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MONODROMY REPRESENTATIONS OF HYPERGEOMETRIC SYSTEMS WITH RESPECT TO FUNDAMENTAL SERIES SOLUTIONS

KEIJI MATSUMOTO

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Abstract. We study the monodromy representation of the generalized hypergeometric differential equation and that of Lauricella's F_C system of hypergeometric differential equations. We use fundamental systems of solutions expressed by the hypergeometric series. We express non-diagonal circuit matrices as reflections with respect to root vectors with all entries 1. We present a simple way to obtain circuit matrices.

1. Introduction. The hypergeometric series ${}_2F_1\left(\begin{matrix} a_1, a_2 \\ b_1 \end{matrix}; x\right)$ satisfies the hypergeometric differential equation, which is second order linear, and with regular singular points at $x = 0, 1, \infty$. There are two natural ways to generalize the hypergeometric differential equations: one is to higher rank ordinary differential equations and the other is to integrable systems of differential equations of multi-variables. As the former, generalized hypergeometric series and equations are well known. As the latter, four kinds of hypergeometric series and systems of hypergeometric differential equations are introduced by P. Appell and G. Lauricella.

In this paper, we study the monodromy representation of the generalized hypergeometric differential equation and that of Lauricella's F_C system of hypergeometric differential equations. We use fundamental systems of solutions expressed by hypergeometric series. We express the circuit matrices along generators of the fundamental group of the complement of the singular locus with respect to each fundamental system of solutions. The aim of this paper is the presentation of a simple way to obtain circuit matrices.

Let us explain our method. For each case of the study of monodromy representations, the problem reduces to determining a circuit matrix M, since the others are trivially given as diagonal matrices. We can regard this target circuit matrix M as a complex reflection with respect to a kind of an inner product, i.e., the eigenspace of M of eigenvalue 1 is the orthogonal complement of an eigenvector v of M of eigenvalue $\lambda(\neq 1)$. Let H be the gram matrix of our fundamental system of solutions with respect to this inner product. We can show that it is diagonal. We normalize our fundamental system so that the λ-eigenvector v of M becomes $(1, \ldots, 1)$. Though the matrix H is changed by this normalization, it is still diagonal. By regarding diagonal entries of H as indeterminants, we set up a system of equations by the Riemann scheme or relations arising from the fundamental group. By solving it, we can

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determine H which yields the circuit M. The point is that the target circuit matrix M can be given by the determination of the gram matrix H. There are many studies to characterize the gram matrix (the invariant hermitian form) by circuit matrices. It is shown by our method that its converse is possible for the generalized hypergeometric equation and Lauricella's F_C system.

There are several studies for the monodromy representation of the generalized hypergeometric differential equation, refer to [BH], [Le], [Mi], [Oh] and [Os]. For that of Lauricella's F_C system in two variables, we have many ways to compute circuit matrices, refer to [GM], [HU], [Kan], [Kat] and [T]. The case of m variables, it was an open problem for a long time to determine the monodromy representation. We did not have a simple system of generators of the fundamental group of the complement of the singular locus. Recently, this open problem is solved in [G]: it is shown that the fundamental group is generated by $m+1$ loops, and that the circuit transformations along them can be expressed by the intersection from on twisted homology groups associated with Euler type integral representations of solutions.

This paper consists of four sections. We determine the monodromy representations of the hypergeometric differential equation, of generalized one, and of Lauricella's F_C system in §2, §3 and §4, respectively. We can obtain the results in §2 from those in §3 by regarding the rank p as 2. However, we describe details in §2 since this section helps readers to understand our method well, and results in §2 are needed when we prove the key proposition in §4 by the induction on the number of variables. Our study in §4 is based on some results in [G]. Lemma 4.3 is an addition to them associated with the fundamental group. This lemma relates a product of loops in $\mathbb{C}^m$ to a loop in $\mathbb{C}^{m-1}$, and enables us to decrease the number of variables. Anyone can simply give an expression of the circuit matrix M for the case of two variables by the reduction to results in §2.

In [KMO], we apply our method to the study of the monodromy representation of a regular holonomic system of rank 9 associated with a hypergeometric series in two variables, which is defined by Kampé de Fériet as a generalization of ${}_3F_2$.

2. Monodromy representation of ${}_2F_1$.

2.1. Hypergeometric differential equation. The hypergeometric series ${}_2F_1\left({a_1, a_2 \atop b_1}; x\right)$ is defined by

$${}_2F_1\left({a_1, a_2 \atop b_1}; x\right) = \sum_{n=0}^{\infty} \frac{(a_1, n)(a_2, n)}{(b_1, n)(1, n)} x^n ,$$

where the main variable x is in $\{x \in \mathbb{C} \mid |x| < 1\}$, a_1, a_2, b_1 are complex parameters with $b_1 \notin -\mathbb{N} = \{0, -1, -2, \dots\}$, and Pochhammer's symbol (a, n) stands for $a(a+1)\cdots(a+n-1)$. This function satisfies the hypergeometric differential equation

$$\left[x(1-x)\left(\frac{d}{dx}\right)^2 + \{b_1 - (a_1 + a_2 + 1)x\}\left(\frac{d}{dx}\right) - a_1 a_2\right] f(x) = 0 . \tag{1}$$

This is a Fuchsian differential equation with regular singular points at $x = 0, 1, \infty$. The Riemann scheme of (1) is

$$\begin{array}{ccc} x=0 & x=1 & x=\infty \\ \hline 0 & 0 & a_1 \\ 1-b_1 & b_1-a_1-a_2 & a_2 \end{array} \tag{2}$$

and a fundamental system of solutions to (1) for $b_1 \notin \mathbb{Z}$ around $\dot{x} = \varepsilon$ is given by the column vector

$$\begin{pmatrix} {}_2F_1\left(\begin{matrix} a_1, a_2 \\ b_1 \end{matrix}; x\right) \\ x^{1-b_1}{}_2F_1\left(\begin{matrix} a_1-b_1+1, a_2-b_1+1 \\ 2-b_1 \end{matrix}; x\right) \end{pmatrix},$$

where ε is a sufficiently small positive real number.

2.2. Circuit matrices M_0 and M_1. In this subsection, we assume that

$$a_1,\ a_2,\ b_1,\ a_1-b_1,\ a_2-b_1,\ a_1+a_2-b_1 \notin \mathbb{Z}. \tag{3}$$

We set

$$A_1 = \exp(2\pi\sqrt{-1}a_1), \quad A_2 = \exp(2\pi\sqrt{-1}a_2), \quad B_1 = \exp(2\pi\sqrt{-1}b_1),$$

which are different from 1 under our assumption. Let ρ_0 and ρ_1 be loops in $X = \mathbb{C} - \{0, 1\}$ with base $\dot{x} = \varepsilon$ represented by

$$\begin{array}{rcl} \rho_0 & : & [0,1] \ni t \mapsto \varepsilon e^{2\pi\sqrt{-1}t} \in X, \\ \rho_1 & : & [0,1] \ni t \mapsto 1-(1-\varepsilon)e^{2\pi\sqrt{-1}t} \in X. \end{array} \tag{4}$$

Note that ρ_0 and ρ_1 turn positively around $x = 0$ and $x = 1$ once, respectively. The fundamental group $\pi_1(X, \dot{x})$ is freely generated by these loops. We set $\rho_\infty = (\rho_0 \cdot \rho_1)^{-1}$, where $\rho_0 \cdot \rho_1$ is a loop joining ρ_0 to ρ_1.

We select a fundamental system of solutions to (1) around $\dot{x}$ as

$$\mathbf{F}_2^g(x) = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \begin{pmatrix} {}_2F_1\left(\begin{matrix} a_1, a_2 \\ b_1 \end{matrix}; x\right) \\ x^{1-b_1}{}_2F_1\left(\begin{matrix} a_1-b_1+1, a_2-b_1+1 \\ 2-b_1 \end{matrix}; x\right) \end{pmatrix}, \tag{5}$$

where g_1 and g_2 are non-zero constants. Let ρ be an element of $\pi_1(X, \dot{x})$. Then there exists $M_\rho^g \in GL_2(\mathbb{C})$ such that the analytic continuation of $\mathbf{F}_2^g(x)$ along ρ is expressed as

$$M_\rho^g \mathbf{F}_2^g(x).$$

We call M_ρ^g the circuit matrix along ρ with respect to the basis $\mathbf{F}_2^g(x)$. We set

$$M_0^g = M_{\rho_0}^g, \quad M_1^g = M_{\rho_1}^g, \quad M_\infty^g = M_{\rho_\infty}^g.$$

By the expression of $\mathbf{F}_2^g(x)$, the following is obvious.

LEMMA 2.1. *For any non-zero constants g_1 and g_2, we have*

$$M_0^g = \begin{pmatrix} 1 & 0 \\ 0 & B_1^{-1} \end{pmatrix}.$$

By using an Euler type integral representation of solutions to (1), we can show the following as is in Lemma 5.2 of [Ma].

