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ATOMIC DECOMPOSITIONS OF WEIGHTED HARDY SPACES WITH VARIABLE EXPONENTS

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Abstract. We establish the atomic decompositions for the weighted Hardy spaces with variable exponents. These atomic decompositions also reveal some intrinsic structures of atomic decomposition for Hardy type spaces.

1. Introduction. There are two main themes for this paper. The first one is to establish the atomic decompositions of weighted Hardy spaces with variable exponents. The second one is the intrinsic structure of the atomic decomposition of Hardy type spaces.

The atomic decomposition is one of the most remarkable results for the study of Hardy spaces. It is impossible to review all the applications and impacts of the atomic decompositions on the theory of function spaces. Thus, to match the main theme of this paper, we briefly review some extensions of the atomic decompositions of Hardy spaces built on some non-Lebesgue spaces on $\mathbb{R}^n$.

Shortly after the introduction of the classical Hardy spaces [49], we already had the study of weighted Hardy spaces and established the corresponding atomic decomposition in [5, 22, 51]. As shown in [51], the weighted Hardy spaces provide an enlarged point of view for the study of function spaces. For instance, it is shown in [51, p. 86] that the Dirac delta function, being one of the most important distributions on the study of partial differential equations, belongs to some weighted Hardy spaces.

Moreover, the atomic decompositions had been extended to the Hardy-Orlicz spaces in [39, 52]. Hardy-Orlicz spaces were introduced in [39] by using maximal functions while Hardy-Orlicz spaces given in [52] is used to study an extension of the function space of bounded mean oscillation. The atomic decomposition for Hardy-Lorentz spaces is given in [1].

Recently, Hardy-Morrey spaces and weighted Hardy-Morrey spaces are introduced in [33, 45, 46, 47] and [29], respectively.

The study of Hardy spaces with variable exponent is inspired by the Lebesgue spaces with variable exponents which recently, gain the attentions of a substantial number of researchers. The Lebesgue spaces with variable exponents were introduced independently by Orlicz and Nakano [40, 41, 44]. For some comprehensive accounts on the study of Lebesgue

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spaces with variable exponents, the reader is referred to [10, 15]. Notice that one of the major breakthroughs on the Lebesgue spaces with variable exponents is the boundedness of the Hardy-Littlewood maximal operator [8, 11, 14, 42]. This study has been extended to weighted Lebesgue spaces with variable exponent $L^{p(\cdot)}_{\omega}$ in [9].

The Hardy spaces with variable exponents are introduced in [38]. The atomic decomposition for the Hardy spaces with variable exponents was also established in [38]. It has been further extended to the Hardy-Morrey spaces with variable exponent in [30]. For the studies of Morrey spaces with variable exponents, the reader is referred to [2, 24, 25, 28, 31, 34].

For the atomic decomposition of the classical Hardy spaces H^p, $0 < p \le 1$, we see that the atom satisfies two essential conditions, namely, the size condition and the vanishing moment condition. More precisely, the atom a with $\operatorname{supp} a \subset Q$ for a cube Q satisfies

$$\|a\|_{L^q} \le |Q|^{\frac{1}{q}-\frac{1}{p}} \tag{1.1}$$

$$\int x^{\gamma} a(x)dx = 0\,, \quad \text{for all multi-indices } \gamma \text{ with } |\gamma| \le \left[\frac{n}{p} - n\right] \tag{1.2}$$

for some $1 < q < \infty$.

In this paper, we are particularly interested in the intrinsic structure of the atomic decomposition. Precisely, the intrinsic structure consists of two questions related to the definition of atoms. How do we determine the order of the vanishing moment condition by the information from the Hardy spaces and how do we identify the range of q from the size condition satisfied by the atom? We find that the answers for both of the above questions are related to the boundedness of the Hardy-Littlewood maximal operator.

The atomic decomposition for classical Hardy spaces is so refined that the relations between the boundedness of the Hardy-Littlewood maximal operator M on Lebesgue spaces and the indices appeared in the atomic decomposition can only be clearly revealed if we chase the details of the proof of the atomic decomposition very carefully.

On the other hand, the atomic decompositions of weighted Hardy spaces with variable exponents $H^{p(\cdot)}_{\omega}$ can fully and easily reveal the connection between the boundedness of M and the indices used in the definition of the atoms for the atomic decomposition.

Roughly speaking, we find that the order of the vanishing moment condition satisfied by the atoms used in the atomic decomposition for $H^{p(\cdot)}_{\omega}$ is determined by the infimum of those r such that the Hardy-Littlewood maximal operator is bounded on the associate space of the r-convexification of $L^{p(\cdot)}_{\omega}$, that is, $(L^{rp(\cdot)}_{\omega^{1/r}})'$ (see [43, Section 2.2] or [37, Volume II, p.53-54] for the definition of r-convexification). In addition, the index q in the size condition for atoms used in the atomic decomposition for $H^{p(\cdot)}_{\omega}$ is related to the left-openness of the boundedness of M on $(L^{rp(\cdot)}_{\omega^{1/r}})'$.

In this paper, we extend the atomic decomposition of weighted Hardy spaces to weighted Hardy spaces with variable exponents. Thus, the main results obtained in this paper, on one hand, generalize the atomic decompositions in [5, 22, 38, 51]. On the other hand, they also clarify the relation between the atomic decompositions of Hardy type spaces and the boundedness of the Hardy-Littlewood maximal operators on function spaces.

This paper is organized as follows. Section 2 gives the definition of weighted Lebesgue spaces with variable exponents and some of their preliminary results. We also introduce indices related to the intrinsic structure of the atomic decomposition and define weighted Hardy spaces with variable exponents in this section. Section 3 presents the Fefferman-Stein vector-valued maximal inequalities on weighted Lebesgue spaces with variable exponents. The smooth atomic decompositions of $H_\omega^{p(\cdot)}$ is given in Section 4. Our main results on the atomic decompositions of $H_\omega^{p(\cdot)}$ are established in Section 5. As an application of atomic decomposition, we show the equivalence of the Littlewood-Paley characterization and the maximal function characterization of weighted Hardy spaces with variable exponents in Section 6.

2. Preliminaries and Definitions. Let $B(z,r) = \{x \in \mathbb{R}^n : |x - z| < r\}$ denote the open ball with center $z \in \mathbb{R}^n$ and radius $r > 0$. Let $\mathbb{B} = \{B(z,r) : z \in \mathbb{R}^n,\ r > 0\}$. Let $\mathcal{M}$ be the class of Lebesgue measurable functions on $\mathbb{R}^n$.

We begin with the definition of the well known Muckenhoupt class of weight functions.

DEFINITION 2.1. For $1 < p < \infty$, a locally integrable function $\omega : \mathbb{R}^n \to [0,\infty)$ is said to be an A_p weight if

$$[\omega]_{A_p} = \sup_{B\in\mathbb{B}} \left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{-\frac{p'}{p}}dx\right)^{\frac{p}{p'}} < \infty$$

where $p' = \frac{p}{p-1}$. A locally integrable function $\omega : \mathbb{R}^n \to [0,\infty)$ is said to be an A_1 weight if for all balls B,

$$\frac{1}{|B|}\int_B \omega(y)dy \le C\omega(x)\,, \quad a.e.\, x \in B$$

for some constant $C > 0$. The infimum of all such C is denoted by $[\omega]_{A_1}$. We define $A_\infty = \bigcup_{p\ge1} A_p$.

For any $B \in \mathbb{B}$ and locally integrable function ω, write $\omega(B) = \int_B \omega(x)dx$.

We recall the definition of Lebesgue spaces with variable exponents and some of theirs properties.

Let $p(\cdot) : \mathbb{R}^n \to (0,\infty]$ be a Lebesgue measurable function, the Lebesgue space with variable exponent $L^{p(\cdot)}$ consists of those Lebesgue measurable function f satisfying

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0 : \rho_{p(\cdot)}(|f(x)|/\lambda) \le 1\right\} < \infty$$

where $\mathbb{R}^n_\infty = \{x \in \mathbb{R}^n : p(x) = \infty\}$ and

$$\rho_{p(\cdot)}(f) = \int_{\mathbb{R}^n\setminus\mathbb{R}^n_\infty} |f(x)|^{p(x)}dx + \operatorname*{ess\,sup}_{\mathbb{R}^n_\infty} |f(x)|\,.$$

For any Lebesgue measurable function $p(\cdot) : \mathbb{R}^n \to (0,\infty]$, define

$$p_- = \operatorname{ess\,inf}\{p(x) : x \in \mathbb{R}^n\}, \quad p_+ = \operatorname{ess\,sup}\{p(x) : x \in \mathbb{R}^n\}$$

and

$$p_* = \min(1, p_-)\,. \tag{2.1}$$

DEFINITION 2.2. Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ be a Lebesgue measurable function and ω be a Lebesgue measurable function such that $0 < \omega(x) < \infty$ almost everywhere. The weighted Lebesgue space with variable exponent $L^{p(\cdot)}_\omega$ consists of all Lebesgue measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ satisfying

$$\|f\|_{L^{p(\cdot)}_\omega} = \|f\omega\|_{L^{p(\cdot)}} < \infty\,.$$

We call $p(\cdot)$ the exponent function of $L^{p(\cdot)}_\omega$.

For any $p(\cdot) : \mathbb{R}^n \to (0,\infty)$, we can also define the weighted Lebesgue spaces with variable exponents by the modular

$$\rho_{p(\cdot),\omega} = \int |f(x)|^{p(x)}\omega(x)dx.$$

Since $\rho_{p(\cdot),\omega^{p(\cdot)}}(f) = \rho_{p(\cdot)}(f\omega)$, for brevity, we study the weighted Lebesgue spaces with variable exponents defined in term of the quasi-norm $\|\cdot\|_{L^{p(\cdot)}_\omega}$.

When $p(\cdot) = p$, $0 < p < \infty$, is a constant function,

$$L^{p(\cdot)}_\omega = L^p(\omega^p) = \left\{ f \in \mathcal{M} : \int |f(x)|^p \omega^p(x)dx < \infty \right\}. \tag{2.2}$$

For any $p(\cdot) : \mathbb{R}^n \to [1,\infty)$, the conjugate function $p'(\cdot)$ is defined by $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Notice that $L^{p(\cdot)}_\omega$, $1 \le p(x) \le \infty$, is not necessarily a Banach function space with respect to the Lebesgue measure. Particularly, when $p(\cdot) = p$, $1 < p < \infty$, for any unbounded Lebesgue measurable E with $|E| < \infty$, $\|\chi_E\|_{L^{p(\cdot)}_\omega} = \omega(E)^{1/p}$ is not necessarily finite.

On the other hand, several crucial properties with respect to the Lebesgue measure are still valid for $L^{p(\cdot)}_\omega$.

The following is the Hölder inequality for the pair $L^{p(\cdot)}_\omega$ and $L^{p'(\cdot)}_{\omega^{-1}}$.

LEMMA 2.1. *Let $p(\cdot) : \mathbb{R}^n \to [1,\infty)$ be a Lebesgue measurable function and ω be a Lebesgue measurable function such that $0 < \omega(x) < \infty$ almost everywhere. We have*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \le 2\|f\|_{L^{p(\cdot)}_\omega}\|g\|_{L^{p'(\cdot)}_{\omega^{-1}}}\,.$$

The proof of the above lemma follows from [15, Lemma 3.2.20].

Next, we have the norm conjugate formula for $L^{p(\cdot)}_\omega$.

PROPOSITION 2.2. *Let $p(\cdot) : \mathbb{R}^n \to [1,\infty)$ be a Lebesgue measurable function and ω be a locally integrable function such that $0 < \omega(x) < \infty$ almost everywhere. We have*

$$\|f\|_{L^{p(\cdot)}_\omega} \approx \sup\left\{ \int_{\mathbb{R}^n} |f(x)g(x)|dx : g \in L^{p'(\cdot)}_{\omega^{-1}},\ \|g\|_{L^{p'(\cdot)}_{\omega^{-1}}} \le 1 \right\}.$$

The proof of the preceding proposition follows from [15, Corollary 3.2.14].

Next, we show that $\|\cdot\|_{L^{p(\cdot)}_\omega}$ is an absolutely continuous quasi-norm. For the definition of absolutely continuous quasi-norm, the reader may consult [4, Chapter 1, Proposition 3.2] or [26, Definition 2.4].

