b)}}}\right)}{t-4C_{(2p,q,a,b)}} & (n > 0) \\ -2C_{(2p,q,a,b)} \dfrac{T_{N+1}\left(\frac{\sqrt{t}}{2\sqrt{C_{(2p,q,a,b)}}}\right) - T_{N-1}\left(\frac{\sqrt{t}}{2\sqrt{C_{(2p,q,a,b)}}}\right)}{t-4C_{(2p,q,a,b)}} & (n < 0). \end{cases}$$

Here $N = |2pqn + 1|$ and a pair of integers (a, b) is satisfying the following conditions;

- $0 < a < 2p, 0 < b < q$,
- $a \equiv b \equiv 1 \ mod\, 2$,
- $0 < k < N, k \equiv n \ mod\, 2$.

PROOF.
Case 1: $n > 0$
We modify one factor $(1 + \cos\frac{2pqk\pi}{N})$ of $\frac{1}{\tau_\rho(M_n)}$ as follows.

LEMMA 4.4. *The set $\{\cos\frac{2pqk\pi}{N} \mid 0 < k < N, k \equiv n \ mod\, 2\}$ is equal to the set $\{\cos\frac{2pk\pi}{N} \mid 0 < k < \frac{N}{2}\}$.*

PROOF. Now $N = 2pqn + 1$ is always an odd integer.
For any $k > \frac{N}{2}$, then clearly $N - k < \frac{N}{2}$. Then

$$\begin{aligned} \cos\frac{2pq(N-k)\pi}{N} &= \cos\left(2pq\pi - \frac{2pqk\pi}{N}\right) \\ &= (-1)^{2pq}\cos\left(-\frac{2pqk\pi}{N}\right) \\ &= \cos\left(\frac{2pqk\pi}{N}\right). \end{aligned}$$

Here if k is even (resp. odd), then $N - k$ is odd (resp. even). Hence it is seen

$$\left\{\cos\frac{2pqk\pi}{N} \mid 0 < k < N, k \equiv n \bmod 2\right\} = \left\{\cos\frac{2pqk\pi}{N} \mid 0 < k < \frac{N}{2}\right\}.$$

For any $k < \frac{N}{2}$, there exists uniquely l such that $-\frac{N}{2} < l < \frac{N}{2}$ and $l \equiv qk \bmod N$. Further there exists uniquely l such that $0 < l < \frac{N}{2}$ and $l \equiv \pm qk \bmod N$. Here $\cos\frac{2pqk\pi}{N} = \cos\frac{2pl\pi}{N}$ if and only if $2pqk \equiv \pm 2pl \bmod N$. Therefore it is seen that the set

$$\left\{\cos\frac{2pqk\pi}{N} \mid 0 < k < \frac{N}{2}\right\} = \left\{\cos\frac{2pk\pi}{N} \mid 0 < k < \frac{N}{2}\right\}.$$

□

Now we can modify

$$\frac{1}{2}\left(1+\cos\frac{2pk\pi}{N}\right) = \frac{1}{2}\cdot 2\cos^2\frac{2pk\pi}{2N} = \cos^2\frac{pk\pi}{N}\,.$$

We put

$$z_k = \cos\frac{pk\pi}{N}\ (0 < k < N)$$

and substitute $x = z_k$ to $T_{N+1}(x)$. Then it holds

$$\begin{aligned} T_{N+1}(z_k) &= \cos\left(\frac{(N+1)(pk\pi)}{N}\right) \\ &= \cos\left(pk\pi + \frac{pk\pi}{N}\right) \\ &= (-1)^{pk} z_k\,. \end{aligned}$$

Similarly it is seen

$$\begin{aligned} T_{N-1}(z_k) &= \cos\left(\frac{(N-1)(pk\pi)}{N}\right) \\ &= \cos\left(pk\pi - \frac{pk\pi}{N}\right) \\ &= (-1)^{pk} z_k\,. \end{aligned}$$

Hence it holds

$$T_{N+1}(z_k) - T_{N-1}(z_k) = 0\,.$$

By properties of Tchebyshev polynomials, it is seen that

- $T_{N+1}(1) - T_{N-1}(1) = 0$,
- $T_{N+1}(-1) - T_{N-1}(-1) = 0$.