LEMMA 2.2. *There exists $H \in GL_2(\mathbb{C})$ such that*

$$M_\rho^g H\, {}^t(M_\rho^g)^\vee = H$$

for any $\rho \in \pi_1(X, \dot{x})$, where $z(a_1, a_2, b_1)^\vee = z(-a_1, -a_2, -b_1)$ for any function z of a_1, a_2, b_1, and $Z^\vee = (z_{ij}^\vee)$ for a matrix $Z = (z_{ij})$.

Since $\tilde{g} \in GL_2(\mathbb{C})$ acts on H as $\tilde{g} H\, {}^t\tilde{g}^\vee$, the matrix H depends on g_1 and g_2. We treat the entries of H as indeterminants. By determining them, we express a representation matrix of the circuit transformation along ρ_1.

LEMMA 2.3. *The matrix H in Lemma 2.2 is diagonal.*

PROOF. We set $H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$. By Lemma 2.2, we have

$$\begin{aligned} M_0^g H\, {}^t(M_0^g)^\vee &= \begin{pmatrix} 1 & 0 \\ 0 & B_1^{-1} \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & B_1 \end{pmatrix} \\ &= \begin{pmatrix} h_{11} & B_1 h_{12} \\ B_1^{-1} h_{21} & h_{22} \end{pmatrix} = H. \end{aligned}$$

Since $B_1 \neq 1$ under our assumption, h_{12} and h_{21} should be 0. □

By the Riemann scheme (2), it is easy to see that the eigenvalues of M_1 are 1 and $\lambda = B_1/(A_1 A_2)$. Note that $\lambda \neq 1$ under our assumption.

LEMMA 2.4. *Let $v = (v_1, v_2)$ be the eigenvector of M_1^g of eigenvalue $\lambda = B_1/(A_1 A_2)$, and w be that of eigenvalue 1. Then we have*

$$wH\, {}^t v^\vee = 0, \quad vH\, {}^t v^\vee \neq 0, \quad v_1 v_2 \neq 0.$$

PROOF. By Lemma 2.2, we have

$$\begin{aligned} wH\, {}^t v^\vee &= w(M_1^g H\, {}^t(M_1^g)^\vee)\, {}^t v^\vee = (w M_1^g) H\, {}^t(v M_1^g)^\vee \\ &= wH\, {}^t(\lambda v)^\vee = \frac{1}{\lambda} wH\, {}^t v^\vee. \end{aligned}$$

Since $\lambda \neq 1$, $wH\, {}^t v^\vee$ vanishes.

Note that

$$\begin{pmatrix} v \\ w \end{pmatrix} H({}^t v^\vee, {}^t w^\vee) = \begin{pmatrix} vH\, {}^t v^\vee & 0 \\ 0 & wH\, {}^t w^\vee \end{pmatrix}.$$

Since v and w are linearly independent, if $vH\, {}^t v^\vee = 0$ then H degenerates. This contradicts to $H \in GL_2(\mathbb{C})$. Thus we have $vH\, {}^t v^\vee \neq 0$.

Suppose that $v_1 = 0$. Then $(0, 1)$ is the eigenvector of M_1^g of eigenvalue λ. By the equality $wH\ {}^t v^\vee = 0$, $(1, 0)$ is the eigenvector of M_1^g of eigenvalue 1. Thus we have

$$M_1^g = \begin{pmatrix} 1 & 0 \\ 0 & B_1/(A_1A_2) \end{pmatrix} .$$

The eigenvalues of

$$(M_\infty^g)^{-1} = M_0^g M_1^g = \begin{pmatrix} 1 & 0 \\ 0 & 1/(A_1A_2) \end{pmatrix}$$

are 1 and $1/(A_1A_2)$; this contradicts to the Riemann scheme (2) under our assumption (3). Hence we have $v_1 \neq 0$. We can similarly show $v_2 \neq 0$. □

Note that the eigenvector v of M_1^g of eigenvalue λ depends on g_1 and g_2. We can choose g_1, g_2 in (5) so that the eigenvector v of M_1^g of eigenvalue λ becomes $\mathrm{v} = (1, 1)$. From now on, we fix the constants g_1 and g_2 as the above values. We denote the fundamental system of solutions to (1) around $\dot{x}$ for these constants in (5) by $\mathbf{F}_2(x)$. The circuit matrices along $\rho_0, \rho_1, \rho_\infty$ with respect to $\mathbf{F}_2(x)$ are denoted by M_0, M_1, M_∞, respectively.

LEMMA 2.5. *The circuit matrix M_1 is expressed as*

$$M_1 = id_2 - \frac{1-\lambda}{\mathrm{v}H\ {}^t\mathrm{v}} H\ {}^t\mathrm{v}\mathrm{v} ,$$

where id_m is the unit matrix of size m, $\lambda = B_1/(A_1A_2)$ and $\mathrm{v} = (1, 1)$.

PROOF. We set

$$M_1' = id_2 - \frac{1-\lambda}{\mathrm{v}H\ {}^t\mathrm{v}} H\ {}^t\mathrm{v}\mathrm{v} .$$

We show that the eigenspaces of M_1' coincides with those of M_1. We have

$$\begin{aligned} \mathrm{v}M_1' &= \mathrm{v}\left(id_2 - \frac{1-\lambda}{\mathrm{v}H\ {}^t\mathrm{v}} H\ {}^t\mathrm{v}\mathrm{v} \right) \\ &= \mathrm{v} - (1-\lambda)\mathrm{v} = \lambda \mathrm{v} , \end{aligned}$$

which means v is an eigenvector of M_1' of eigenvalue λ. Let w be a vector satisfying $wH\ {}^t\mathrm{v} = 0$. Then we have

$$\begin{aligned} wM_1' &= w\left(id_2 - \frac{1-\lambda}{\mathrm{v}H\ {}^t\mathrm{v}} H\ {}^t\mathrm{v}\mathrm{v} \right) \\ &= w - \frac{(1-\lambda)wH\ {}^t\mathrm{v}}{\mathrm{v}H\ {}^t\mathrm{v}} \mathrm{v} = w, \end{aligned}$$

which means w is an eigenvector of M_1' of eigenvalue 1. Since M_1 and M_1' have the same eigenspaces, they coincide as matrices. □

We regard the diagonal entries of H as indeterminants in the expression of M_1 in Lemma 2.5. By evaluating them, we determine the circuit matrix M_1. Note that the expression of M_1 in Lemma 2.5 is invariant under a scalar multiple to H. We can assume that

$$H = \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}.$$

PROPOSITION 2.6. *We have*

$$h = -\frac{(B_1 - A_1)(B_1 - A_2)}{B_1(A_1 - 1)(A_2 - 1)},$$

$$M_1 = id_2 - \begin{pmatrix} \dfrac{B_1(A_1-1)(A_2-1)}{A_1A_2(B_1-1)} & \dfrac{B_1(A_1-1)(A_2-1)}{A_1A_2(B_1-1)} \\ -\dfrac{(B_1-A_1)(B_1-A_2)}{A_1A_2(B_1-1)} & -\dfrac{(B_1-A_1)(B_1-A_2)}{A_1A_2(B_1-1)} \end{pmatrix}.$$

PROOF. We compute the trace of M_0M_1, which should be $1/A_1 + 1/A_2$ by the Riemann scheme (2). Since

$$M_0M_1 = \begin{pmatrix} 1 & 0 \\ 0 & B_1^{-1} \end{pmatrix} - \frac{1-\lambda}{1+h}\begin{pmatrix} 1 & 1 \\ B_1^{-1}h & B_1^{-1}h \end{pmatrix},$$

we have

$$\begin{aligned}
\mathrm{tr}(M_0M_1) &= 1 + B_1^{-1} + \frac{(\lambda-1)(1+B_1^{-1}h)}{1+h} \\
&= \frac{(A_1A_2+1)B_1h + A_1A_2 + B_1^2}{A_1A_2B_1(1+h)} = \frac{1}{A_1} + \frac{1}{A_2}.
\end{aligned}$$

We can reduce the last equation to a linear equation with respect to h, which is solved as

$$h = -\frac{(A_1 - B_1)(A_2 - B_1)}{B_1(A_1 - 1)(A_2 - 1)}.$$

We obtain the expression of M_1 by the substitution of this solution into Lemma 2.5. □

REMARK 2.7. Note that

$$\mathrm{v}H\,{}^t\mathrm{v} = \mathrm{tr}(H) = \frac{(A_1A_2 - B_1)(B_1 - 1)}{(A_1 - 1)(A_2 - 1)B_1}.$$

We have

$$\frac{1-\lambda}{\mathrm{v}H\,{}^t\mathrm{v}} = \frac{A_1A_2 - B_1}{A_1A_2} \times \frac{(A_1-1)(A_2-1)B_1}{(A_1A_2 - B_1)(B_1 - 1)} = \frac{(A_1-1)(A_2-1)B_1}{A_1A_2(B_1-1)},$$

in which the factor $A_1A_2 - B_1$ is canceled.