LEMMA 2.3. *Let* $p(\cdot):\mathbb{R}^n\to(0,\infty)$ *be a Lebesgue measurable function with* $0<p_-\le p_+<\infty$ *and* $\omega\in\mathcal{W}_{p(\cdot)}$. *Let* $\{f_j\}_{j\in\mathbb{N}}$ *be Lebesgue measurable functions with* $f_j\downarrow 0$. *If* $f_1\in L^{p(\cdot)}_\omega$, *then* $\|f_j\|_{L^{p(\cdot)}_\omega}\downarrow 0$.

PROOF. We have $\{f_j\}_{j\in\mathbb{N}}\subset L^{p(\cdot)}_\omega$ and $\omega f_j\downarrow 0$. As $L^{p(\cdot)}$ possesses absolutely continuous quasi-norm, $\|\cdot\|_{L^{p(\cdot)}}$ is absolutely continuous. Thus, $\|f_j\|_{L^{p(\cdot)}_\omega}=\|\omega f_j\|_{L^{p(\cdot)}}\downarrow 0$. □

The convergence of the atomic decompositions of $H^{p(\cdot)}_\omega$ in the topology of $H^{p(\cdot)}_\omega$ is guaranteed by the above lemma.

We now introduce weights that we use to define weighted Hardy spaces with variable exponents.

DEFINITION 2.3. Let $p(\cdot):\mathbb{R}^n\to(0,\infty)$ be a Lebesgue measurable function with $0<p_-\le p_+<\infty$. Let $\mathcal{W}_{p(\cdot)}$ consist of those Lebesgue measurable function ω satisfying;

(1) $\|\chi_B\|_{L^{p(\cdot)/p_*}_{\omega^{p_*}}}<\infty$ and $\|\chi_B\|_{L^{(p(\cdot)/p_*)'}_{\omega^{-p_*}}}<\infty,\quad \forall B\in\mathbb{B}$,

(2) there exist $\kappa>1$ and $s>1$ such that the Hardy-Littlewood maximal operator is bounded on $L^{(sp(\cdot))'/\kappa}_{\omega^{-\kappa/s}}$.

Notice that $L^{sp(\cdot)}_{\omega^{1/s}}$ is the s-convexification of $L^{p(\cdot)}_\omega$.

It is necessary to introduce s since the Hardy-Littlewood operator is not bounded on those Lebesgue spaces with variable exponent $L^{p(\cdot)}_\omega$ with $p_-\le 1$.

The introduction of κ is inspired by the left-openness property from the Muckenhoupt class and the class $\mathcal{A}$ defined and studied in [15, Chapter 5]. For the left-openness of the class $\mathcal{A}$, the reader may consult [15, Theorem 5.4.15].

In fact, the κ is also used to determine the size condition satisfied by the atoms for the atomic decompositions of the weighted Hardy spaces with variable exponents.

We introduce the following indices so that the intrinsic structure of the atomic decompositions of weighted Hardy spaces with variable exponents can be precisely stated. For any $\omega\in\mathcal{W}_{p(\cdot)}$, write

$$s_\omega=\inf\{s\ge 1: M \text{ is bounded on } L^{(sp(\cdot))'}_{\omega^{-1/s}}\}\quad\text{and}\tag{2.3}$$

$$\mathbb{S}_\omega=\{s: s\ge 1,\ M \text{ is bounded on } L^{(sp(\cdot))'/\kappa}_{\omega^{-\kappa/s}} \text{ for some } \kappa>1\}.\tag{2.4}$$

By using Jensen's inequality, we find that for any $s\in\mathbb{S}_\omega$, we have $s\ge s_\omega$.

For any fixed $s\in\mathbb{S}_\omega$, define

$$\kappa^s_\omega=\sup\{\kappa>1: M \text{ is bounded on } L^{(sp(\cdot))'/\kappa}_{\omega^{-\kappa/s}}\}.$$

The index κ_ω^s is used to measure the left-openness of the boundedness of M on the family $\{L^{(sp(\cdot))'/\kappa}_{\omega^{-\kappa/s}}\}_{\kappa>1}$.

The indices s_ω and κ_ω^s are defined for presenting the atomic decompositions of $H^{p(\cdot)}_\omega$. They are also related to the intrinsic structure of the atomic decompositions. The index s_ω is related to the vanishing moment condition and the index κ_ω^s is related to the size condition.

When $p(\cdot) = p$, $0 < p < \infty$, is a constant function and $\omega \equiv 1$, we have $s_\omega = 1/p$ and $\kappa_\omega^{1/p} = \infty$.

Furthermore, by using Jensen's inequality, for any $1 < r < \infty$ we have

$$(Mf)^r \leq M(|f|^r)\,. \tag{2.5}$$

Therefore, when ω fulfills Definition 2.3 (2), the Hardy-Littlewood operator is also bounded on $L^{(sp(\cdot))'}_{\omega^{-1/s}}$.

Since for any $s \in \mathbb{S}_\omega$, $s \geq s_\omega \geq \frac{1}{p_*}$ and $L^{p(\cdot)/p_*}_{\omega^{p_*}}$ is a Banach lattice, Lemma 2.1 and the Hölder inequality for Banach lattice [37, Volume II, Proposition 1.d.2] yield that for any $B \in \mathbb{B}$ and $f \in L^{sp(\cdot)}_{\omega^{1/s}}$

$$\begin{aligned}\int \chi_B(x)|f(x)|dx &\leq \|\chi_B\|_{L^{(p(\cdot)/p_*)'}_{\omega^{-p_*}}} \|\chi_B f\|_{L^{p(\cdot)/p_*}_{\omega^{p_*}}} \\ &\leq \|\chi_B\|_{L^{(p(\cdot)/p_*)'}_{\omega^{-p_*}}} \||f|^{sp_*}\|^{\frac{1}{sp_*}}_{L^{p(\cdot)/p_*}_{\omega^{p_*}}} \|\chi_B\|^{1-\frac{1}{sp_*}}_{L^{p(\cdot)/p_*}_{\omega^{p_*}}} \\ &= \|\chi_B\|_{L^{(p(\cdot)/p_*)'}_{\omega^{-p_*}}} \|f\|_{L^{sp(\cdot)}_{\omega^{1/s}}} \|\chi_B\|^{1-\frac{1}{sp_*}}_{L^{p(\cdot)/p_*}_{\omega^{p_*}}}\,. \end{aligned} \tag{2.6}$$

In view of the definition of $L^{(sp(\cdot))'}_{\omega^{-1/s}}$, $\chi_B \in L^{(sp(\cdot))'}_{\omega^{-1/s}}$ for any $B \in \mathbb{B}$.

Thus, when ω satisfies Definition 2.3 (1), we have

$$\|\chi_B\|_{L^{(sp(\cdot))'/\kappa}_{\omega^{-\kappa/s}}} = \|\chi_B\|^\kappa_{L^{(sp(\cdot))'}_{\omega^{-1/s}}} = \|\chi_B\|_{L^{(sp(\cdot))'}_{\omega^{-1/s}}} < \infty\,, \quad \forall B \in \mathbb{B}\,. \tag{2.7}$$

That is, Definition 2.3 (1) guarantees that $L^{(sp(\cdot))'/\kappa}_{\omega^{-\kappa/s}}$ is non-trivial and it does make sense to assume the boundedness of the Hardy-Littlewood maximal operator on $L^{(sp(\cdot))'/\kappa}_{\omega^{-\kappa/s}}$.

When $p(\cdot) = p$, $1 < p < \infty$, is a constant function, Definition 2.3 (1) is equivalent to the assumption that ω^p and $\omega^{-p'}$ are locally integrable functions.

When $p(\cdot) = p$, $0 < p \leq 1$, is a constant function, Definition 2.3 (1) is equivalent to the assumption that ω is locally integrable and ω^{-1} is locally bounded.

Furthermore, for Definition 2.3 (2), we have the following result:

PROPOSITION 2.4. *Let $0 < p < \infty$. If $p(\cdot) = p$, then a Lebesgue measurable function $\omega : \mathbb{R}^n \to (0, \infty)$ satisfies Definition 2.3 (2) if and only if $\omega^p \in A_\infty$.*

PROOF. Let $\omega^p \in A_\infty$. Then, for some large s, we have $\omega^p \in A_{sp}$ and $sp > 1$. In view of [23, Proposition 9.1.5 (4)], $\omega^{-\frac{p}{sp-1}} \in A_{(sp)'}$.

As

$$-\frac{p}{sp-1} = -\frac{1}{s}\frac{sp}{sp-1} = -\frac{1}{s}(sp)',$$

$\omega^{-\frac{1}{s}(sp)'} \in A_{(sp)'}$. By using the left-openness property of $A_{(sp)'}$ [23, Corollary 9.2.6]. There is a $\kappa > 1$ such that $\omega^{-\frac{1}{s}(sp)'} \in A_{(sp)'/\kappa}$. That is, M is bounded on $L^{(sp)'/\kappa}_{\omega^{-\kappa/s}}$.

Next, let M is bounded on $L^{(sp)'/\kappa}_{\omega^{-\kappa/s}}$ for some $\kappa, s > 1$. The Jensen inequality assures that M is bounded on $L^{(sp(\cdot))'}_{\omega^{-1/s}}$. That is, $\omega^{-\frac{1}{s}(sp)'} \in A_{(sp)'}$.

Thus, by [23, Proposition 9.1.5 (4)] again, we find that

$$\omega^{(-\frac{1}{s}(sp)')(-\frac{1}{(sp)'-1})} \in A_{sp}.$$

Since

$$\Big(-\frac{1}{s}(sp)'\Big)\Big(-\frac{1}{(sp)'-1}\Big) = \Big(-\frac{p}{sp-1}\Big)(-(sp-1)) = p,$$

we have $\omega^p \in A_{sp} \subset A_\infty$. □

The above proposition and (2.2) show that when $p(\cdot) = p$, $0 < p < \infty$, $L^{p(\cdot)}_\omega$ becomes the weighted Lebesgue spaces with weight belonging to A_∞.

For a general Lebesgue measurable function $p(\cdot) : \mathbb{R}^n \to (0,\infty)$, we have the following result which guarantees ω satisfies the first condition in Definition 2.3 (1).

LEMMA 2.5. *Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ be a Lebesgue measurable function with $0 < p_- \le p_+ < \infty$. If ω^{p_+} is locally integrable, then for any $B \in \mathbb{B}$, $\|\chi_B\|_{L^{p(\cdot)/p_*}_{\omega^{p_*}}} < \infty$.*

PROOF. Since ω^{p_+} is locally integrable, we have

$$\begin{aligned}\rho_{p(\cdot)/p_*}(\chi_B\omega^{p_*}) = \int_B (\omega(x))^{p(x)}dx &\le |\{x \in B : \omega(x) \le 1\}| + \int_B (\omega(x))^{p_+}dx \\ &\le |B| + \int_B (\omega(x))^{p_+}dx < \infty.\end{aligned}$$

As $p(\cdot)/p_* : \mathbb{R}^n \to [1,\infty)$, [10, Proposition 2.12] ensures that $\chi_B\omega^{p_*} \in L^{p(\cdot)/p_*}$. That is, $\|\chi_B\|_{L^{p(\cdot)/p_*}_{\omega^{p_*}}} < \infty$. □

Whenever $p(\cdot)$ is log-Hölder continuous and satisfies log-Hölder decay condition [10, Definition 2.2] and [15, Definitions 4.1.1 and 4.1.4], a necessary and sufficient condition for the boundedness of M on $L^{p(\cdot)}_\omega$ is given in [8, Definition 1.4 and Theroem 1.5].

Since our results for the Hardy spaces with variable exponent are valid for exponent function $p(\cdot)$ which is not necessarily log-Hölder continuous nor satisfying log-Hölder decay condition, we refer the reader to [8] for the boundedness of M on $L^{p(\cdot)}_\omega$ with $p(\cdot)$ being log-Hölder continuous and satisfying log-Hölder decay condition.

Furthermore, the main results obtained in this paper also generalize the atomic decompositions given in [38] since the atomic decompositions obtained in [38] apply to the Hardy spaces with variable exponent with the exponent function being log-Hölder continuous and satisfying the log-Hölder decay condition.

At the end of this section, we use the Littlewood-Paley function to define weighted Hardy spaces with variable exponents.

Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ denote the classes of tempered functions and Schwartz distributions, respectively. Let $\mathcal{P}$ denote the class of polynomials in $\mathbb{R}^n$.