Therefore we consider the following;

$$X_n(x) = \begin{cases} \frac{T_{N+1}(x)-T_{N-1}(x)}{2(x^2-1)} & (n > 0) \\ -\frac{T_{N+1}(x)-T_{N-1}(x)}{2(x^2-1)} & (n < 0)\,. \end{cases}$$

We mention that the degree of $X_n(x)$ is $N-1$.

By the above computation, $z_1, \dots, z_{N-1}$ are the zeros of $X_n(x)$. Further we can see

$$\begin{aligned} z_{N-k} &= \cos \frac{p(N-k)\pi}{N} \\ &= \cos\Big(p\pi - \frac{pk\pi}{N}\Big) \\ &= (-1)^p \cos\Big(-\frac{pk\pi}{N}\Big) \\ &= (-1)^p \cos\Big(\frac{pk\pi}{N}\Big) \\ &= -z_k\,. \end{aligned}$$

This means $N-1$ roots $z_1, \dots, z_{N-1}$ of $X_n(x) = 0$ occur in a pairs. Because $T_{N+1}(x)$, $T_{N-1}(x)$ are even functions, they are functions of x^2. Hence $X_n(x)$ is also an even function. Here by replacing x^2 by $\frac{t}{4C_{(2p,q,a,b)}}$, namely x by $\frac{\sqrt{t}}{2\sqrt{C_{(2p,q,a,b)}}}$, we put

$$\begin{aligned} Y_{(n,a,b)}(t) &= X_n\left(\frac{\sqrt{t}}{2\sqrt{C_{(2p,q,a,b)}}}\right) \\ &= \frac{T_{N+1}\left(\frac{\sqrt{t}}{2\sqrt{C_{(2p,q,a,b)}}}\right) - T_{N-1}\left(\frac{\sqrt{t}}{2\sqrt{C_{(2p,q,a,b)}}}\right)}{2\left(\left(\frac{\sqrt{t}}{2\sqrt{C_{(2p,q,a,b)}}}\right)^2 - 1\right)} \\ &= \frac{T_{N+1}\left(\frac{\sqrt{t}}{2\sqrt{C_{(2p,q,a,b)}}}\right) - T_{N-1}\left(\frac{\sqrt{t}}{2\sqrt{C_{(2p,q,a,b)}}}\right)}{2\left(\frac{t}{4C_{(2p,q,a,b)}} - 1\right)} \\ &= 2C_{(2p,q,a,b)} \frac{T_{N+1}\left(\frac{\sqrt{t}}{2\sqrt{C_{(2p,q,a,b)}}}\right) - T_{N-1}\left(\frac{\sqrt{t}}{2\sqrt{C_{(2p,q,a,b)}}}\right)}{t - 4C_{(2p,q,a,b)}}\,. \end{aligned}$$

Here it holds that its degree of $Y_{(n,a,b)}(s)$ is $\frac{N-1}{2}$, and the roots of $Y_{(n,a,b)}(t)$ are $4C_{(2p,q,a,b)}z_k^2$ $= 4C_{(2p,q,a,b)} \cos^2 \frac{\pi k}{2pqn+1}$ $(0 < k < \frac{N-1}{2})$, which are all non trivial values of $\frac{1}{\tau_{\rho_{(a,b,k)}}(M_n)}$. Therefore we obtain the formula.

Case 2: $n < 0$
In this case we modify $N = |2pqn + 1| = 2pq|n| - 1$. By the same arguments, it is easy to see the claim of the theorem can be proved. Therefore the proof completes. □

REMARK 4.5. By defining as $X_0(t) = 1$, it implies $Y_{(0,a,b)}(t) = 1$. Then the above statement is true for $n = 0$.

COROLLARY 4.6. *The degree of* $\sigma_{(2p,q,n)}(t)$ *is given by* $\frac{(N-1)p(q-1)}{4}$.

PROOF. The number of the pairs (a, b) is given by $\frac{p(q-1)}{2}$. As the degree of $Y_{(n,a,b)}(t)$ is $\frac{N-1}{2}$, then the degree of of $\sigma_{(2p,q,n)}(t)$ is given by $\frac{(N-1)p(q-1)}{4}$. □

5. 3-term relations. Finally we prove 3-term realtions for each factor $Y_{(n,a,b)}(t)$ as follows.