We conclude this subsection by the following.

THEOREM 2.8. *Suppose the non-integral condition* (3) *for* a_1, a_2 *and* b_1. *Then there exists a fundamental system* $\mathbf{F}_2(x)$ *of solutions to the hypergeometric differential equation* (1) *around* $\dot{x} = \varepsilon$ *such that the circuit matrices* M_0 *and* M_1 *along the loops* ρ_0 *and* ρ_1 *in* (4) *are expressed as*

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & B_1^{-1} \end{pmatrix}, \quad M_1 = id_2 - \frac{1-\lambda}{\mathrm{v} H\, {}^t\mathrm{v}} H\, {}^t\mathrm{v}\mathrm{v},$$

where ε *is a sufficiently small positive real number,* $A_1 = e^{2\pi\sqrt{-1}a_1}$, $A_2 = e^{2\pi\sqrt{-1}a_2}$, $B_1 = e^{2\pi\sqrt{-1}b_1}$, $\lambda = B_1/(A_1A_2)$, $\mathrm{v} = (1, 1)$ *and*

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -\dfrac{(A_1 - B_1)(A_2 - B_1)}{B_1(A_1 - 1)(A_2 - 1)} \end{pmatrix}.$$

3. Monodromy representation of ${}_pF_{p-1}$.

3.1. Generalized hypergeometric differential equation. The generalized hypergeometric series is defined by

$${}_pF_{p-1}\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_{p-1} \end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(a_1, n) \cdots (a_p, n)}{(b_1, n) \cdots (b_{p-1}, n)(1, n)} x^n,$$

where the main variable x is in $\{x \in \mathbb{C} \mid |x| < 1\}$, $a_1, \dots, a_p, b_1, \dots, b_{p-1}$ are complex parameters with $b_1, \dots, b_{p-1} \notin -\mathbb{N}$. This series satisfies the differential equation of rank p:

$$\begin{aligned} &\left(x\frac{d}{dx} + a_1\right) \cdots \left(x\frac{d}{dx} + a_p\right) f(x) \\ &= \frac{d}{dx}\left(x\frac{d}{dx} + b_1 - 1\right) \cdots \left(x\frac{d}{dx} + b_{p-1} - 1\right) f(x). \end{aligned} \tag{6}$$

This is a Fuchsian differential equation with regular singular points at $x = 0, 1, \infty$. The Riemann scheme of (6) is

(7)

$x = 0$	$x = 1$	$x = \infty$
0	0	a_1
$1 - b_1$	1	a_2
$\vdots$	$\vdots$	$\vdots$
$1 - b_{p-2}$	$p - 2$	a_{p-1}
$1 - b_{p-1}$	$\sum_{j=1}^{p-1} b_j - \sum_{i=1}^{p} a_i$	a_p

and a fundamental system of solutions to (6) for $b_1, \dots, b_{p-1} \notin \mathbb{Z}$ around $\dot{x} = \varepsilon$ is given by

$$
\begin{pmatrix}
{}_pF_{p-1}\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_{p-1} \end{matrix}; x\right) \\
x^{1-b_1}\,{}_pF_{p-1}\left(\begin{matrix} a_1 - b_1 + 1, \dots, a_p - b_1 + 1 \\ 2 - b_1, b_2 - b_1 + 1, \dots, b_{p-1} - b_1 + 1 \end{matrix}; x\right) \\
\vdots \\
x^{1-b_{p-1}}\,{}_pF_{p-1}\left(\begin{matrix} a_1 - b_{p-1} + 1, \dots, a_p - b_{p-1} + 1 \\ b_1 - b_{p-1} + 1, \dots, b_{p-2} - b_{p-1} + 1, 2 - b_{p-1} \end{matrix}; x\right)
\end{pmatrix}, \tag{8}
$$

where ε is a sufficiently small positive real number. Note that there are $p-1$ linearly independent holomorphic solutions to (6) on an annulus $\{x \in \mathbb{C} \mid 0 < |x-1| < \varepsilon\}$.

3.2. Circuit matrices M_0 and M_1. In this subsection, we assume that

$$
a_i\,,\;\; b_j\,,\;\; a_i - b_j\,,\;\; a_i - a_{i'}\,,\;\; b_j - b_{j'}\,,\;\; \sum_{i=1}^{p} a_i - \sum_{j=1}^{p-1} b_j \notin \mathbb{Z}\,, \tag{9}
$$

where $1 \le i, i' \le p$, $1 \le j, j' \le p-1$, $i \ne i'$ and $j \ne j'$. We set

$$
A_i = \exp(2\pi\sqrt{-1}a_i)\,, \quad B_j = \exp(2\pi\sqrt{-1}b_j)\,,
$$

for $1 \le i \le p$ and $1 \le j \le p-1$. We choose a fundamental system $\mathbf{F}_p^g(x)$ of solutions to (6) around $\dot{x} = \varepsilon$ as the left multiplication of the diagonal matrix

$$
g = \begin{pmatrix} g_1 & & & \\ & g_2 & & \\ & & \ddots & \\ & & & g_p \end{pmatrix} \in GL_p(\mathbb{C})
$$

to the column vector (8).

Let M_0^g and M_1^g be the circuit matrices along the loops ρ_0 and ρ_1 in (4) with respect to $\mathbf{F}_p^g(x)$. We set $M_\infty^g = (M_0^g M_1^g)^{-1}$.

LEMMA 3.1. *For any diagonal matrix $g \in GL_p(\mathbb{C})$, the circuit matrix M_0^g is*

$$
\begin{pmatrix} 1 & & & \\ & B_1^{-1} & & \\ & & \ddots & \\ & & & B_{p-1}^{-1} \end{pmatrix}.
$$

PROOF. It is clear by (8). □

As is in Subsection 2.2, we have the following lemma.

LEMMA 3.2. *Let M^g_ρ be the circuit matrix along $\rho \in \pi_1(X, \dot{x})$ with respect to $\mathbf{F}^g_p(x)$. Then there exists a diagonal matrix $H \in GL_p(\mathbb{C})$ such that*

$$M^g_\rho H\, {}^t(M^g_\rho)^\vee = H\,,$$

where $z(a_1, \dots, a_p, b_1, \dots, b_{p-1})^\vee = z(-a_1, \dots, -a_p, -b_1, \dots, -b_{p-1})$ for any function z of the parameters.

The matrix H depends on $g_1, \dots, g_p$. We treat the entries of H as indeterminants.

By the Riemann scheme (7) and our assumption (9), the eigenvalues of M^g_1 are 1 and

$$\lambda = \Big(\prod_{j=1}^{p-1} B_j\Big) \Big/ \Big(\prod_{i=1}^{p} A_i\Big);$$

the eigenspace of M^g_1 of eigenvalue 1 is $p-1$ dimensional and that of eigenvalue λ is one dimensional.

LEMMA 3.3. *Let $v = (v_1, \dots, v_p)$ be an eigenvector of M^g_1 of eigenvalue λ. Then the eigenspace of M^g_1 of eigenvalue 1 is characterized as*

$$\{w \in \mathbb{C}^p \mid wH\, {}^t v^\vee = 0\}\,.$$

Moreover, the vector v satisfies

$$vH\, {}^t v^\vee \neq 0\,.$$

PROOF. Trace the proof of Lemma 2.4. □

LEMMA 3.4. *Let $v = (v_1, \dots, v_p)$ be an eigenvector of M^g_1 of eigenvalue λ. Then the circuit matrix M^g_1 is expressed as*

$$M^g_1 = id_p - \frac{1-\lambda}{vH\, {}^t v^\vee} H\, {}^t v^\vee v\,.$$

Moreover, none of $v_1, \dots, v_p$ vanishes.

PROOF. We set

$$M'_1 = id_p - \frac{1-\lambda}{vH\, {}^t v^\vee} H\, {}^t v^\vee v\,.$$

We show that the eigenspaces of M'_1 coincides with those of M^g_1. Note that

$$vM'_1 = v\left(id_p - \frac{1-\lambda}{vH\, {}^t v^\vee} H\, {}^t v^\vee v\right) = v - (1-\lambda)v = \lambda v\,,$$
$$wM'_1 = w\left(id_p - \frac{1-\lambda}{vH\, {}^t v^\vee} H\, {}^t v^\vee v\right) = w - \frac{(1-\lambda)wH\, {}^t v^\vee}{vH\, {}^t v^\vee} v = w\,,$$

for any element w satisfying $wH\, {}^t v^\vee = 0$. By Lemma 3.3, we have $M'_1 = M^g_1$.