DEFINITION 2.4. Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a Lebesgue measurable function with $0 < p_- \le p_+ < \infty$ and $\omega \in \mathcal{W}_{p(\cdot)}$. The weighted Hardy space with variable exponent $H_\omega^{p(\cdot)}$ consists of those $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$ such that

$$\|f\|_{H_\omega^{p(\cdot)}} = \left\| \left(\sum_{\nu \in \mathbb{Z}} |\varphi_\nu * f|^2 \right)^{1/2} \right\|_{L_\omega^{p(\cdot)}} < \infty$$

where $\varphi_\nu(x) = 2^{\nu n}\varphi(2^\nu x)$, $\nu \in \mathbb{Z}$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies

$$\text{supp } \hat{\varphi} \subseteq \{x \in \mathbb{R}^n \ : 1/2 \le |x| \le 2\} \quad \text{and} \quad |\hat{\varphi}(\xi)| \ge C, \quad 3/5 \le |x| \le 5/3 \tag{2.8}$$

for some $C > 0$.

Hardy spaces with variable exponents can also be defined via the maximal functions. In Section 6 of this paper, as an application of the atomic decompositions of $H_\omega^{p(\cdot)}$, we establish the equivalence of these two characterizations of $H_\omega^{p(\cdot)}$.

3. Vector-valued maximal inequalities. We apply the extrapolation theory to obtain the Fefferman-Stein vector-valued maximal inequalities on $L_\omega^{p(\cdot)}$ in this section.

THEOREM 3.1. *Let $1 < q < \infty$ and $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a Lebesgue measurable function with $0 < p_- \le p_+ < \infty$. If $\omega \in \mathcal{W}_{p(\cdot)}$, then for any $r > s_\omega$, we have*

$$\left\| \left(\sum_{i \in \mathbb{N}} (Mf_i)^q \right)^{1/q} \right\|_{L_{\omega^{1/r}}^{rp(\cdot)}} \le C \left\| \left(\sum_{i \in \mathbb{N}} |f_i|^q \right)^{1/q} \right\|_{L_{\omega^{1/r}}^{rp(\cdot)}} \tag{3.1}$$

for some $C > 0$.

PROOF. According to the definition of s_ω, we have $s > s_\omega$ satisfying $s < r$ such that M is bounded on $L_{\omega^{-1/s}}^{(sp(\cdot))'}$.

We follow the idea from the extrapolation theory, see [7]. For any non-negative function h, define

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k \|M\|_{L_{\omega^{-1/s}}^{(sp(\cdot))'}}^k}$$

where $\|M\|_{L_{\omega^{-1/s}}^{(sp(\cdot))'}}$ is the operator norm of the Hardy-Littlewood maximal operator on $L_{\omega^{-1/s}}^{(sp(\cdot))'}$.

We find that

$$h(x) \le \mathcal{R}h(x)\,, \tag{3.2}$$

$$\|\mathcal{R}h\|_{L^{(sp(\cdot))'}_{\omega^{-1/s}}} \leq 2\|h\|_{L^{(sp(\cdot))'}_{\omega^{-1/s}}}, \tag{3.3}$$

$$[\mathcal{R}h]_{A_1} \leq 2\|M\|_{L^{(sp(\cdot))'}_{\omega^{-1/s}}}. \tag{3.4}$$

Write

$$\mathcal{F} = \left\{ \left(\left(\sum_{i=0}^{K} (Mf_i)^q \right)^{1/q}, \left(\sum_{i=0}^{K} |f_i|^q \right)^{1/q} \right) : K \in \mathbb{N},\ \{f_i\}_{i=0}^{K} \subset L^\infty_{\text{comp}} \right\}$$

where L^∞_{comp} denotes the set of bounded functions with compact support.

Let $\theta = r/s > 1$. According to the weighted norm inequalities for Lebesgue spaces obtained in [3], for any $(F, G) \in \mathcal{F}$ and $w \in A_1$, we have

$$\int F(x)^\theta w(x)dx \leq C \int G(x)^\theta w(x)dx\,. \tag{3.5}$$

In view of Proposition 2.2, we find that

$$\begin{aligned} \|F\|^\theta_{L^{rp(\cdot)}_{\omega^{1/r}}} &= \|F^\theta\|_{L^{sp(\cdot)}_{\omega^{1/s}}} \\ &\leq C \sup\left\{ \int_{\mathbb{R}^n} |F(x)^\theta g(x)|dx : g \in L^{(sp(\cdot))'}_{\omega^{-1/s}},\ \|g\|_{L^{(sp(\cdot))'}_{\omega^{-1/s}}} \leq 1 \right\} \end{aligned} \tag{3.6}$$

for some $C > 0$.

Since F is non-negative, we are allowed to taking over those non-negative g only. For any fixed non-negative $g \in L^{(sp(\cdot))'}_{\omega^{-1/s}}$ with $\|g\|_{L^{(sp(\cdot))'}_{\omega^{-1/s}}} \leq 1$, (3.2) assures that

$$\int F(x)^\theta g(x)dx \leq \int F(x)^\theta \mathcal{R}g(x)dx \tag{3.7}$$

for some $C > 0$.

Property (3.4) assures that $\mathcal{R}g \in A_1$. Therefore, (3.3), (3.5) and Lemma 2.1 give

$$\begin{aligned} \int F(x)^\theta \mathcal{R}g(x)dx &\leq C \int G(x)^\theta \mathcal{R}g(x)dx \leq C\|G^\theta\|_{L^{sp(\cdot)}_{\omega^{1/s}}} \|\mathcal{R}g\|_{L^{(sp(\cdot))'}_{\omega^{-1/s}}} \\ &\leq C\|G\|^\theta_{L^{rp(\cdot)}_{\omega^{1/r}}} \|g\|_{L^{(sp(\cdot))'}_{\omega^{-1/s}}} \leq C\|G\|^\theta_{L^{rp(\cdot)}_{\omega^{1/r}}} \end{aligned} \tag{3.8}$$

for some $C > 0$.

Thus, (3.6), (3.7) and (3.8) yield (3.1) when $\{f_i\}_{i\in\mathbb{N}} \in L^\infty_{comp}$. The validity of (3.1) for all $f \in L^{rp(\cdot)}_{\omega^{1/r}}$ follows from the fact that $f_N \uparrow f$ and $Mf_N \uparrow Mf$ as $N \to \infty$ where $f_N = f\chi_{\{x\in\mathbb{R}^n:|x|<N,|f(x)|<N\}}$. □

Theorem 3.1 is a key component for establishing the atomic decomposition for $H^{p(\cdot)}_\omega$. Moreover, the above result also has its own independent interest. It extends several existing results on vector-valued maximal inequalities. It covers the vector-valued maximal inequalities for weighted Lebesgue spaces in [3]. It also generalizes the vector-valued maximal inequalities for Lebesgue spaces with variable exponent in [6].

4. Smooth atomic decompositions. In this section, we establish the smooth atomic decomposition for $H^{p(\cdot)}_{\omega}$. In [26], a general approach is given for the study of function spaces defined via the Littlewood-Paley function. Thus, in this section, we recall the results from [26] and apply it directly to $H^{p(\cdot)}_{\omega}$. Some similar approaches for studying function spaces are given in [36, 53].

For any $j \in \mathbb{Z}$ and $k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$, $Q_{j,k} = \{(x_1, x_2 \dots, x_n) \in \mathbb{R}^n : k_i \leq 2^j x_i \leq k_i + 1, i = 1, 2, \dots, n\}$. We write $|Q|$ and $l(Q)$ to be the Lebesgue measure of Q and the side length of Q, respectively. We denote the set of dyadic cubes $\{Q_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ by $\mathcal{Q}$.

DEFINITION 4.1. Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a Lebesgue measurable function with $0 < p_- \leq p_+ < \infty$ and $\omega \in \mathcal{W}_{p(\cdot)}$. The sequence space $h^{p(\cdot)}_{\omega}$ is the collection of all complex-valued sequences $s = \{s_Q\}_{Q \in \mathcal{Q}}$ such that

$$\|s\|_{h^{p(\cdot)}_{\omega}} = \left\| \left(\sum_{Q} (|s_Q| \tilde{\chi}_Q)^2 \right)^{1/2} \right\|_{L^{p(\cdot)}_{\omega}} < \infty ,$$

where $\tilde{\chi}_Q = |Q|^{-1/2} \chi_Q$.

We restate the definition of the ϕ-ψ transform introduced by Frazier and Jawerth in [16, 18, 19]. Let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy

$$\operatorname{supp} \hat{\varphi}, \operatorname{supp} \hat{\psi} \subseteq \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\} , \tag{4.1}$$

$$|\hat{\varphi}(\xi)|, |\hat{\psi}(\xi)| \geq C \quad \text{if} \quad 3/5 \leq |\xi| \leq 5/3 \quad \text{for some } C > 0 , \tag{4.2}$$

$$\sum_{\nu \in \mathbb{Z}} \overline{\hat{\varphi}(2^{-\nu}\xi)} \hat{\psi}(2^{-\nu}\xi) = 1 \quad \text{if} \quad \xi \neq 0 \tag{4.3}$$

where $\hat{\varphi}$ denotes the Fourier transform of φ and similarly for $\hat{\psi}$.

Define $\tilde{\varphi}(x) = \overline{\varphi(-x)}$. Write $\varphi_\nu(x) = 2^{\nu n}\varphi(2^\nu x)$, $\psi_\nu(x) = 2^{\nu n}\psi(2^\nu x)$ and

$$\varphi_Q(x) = |Q|^{-1/2}\varphi(2^\nu x - k) , \quad \psi_Q(x) = |Q|^{-1/2}\psi(2^\nu x - k) , \quad \nu \in \mathbb{Z} , \quad k \in \mathbb{Z}^n$$

for $Q = Q_{\nu,k} \in \mathcal{Q}$. For any $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$ and for any complex-valued sequences $s = \{s_Q\}$, we define

$$\mathrm{S}_\varphi(f) = \{(\mathrm{S}_\varphi f)_Q\}_{Q \in \mathcal{Q}} = \{\langle f, \varphi_Q \rangle\}_{Q \in \mathcal{Q}} \quad \text{and} \quad \mathrm{T}_\psi(s) = \sum_Q s_Q \psi_Q .$$

We find that $\mathrm{T}_\psi \circ \mathrm{S}_\varphi = \mathrm{id}$ in $H^{p(\cdot)}_{\omega}$ because $H^{p(\cdot)}_{\omega}$ is a subspace of $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$ [18, Theorem 2.2].

By using the terminologies given in [26, Definition 1.8], Theorem 3.1 guarantees that the pair $(l^q, L^{p(\cdot)}_{\omega})$, $1 < q < \infty$, is an a-admissible pair when $0 < a < \frac{1}{s_\omega}$.

THEOREM 4.1. *Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a Lebesgue measurable function with $0 < p_- \leq p_+ < \infty$ and $\omega \in \mathcal{W}_{p(\cdot)}$. The weighted Hardy space with variable exponent $H^{p(\cdot)}_{\omega}$ is well defined. That is, it is independent of the function φ in Definition* 2.4.

Moreover, the operators S_φ *and* T_ψ *are bounded operators on* $H^{p(\cdot)}_\omega$ *and* $h^{p(\cdot)}_\omega$*, respectively. In addition, we have constants* $C_1 > C_2 > 0$ *such that*

$$C_2\|f\|_{H^{p(\cdot)}_\omega} \le \|\mathrm{S}_\varphi(f)\|_{h^{p(\cdot)}_\omega} \le C_1\|f\|_{H^{p(\cdot)}_\omega}, \quad \forall f \in H^{p(\cdot)}_\omega . \tag{4.4}$$

PROOF. We apply the general approach given in [26, Theorem 3.1]. According to [26, Definition 1.2], we have to show that

$$(1+|x|)^{-L} \in L^{p(\cdot)}_\omega \tag{4.5}$$

for some $L > 0$.

According to [21, Chapter II, Theorem 2.12], for any $1 < p < \infty$ and for any Lebesgue measurable functions $\phi \ge 0$ and f on $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} (M\chi_{B(0,1)}(x))^p \phi(x)dx \le C_p \int_{\mathbb{R}^n} |\chi_{B(0,1)}(x)|^p M(\phi)(x)dx \tag{4.6}$$

for some $C_p > 0$ independent of f and ϕ.