PROPOSITION 5.1. *For any n, it holds that*

$$Y_{(n+1,a,b)}(t) = D(t)Y_{(n,a,b)}(t) - Y_{(n-1,a,b)}(t)$$

where $D(t) = 2T_{2pq}\left(\frac{\sqrt{t}}{2\sqrt{C_{2p,q,a,b}}}\right)$.

PROOF. Recall Prop. 3.6 (7);

$$2T_m(x)T_n(x) = T_{m+n}(x) + T_{m-n}(x)\,.$$

Then if $n > 0$ we have

$$\begin{aligned}
2T_{2pq}(x)X_n(x) &= 2T_{2pq}(x)\left(\frac{T_{2pqn+2}(x) - T_{2pqn}(x)}{2(x^2-1)}\right) \\
&= \frac{(T_{2pq+2pqn+2}(x) + T_{2pqn+2-2pq}(x)) - (T_{2pq+2pqn}(x) + T_{2pqn-2pq}(x))}{2(x^2-1)} \\
&= \frac{T_{2pq(n+1)+2}(x) - T_{2pq(n+1)}(x) + T_{2pq(n-1)+2}(x) - T_{2pq(n-1)}(x))}{2(x^2-1)} \\
&= X_{n+1}(x) + X_{n-1}(x)\,.
\end{aligned}$$

Therefore it can be seen that

$$X_{n+1}(x) = 2T_{2pq}(x)X_n(x) - X_{n-1}(x)$$

and

$$Y_{(n+1,a,b)}(t) = 2T_{2pq}\left(\frac{\sqrt{t}}{2\sqrt{C_{(2p,q,a,b)}}}\right)Y_{(n,a,b)}(t) - Y_{(n-1,a,b)}(t)\,.$$

If $n = 0$, 3-term relation is

$$Y_{(1,a,b)}(t) = D(t)Y_{(0,a,b)}(t) - Y_{(-1,a,b)}(t)\,.$$

It can be seen by direct computation

$$\begin{aligned}
2T_{2pq}(x)X_0(x) - X_{-1}(x) &= 2T_{2pq}(x) - X_{-1}(x) \\
&= X_1(x)\,.
\end{aligned}$$

If $n < 0$, it can be also proved. □

We show some examples. First we treat (2, 3)-torus knot again.

EXAMPLE 5.2. Put $p = 1, q = 3$. In this case $a = b = 1$. Then we see

$$C_{2,3,1,1} = \left(1 - \cos\frac{\pi}{2}\right)\left(1 - \cos\frac{\pi}{3}\right) = \frac{1}{2}\,.$$

By applyng Theorem 4.3 and Proposition 5.1,

$$\begin{aligned}
\sigma_{(2,3,-1)}(t) &= \frac{T_6\left(\frac{\sqrt{t}}{\sqrt{2}}\right) - T_4\left(\frac{\sqrt{t}}{\sqrt{2}}\right)}{2\left(1 - \left(\frac{\sqrt{t}}{\sqrt{2}}\right)^2\right)} \\
&= -4t^2 + 6t - 1\,. \\
\sigma_{(2,3,0)}(t) &= 1\,. \\
\sigma_{(2,3,1)}(t) &= 8t^3 - 20t^2 + 12t - 1\,.
\end{aligned}$$

We show one more example.

EXAMPLE 5.3. Here put $(2p, q) = (2, 5)$. In this case $(a, b) = (1, 1)$ or $(1, 3)$ and the constants $C_{(2,5,1,1)}, C_{(2,5,1,3)}$ are given as follows:

$$\begin{aligned}
C_{(2,5,1,1)} &= \left(1 - \cos\frac{\pi}{2}\right)\left(1 - \cos\frac{\pi}{5}\right) \\
&= 1 - \cos\frac{\pi}{5} \\
&= \frac{1}{4}\left(3 - \sqrt{5}\right)\,.
\end{aligned}$$

$$\begin{aligned}
C_{(2,5,1,3)} &= \left(1 - \cos\frac{\pi}{2}\right)\left(1 - \cos\frac{3\pi}{5}\right) \\
&= 1 - \cos\frac{3\pi}{5} \\
&= \frac{1}{4}\left(3 + \sqrt{5}\right)\,.
\end{aligned}$$