Suppose that $v_i = 0$. Then the matrix M_1^g takes the form

$$\begin{array}{cc} & \begin{array}{ccc} & i & \end{array} \\ i & \begin{pmatrix} * & {}^t\mathbf{0} & * \\ \mathbf{0} & 1 & \mathbf{0}' \\ * & {}^t\mathbf{0}' & * \end{pmatrix} \end{array}$$

by its expression, where $\mathbf{0}$ and $\mathbf{0}'$ are zero vectors. Since M_0^g is diagonal, we have

$$M_0^g M_1^g = \begin{array}{cc} & \begin{array}{ccc} & i & \end{array} \\ i & \begin{pmatrix} * & {}^t\mathbf{0} & * \\ \mathbf{0} & B_{i-1}^{-1} & \mathbf{0}' \\ * & {}^t\mathbf{0}' & * \end{pmatrix} \end{array},$$

where we regard B_0 as 1. Hence M_∞ has an eigenvalue B_{i-1}, which contradicts to the Riemann scheme (7) under our assumption (9). Therefore, we have $v_i \neq 0$ for $1 \leq i \leq p$. □

We choose $g_1, \ldots, g_p$ so that the eigenvector of eigenvalue λ becomes $\mathrm{v} = (1, \ldots, 1)$. From now on, we fix the constants $g_1, \ldots, g_p$ as the above values. We denote the fundamental system of solutions to (6) around $\dot{x}$ for these constants in $\mathbf{F}_p^g(x)$ by $\mathbf{F}_p(x)$. The circuit matrices with respect to $\mathbf{F}_p(x)$ are expressed by

$$(10) \qquad M_0 = \begin{pmatrix} 1 & & & \\ & B_1^{-1} & & \\ & & \ddots & \\ & & & B_{p-1}^{-1} \end{pmatrix}, \quad M_1 = id_p - \frac{1-\lambda}{\mathrm{v}H\,{}^t\mathrm{v}} H\,{}^t\mathrm{v}\mathrm{v}\,.$$

Here we regard the diagonal entries of H as indeterminants in the expression of M_1. By evaluating them, we determine the expression of M_1. Note that the expression of M_1 is invariant under a scalar multiple to H. We can assume that

$$(11) \qquad H = \begin{pmatrix} 1 & & & \\ & h_1 & & \\ & & \ddots & \\ & & & h_{p-1} \end{pmatrix}.$$

Note that the matrix H is unique after this normalization.

PROPOSITION 3.5. *For $1 \leq k \leq p-1$, we have*

$$h_k = \frac{-\left(\prod_{1 \leq j \leq p-1}^{j \neq k} (B_j - 1)\right)\left(\prod_{i=1}^{p} (A_i - B_k)\right)}{B_k \left(\prod_{1 \leq j \leq p-1}^{j \neq k} (B_j - B_k)\right)\left(\prod_{i=1}^{p} (A_i - 1)\right)}.$$

PROOF. We consider the eigen polynomial

$$Q(t) = \det(t \cdot id_p - M_0 M_1)$$

of the matrix $M_0M_1 = M_\infty^{-1}$. By the Riemann scheme (7), $1/A_1, \dots, 1/A_p$ are solutions to the equation $Q(t) = 0$. Thus we have

$$\begin{aligned}
&\det(M_0M_1 - id_p/A_\ell)\\
&= \begin{vmatrix} d_0+\mu & \mu & \mu & \cdots & \mu \\ \mu B_1^{-1}h_1 & d_1+\mu B_1^{-1}h_1 & \mu B_1^{-1}h_1 & \cdots & \mu B_1^{-1}h_1 \\ \mu B_2^{-1}h_2 & \mu B_2^{-1}h_2 & d_2+\mu B_2^{-1}h_2 & \cdots & \mu B_2^{-1}h_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu B_{p-1}^{-1}h_{p-1} & \mu B_{p-1}^{-1}h_{p-1} & \mu B_{p-1}^{-1}h_{p-1} & \cdots & d_{p-1}+\mu B_{p-1}^{-1}h_{p-1} \end{vmatrix}\\
&= \begin{vmatrix} d_0+\mu & \mu & \mu & \cdots & \mu \\ -d_0B_1^{-1}h_1 & d_1 & 0 & \cdots & 0 \\ -d_0B_2^{-1}h_2 & 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -d_0B_{p-1}^{-1}h_{p-1} & 0 & 0 & \cdots & d_{p-1} \end{vmatrix}\\
&= \frac{1}{1+h_1+\cdots+h_{p-1}} \begin{vmatrix} \nu & \lambda-1 & \lambda-1 & \cdots & \lambda-1 \\ -d_0B_1^{-1}h_1 & d_1 & 0 & \cdots & 0 \\ -d_0B_2^{-1}h_2 & 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -d_0B_{p-1}^{-1}h_{p-1} & 0 & 0 & \cdots & d_{p-1} \end{vmatrix} = 0\,,
\end{aligned}$$

where $\mu = \frac{\lambda-1}{1+h_1+\cdots+h_{p-1}}$, $\nu = d_0(h_1+\cdots+h_{p-1}) + \lambda - 1/A_\ell$ and

$$d_0 = \frac{A_\ell - 1}{A_\ell}, \quad d_1 = \frac{A_\ell - B_1}{A_\ell B_1}, \dots, \ d_{p-1} = \frac{A_\ell - B_{p-1}}{A_\ell B_{p-1}}\,.$$

The last determinant is linear with respect to $h_1, \dots, h_{p-1}$ since these variables appear only in the first column as linear terms. By the cofactor expansion with respect to the first column, we can evaluate its coefficient of h_k and its constant term. By multiplying $A_\ell^{p-1}\big(\prod_{i=1}^p A_i\big)\big(\prod_{j=1}^{p-1} B_j\big)$ to them, we have a linear equation

$$\begin{aligned}
&-\sum_{k=1}^{p-1} B_k(A_\ell - 1)\Bigg(\prod_{1\le j\le p-1}^{j\ne k} (A_\ell - B_j)\Bigg)\Bigg(\prod_{1\le i\le p}^{i\ne\ell} A_i - \prod_{1\le j\le p-1}^{j\ne k} B_j\Bigg)h_k\\
&= \Bigg(\prod_{j=1}^{p-1}(A_\ell - B_j)\Bigg)\Bigg(\prod_{1\le i\le p}^{i\ne\ell} A_i - \prod_{j=1}^{p-1} B_j\Bigg)
\end{aligned}$$

from $Q(1/A_\ell) = 0$. By letting ℓ vary from 1 to p, we have a system of p linear equations with respect to $h_1, \dots, h_{p-1}$ under the assumption (9). Since $\det(M_0M_1) = 1/\prod_{i=1}^p A_i$ for any $h_1, \dots, h_{p-1}$, the last equation $Q(1/A_p) = 0$ is not independent of the others. Thus this system is of rank $p-1$ and has a unique solution. We can check that

$$h_k = \frac{-\left(\prod_{1\le j\le p-1}^{j\neq k}(B_j-1)\right)\left(\prod_{i=1}^{p}(A_i-B_k)\right)}{B_k\left(\prod_{1\le j\le p-1}^{j\neq k}(B_j-B_k)\right)\left(\prod_{i=1}^{p}(A_i-1)\right)} \quad (1 \le k \le p-1)$$

satisfy this system. The uniqueness of H completes this proposition. □

REMARK 3.6. Note that

$$\mathrm{v}H\,{}^t\mathrm{v} = \mathrm{tr}(H) = \frac{\left(\prod_{i=1}^{p}A_i - \prod_{j=1}^{p-1}B_j\right)\prod_{j=1}^{p-1}(B_j-1)}{\prod_{i=1}^{p}(A_i-1)\prod_{j=1}^{p-1}B_j}.$$

We have

$$\frac{1-\lambda}{\mathrm{v}H\,{}^t\mathrm{v}} = \frac{\prod_{i=1}^{p}(A_i-1)\prod_{j=1}^{p-1}B_j}{\prod_{i=1}^{p}A_i\prod_{j=1}^{p-1}(B_j-1)},$$

in which the factor $\prod_{i=1}^{p} A_i - \prod_{j=1}^{p-1} B_j$ in $1-\lambda$ and $\mathrm{v}H\,{}^t\mathrm{v}$ is canceled.

We conclude this subsection by the following.