We have

$$\frac{Cr^n}{(r+|x-y|)^n} \le (M\chi_{B(y,r)})(x) \tag{4.7}$$

for some $C > 0$ independent of $x, y \in \mathbb{R}^n$ and $r > 0$.

Consequently, for any $\phi \in L^{(sp(\cdot))'}_{\omega^{-1/s}}$,

$$\begin{aligned}\int_{\mathbb{R}^n} (1+|x|)^{-np}\phi(x)dx &\le C_p \int_{B(0,1)} M(\phi)(x)dx \\ &\le C\|\chi_{B(0,1)}\|_{L^{sp(\cdot)}_{\omega^{1/s}}} \|\phi\|_{L^{(sp(\cdot))'}_{\omega^{-1/s}}} .\end{aligned} \tag{4.8}$$

Since Definition 2.3 (1) assures that $\|\chi_B\|_{L^{p(\cdot)}_\omega} = \|\chi_B\|^s_{L^{sp(\cdot)}_{\omega^{1/s}}} < \infty$, $\forall B \in \mathbb{B}$, by taking supreme over all $\phi \in L^{(sp(\cdot))'}_{\omega^{-1/s}}$ with $\|\phi\|_{L^{(sp(\cdot))'}_{\omega^{-1/s}}} \le 1$ on (4.8), we obtain

$$\|(1+|x|)^{-snp}\|^{1/s}_{L^{p(\cdot)}_\omega} = \|(1+|x|)^{-np}\|_{L^{sp(\cdot)}_{\omega^{1/s}}} \le C\|\chi_{B(0,1)}\|_{L^{sp(\cdot)}_{\omega^{1/s}}} < \infty .$$

Therefore, (4.5) is valid with $L = snp$. Finally, our claimed results follow from Theorem 3.1 and [26, Theorem 3.1]. □

We state the definition of smooth atoms from [19, p.46].

DEFINITION 4.2. For each dyadic cube Q, A_Q is a smooth N-atom for $H^{p(\cdot)}_\omega$, $N \in \mathbb{N}$, if it satisfies

$$\int x^\gamma A_Q(x)dx = 0 \quad \text{for} \quad 0 \le |\gamma| \le N, \gamma \in \mathbb{N}^n , \tag{4.9}$$

$$\operatorname{supp} A_Q \subseteq 3Q , \tag{4.10}$$

and for $\gamma \in \mathbb{N}^n$,

$$|\partial^\gamma A_Q(x)| \le C_\gamma |Q|^{-1/2-|\gamma|/n}. \tag{4.11}$$

In view of [27, Theorem 2.1], we have the smooth atomic decomposition for $H^{p(\cdot)}_\omega$.

THEOREM 4.2 (Smooth Atomic Decomposition). *Let $N \in \mathbb{N}$, $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a Lebesgue measurable function with $0 < p_- \leq p_+ < \infty$ and $\omega \in \mathcal{W}_{p(\cdot)}$. For any $f \in H^{p(\cdot)}_\omega$, there exist a sequence $s = \{s_Q\}_{Q\in\mathcal{Q}} \in h^{p(\cdot)}_\omega$ and a family of smooth N-atoms $\{A_Q\}_{Q\in\mathcal{Q}}$ such that $f = \sum_{Q\in\mathcal{Q}} s_Q A_Q$ and $\|s\|_{h^{p(\cdot)}_\omega} \leq C\|f\|_{H^{p(\cdot)}_\omega}$ for some constant $C > 0$.*

5. Non-smooth atomic decompositions. In this section, we establish the non-smooth atomic decomposition for weighted Hardy spaces with variable exponent. It consists of a decomposition theorem and a reconstruction theorem. They extend the atomic decompositions of the weighted Hardy spaces and the Hardy spaces with variable exponent obtained in [51] and [38], respectively.

We obtain our non-atomic decomposition of $H^{p(\cdot)}_\omega$ by using the smooth atomic decomposition of $H^{p(\cdot)}_\omega$ given in Theorem 4.2. Theorem 4.2 exhibits a connection between the weighted Hardy space with variable exponent and the sequence space $h^{p(\cdot)}_\omega$. In this section, we first obtain an atomic decomposition for the sequence space $h^{p(\cdot)}_\omega$. Then, we rearrange the atomic decomposition of $h^{p(\cdot)}_\omega$ and reassemble it into the non-smooth atomic decomposition of $H^{p(\cdot)}_\omega$. The reader may refer [18, Section 7] and [29] for using some similar ideas to study the atomic decompositions for Triebel-Lizorkin spaces and weighted Hardy-Morrey spaces, respectively.

In addition, in this section, the intrinsic structure of the atomic decompositions of $H^{p(\cdot)}_\omega$ is presented explicitly in the statement of Theorem 5.3.

For any sequence $s = \{s_Q\}_{Q\in\mathcal{Q}}$, write

$$g(s) = \left(\sum_{Q\in\mathcal{Q}} (|s_Q|\tilde{\chi}_Q)^2 \right)^{1/2}.$$

We call $g(s)$ the Littlewood-Paley function of s. According to the definition of $h^{p(\cdot)}_\omega$, we have $\|s\|_{h^{p(\cdot)}_\omega} = \|g(s)\|_{L^{p(\cdot)}_\omega}$.

DEFINITION 5.1. Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a Lebesgue measurable function with $0 < p_- \leq p_+ < \infty$ and $\omega \in \mathcal{W}_{p(\cdot)}$. A sequence $r = \{r_Q\}_{Q\in\mathcal{Q}}$ is an ∞-atom for $h^{p(\cdot)}_\omega$ if there exists a dyadic cube $P \in \mathcal{Q}$ such that $r_Q = 0$ if $Q \not\subset P$ and $\|g(r)\|_{L^\infty} \leq \frac{1}{\|\chi_P\|_{L^{p(\cdot)}_\omega}}$.

We call P the support of r and write $\text{supp}(r) = P$.

Moreover, a family of ∞-atoms indexed by $\mathcal{Q}$, $\{r_J\}_{J\in\mathcal{Q}}$, is called an ∞-atomic family for $h^{p(\cdot)}_\omega$ if $\text{supp}(r_J) = J$.

The reader is referred to [16, p.403] for the definition of ∞-atom for the classical Hardy space.

We now establish the atomic decomposition of the sequence space $h^{p(\cdot)}_\omega$.

THEOREM 5.1. *Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a Lebesgue measurable function with $0 < p_- \leq p_+ < \infty$ and $\omega \in \mathcal{W}_{p(\cdot)}$. For any $s \in h^{p(\cdot)}_\omega$, there exist a family of ∞-atomic* *family for* $h^{p(\cdot)}_\omega$, $\{r_J\}_{J\in\mathcal{Q}}$, *and a sequence of scalars* $\{t_J\}_{J\in\mathcal{Q}}$ *such that*

$$s=\sum_{J\in\mathcal{Q}} t_J r_J\,, \quad \textit{and} \tag{5.1}$$

$$\Bigg\|\sum_{J\in\mathcal{Q}}\left(\frac{|t_J|}{\|\chi_J\|_{L^{p(\cdot)}_\omega}}\right)^{\theta}\chi_J\Bigg\|^{\frac{1}{\theta}}_{L^{p(\cdot)/\theta}_{\omega^\theta}} \le C\|s\|_{h^{p(\cdot)}_\omega}\,, \quad \forall\, 0<\theta<\infty\,, \tag{5.2}$$

for some $C>0$ *independent of* s.

PROOF. For any $P\in\mathcal{Q}$, define

$$g_P(s)=\Bigg(\sum_{Q\in\mathcal{Q},\,P\subseteq Q}(|Q|^{-\frac{1}{2}}|s_Q|)^2\Bigg)^{1/2}.$$

Whenever $P_1\subseteq P_2$, we have $0\le g_{P_2}(s)\le g_{P_1}(s)$. We also find that, for any given $x\in\mathbb{R}^n$, $g_P(s)$ satisfies

$$\lim_{l(P)\to\infty,\,x\in P} g_P(s)=0\,, \tag{5.3}$$

$$\lim_{l(P)\to 0,\,x\in P} g_P(s)=g(s)(x)\,. \tag{5.4}$$

For any $k\in\mathbb{Z}$, define $\mathcal{A}_k=\{P\in\mathcal{Q}: g_P(s)>2^k\}$. Identity (5.4) guarantees that

$$\{x\in\mathbb{R}^n: g(s)(x)>2^k\}=\bigcup_{P\in\mathcal{A}_k}P\,. \tag{5.5}$$

According to the proof of [29, (4.4)], we have

$$\Bigg(\sum_{P\in\mathcal{Q}\setminus\mathcal{A}_k}(|s_P|\tilde{\chi}_P(x))^2\Bigg)^{\frac{1}{2}}\le 2^k\,, \quad \forall x\in\mathbb{R}^n\,. \tag{5.6}$$

For any $k\in\mathbb{Z}$, let $\mathcal{B}_k$ denote the set of maximal dyadic cubes in $\mathcal{A}_k\setminus\mathcal{A}_{k+1}$. As maximal dyadic cubes exist in $\mathcal{A}_k$, $\mathcal{B}_k$ is well defined. According to the proof of [20, Theorem 7.3], for any $J\in\mathcal{B}_k$, the family of sequences $\beta_J=\{(\beta_J)_Q\}_{Q\in\mathcal{Q}}$ defined by

$$(\beta_J)_Q=\begin{cases} s_Q\,, & Q\subseteq J \quad \text{and} \quad Q\in\mathcal{A}_k\setminus\mathcal{A}_{k+1}\,,\\ 0\,, & \text{otherwise}\,,\end{cases}$$

satisfy $s=\sum_{J\in\mathcal{Q}}\beta_J$ and $|g(\beta_J)|\le 2^{k+1}$.

Let $r_J=2^{-k-1}\|\chi_J\|^{-1}_{L^{p(\cdot)}_\omega}\beta_J$ and $t_J=2^{k+1}\|\chi_J\|_{L^{p(\cdot)}_\omega}$. As

$$\mathcal{Q}=\Bigg(\bigcup_{k=-\infty}^{\infty}\Bigg(\bigcup_{J\in\mathcal{B}_k}\{Q\in\mathcal{Q}: Q\subset J\}\Bigg)\Bigg)\bigcup\{Q\in\mathcal{Q}: s_Q=0\} \tag{5.7}$$

is a disjoint union, we find that $s=\sum_{J\in\mathcal{Q}}t_J r_J$ and $\{r_J\}_{J\in\mathcal{Q}}$ is an ∞-atomic family for $h^{p(\cdot)}_\omega$.

In view of the disjoint union in (5.7), we find that

$$\sum_{J\in\mathcal{Q}}\left(\frac{|t_J|}{\|\chi_J\|_{L^{p(\cdot)}_{\omega}}}\right)^{\theta}\chi_J \le \sum_{k\in\mathbb{Z}}2^{(k+1)\theta}\sum_{J\in\mathcal{B}_k}\chi_J \le C(g(s))^{\theta}$$

for some $C > 0$. Applying the quasi-norm $\|\cdot\|_{L^{p(\cdot)/\theta}_{\omega^{\theta}}}$ on both sides of the above inequalities, we obtain

$$\left\|\sum_{J\in\mathcal{Q}}\left(\frac{|t_J|}{\|\chi_J\|_{L^{p(\cdot)}_{\omega}}}\right)^{\theta}\chi_J\right\|_{L^{p(\cdot)/\theta}_{\omega^{\theta}}} \le C\|(g(s))^{\theta}\|_{L^{p(\cdot)/\theta}_{\omega^{\theta}}} = C\|s\|^{\theta}_{h^{p(\cdot)}_{\omega}}.$$

□

Next, we transfer the result from the atomic decomposition of $h^{p(\cdot)}_{\omega}$ to the atomic decomposition of the weighted Hardy spaces with variable exponent. We begin with the definition of the non-smooth atoms for $H^{p(\cdot)}_{\omega}$.

DEFINITION 5.2. Let $1 < r < \infty$, $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ be a Lebesgue measurable function with $0 < p_- \le p_+ < \infty$ and $\omega \in \mathcal{W}_{p(\cdot)}$. For any $N \in \mathbb{N}$, a family of functions $\{a_Q\}_{Q\in\mathcal{Q}}$ is called a $(p(\cdot), r, N)$-atomic family with respect to ω if

$$\begin{aligned}
&\operatorname{supp} a_Q \subseteq 3Q, \quad \forall Q \in \mathcal{Q},\\
&\int x^{\gamma} a_Q(x)dx = 0, \quad \forall \gamma \in \mathbb{N}^n \text{ with } 0 \le |\gamma| \le N,\\
&\|a_Q\|_{L^r} \le \frac{|Q|^{\frac{1}{r}}}{\|\chi_Q\|_{L^{p(\cdot)}_{\omega}}}.
\end{aligned}$$

We now ready to use the atomic decompositions for the sequence spaces $h^{p(\cdot)}_{\omega}$ to establish the atomic decomposition of weighted Hardy spaces with variable exponent $H^{p(\cdot)}_{\omega}$.