First we put $n = -1$. By Theorem 4.3,

$$\begin{aligned}
\sigma_{(2,5,-1)}(t) &= Y_{(-1,1,1)}(t) Y_{(-1,1,3)}(t) \\
&= X_{-1}\left(\frac{\sqrt{t}}{2\sqrt{C_{2,5,1,1}}}\right) X_{-1}\left(\frac{\sqrt{t}}{2\sqrt{C_{2,5,1,3}}}\right) \\
&= 4C_{(2,5,1,1)}C_{(2,5,1,3)} \frac{T_{10}\left(\frac{\sqrt{t}}{2\sqrt{C_{2,5,1,1}}}\right) - T_8\left(\frac{\sqrt{t}}{2\sqrt{C_{2,5,1,1}}}\right)}{t - 4C_{(2,5,1,1)}}
\end{aligned}$$

$$\cdot \frac{T_{10}\left(\frac{\sqrt{t}}{2\sqrt{C_{2,5,1,3}}}\right) - T_8\left(\frac{\sqrt{t}}{2\sqrt{C_{2,5,1,3}}}\right)}{t - 4C_{(2,5,1,3)}}$$
$$= 64t^{10} + 384t^9 - 2880t^8 + 5952t^7 + 2336t^6$$
$$- 14856t^5 + 12192t^4 - 4608t^3 + 820t^2 - 60t + 1\,.$$

By the definition,

$$\sigma_{(2p,q,0)}(t) = 1\,.$$

By applying the 3-term realtion

$$Y_{(1,a,b)}(t) = 2T_{10}\left(\frac{\sqrt{t}}{2C_{(2,5,a,b)}}\right) Y_{(0,2p,q)}(t) - Y_{(-1,a,b)}(t)\,,$$

we obtain

$$\sigma_{(2,5,1)}(t) = 256t^{12} + 384t^{11} - 16064t^{10} + 61056t^9 - 72000t^8$$
$$- 57888t^7 + 197424t^6 - 172824t^5 + 273408t^4$$
$$- 16632t^3 + 1880t^2 - 90t + 1\,.$$

REMARK 5.4. For the set of diffeomorphism classes of these homology spheres $M_n = \Sigma(2p, q, N)$, the set $\{\tau_\rho(M_n)\}$ of the values is a perfect invariant. Then the torsion polynomial $\sigma_{(2p,q,n)}(t)$ is also a prefect invariant. That is to say, the set $\{\tau_\rho(M_n)\}$ or the torsion polynomial $\sigma_{(2p,q,n)}(t)$ decides the triple $(2p, q, n)$.

Finally we mention some problems.

PROBLEM 5.5.

- How strong the set of Reidemeister torsions and the torsion polynomial are in general?
- Can this torsion polynomial be computed for any torus knot? The assumption on p is coming from a technical reason to prove.
- Can this torsion polynomial be computed for any homology 3-sphere with the finite set of $\{\tau_\rho\}$?
- How it can be treated for a 3-manifold with the infinite set of $\{\tau_\rho\}$?

REMARK 5.6. Recently we proved the formula of Reidemeister torsion for any homology 3-sphere along the figure-eight knot in [4].

REFERENCES

[1] D. JOHNSON, A geometric form of Casson's invariant and its connection to Reidemeister torsion, unpublished lecture notes.

[2] T. KITANO, Reidemeister torsion of Seifert fibered spaces for $SL(2;\mathbb{C})$ representations, Tokyo J. Math. 17 (1994), 59–75.

[3] T. KITANO, Reidemeister torsion of the figure-eight knot exterior for $SL(2;\mathbb{C})$-representations, Osaka J. Math. 31, (1994), 523–532.

[4] T. KITANO, Reidemeister torsion of a 3-manifold obtained by an integral Dehn- surgery along the figure-eight kno, Kodai Math. J. 39 (2016), 290–296.
[5] J. MILNOR, Two complexes which are homeomorphic but combinatorially distinct, Ann. of Math. 74 (1961), 575–590.
[6] J. MILNOR, A duality theorem for Reidemeister torsion, Ann. of Math. 76 (1962), 137–147.
[7] J. MILNOR, Whitehead torsion, Bull. Amer. Math. Soc. 72 (1966), 348–426.

DEPARTMENT OF INFORMATION SYSTEMS SCIENCE
FACULTY OF SCIENCE AND ENGINEERING
SOKA UNIVERSITY
TANGI-CHO 1–236, HACHIOJI
TOKYO 192–8577
JAPAN

E-mail address: kitano@soka.ac.jp