THEOREM 3.7. *Suppose the non-integral condition* (9) *for* $a_1, \dots, a_p$, $b_1, \dots, b_{p-1}$. *Then there exists a fundamental system* $\mathbf{F}_p(x)$ *of solutions to the hypergeometric differential equation* (6) *around* $\dot{x} = \varepsilon$ *such that the circuit matrices* M_0 *and* M_1 *along the loops* ρ_0 *and* ρ_1 *in* (4) *are expressed as*

$$M_0 = \begin{pmatrix} 1 & & & \\ & B_1^{-1} & & \\ & & \ddots & \\ & & & B_{p-1}^{-1} \end{pmatrix}, \quad M_1 = id_p - \frac{1-\lambda}{\mathrm{v}H\,{}^t\mathrm{v}} H\,{}^t\mathrm{v}\mathrm{v},$$

where ε *is a sufficiently small positive real number,* $A_i = e^{2\pi\sqrt{-1}a_i}$ $(1 \le i \le p)$, $B_j = e^{2\pi\sqrt{-1}b_j}$ $(1 \le j \le p-1)$, $\lambda = \left(\prod_{j=1}^{p-1} B_j\right)/\left(\prod_{i=1}^{p} A_i\right)$, $\mathrm{v} = (1, \dots, 1)$ *and*

$$H = \begin{pmatrix} 1 & & & \\ & h_1 & & \\ & & \ddots & \\ & & & h_{p-1} \end{pmatrix},$$

$$h_k = \frac{-\left(\prod_{1\le j\le p-1}^{j\ne k}(B_j-1)\right)\left(\prod_{i=1}^{p}(A_i-B_k)\right)}{B_k\left(\prod_{1\le j\le p-1}^{j\ne k}(B_j-B_k)\right)\left(\prod_{i=1}^{p}(A_i-1)\right)} \quad (1\le k\le p-1)\,.$$

4. Monodromy representation of F_C.

4.1. Lauricella's F_C system. In this subsection, we refer to [AK], [HT] and [La] for fundamental properties of Lauricella's F_C system. Lauricella's hypergeometric series F_C is defined by

$$F_C\left(\begin{matrix}a_1, a_2\\ b_1,\ldots,b_m\end{matrix}; x_1,\ldots,x_m\right) = \sum_{n_1,\ldots,n_m\in\mathbb{N}^m} \frac{(a_1, n_1+\cdots+n_m)(a_2, n_1+\cdots+n_m)}{(b_1,n_1)\cdots(b_m,n_m)(1,n_1)\cdots(1,n_m)} x_1^{n_1}\cdots x_m^{n_m}\,,$$

where the vector $x=(x_1,\ldots,x_m)$ consisting of the main variables is in

$$\{x\in\mathbb{C}^m \mid \sqrt{|x_1|}+\cdots+\sqrt{|x_m|}<1\}\,,$$

and $a_1, a_2, b_1, \ldots, b_m$ are complex parameters with $b_1,\ldots,b_m\notin -\mathbb{N}$. This series satisfies differential equations

$$\Bigg[x_i(1-x_i)\partial_i^2 - x_i\sum_{1\le j\le m}^{j\ne i} x_j\partial_i\partial_j - \sum_{1\le j_1,j_2\le m}^{j_1\ne i} x_{j_1}x_{j_2}\partial_{j_1}\partial_{j_2} + \{b_i-(a_1+a_2+1)x_i\}\partial_i - (a_1+a_2+1)\sum_{1\le j\le m}^{j\ne i} x_j\partial_j - a_1a_2\Bigg] f(x) = 0\,,$$

$(i=1,\ldots,m)$, which generate Lauricella's F_C system of hypergeometric differential equations. Here ∂_i is the partial differential operator with respect to x_i. Lauricella's F_C system is integrable of rank 2^m and regular singular with singular locus

$$S_m = \{x\in\mathbb{C}^m \mid x_1\cdots x_m R_m(X) = 0\}\,,$$

where $R_m(x)$ is a polynomial of degree 2^{m-1} given by

$$\prod_{\sigma_1,\ldots,\sigma_m=\pm 1}(1+\sigma_1\sqrt{x_1}+\cdots+\sigma_m\sqrt{x_m})\,.$$

FACT 4.1 ([La]). *If $b_1, \dots, b_m \notin \mathbb{Z}$ then a fundamental system of solutions to Lauricella's F_C system around $\dot{x} = (\varepsilon_1, \dots, \varepsilon_m)$ is given as follows:*

1	$F_C\left(\begin{matrix} a_1, a_2 \\ b_1, \dots, b_m \end{matrix}; x\right)$
m	$\vdots$ $x_j^{1-b_j} F_C\left(\begin{matrix} a_1 - b_j + 1, a_2 - b_j + 1 \\ b_1, \dots, 2 - b_j, \dots, b_m \end{matrix}; x\right)$ $\vdots$
$\vdots$	$\vdots$
$\binom{m}{r}$	$\vdots$ $\left[\prod_{j \in J_r} x_j^{1-b_j}\right] F_C\left(\begin{matrix} a_1 + \sum_{j \in J_r}(1-b_j), a_2 + \sum_{j \in J_r}(1-b_j) \\ b_1 + 2\delta_{1,J_r}(1-b_1), \dots, b_m + 2\delta_{m,J_r}(1-b_m) \end{matrix}; x\right)$ $\vdots$
$\vdots$	$\vdots$
1	$\left[\prod_{j=1}^{m} x_j^{1-b_j}\right] F_C\left(\begin{matrix} a_1 + \sum_{j=1}^{m}(1-b_j), a_2 + \sum_{j=1}^{m}(1-b_j) \\ 2 - b_1, \dots, 2 - b_m \end{matrix}; x\right)$

where $\varepsilon_1, \dots, \varepsilon_m$ are sufficiently small positive real numbers satisfying

$$\varepsilon_1 \gg \cdots \gg \varepsilon_m ,$$

and J_r is a subset of $\{1, \dots, m\}$ of cardinality r, and

$$\delta_{i,J_r} = \begin{cases} 1 & \text{if} \quad i \in J_r , \\ 0 & \text{if} \quad i \notin J_r . \end{cases} \tag{12}$$

We denote the solution with the factor $\prod_{j \in J_r} x_j^{1-b_j}$ in Fact 4.1 by $F_C^{J_r}(x)$. For the empty set $J_0 = \phi$, we omit J_0 from this expression, i.e.,

$$F_C^{J_0}(x) = F_C^{\phi}(x) = F_C\left(\begin{matrix} a_1, a_2 \\ b_1, \dots, b_m \end{matrix}; x\right).$$

4.2. Circuit matrices of Lauricella's F_C. In this subsection, we assume that

$$b_1, \dots, b_m,\ a_1 - \sum_{j \in J} b_j,\ a_2 - \sum_{j \in J} b_j,\ a_1 + a_2 - \sum_{j=1}^{m} b_j + \frac{m+1}{2} \notin \mathbb{Z}, \tag{13}$$

where J runs over the subsets of $\{1, \dots, m\}$. We set

$$A_i = \exp(2\pi\sqrt{-1}a_i)\ (i = 1, 2), \quad B_j = \exp(2\pi\sqrt{-1}b_j)\ (1 \le j \le m).$$

We choose a fundamental system $\mathbf{F}_C^g(x)$ of solutions to Lauricella's system of F_C around $\dot{x} = (\varepsilon_1, \dots, \varepsilon_m)$ as

$$\mathbf{F}_C^g(x) = g \begin{pmatrix} F_C(x) \\ \vdots \\ F_C^J(x) \\ \vdots \\ F_C^{J_m}(x) \end{pmatrix}, \quad g = \mathrm{diag}(g_\phi, \dots, g_J, \dots, g_{J_m}) \in GL_{2^m}(\mathbb{C}),$$

where $\mathrm{diag}(z_1, \dots, z_m)$ denotes the diagonal matrix with diagonal entries $z_1, \dots, z_m$, $J \subset \{1, \dots, m\}$ are arranged lexicographically, i.e,

$$J_0 = \phi, \{1\}, \{2\}, \{1,2\}, \{3\}, \dots, \{1,2,3\}, \{4\}, \dots, \{1, \dots, m\} = J_m .$$

Note that the order of J from the smallest is

$$2^J = 1 + \sum_{i=1}^{m} \delta_{i,J} 2^{i-1} = 1 + \delta_{1,J}2^0 + \delta_{2,J}2^1 + \delta_{3,J}2^2 + \cdots + \delta_{m,J}2^{m-1},$$

where $\delta_{i,J}$ is given in (12).

Let X be the complement of the singular locus S_m in $\mathbb{C}^m$. Let ρ be a loop in X with base point $\dot{x} = (\varepsilon_1, \dots, \varepsilon_m)$. Then there exists $M_\rho \in GL_{2^m}(\mathbb{C})$ such that the analytic continuation of $\mathbf{F}_C^g(x)$ along ρ is expressed as $M_\rho^g \mathbf{F}_C^g(x)$. We call M_ρ^g the circuit matrix of Lauricella's system F_C with respect to the fundamental system $\mathbf{F}_C^g(x)$.

We give a system of generators of the fundamental group $\pi_1(X, \dot{x})$.