THEOREM 5.2. *Let $1 < q < \infty$, $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ be a Lebesgue measurable function with $0 < p_- \le p_+ < \infty$ and $\omega \in \mathcal{W}_{p(\cdot)}$. For any $f \in H^{p(\cdot)}_{\omega}$ and any positive integer N, there exist a $(p(\cdot), q, N)$-atomic family with respect to ω, $\{a_Q\}_{Q\in\mathcal{Q}}$, and a sequence $t = \{t_Q\}_{Q\in\mathcal{Q}}$ such that $f = \sum_{Q\in\mathcal{Q}} t_Q a_Q$ and*

$$\left\|\sum_{J\in\mathcal{Q}}\left(\frac{|t_J|}{\|\chi_J\|_{L^{p(\cdot)}_{\omega}}}\right)^{\theta}\chi_J\right\|^{\frac{1}{\theta}}_{L^{p(\cdot)/\theta}_{\omega^{\theta}}} \le C\|f\|_{H^{p(\cdot)}_{\omega}}, \quad \forall 0 < \theta < \infty$$

for some $C > 0$.

PROOF. Theorem 4.2 assures that, for any $f \in H^{p(\cdot)}_{\omega}$, there exist a family of smooth N-atoms $\{A_Q\}_{Q\in\mathcal{Q}}$ and a sequence $s = \{s_Q\}_{Q\in\mathcal{Q}} \in h^{p(\cdot)}_{\omega}$ such that $f = \sum_{Q\in\mathcal{Q}} s_Q A_Q$ and $\|s\|_{h^{p(\cdot)}_{\omega}} \le C\|f\|_{H^{p(\cdot)}_{\omega}}$.

According to Theorem 5.1, we have $t = \{t_J\}_{J\in\mathcal{Q}}$ and an ∞-atomic family for $h_\omega^{p(\cdot)}$, $\{r_J\}_{J\in\mathcal{Q}}$, such that $s = \sum_{J\in\mathcal{Q}} t_J r_J$ and

$$\left\| \sum_{J\in\mathcal{Q}} \left(\frac{|t_J|}{\|\chi_J\|_{L_\omega^{p(\cdot)}}} \right)^\theta \chi_J \right\|_{L_{\omega^\theta}^{p(\cdot)/\theta}}^{\frac{1}{\theta}} \leq C\|s\|_{h_\omega^{p(\cdot)}}, \quad \forall 0<\theta<\infty$$

for some $C > 0$.

Consequently, we rewrite f as

$$f = \sum_{Q\in\mathcal{Q}} s_Q A_Q = \sum_{Q\in\mathcal{Q}} \left(\sum_{J\in\mathcal{Q}} t_J r_J \right)_Q A_Q = \sum_{J\in\mathcal{Q}} t_J a_J$$

where $a_J = \sum_{Q\subseteq J} (r_J)_Q A_Q$. We have supp$a_J \subseteq 3J$ because supp$A_Q \subseteq 3Q$ and $Q \subseteq J$.

In view of the Littlewood-Paley characterization of Lebesgue spaces L^q, $1 < q < \infty$ and the boundedness of the φ-ψ transforms on $\dot{F}_q^{02} = L^q$ and $\dot{f}_q^{02}$, respectively [18, Theorem 2.2], we obtain

$$\|a_J\|_{L^q} \leq C\|g(r_J)\|_{L^q} \leq C \frac{|J|^{\frac{1}{q}}}{\|\chi_J\|_{L_\omega^{p(\cdot)}}}$$

for some $C > 0$. The vanishing moment conditions for a_J are inherited from the corresponding conditions from $\{A_Q\}_{Q\in\mathcal{Q}}$. Thus, $\{a_J\}_{J\in\mathcal{Q}}$ is a $(p(\cdot), q, N)$-atomic family with respect to ω and

$$\left\| \sum_{J\in\mathcal{Q}} \left(\frac{|t_J|}{\|\chi_J\|_{L_\omega^{p(\cdot)}}} \right)^\theta \chi_J \right\|_{L_{\omega^\theta}^{p(\cdot)/\theta}}^{\frac{1}{\theta}} \leq C\|f\|_{H_\omega^{p(\cdot)}}.$$

□

The following is the reconstruction theorem for the atomic decompositions of weighted Hardy spaces with variable exponents.

THEOREM 5.3. *Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ be a Lebesgue measurable function with $0 < p_- \leq p_+ < \infty$ and $\omega \in \mathcal{W}_{p(\cdot)}$. Suppose that $0 < \theta \leq 1$ satisfies $\frac{1}{\theta} \in \mathbb{S}_\omega$.*

For any $(p(\cdot), q, [ns_\omega - n])$-atomic family with respect to ω, $\{a_j\}_{j\in\mathbb{N}}$, with $q > \theta(\kappa_\omega^{1/\theta})'$, supp $a_j \subset Q_j$ and sequence of scalars $\{\lambda_j\}_{j\in\mathbb{N}}$ satisfying

$$\left\| \sum_{j\in\mathbb{N}} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L_\omega^{p(\cdot)}}} \right)^\theta \chi_{Q_j} \right\|_{L_{\omega^\theta}^{p(\cdot)/\theta}}^{\frac{1}{\theta}} < \infty, \tag{5.8}$$

the series $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ and $f \in H_\omega^{p(\cdot)}$ with

$$\|f\|_{H_\omega^{p(\cdot)}} \leq C \left\| \sum_{j\in\mathbb{N}} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L_\omega^{p(\cdot)}}} \right)^\theta \chi_{Q_j} \right\|_{L_{\omega^\theta}^{p(\cdot)/\theta}}^{\frac{1}{\theta}} \tag{5.9}$$

for some $C > 0$ independent of f.

Moreover, $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$ also converges in $H_\omega^{p(\cdot)}$.

Theorems 5.2 and 5.3 extend the atomic decompositions for weighted Hardy space [5, 22, 51] to $H^{p(\cdot)}_\omega$. They also generalize the atomic decompositions for the Hardy spaces with variable exponents in [38, 48] to $H^{p(\cdot)}_\omega$.

The intrinsic structure of the atomic decomposition of $H^{p(\cdot)}_\omega$ is clearly presented in the above theorem. The order of the vanishing moment conditions satisfied by the atoms is $[ns_\omega - n]$. It is determined by the boundedness of M on $L^{(sp(\cdot))'}_{\omega^{-1/s}}$. The index q for the size condition satisfied by the atoms is given by $q > \theta(\kappa^{1/\theta}_\omega)'$. It is related to $\kappa^{1/\theta}_\omega$ and condition (5.8).

When $\omega \equiv 1$ and $p(\cdot) = p$, $0 < p < 1$, is a constant function, we have $\theta = p$, $s_\omega = 1/p$ and $\kappa^{1/p}_\omega = \infty$. Therefore, Theorem 5.3 becomes the atomic decomposition of the classical Hardy spaces.

We need the subsequent supporting results to obtain Theorem 5.3.

LEMMA 5.4. *Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ be a Lebesgue measurable function with $0 < p_- \le p_+ < \infty$ and $\omega \in \mathcal{W}_{p(\cdot)}$. Let $s \in \mathbb{S}_\omega$ and $\{\lambda_k\}_{k\in\mathbb{N}}$ be a sequence of scalars. For any $r > (\kappa^s_\omega)'$, $\{b_k\}_{k\in\mathbb{N}} \subset L^r$ with supp $b_k \subseteq Q_k \in \mathcal{Q}$ and*

$$\|b_k\|_{L^r} \le A_k|Q_k|^{\frac{1}{r}}, \tag{5.10}$$

where $A_k > 0$, $\forall k \in \mathbb{N}$, we have

$$\Big\|\sum_{k\in\mathbb{N}} \lambda_k b_k\Big\|_{L^{sp(\cdot)}_{\omega^{1/s}}} \le C\Big\|\sum_{k\in\mathbb{N}} A_k|\lambda_k|\chi_{Q_k}\Big\|_{L^{sp(\cdot)}_{\omega^{1/s}}} \tag{5.11}$$

for some $C > 0$ independent of $\{A_k\}_{k\in\mathbb{N}}$, $\{b_k\}_{k\in\mathbb{N}}$ and $\{\lambda_k\}_{k\in\mathbb{N}}$.

PROOF. Fix an $s \in \mathbb{S}_\omega$. For any $g \in L^{(sp(\cdot))'}_{\omega^{-1/s}}$ with $\|g\|_{L^{(sp(\cdot))'}_{\omega^{-1/s}}} \le 1$, we find that

$$\left|\int_{\mathbb{R}^n} b_k(x)g(x)dx\right| \le \|b_k\|_{L^r}\|\chi_{Q_k}g\|_{L^{r'}} \le A_k|Q_k|^{\frac{1}{r}}\left(\int_{Q_k}|g(x)|^{r'}dx\right)^{\frac{1}{r'}}$$

where r' is the conjugate of r. Consequently,

$$\begin{aligned}\left|\int_{\mathbb{R}^n} b_k(x)g(x)dx\right| &\le A_k|Q_k|\left(\frac{1}{|Q_k|}\int_{Q_k}|g(x)|^{r'}dx\right)^{\frac{1}{r'}}\\ &\le CA_k|Q_k|\inf_{x\in Q_k}(M(|g|^{r'})(x))^{\frac{1}{r'}}\\ &\le CA_k\int_{Q_k}(M(|g|^{r'})(x))^{\frac{1}{r'}}dx\end{aligned}$$

for some $C > 0$.

Therefore, Lemma 2.1 gives

$$\left|\int_{\mathbb{R}^n}\Big(\sum_{k\in\mathbb{N}}\lambda_k b_k(x)\Big)g(x)dx\right|$$

$$
\begin{aligned}
&\le C\sum_{k\in\mathbb{N}} A_k|\lambda_k|\int_{Q_k}(M(|g|^{r'})(x))^{\frac{1}{r'}}dx\\
&\le C\int_{\mathbb{R}^n}\Big(\sum_{k\in\mathbb{N}} A_k|\lambda_k|\chi_{Q_k}(x)\Big)(M(|g|^{r'})(x))^{\frac{1}{r'}}dx\\
&\le C\Big\|\sum_{k\in\mathbb{N}} A_k|\lambda_k|\chi_{Q_k}\Big\|_{L^{sp(\cdot)}_{\omega^{1/s}}}\|(M(|g|^{r'}))^{\frac{1}{r'}}\|_{L^{(sp(\cdot))'}_{\omega^{-1/s}}}\\
&\le C\Big\|\sum_{k\in\mathbb{N}} A_k|\lambda_k|\chi_{Q_k}\Big\|_{L^{sp(\cdot)}_{\omega^{1/s}}}\|M(|g|^{r'})\|^{1/r'}_{L^{(sp(\cdot))'/r'}_{\omega^{-r'/s}}}.
\end{aligned}
$$

As $r' < \kappa^s_\omega$, the definition of κ^s_ω guarantees that there exists $r' < \kappa < \kappa^s_\omega$ such that M is bounded on $L^{(sp(\cdot))'/\kappa}_{\omega^{-\kappa/s}}$. Thus, (2.5) asserts that M is bounded on $L^{(sp(\cdot))'/r'}_{\omega^{-r'/s}}$.

Finally, Proposition 2.2 yields (5.11). □

The reader is referred to [30, Proposition 5.8] for a similar result of the above lemma on Morrey spaces with variable exponents.