FACT 4.2 ([G]). *Let ρ_i ($1 \le i \le m$) be a loop defined by*

$$\rho_i : [0,1] \ni t \mapsto \Big(\varepsilon_1, \dots, \varepsilon_{i-1}, \overset{i\text{-}th}{\varepsilon_i e^{2\pi\sqrt{-1}t}}, \varepsilon_{i+1}, \dots, \varepsilon_m\Big) \in X,$$

and let ρ_{m+1} be a loop in the intersection of X and the line

$$L = \{\dot{x} \cdot t \in \mathbb{C}^m \mid t \in \mathbb{C}\}$$

starting from $\dot{x}$, turning around the nearest point of the intersection $S_m \cap L$ to $\dot{x}$ once positively, and tracing back to $\dot{x}$. Then these loops generate the fundamental group $\pi_1(X, \dot{x})$, and satisfy the relations

$$\rho_j \rho_i = \rho_i \rho_j, \quad (\rho_i \rho_{m+1})^2 = (\rho_{m+1}\rho_i)^2, \quad (1 \le i < j \le m).$$

LEMMA 4.3. *We have*

$$(\rho_{m+1} \cdot \rho_m \cdot \rho_{m+1} \cdot \rho_m^{-1}) \cdot \rho_m = \rho_m \cdot (\rho_{m+1} \cdot \rho_m \cdot \rho_{m+1} \cdot \rho_m^{-1}),$$
$$\rho_{m+1} \cdot \rho_m \cdot \rho_{m+1} \cdot \rho_m^{-1} \overset{\widehat{X}}{\sim} \rho_m',$$

where ρ'_m is the generator of $\pi_1(X', \dot{x}')$ for $X' = \mathbb{C}^{m-1} - S_{m-1}$ embedded into the hyperplane $x_m = 0$ in the space $\widehat{X} = \{x \in \mathbb{C}^m \mid x_1 \cdots x_{m-1} R_m(x) \neq 0\}$ with base point $\dot{x}' = (\varepsilon_1, \dots, \varepsilon_{m-1}) \in X'$, and $\overset{\widehat{X}}{\sim}$ denotes the homotopy equivalence in $\widehat{X}$.

PROOF. It is a direct consequence from Fact 4.2 that $\rho_{m+1} \cdot \rho_m \cdot \rho_{m+1} \cdot \rho_m^{-1}$ commutes with ρ_m. Let the line L move along ρ_m. By tracing the deformation of ρ_{m+1}, we have a loop starting from $\dot{x}$, turning around the second nearest point $S_m \cap L$ to $\dot{x}$ once positively, and tracing back to $\dot{x}$. Since the base point $\dot{x}$ moves along ρ_m, this deformation is homotopic to $\rho_m \cdot \rho_{m+1} \cdot \rho_m^{-1}$. Thus the loop

$$\rho_{m+1} \cdot (\rho_m \cdot \rho_{m+1} \cdot \rho_m^{-1})$$

turns around the first and second nearest points $S_m \cap L$ to $\dot{x}$ once positively. Consider the limit as $x_m \to 0$. These points meets and the polynomial $R_m(x_1, \dots, x_m)$ reduces to $R_{m-1}(x_1, \dots, x_{m-1})^2$. Moreover the duplicated point is the nearest point of the intersection $S_{m-1} \cap L'$ to $\dot{x}'$. Hence the loop $\rho_{m+1} \cdot \rho_m \cdot \rho_{m+1} \cdot \rho_m^{-1}$ is homotopic to ρ'_m. □

REMARK 4.4. By the symmetry of the space X, Lemma 4.3 is valid for the replacement $\rho_m \to \rho_i$ $(i = 1, \dots, m-1)$ with changing the embedding $X' \hookrightarrow \mathbb{C}^m$.

We set

$$M_i^g = M_{\rho_i}^g \quad (1 \le i \le m+1)\,.$$

LEMMA 4.5. *The circuit matrix M_i^g of Lauricella's system F_C is a diagonal matrix whose entry corresponding to a subset J of $\{1, \dots, m\}$ is*

$$B_i^{-\delta_{i,J}} = \begin{cases} \dfrac{1}{B_i} & \text{if} \quad i \in J\,, \\ 1 & \text{if} \quad i \notin J\,, \end{cases}$$

where $B_i = \exp(2\pi\sqrt{-1} b_i)$. It is independent of the diagonal matrix $g \in GL_{2^m}(\mathbb{C})$.

PROOF. We have only to note that the solution F_C^J has a factor $x_i^{1-b_i}$ if and only if $i \in J$. □

There are 2^{m-1} subsets J's such that $i \in J$ for any $1 \le i \le m$. The both eigenspaces of M_i of eigenvalue $1/B_i$ and of eigenvalue 1 are 2^{m-1} dimensional.

We need the following two facts given in [G].

FACT 4.6. *Let M_ρ^g be the circuit matrix along $\rho \in \pi_1(X, \dot{x})$ with respect to $\mathbf{F}_C^g(x)$. Then there exists a diagonal matrix $H \in GL_{2^m}(\mathbb{C})$ such that*

$$M_\rho^g H\, {}^t(M_\rho^g)^\vee = H\,,$$

where $z(a_1, a_2, b_1, \dots, b_m)^\vee = z(-a_1, -a_2, -b_1, \dots, -b_m)$ for any function z of the parameters.

Note that the matrix H depends on the diagonal matrix $g \in GL_{2^m}(\mathbb{C})$. We treat the entries of H as indeterminants.

FACT 4.7. *The eigenvalues of the circuit matrix M^g_{m+1} consists of 1 and λ. The eigenspace of eigenvalue λ is spanned by a row vector v. The eigenspace of eigenvalue 1 is $2^m - 1$ dimensional.*

REMARK 4.8. It is shown in [G] that the eigenvalue λ of the circuit matrix M_{m+1} is

$$(-1)^{m+1}\left(\prod_{j=1}^{m} B_j\right)\Big/(A_1A_2)\,,$$

which is different from 1 under our assumption, where $A_i = \exp(2\pi\sqrt{-1}a_i)$ $(i = 1, 2)$. In this subsection, we treat λ as an indeterminant different from 1, and we show that λ should take the above value.

LEMMA 4.9. *Let $v = (\dots, v_{J_r}, \dots)$ be an eigenvector of M^g_{m+1} of eigenvalue λ. Then the eigenspace of M^g_{m+1} of eigenvalue 1 is characterized as*

$$\{w \in \mathbb{C}^{2^m} \mid wH\,{}^t v^\vee = 0\}\,.$$

Moreover, the vector v satisfies

$$vH\,{}^t v^\vee \neq 0\,.$$

PROOF. Trace the proof of Lemma 2.4. □

LEMMA 4.10. *Let v be an eigenvector of M^g_{m+1} of eigenvalue λ. Then the circuit matrix M^g_{m+1} is expressed as*

$$M^g_{m+1} = id_{2^m} - \frac{1-\lambda}{vH\,{}^t v^\vee} H\,{}^t v^\vee v\,.$$

Moreover, no entry of v vanishes.

PROOF. For the expression of M^g_{m+1}, trace the proof of Lemma 3.4. We show that the j-th entry v_j of v does not vanish. Under our assumption (13), Lauricella's F_C system is irreducible by Theorem 13 in [HT]. Suppose that $v_j = 0$. Then the matrix M^g_{m+1} takes the form

$$\begin{array}{cc} & \begin{array}{ccc} & j & \end{array} \\ j & \begin{pmatrix} * & {}^t\mathbf{0} & * \\ \mathbf{0} & 1 & \mathbf{0}' \\ * & {}^t\mathbf{0}' & * \end{pmatrix} \end{array}$$

by its expression, where $\mathbf{0}$ and $\mathbf{0}'$ are zero vectors. Since M^g_i $(1 \le i \le m)$ are diagonal, the space spanned by the j-th unit vector is invariant under the actions of circuit matrices. This contradicts to the irreducibility of the system. Therefore, we have $v_j \neq 0$. □

We choose $g \in GL_{2^m}(\mathbb{C})$ so that the eigenvector of eigenvalue λ becomes $\mathrm{v} = (1, \dots, 1)$. From now on, we fix the entries of g as above values. We denote the fundamental system of solutions to Lauricella's F_C around $\dot{x}$ for this g in $\mathbf{F}_C^g(x)$ by $\mathbf{F}_C(x)$. We denote the circuit matrices with respect to $\mathbf{F}_C(x)$ by $M_1, \dots, M_m$ and M_{m+1}. Explicit forms of $M_1, \dots, M_m$ are given in Lemma 4.5, and we have

$$M_{m+1} = id_{2^m} - \frac{1-\lambda}{\mathrm{v}H\,{}^t\mathrm{v}} H\,{}^t\mathrm{v}\mathrm{v}\,,$$

where we regard λ and the entries of H as indeterminants. By evaluating them, we determine the expression of M_{m+1}. By a scalar multiplication to H, we can assume that

$$H = \mathrm{diag}(1, \dots, h_J, \dots)\,,$$

where J runs over the non-empty subsets of $\{1, \dots, m\}$ arranged lexicographically. Note that the matrix H is unique after this normalization.

LEMMA 4.11. *The eigenspace of M_{m+1} of eigenvalue* 1 *is spanned by row vectors*

$$h_J e_\phi - e_J, \quad \phi \neq J \subset \{1, \dots, m\}\,,$$

where $e_\phi = (1, 0, \dots, 0) \in \mathbb{N}^{2^m}$ and e_J is the 2^J-th unit vector of size 2^m.