PROOF OF THEOREM 5.3. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy the conditions in Definition 2.4. For any $h \in \mathcal{S}'(\mathbb{R}^n)$, define the Lebesgue measurable function $\mathcal{G}(h)$ by

$$
\mathcal{G}(h) = \Big(\sum_{\nu\in\mathbb{Z}} |(h*\varphi_\nu)|^2\Big)^{1/2}.
$$

Write $N = [ns_\omega - n]$. According to the proof of [29, Theroem 4.4], we find that for any $(p(\cdot), q, N)$-atomic family with respect to ω, $\{a_j\}_{j\in\mathbb{N}}$, with $\operatorname{supp} a_j \subset Q_j$,

$$
\begin{aligned}
|(a_j*\varphi_\nu)(x)| &\le C2^{(N+n+1)\nu}|Q_j|^{\frac{N+1}{n}}(1+2^\nu|x-x_{Q_j}|)^{-L}\int_{3Q_j}|a_j(y)|dy\\
&\le C2^{(N+n+1)\nu}|Q_j|^{\frac{N+1}{n}}(1+2^\nu|x-x_{Q_j}|)^{-L}\|a_j\|_{L^q}|Q_j|^{1/q'}\\
&\le C2^{(N+n+1)\nu}|Q_j|^{\frac{N+1}{n}}(1+2^\nu|x-x_{Q_j}|)^{-L}\frac{|Q_j|^{1/q'}|Q_j|^{1/q}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}}\\
&= C2^{(N+n+1)\nu}|Q_j|^{\frac{N+1}{n}+1}(1+2^\nu|x-x_{Q_j}|)^{-L}\frac{1}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}}
\end{aligned}
$$

for some sufficient large $L > 0$.

By using the embedding $l^1 \hookrightarrow l^2$ and the inequality

$$
\sum_{\nu\in\mathbb{Z}} 2^{(N+n+1)\nu}(1+2^\nu|x-x_Q|)^{-L} \le C|x-x_Q|^{-N-n-1},
$$

we find that

$$
\mathcal{G}(f) \le C\sum_{j\in\mathbb{N}}|\lambda_j|X_j + C\sum_{j\in\mathbb{N}}|\lambda_j|Y_j = X+Y
$$

for some $C > 0$ where

$$X_j(x) = \mathcal{G}(a_j)(x)\chi_{4Q_j}(x)$$
$$Y_j(x) = \frac{1}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_{\omega}}}\chi_{\mathbb{R}^n\setminus 4Q_j}(x)\left(1+\frac{|x-x_{Q_j}|}{l(Q_j)}\right)^{-N-n-1}.$$

We first consider X. Since $\theta \le 1$, by using the θ-inequality, we obtain

$$X^\theta \le C\sum_{j\in\mathbb{N}}|\lambda_j|^\theta|X_j|^\theta$$

for some $C > 0$.

The Littlewood-Paley characterization of L^q gives

$$\|X_j^\theta\|_{L^{q/\theta}} \le C\|\mathcal{G}(a_j)\|^\theta_{L^q} \le C\|a_j\|^\theta_{L^q} \le C\frac{|Q_j|^{\frac{\theta}{q}}}{\|\chi_{Q_j}\|_{L^{p(\cdot)/\theta}_{\omega^\theta}}}$$

for some $C > 0$.

Since $q > \theta(\kappa^s_\omega)'$, X_j satisfies (5.10) with $\operatorname{supp} X_j \subseteq 4Q_j$ and $A_j = \|\chi_{Q_j}\|^{-\theta}_{L^{p(\cdot)}_{\omega}}$. Furthermore, since $\frac{1}{\theta} \in \mathbb{S}_\omega$, we are allowed to apply Lemma 5.4 with $r = q/\theta$ to obtain

$$\|X\|_{L^{p(\cdot)}_{\omega}} = \|X^\theta\|^{1/\theta}_{L^{p(\cdot)/\theta}_{\omega^\theta}} \le C\left\|\sum_{j\in\mathbb{N}}\left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_{\omega}}}\right)^\theta\chi_{4Q_j}\right\|^{1/\theta}_{L^{p(\cdot)/\theta}_{\omega^\theta}}.$$

Since

$$\chi_{4Q_j} \le C(M\chi_{Q_j})^2 \tag{5.12}$$

for some $C > 0$ independent of j, we get

$$\|X\|_{L^{p(\cdot)}_{\omega}} \le C\left\|\sum_{j\in\mathbb{N}}\left(\frac{|\lambda_j|^{\theta/2}}{\|\chi_{Q_j}\|^{\theta/2}_{L^{p(\cdot)}_{\omega}}}M\chi_{Q_j}\right)^2\right\|^{1/\theta}_{L^{p(\cdot)/\theta}_{\omega^\theta}}$$
$$= C\left\|\left(\sum_{j\in\mathbb{N}}\left(\frac{|\lambda_j|^{\theta/2}}{\|\chi_{Q_j}\|^{\theta/2}_{L^{p(\cdot)}_{\omega}}}M\chi_{Q_j}\right)^2\right)^{\frac{1}{2}}\right\|^{2/\theta}_{L^{2p(\cdot)/\theta}_{\omega^{\theta/2}}}$$

for some $C > 0$.

Moreover, as $\frac{1}{\theta} \in \mathbb{S}_\omega$, $s_\omega \le \frac{1}{\theta} < \frac{2}{\theta}$. The Fefferman-Stein vector-valued maximal inequality, Theorem 3.1, yields

$$\|X\|_{L^{p(\cdot)}_{\omega}} \le C\left\|\left(\sum_{j\in\mathbb{N}}\left(\frac{|\lambda_j|^{\theta/2}}{\|\chi_{Q_j}\|^{\theta/2}_{L^{p(\cdot)}_{\omega}}}\chi_{Q_j}\right)^2\right)^{\frac{1}{2}}\right\|^{2/\theta}_{L^{2p(\cdot)/\theta}_{\omega^{\theta/2}}}$$
$$= C\left\|\sum_{j\in\mathbb{N}}\frac{|\lambda_j|^\theta}{\|\chi_{Q_j}\|^\theta_{L^{p(\cdot)}_{\omega}}}\chi_{Q_j}\right\|^{1/\theta}_{L^{p(\cdot)/\theta}_{\omega^\theta}}$$

for some $C > 0$.

Then, we deal with the function Y.

By using (4.7), we have

$$Y \le \sum_{j\in\mathbb{N}} \frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}} \chi_{\mathbb{R}^n\setminus 4Q_j}(x) \left(1+\frac{|x-x_{Q_j}|}{l(Q_j)}\right)^{-N-n-1}$$

$$\le C \sum_{j\in\mathbb{N}} \left(M\left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}} \chi_{Q_j}\right)^{1/\beta}(x)\right)^{\beta}$$

for any $1<\beta<\frac{N+n+1}{n}$.

As $N=[ns_\omega - n]$, we have

$$\frac{N+n+1}{n} > \frac{ns_\omega - n - 1 + n + 1}{n} = s_\omega\,. \tag{5.13}$$

Therefore, we can select β satisfying

$$1 \le s_\omega < \beta < \frac{N+n+1}{n}\,.$$

Applying the quasi-norm $\|\cdot\|_{L^{p(\cdot)}_\omega}$ on both sides of the above inequality, we find that

$$\|Y\|_{L^{p(\cdot)}_\omega} \le C \left\| \sum_{j\in\mathbb{N}} \left(M\left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}} \chi_{Q_j}\right)^{1/\beta}\right)^{\beta} \right\|_{L^{p(\cdot)}_\omega}$$

$$= C \left\| \left(\sum_{j\in\mathbb{N}} \left(M\left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}} \chi_{Q_j}\right)^{1/\beta}\right)^{\beta} \right)^{1/\beta} \right\|_{L^{\beta p(\cdot)}_{\omega^{1/\beta}}}.$$

As $\beta > s_\omega$, the θ-inequality and Theorem 3.1 yield

$$\|Y\|_{L^{p(\cdot)}_\omega} \le C \left\| \left(\sum_{j\in\mathbb{N}} \left(\left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}} \chi_{Q_j}\right)^{1/\beta}\right)^{\beta} \right)^{1/\beta} \right\|_{L^{\beta p(\cdot)}_{\omega^{1/\beta}}}$$

$$= C \left\| \sum_{j\in\mathbb{N}} \frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}} \chi_{Q_j} \right\|_{L^{p(\cdot)}_\omega} \le C \left\| \sum_{j\in\mathbb{N}} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}}\right)^{\theta} \chi_{Q_j} \right\|^{\frac{1}{\theta}}_{L^{p(\cdot)/\theta}_{\omega^\theta}}$$

for some $C>0$.

The above estimates for $\|X\|_{L^{p(\cdot)}_\omega}$ and $\|Y\|_{L^{p(\cdot)}_\omega}$ yield

$$\|f\|_{H^{p(\cdot)}_\omega} = \|\mathcal{G}(f)\|_{L^{p(\cdot)}_\omega} \le C \left\| \sum_{j\in\mathbb{N}} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}}\right)^{\theta} \chi_{Q_j} \right\|^{\frac{1}{\theta}}_{L^{p(\cdot)/\theta}_{\omega^\theta}}$$

for some $C>0$ independent of $f\in H^{p(\cdot)}_\omega$.

Since $S_N = \sum_{j=N}^{\infty} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}}\right)^{\theta} \chi_{Q_j} \downarrow 0$ as N goes to infinity, Lemma 2.3 assures that

$$\lim_{N\to\infty} \left\| \sum_{j=N}^{\infty} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}}\right)^{\theta} \chi_{Q_j} \right\|_{L^{p(\cdot)/\theta}_{\omega^\theta}} = 0\,.$$

Write $f_N = \sum_{j=0}^{N-1} \lambda_j a_j$. Then (5.9) yields

$$\lim_{N\to\infty} \|f - f_N\|_{H_\omega^{p(\cdot)}} \le C \lim_{N\to\infty} \left\| \sum_{j=N}^{\infty} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L_\omega^{p(\cdot)}}} \right)^{\theta} \chi_{Q_j} \right\|_{L_{\omega^\theta}^{p(\cdot)/\theta}}^{\frac{1}{\theta}} = 0$$

which asserts the convergence of the atomic decomposition in $H_\omega^{p(\cdot)}$. □

6. Characterization by maximal functions. In this section, we present an application of the atomic decompositions. We show that $H_\omega^{p(\cdot)}$ possesses the maximal function characterization. That is, the definitions of $H_\omega^{p(\cdot)}$ via the Littlewood-Paley function and the maximal functions are equivalent.

For classical Hardy spaces, this equivalence can be obtained by studying the boundedness of singular integral operators on vector-valued Hardy spaces [23, Sections 6.4.4-6.4.6]. This idea is also used in [38] for Hardy spaces with variable exponents.

However, in this paper, we establish this equivalence for $H_\omega^{p(\cdot)}$ by atomic decompositions.

We first recall some terminologies and notations from the study of maximal functions.

We say that $f \in \mathcal{S}'(\mathbb{R}^n)$ is a bounded tempered distribution if $\varphi * f \in L^\infty(\mathbb{R}^n)$ for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

For any $N \in \mathbb{N}$, define

$$\mathfrak{N}_N(\phi) = \sup_{x\in\mathbb{R}^n} (1+|x|)^N \sum_{|\gamma|\le N+1} |\partial^\gamma \phi(x)|, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n).$$

Write

$$\mathcal{F}_N = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \mathfrak{N}_N(\phi) \le 1\}.$$

For any $t > 0$ and $\Phi \in \mathcal{S}(\mathbb{R}^n)$, write $\Phi_t(x) = t^{-n}\Phi(x/t)$.

For any $f \in \mathcal{S}'(\mathbb{R}^n)$, the grand maximal function of f is given by

$$(\mathcal{M} f)(x) = \sup_{\Phi\in\mathcal{F}_N} \sup_{t>0} |(\Phi_t * f)(x)|,$$

see [50, Chapter III, (2)].

The grand maximal function depends on N, for simplicity, we use the abused notion $\mathcal{M}$.

DEFINITION 6.1. Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ be a Lebesgue measurable function with $0 < p_- \le p_+ < \infty$ and $\omega \in \mathcal{W}_{p(\cdot)}$. The weighted Hardy space with variable exponent $\mathcal{H}_\omega^{p(\cdot)}$ consists of all bounded $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying

$$\|f\|_{\mathcal{H}_\omega^{p(\cdot)}} = \|\mathcal{M} f\|_{L_\omega^{p(\cdot)}} < \infty.$$

The main result of this section is the equivalence of the definitions of the weighted Hardy space with variable exponents by using the Littlewood-Paley characterization and the grand maximal function characterization.

THEOREM 6.1. *Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ be a Lebesgue measurable function with $0 < p_- \le p_+ < \infty$ and $\omega \in \mathcal{W}_{p(\cdot)}$. The quasi-norms $\|\cdot\|_{H^{p(\cdot)}_\omega}$ and $\|\cdot\|_{\mathcal{H}^{p(\cdot)}_\omega}$ are mutually equivalent.*

We prove the above result by showing that $\mathcal{H}^{p(\cdot)}_\omega$ also possesses atomic decompositions as what we obtain in the previous section for $H^{p(\cdot)}_\omega$.

Even though the statement of the atomic decomposition for $\mathcal{H}^{p(\cdot)}_\omega$ is precisely the same as Theorems 5.2 and 5.3, the proofs are different. For the sake of completeness, we present the atomic decompositions for $\mathcal{H}^{p(\cdot)}_\omega$ in the following.

THEOREM 6.2. *Let $1 < q < \infty$, $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ be a Lebesgue measurable function with $0 < p_- \le p_+ < \infty$ and $\omega \in \mathcal{W}_{p(\cdot)}$. For any $f \in \mathcal{H}^{p(\cdot)}_\omega$ and any positive integer N, there exist a $(p(\cdot), q, N)$-atomic family with respect to ω, $\{a_Q\}_{Q\in\mathcal{Q}}$, and a sequence $t = \{t_Q\}_{Q\in\mathcal{Q}}$ such that $f = \sum_{Q\in\mathcal{Q}} t_Q a_Q$ and*

$$\left\| \sum_{J\in\mathcal{Q}} \left(\frac{|t_J|}{\|\chi_J\|_{L^{p(\cdot)}_\omega}} \right)^\theta \chi_J \right\|^{\frac{1}{\theta}}_{L^{p(\cdot)/\theta}_{\omega^\theta}} \le C\|f\|_{\mathcal{H}^{p(\cdot)}_\omega}, \quad \forall 0 < \theta < \infty$$

for some $C > 0$.

We also have the reconstruction theorem for the atomic decomposition of $\mathcal{H}^{p(\cdot)}_\omega$.

THEOREM 6.3. *Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ be a Lebesgue measurable function with $0 < p_- \le p_+ < \infty$ and $\omega \in \mathcal{W}_{p(\cdot)}$. Suppose that $0 < \theta \le 1$ satisfies $\frac{1}{\theta} \in \mathbb{S}_\omega$.*

For any $(p(\cdot), q, [ns_\omega - n])$-atomic family with respect to ω, $\{a_j\}_{j\in\mathbb{N}}$, with $q > \theta(\kappa_\omega^{1/\theta})'$, $\operatorname{supp} a_j \subset Q_j$ *and sequence of scalars $\{\lambda_j\}_{j\in\mathbb{N}}$ satisfying*

$$\left\| \sum_{j\in\mathbb{N}} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}} \right)^\theta \chi_{Q_j} \right\|^{\frac{1}{\theta}}_{L^{p(\cdot)/\theta}_{\omega^\theta}} < \infty, \tag{6.1}$$

the series

$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j$$

converges in $\mathcal{S}'(\mathbb{R}^n)$ and $f \in H^{p(\cdot)}_\omega$ with

$$\|f\|_{\mathcal{H}^{p(\cdot)}_\omega} \le C \left\| \sum_{j\in\mathbb{N}} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}} \right)^\theta \chi_{Q_j} \right\|^{\frac{1}{\theta}}_{L^{p(\cdot)/\theta}_{\omega^\theta}} \tag{6.2}$$

for some $C > 0$ independent of f.

Theorem 6.1 follows from Theorems 5.2, 5.3, 6.2 and 6.3. Thus, it remains to establish Theorems 6.2 and 6.3.

We use the ideas from [50, Chapter III, Section 2] and [30, Section 5] to obtain Theorems 6.2 and 6.3.

We recall a crucial supporting result for the atomic decomposition [50, Chapter III, Section 2.1] and [51, Chapter VIII, Lemma 3]. We use the presentation given in [30, Proposition 5.4] and [38, Lemma 4.7] .

For any $d \in \mathbb{N}$, let $\mathcal{P}_d$ denote the class of polynomials in $\mathbb{R}^n$ of degree less than or equal to d.

PROPOSITION 6.4. *Let $d \in \mathbb{N}$ and $\sigma > 0$. For any $f \in \mathcal{S}'(\mathbb{R}^n)$, there exist $g \in \mathcal{S}'(\mathbb{R}^n)$, $\{b_k\}_{k\in\mathbb{N}} \subset \mathcal{S}'(\mathbb{R}^n)$, a collection of cubes $\{Q_k\}_{k\in\mathbb{N}}$ and a family of smooth functions with compact supports $\{\eta_k\}$ such that*

(1) $f = g + b$ *where* $b = \sum_{k\in\mathbb{N}} b_k$,

(2) *the family* $\{Q_k\}_{k\in\mathbb{N}}$ *has bounded intersection property and*

$$\bigcup_{k\in\mathbb{N}} Q_k = \{x \in \mathbb{R}^n : (\mathcal{M}f)(x) > \sigma\},$$

(3) *supp*$\eta_k \subset Q_k$, $0 \leq \eta_k \leq 1$ *and*

$$\sum_{k\in\mathbb{N}} \eta_k = \chi_{\{x\in\mathbb{R}^n:(\mathcal{M}f)(x)>\sigma\}},$$

(4) *the tempered distribution g satisfies*

$$(\mathcal{M}g)(x) \leq (\mathcal{M}f)(x)\chi_{\{x\in\mathbb{R}^n:(\mathcal{M}f)(x)\leq\sigma\}}(x) + \sigma \sum_{k\in\mathbb{N}} \frac{l(Q_k)^{n+d+1}}{(l(Q_k) + |x - x_k|)^{n+d+1}},$$

where x_k denotes the center of the cube Q_k,

(5) *the tempered distribution b_k is given by $b_k = (f - c_k)\eta_k$ where $c_k \in \mathcal{P}_d$ satisfying*

$$\langle f - c_k, q \cdot \eta_k\rangle = 0, \quad \forall q \in \mathcal{P}_d,$$

and

$$(\mathcal{M}b_k)(x) \leq C(\mathcal{M}f)(x)\chi_{Q_k}(x) + C\sigma \frac{l(Q_k)^{n+d+1}}{|x - x_k|^{n+d+1}} \chi_{\mathbb{R}^n\setminus Q_k}(x) \tag{6.3}$$

for some $C > 0$.

For brevity, we refer the reader to [50, Chapter III, Section 2.1] for the proof of the above proposition.

PROPOSITION 6.5. *Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a Lebesgue measurable function with $0 < p_- \leq p_+ < \infty$ and $\omega \in \mathcal{W}_{p(\cdot)}$. If $f \in H^{p(\cdot)}_\omega$, then the distribution g given in Proposition 6.4 is locally integrable.*

PROOF. We first show that $\mathcal{M}g \in L^1_{\mathrm{loc}}$. In view of Proposition 6.4 (4) and (4.7), it suffices to show that $F = \sum_{k\in\mathbb{N}}(M\chi_{Q_k})^{\frac{n+d+1}{n}} \in L^1_{\mathrm{loc}}$.

For any $B \in \mathbb{B}$, by [21, Chapter II, Theorem 2.12], we have

$$\int_B |F(x)|dx \leq \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^n} (M\chi_{Q_k}(x))^{\frac{n+d+1}{n}} \chi_B(x)dx$$

$$\leq \int_{\mathbb{R}^n} \left(\sum_{k \in \mathbb{N}} \chi_{Q_k}(x) \right) (M\chi_B)(x)dx$$

because $\frac{n+d+1}{n} > 1$.

The definition of s_ω (2.3), there exists an r such that $s_\omega < r$ and the Hardy-Littlewood maximal operator M is bounded on $L^{(rp(\cdot))'}_{\omega^{-1/r}}$. Therefore, the bounded intersection property and Lemma 2.1 yield

$$\begin{aligned}
\int_B |F(x)|dx &\leq C \int_{\mathbb{R}^n} \chi_{\{x \in \mathbb{R}^n : (\mathcal{M}f)(x) > \sigma\}}(x)(M\chi_B)(x)dx \\
&\leq C \|\chi_{\{x \in \mathbb{R}^n : (\mathcal{M}f)(x) > \sigma\}}\|_{L^{rp(\cdot)}_{\omega^{1/r}}} \|M\chi_B\|_{L^{(rp(\cdot))'}_{\omega^{-1/r}}} \\
&\leq C \|\chi_{\{x \in \mathbb{R}^n : (\mathcal{M}f)(x) > \sigma\}}\|^{1/r}_{L^{p(\cdot)}_\omega} \|\chi_B\|_{L^{(rp(\cdot))'}_{\omega^{-1/r}}} \\
&\leq C\sigma^{-1/r} \|\mathcal{M}f\|^{1/r}_{L^{p(\cdot)}_\omega} \|\chi_B\|_{L^{(rp(\cdot))'}_{\omega^{-1/r}}} < \infty. \qquad (6.4)
\end{aligned}$$

That is, $F \in L^1_{\text{loc}}$ and, hence, $\mathcal{M}g \in L^1_{\text{loc}}$. By using the idea from [50, Chapter III, 2.3.3], we now prove that $g \in L^1_{\text{loc}}$.

For any $B \in \mathbb{B}$, let A_B be the space of finite Borel measures on B. A_B is the dual of the space of continuous functions on B and $\mathcal{M}g \in L^1_{\text{loc}} \subset A_B$. Taking an approximate of identity Φ, we have $|\Phi_i * g| \leq \mathcal{M}g$ and $\Phi_i * g \to g$ in $\mathcal{S}'(\mathbb{R}^n)$. The Banach-Alaoglu theorem assures that there exists a subsequence of $\Phi_i * g$ converges weakly to a measure $d\mu \in A_B$.

Since $|\Phi_i * g| \leq \mathcal{M}g$, we find that $d\mu = hdx$ is absolutely continuous with $\int_B |h(x)|dx < \infty$ and, hence, $g = h$. Therefore, $g \in L^1_{\text{loc}}$. □

The proof of the following proposition also provides a supporting result for the proof of Theorem 6.2.

PROPOSITION 6.6. *Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a Lebesgue measurable function with $0 < p_- \leq p_+ < \infty$ and $\omega \in \mathcal{W}_{p(\cdot)}$. Then $\mathcal{H}^{p(\cdot)}_\omega \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{H}^{p(\cdot)}_\omega \cap L^1_{\text{loc}}$ is dense in $\mathcal{H}^{p(\cdot)}_\omega$.*

PROOF. According to [50, Chapter III, (21)], for any $\phi \in \mathcal{S}(\mathbb{R}^n)$, there exists $B_0 \in \mathbb{B}$ such that

$$|\langle f, \phi \rangle| \leq C\mathcal{M}f(x), \quad \forall x \in B_0$$

for some $C > 0$. That is,

$$|\langle f, \phi \rangle|^{p_*} \leq \frac{C}{|B_0|} \int_{B_0} |\mathcal{M}f(x)|^{p_*}dx \leq \frac{C}{|B_0|} \|(\mathcal{M}f)^{p_*}\|_{L^{p(\cdot)/p_*}_{\omega^{p_*}}} \|\chi_{B_0}\|_{L^{(p(\cdot)/p_*)'}_{\omega^{-p_*}}}.$$

Definition 2.3 (1) assures that $\|\chi_{B_0}\|_{L^{(p(\cdot)/p_*)'}_{\omega^{-p_*}}} < \infty$. Thus,

$$|\langle f, \phi \rangle| \leq \frac{C}{|B_0|} \|\mathcal{M}f\|_{L^{p(\cdot)}_\omega} \leq C\|f\|_{\mathcal{H}^{p(\cdot)}_\omega}$$

for some $C > 0$ independent of $f \in \mathcal{H}^{p(\cdot)}_\omega$. That is, $\mathcal{H}^{p(\cdot)}_\omega \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$.

Next, we show that $\mathcal{H}^{p(\cdot)}_\omega \cap L^1_{\mathrm{loc}}$ is dense in $\mathcal{H}^{p(\cdot)}_\omega$.

For any $f \in \mathcal{H}^{p(\cdot)}_\omega$, by applying Proposition 6.4 with $d = d_\omega = [ns_\omega - n]$ and $\sigma = 2^j$, $j \in \mathbb{Z}$, we have $f = g^j + b^j$ with $b^j = \sum_{k\in\mathbb{N}} b^j_k$. The b^j_k are supported in the cubes Q^j_k where these cubes satisfy

$$\bigcup_{k\in\mathbb{N}} Q^j_k = \{x \in \mathbb{R}^n : (\mathcal{M}f)(x) > 2^j\} = O^j. \tag{6.5}$$

We have $O^j \downarrow \emptyset$, as $j \to \infty$.