PROOF. Since $\mathrm{v} = (1, \dots, 1)$, and $H = \mathrm{diag}(1, \dots, h_J, \dots)$, we have

$$(h_J e_\phi - e_J)H\,{}^t\mathrm{v}^\vee = (h_J e_\phi - h_J e_J)\,{}^t\mathrm{v} = h_J - h_J = 0\,.$$

By Lemma 4.9, these vectors span the eigenspace of M_{m+1} of eigenvalue 1. □

PROPOSITION 4.12. *We have*

$$h_J = (-1)^{|J|} \frac{\left(A_1 - \prod_{j\in J} B_j\right)\left(A_2 - \prod_{j\in J} B_j\right)}{(A_1-1)(A_2-1)\prod_{j\in J} B_j}\,,$$

$$\mathrm{tr}(H) = \frac{\left(A_1A_2 + (-1)^m \prod_{j=1}^m B_j\right)\prod_{j=1}^m (B_j - 1)}{(A_1-1)(A_2-1)\prod_{j=1}^m B_j}\,,$$

$$\lambda = (-1)^{m+1}\left(\prod_{j=1}^m B_j\right)\Big/(A_1A_2)\,,$$

where $|J|$ is the cardinality of J.

PROOF. At first, we determine the entries of H. We use the induction on m. We have shown in Proposition 2.6 that our assertion holds for $m = 1$.

Assume that our assertion holds for $m-1$. From our fundamental system $\mathbf{F}_C(x)$ to Lauricella's system F_C, we choose the 2^{m-1} solutions corresponding to the subsets of $\{1, \dots, m-1\}$ and restrict to the hyperplane $x_m = 0$. Then we have the fundamental system $\mathbf{F}'_C(x)$ to Lauricella's system F_C of the $m-1$ variables $x_1, \dots, x_{m-1}$. Note that the top-left block matrix of M_i $(1 \le i \le m-1)$ of size 2^{m-1} coincides with the circuit matrix M_i' for this fundamental system $\mathbf{F}_C'(x)$. By Lemma 4.3, the matrix $M_{m+1}M_mM_{m+1}M_m^{-1}$ commutes with M_m. Thus it is block diagonal with block size 2^{m-1}, i.e.,

$$M_{m+1}M_mM_{m+1}M_m^{-1} = \begin{pmatrix} M_m' & O \\ O & M_m'' \end{pmatrix}.$$

We consider its top-left block matrix M_m' of size 2^{m-1}. By Lemma 4.3, this can be regarded as the circuit matrix of $\rho_m' \in \pi_1(X', \dot{x}')$ with respect to the restriction of chosen 2^{m-1} solutions to $x_m = 0$. The eigenspace of M_m' of eigenvalue 1 is $2^{m-1}-1$ dimensional by Fact 4.7. By the assumption of the induction, the other eigenvalue of M_m' is $\lambda' = (-1)^m \big(\prod_{j=1}^{m-1} B_j \big) \big/ (A_1A_2)$. We show that $\mathrm{v}' = (1, \dots, 1) \in \mathbb{N}^{2^{m-1}}$ is its eigenvector. This is equivalent to show that the top-left block of the normalizing matrix $g \in GL_{2^m}(\mathbb{C})$ coincides with the normalizing matrix $g' \in GL_{2^{m-1}}(\mathbb{C})$ for the $m-1$ variables case modulo non-zero scalar multiplication. Let $e_{J'}$ and $e_{J'}'$ be the $e^{J'}$-th unit vector of size 2^m and that of size 2^{m-1} for a subset J' of $\{1, \dots, m-1\}$. Then we have $e_{J'}M_m = e_{J'}$ by Lemma 4.5. Lemma 4.11 yields that

$$(h_{J'}e_\phi - e_{J'})M_{m+1}M_mM_{m+1}M_m^{-1} = h_{J'}e_\phi - e_{J'}$$

for any non-empty set J' of $\{1, \dots, m-1\}$. Thus

$$h_{J'}e_\phi' - e_{J'}' \quad (\phi \neq J' \subset \{1, \dots, m-1\})$$

span the eigenspace of M_m' of eigenvalue 1. Since

$$(h_{J'}e_\phi' - e_{J'}')H'\mathrm{v}' = 0$$

for the top-left block matrix H' of H of size 2^{m-1}, v' is an eigenvector of M_m' of eigenvalue λ' by Lemma 4.9. Hence H' coincides with the matrix for the case of $m-1$ variables, i.e., $h_{J'}$ for any subset of $\{1, \dots, m-1\}$ should be equal to

$$(-1)^{|J'|} \frac{\left(A_1 - \prod_{j \in J'} B_j\right)\left(A_2 - \prod_{j \in J'} B_j\right)}{\left(\prod_{j \in J'} B_j\right)(A_1 - 1)(A_2 - 1)}.$$

From our fundamental system $\mathbf{F}_C(x)$ to Lauricella's system F_C, we choose the 2^{m-1} solutions corresponding to the subsets of $\{1, \dots, m-2, m\}$ and restrict to the hyperplane $x_{m-1} = 0$. Then we can lead $h_{J'}$ for any subset J' of $\{1, \dots, m-2, m\}$ similarly to the previous way by the symmetry of the Lauricella' system F_C. Especially, we have

$$h_{\{m\}} = -\frac{(A_1 - B_m)(A_2 - B_m)}{(A_1 - 1)(A_2 - 1)B_m}.$$

From our fundamental system $\mathbf{F}_C(x)$ to Lauricella's system F_C, we choose the 2^{m-1} solutions corresponding to the subsets of $\{1, \dots, m\}$ including the index m. Note that these solutions include the factor $x_m^{1-b_m}$. We consider the ratio of them and restrict it to $x_m = 0$.

This restriction of the ratio coincides with the ratio of the fundamental system $\mathbf{F}_C(x)$ to Lauricella's system F_C of the $m-1$ variables $x_1, \dots, x_{m-1}$ with parameters $a_1 - b_m + 1$, $a_2 - b_m + 1$, $b_1, \dots, b_{m-1}$ by Fact 4.1. Its circuit matrices appear in the bottom-right blocks of M_i $(1 \le i \le m-1)$ and of $M_{m+1} M_m M_{m+1} M_m^{-1}$. We can show similarly to the previous that $\mathrm{v}' = (1, \dots, 1) \in \mathbb{N}^{2^{m-1}}$ is an eigenvector of the bottom-right block matrix M_m'' of $M_{m+1} M_m M_{m+1} M_m^{-1}$ of non-one eigenvalue. By the assumption of the induction, for any subset J' of $\{1, \dots, m-1\}$, the ratio of $h_{J' \cup \{m\}}$ and $h_{\{m\}}$ coincides with $h_{J'}|_{(A_1,A_2) \to (A_1/B_m, A_2/B_m)}$, which is the transformed $h_{J'}$ by the replacement

$$(A_1, A_2) \to (A_1/B_m, A_2/B_m).$$

Hence we have

$$\begin{aligned} h_{J' \cup \{m\}} &= h_{\{m\}} \cdot h_{J'}|_{(A_1,A_2) \to (A_1/B_m, A_2/B_m)} \\ &= -\frac{(A_1 - B_m)(A_2 - B_m)}{(A_1 - 1)(A_2 - 1)B_m} \cdot (-1)^{|J'|} \frac{\left(\dfrac{A_1}{B_m} - \prod_{j \in J'} B_j\right)\left(\dfrac{A_2}{B_m} - \prod_{j \in J'} B_j\right)}{\left(\dfrac{A_1}{B_m} - 1\right)\left(\dfrac{A_2}{B_m} - 1\right) \prod_{j \in J'} B_j} \\ &= (-1)^{|J' \cup \{m\}|} \frac{\left(A_1 - \prod_{j \in J' \cup \{m\}} B_j\right)\left(A_2 - \prod_{j \in J' \cup \{m\}} B_j\right)}{(A_1 - 1)(A_2 - 1) \prod_{j \in J' \cup \{m\}} B_j}. \end{aligned}$$

Next we compute the trace of H. We have seen that our assertion on $\mathrm{tr}(H)$ holds for $m = 1$ in Remark 2.7. Suppose that our assertion on $\mathrm{tr}(H)$ holds for $m-1$. Let H' be the top-left block matrix of H of size 2^{m-1}. By the previous consideration and the assumption of the induction, we have

$$\begin{aligned} \mathrm{tr}(H) &= \mathrm{tr}(H') + h_{\{m\}} \cdot \mathrm{tr}(H')|_{(A_1,A_2) \to (A_1/B_m, A_2/B_m)} \\ &= \frac{\left(A_1 A_2 + (-1)^{m-1} \prod_{j=1}^{m-1} B_j\right) \prod_{j=1}^{m-1} (B_j - 1)}{(A_1 - 1)(A_2 - 1) \prod_{j=1}^{m-1} B_j} \\ &\quad - \frac{(A_1 - B_m)(A_2 - B_m)}{(A_1 - 1)(A_2 - 1)B_m} \cdot \frac{\left(\dfrac{A_1 A_2}{B_m^2} + (-1)^{m-1} \prod_{j=1}^{m-1} B_j\right) \prod_{j=1}^{m-1} (B_j - 1)}{\left(\dfrac{A_1}{B_m} - 1\right)\left(\dfrac{A_2}{B_m} - 1\right) \prod_{j=1}^{m-1} B_j}. \end{aligned}$$

By taking out the common factor

$$\prod_{j=1}^{m-1} (B_j - 1) \Big/ \left[(A_1 - 1)(A_2 - 1) \prod_{j=1}^{m} B_j\right]$$

from the above, we have

$$\left(A_1A_2B_m+(-1)^{m-1}\prod_{j=1}^{m}B_j\right)-\left(A_1A_2+(-1)^{m-1}B_m\prod_{j=1}^{m}B_j\right)$$
$$=\left(A_1A_2+(-1)^{m}\prod_{j=1}^{m}B_j\right)(B_m-1)\,,$$

which yields our assertion on $\mathrm{tr}(H)$ for m.

Finally, we determine the eigenvalue λ so that $u=(1,\dots,1,0,\dots,0)$ is an eigenvector of $M_{m+1}M_mM_{m+1}M_m^{-1}$. Note that

$$M_mM_{m+1}M_m^{-1}=id_{2^m}-\frac{1-\lambda}{wH\,{}^tw^{\vee}}H\,{}^tw^{\vee}w$$

for $w=\mathrm{v}M_m^{-1}=(1,\dots,1,B_m,\dots,B_m)$. Note also that

$$\mathrm{v}H\,{}^t\mathrm{v}=wH\,{}^tw^{\vee}=\mathrm{tr}(H)\,,\quad uH\,{}^t\mathrm{v}=uH\,{}^tw^{\vee}=\mathrm{tr}(H'),$$

$$\begin{aligned}\mathrm{v}H\,{}^tw^{\vee}&=\mathrm{tr}(H')+B_m^{-1}h_{\{m\}}\mathrm{tr}(H')|_{(A_1,A_2)\to(A_1/B_m,A_2/B_m)}\\&=\frac{A_1A_2(B_m+1)\prod_{j=1}^{m}(B_j-1)}{(A_1-1)(A_2-1)B_m\prod_{j=1}^{m}B_j},\\ \frac{\mathrm{v}H\,{}^tw^{\vee}}{\mathrm{v}H\,{}^t\mathrm{v}}&=\frac{A_1A_2(B_m+1)}{(A_1A_2+(-1)^m\prod_{j=1}^{m}B_j)B_m}.\end{aligned}$$

Thus we have

$$\begin{aligned}&uM_{m+1}M_mM_{m+1}M_m^{-1}\\&=u\left(id_{2^m}-\frac{1-\lambda}{\mathrm{v}H\,{}^t\mathrm{v}}H\,{}^t\mathrm{v}\mathrm{v}\right)\left(id_{2^m}-\frac{1-\lambda}{wH\,{}^tw^{\vee}}H\,{}^tw^{\vee}w\right)\\&=u-\frac{(1-\lambda)uH\,{}^t\mathrm{v}}{\mathrm{v}H\,{}^t\mathrm{v}}\mathrm{v}-\frac{(1-\lambda)uH\,{}^t\mathrm{v}}{\mathrm{v}H\,{}^t\mathrm{v}}w+\frac{(1-\lambda)^2(uH\,{}^t\mathrm{v})(\mathrm{v}H\,{}^tw^{\vee})}{(\mathrm{v}H\,{}^t\mathrm{v})^2}w\\&=u-\frac{uH\,{}^t\mathrm{v}}{\mathrm{v}H\,{}^t\mathrm{v}}(1-\lambda)(\mathrm{v}+w)+\frac{(uH\,{}^t\mathrm{v})(\mathrm{v}H\,{}^tw^{\vee})}{(\mathrm{v}H\,{}^t\mathrm{v})^2}(1-\lambda)^2w\,,\end{aligned}$$

which should be a scalar multiple of u. Since its 2^m entry vanishes, λ satisfies the quadratic equation

$$(1+B_m)(1-\lambda)=\frac{\mathrm{v}H\,{}^tw^{\vee}}{\mathrm{v}H\,{}^t\mathrm{v}}B_m(1-\lambda)^2.$$

Hence we have

$$1-\lambda=\frac{(B_m+1)\mathrm{v}H\,{}^t\mathrm{v}}{B_m\mathrm{v}H\,{}^tw^\vee}=1+(-1)^m\Bigg(\prod_{j=1}^m B_j\Bigg)\Bigg/(A_1A_2)\,,$$

under the assumption $\lambda \neq 1$. □

REMARK 4.13. It is easy to obtain

$$\lambda=\pm\Bigg(\prod_{j=1}^m B_j\Bigg)\Bigg/(A_1A_2)$$

inductively. In fact, the determinant of $M_{m+1}M_mM_{m+1}M_m^{-1}$ is λ^2. On the other hand, the determinants of its top-left block matrix and bottom-right one are

$$\det(M'_m)=(-1)^m\Bigg(\prod_{j=1}^{m-1} B_j\Bigg)\Bigg/(A_1A_2)\,,$$

$$\det(M'_m)|_{(A_1,A_2)\to(A_1/B_m,A_2/B_m)}=(-1)^m\prod_{j=1}^{m-1} B_j\Bigg/[(A_1/B_m)(A_2/B_m)]\,,$$

respectively. These product is equal to λ^2.

REMARK 4.14. We have

$$\frac{1-\lambda}{\mathrm{v}H\,{}^t\mathrm{v}}=\frac{(A_1-1)(A_2-1)\prod_{j=1}^m B_j}{A_1A_2\prod_{j=1}^m(B_j-1)}\,,$$

in which the factor $A_1A_2+(-1)^m\prod_{j=1}^m B_j$ in $1-\lambda$ and $\mathrm{v}H\,{}^t\mathrm{v}$ is canceled.

We combine results as follows.

THEOREM 4.15. *Suppose the non-integral condition* (13) *for* a_1, a_2 *and* $b_1,\dots,b_m$. *Then there exists a fundamental system* $\mathbf{F}_C(x)$ *of solutions to Lauricella's* F_C *system around* $\dot{x}=(\varepsilon_1,\dots,\varepsilon_m)$ *such that the circuit matrices* $M_1,\dots,M_m$ *and* M_{m+1} *along the loops* $\rho_1,\dots,\rho_m$ *and* ρ_{m+1} *in Fact* 4.2 *are expressed as*

$$M_i=\mathrm{diag}(1,\dots,B_i^{-\delta_{i,J}},\dots)\quad(1\le i\le m)\,,$$
$$M_{m+1}=id_{2^m}-\frac{1-\lambda}{\mathrm{v}H\,{}^t\mathrm{v}}H\,{}^t\mathrm{v}\mathrm{v}\,,$$

where $\varepsilon_1,\dots,\varepsilon_m$ *are sufficiently small positive real numbers,* $A_i=e^{2\pi\sqrt{-1}a_i}$ $(i=1,2)$, $B_j=e^{2\pi\sqrt{-1}b_j}$ $(1\le j\le m)$, $\mathrm{v}=(1,\dots,1)\in\mathbb{N}^{2^m}$,

$$\delta_{i,J}=\begin{cases}1 & \text{if} \quad i\in J,\\ 0 & \text{if} \quad i\notin J,\end{cases}\qquad \lambda=(-1)^{m+1}\Bigg(\prod_{j=1}^m B_j\Bigg)\Bigg/(A_1A_2),$$

$$H=\mathrm{diag}(1,\ldots,h_J,\ldots)\,,\quad h_J=(-1)^{|J|}\frac{\Big(A_1-\prod_{j\in J}B_j\Big)\Big(A_2-\prod_{j\in J}B_j\Big)}{\Big(\prod_{j\in J}B_j\Big)(A_1-1)(A_2-1)}\,,$$

J runs over the non-empty subsets of $\{1,\ldots,m\}$ arranged lexicographically, and $|J|$ is the cardinality of J.

REMARK 4.16. We have seen that $N_m=M_{m+1}M_mM_{m+1}M_m^{-1}$ is block diagonal with block size 2^{m-1}. We inductively define matrices N_{m-k} as

$$N_{m-k}=N_{m-k+1}M_{m-k}N_{m-k+1}M_{m-k}^{-1}\,,\quad k=1,\ldots,m-2\,.$$

Then the matrix N_{m-k} is block diagonal with block size 2^{m-k-1}.

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DEPARTMENT OF MATHEMATICS
HOKKAIDO UNIVERSITY
SAPPORO 060–0810
JAPAN

E-mail address: matsu@math.sci.hokudai.ac.jp