We now show that $b^j \to 0$ in $\mathcal{H}^{p(\cdot)}_\omega$ when $j \to \infty$. The definition of s_ω (2.3) and the inequality (5.13) assure the existence of r such that

$$s_\omega < r < \frac{[ns_\omega - n] + n + 1}{n} = \frac{d_\omega + n + 1}{n}$$

and the Hardy-Littlewood maximal operator M is bounded on $L^{(rp(\cdot))'}_{\omega^{-1/r}}$.

In view of (4.7) and (6.3), for any $h \in L^{(rp(\cdot))'}_{\omega^{-1/r}}$ with $\|h\|_{L^{(rp(\cdot))'}_{\omega^{-1/r}}} \le 1$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |(\mathcal{M}b^j)(x)|^{1/r}|h(x)|dx \\
&\quad\le C\int_{\mathbb{R}^n}\sum_{k\in\mathbb{N}} |(\mathcal{M}f)(x)|^{1/r}|h(x)|\chi_{Q^j_k}(x)dx \\
&\qquad + C2^{j/r}\int_{\mathbb{R}^n}|h(x)|\sum_{k\in\mathbb{N}}\left(\frac{l(Q^j_k)^{n+d_\omega+1}\chi_{\mathbb{R}^n\setminus Q^j_k}(x)}{(l(Q^j_k)+|x-x^j_k|)^{n+d_\omega+1}}\right)^{1/r}dx \\
&\quad\le C\int_{O^j}|(\mathcal{M}f)(x)|^{1/r}|h(x)|dx + C2^{j/r}\sum_{k\in\mathbb{N}}\int_{\mathbb{R}^n}((M\chi_{Q^j_k})(x))^{(n+d_\omega+1)/rn}|h(x)|dx\,.
\end{aligned}$$

By using [21, Chapter II, Theorem 2.12], we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n}((M\chi_{Q^j_k})(x))^{\frac{n+d_\omega+1}{rn}}|h(x)|dx &\le \int_{\mathbb{R}^n}(\chi_{Q^j_k}(x))^{\frac{n+d_\omega+1}{rn}}(Mh)(x)dx \\
&= \int_{\mathbb{R}^n}\chi_{Q^j_k}(x)(Mh)(x)dx = \int_{Q^j_k}(Mh)(x)dx
\end{aligned}$$

as $\frac{n+d_\omega+1}{rn} > 1$.

Lemma 2.1, the bounded intersection property satisfied by $\{Q^j_k\}_{k\in\mathbb{N}}$ and (6.5) assure that

$$\begin{aligned}
\int_{\mathbb{R}^n}|(\mathcal{M}b^j)(x)|^{1/r}|h(x)|dx &\le C\int_{O^j}|(\mathcal{M}f)(x)|^{1/r}(Mh)(x)dx \\
&\le C\|\chi_{O^j}(\mathcal{M}f)^{1/r}\|_{L^{rp(\cdot)}_{\omega^{1/r}}}\|Mh\|_{L^{(rp(\cdot))'}_{\omega^{-1/r}}}
\end{aligned}$$

for some $C > 0$.

Since M is bounded on $L^{(rp(\cdot))'}_{\omega^{-1/r}}$ and $\|h\|_{L^{(rp(\cdot))'}_{\omega^{-1/r}}} \leq 1$, we obtain

$$\int |(\mathcal{M}b^j)(x)|^{1/r}|h(x)|dx \leq C\|\chi_{O^j}(\mathcal{M}f)^{1/r}\|_{L^{rp(\cdot)}_{\omega^{1/r}}}\|h\|_{L^{(rp(\cdot))'}_{\omega^{-1/r}}}$$
$$\leq C\|\chi_{O^j}(\mathcal{M}f)^{1/r}\|_{L^{rp(\cdot)}_{\omega^{1/r}}}$$

for some $C > 0$.

By taking supremum over those $h \in L^{(rp(\cdot))'}_{\omega^{-1/r}}$ with $\|h\|_{L^{(rp(\cdot))'}_{\omega^{-1/r}}} \leq 1$, Proposition 2.2 yields

$$\|\mathcal{M}b^j\|^{1/r}_{L^{p(\cdot)}_{\omega}} = \|(\mathcal{M}b^j)^{1/r}\|_{L^{rp(\cdot)}_{\omega^{1/r}}} \leq C\|\chi_{O^j}(\mathcal{M}f)^{1/r}\|_{L^{rp(\cdot)}_{\omega^{1/r}}} = C\|\chi_{O^j}\mathcal{M}f\|^{1/r}_{L^{p(\cdot)}_{\omega}}.$$

Thus, $b^j \in \mathcal{H}^{p(\cdot)}_{\omega}$.

Since Lemma 2.3 asserts that $\|\cdot\|_{L^{p(\cdot)}_{\omega}}$ is an absolutely continuous quasi-norm, $\mathcal{M}f \in L^{p(\cdot)}_{\omega}$ and $\chi_{O^j}\mathcal{M}f \downarrow 0$ as $j \to \infty$, the above inequality gives

$$\lim_{j\to\infty}\|b^j\|_{\mathcal{H}^{p(\cdot)}_{\omega}} = \lim_{j\to\infty}\|\mathcal{M}b^j\|_{L^{p(\cdot)}_{\omega}} \leq C\lim_{j\to\infty}\|\chi_{O^j}\mathcal{M}f\|_{L^{p(\cdot)}_{\omega}} = 0.$$

Consequently, $g^j = f - b^j \in \mathcal{H}^{p(\cdot)}_{\omega}$. Since $g^j \in \mathcal{H}^{p(\cdot)}_{\omega} \cap L^1_{\mathrm{loc}}$, we find that $\lim_{j\to\infty}\|f - g^j\|_{\mathcal{H}^{p(\cdot)}_{\omega}} = \lim_{j\to\infty}\|b^j\|_{\mathcal{H}^{p(\cdot)}_{\omega}} = 0$. Therefore, $\mathcal{H}^{p(\cdot)}_{\omega} \cap L^1_{\mathrm{loc}}$ is dense in $\mathcal{H}^{p(\cdot)}_{\omega}$. □

We now ready to prove Theorem 6.2.

PROOF OF THEOREM 6.2. It suffices to establish the atomic decomposition for $(p(\cdot), \infty, d)$ atoms with $d \geq d_\omega$.

In view of Proposition 6.6, $\mathcal{H}^{p(\cdot)}_{\omega} \cap L^1_{\mathrm{loc}}$ is dense in $\mathcal{H}^{p(\cdot)}_{\omega}$. Therefore, by using the density argument, it suffices to assume that $f \in \mathcal{H}^{p(\cdot)}_{\omega} \cap L^1_{\mathrm{loc}}$.

For any $d \geq d_\omega$ and $f \in \mathcal{H}^{p(\cdot)}_{\omega} \cap L^1_{\mathrm{loc}}$, by applying Proposition 6.4 with $\sigma = 2^j$, $j \in \mathbb{Z}$, we have $f = g^j + b^j$ with $b^j = \sum_{k\in\mathbb{N}} b^j_k$. The b^j_k are supported in the cubes Q^j_k where these cubes satisfy (6.5).

Let $\{\eta^j_k\}$ be the family of smooth functions given in Proposition 6.4 (3) for the collection of cube $\{Q^j_k\}$.

In view of (4.7) and (6.4), there exists a $x_j \in B(0, 1)$ such that

$$\sum_{k\in\mathbb{N}} \frac{l(Q^j_k)^{n+d+1}}{(l(Q^j_k) + |x_j - x^j_k|)^{n+d+1}} \leq \frac{C}{|B(0,1)|}\int_{B(0,1)} \sum_{k\in\mathbb{N}}(M\chi_{Q^j_k}(x))^{\frac{n+d+1}{n}}dx \leq C2^{-j/r}$$

for some $C > 0$ independent of $j \in \mathbb{Z}$. For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, write $\varphi^j(\cdot) = \varphi(\cdot - x_j)$, we have $(\varphi * g^j)(0) = (\varphi^j * g^j)(x_j)$. As $x_j \in B(0, 1)$, $\mathfrak{N}_N(C\varphi^j) \leq \mathfrak{N}_N(\varphi)$ for some $C > 0$ independent of $j \in \mathbb{Z}$. Proposition 6.4 (4) ensures that

$$|\varphi * g^j(0)| = |(\varphi^j * g^j)(x_j)| \leq \mathcal{M}(g^j)(x_j) \leq C2^{j(1-\frac{1}{r})}$$

for some $C > 0$. As $r > s_\omega \geq 1$, we obtain $g^j \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $j \to -\infty$.

In addition, Proposition 6.6 ensures that $b^j \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $j \to \infty$. The convergence of g^j and b^j guarantees that $f = \sum_{j\in\mathbb{Z}}(g^{j+1} - g^j)$ converges in $\mathcal{S}'(\mathbb{R}^n)$.

Moreover, Item (5) of Proposition 6.4 gives

$$g^{j+1} - g^j = b^{j+1} - b^j = \sum_{k\in\mathbb{N}}((f - c_k^{j+1})\eta_k^{j+1} - (f - c_k^j)\eta_k^j)$$

where $c_k^j \in \mathcal{P}_d$ satisfies

$$\int_{\mathbb{R}^n}(f(x) - c_k^j(x))q(x)\eta_k^j(x)dx = 0\,, \quad \forall q \in \mathcal{P}_d\,.$$

Consequently, we have $f = \sum_{j,k} A_k^j$ where

$$A_k^j = (f - c_k^j)\eta_k^j - \sum_{l\in\mathbb{N}}(f - c_l^{j+1})\eta_l^{j+1}\eta_k^j + \sum_{l\in\mathbb{N}} c_{k,l}\eta_l^{j+1}$$

and $c_{k,l} \in \mathcal{P}_d$ fulfills

$$\int_{\mathbb{R}^n}((f(x) - c_l^{j+1}(x))\eta_k^j(x) - c_{k,l}(x))q(x)\eta_l^{j+1}(x)dx = 0, \quad \forall q \in \mathcal{P}_d\,.$$

Define

$$a_k^j = \lambda_{j,k}^{-1}A_k^j \quad \text{and} \quad \lambda_{j,k} = c2^j\|\chi_{Q_k^j}\|_{L_\omega^{p(\cdot)}}$$

where c is a constant determined by the family $\{A_k^j\}_{j,k}$. Most importantly, the constant c is independent of j and k, see [50, p.108-109].

The proof for the classical Hardy space [50, Chapter III, Section 2] assures that a_k^j is a $(p(\cdot), \infty, d)$ atom.

The definition of Q_k^j and the finite intersection property of the family $\{Q_k^j\}_{k\in\mathbb{N}}$ yield that for any $0 < \theta < \infty$

$$\sum_{k\in\mathbb{N}}\left(\frac{|\lambda_{j,k}|}{\|\chi_{Q_k^j}\|_{L_\omega^{p(\cdot)}}}\right)^\theta \chi_{Q_k^j}(x) \le C2^{\theta j}\chi_{O^j}(x)$$

for some $C > 0$.

That is,

$$\sum_{j,k}\left(\frac{|\lambda_{j,k}|}{\|\chi_{Q_k^j}\|_{L_\omega^{p(\cdot)}}}\right)^\theta \chi_{Q_k^j}(x) \le C\sum_{j\in\mathbb{Z}}2^{\theta j}\chi_{O^j}(x) \le C(\mathcal{M}f)(x)^\theta\,.$$

Applying the quasi-norm $\|\cdot\|_{L_{\omega^\theta}^{p(\cdot)/\theta}}^{1/\theta}$ on both sides of the above inequality, we find that

$$\left\|\sum_{j,k}\left(\frac{|\lambda_{j,k}|}{\|\chi_{Q_k^j}\|_{L_\omega^{p(\cdot)}}}\right)^\theta \chi_{Q_k^j}\right\|_{L_{\omega^\theta}^{p(\cdot)/\theta}}^{\frac{1}{\theta}} \le C\|f\|_{\mathcal{H}_\omega^{p(\cdot)}}\,, \quad 0 < \theta < \infty$$

for some $C > 0$ independent of f. □

PROOF OF THEOREM 6.3. Let $\{a_j\}_{j\in\mathbb{N}}$ be a family of $(p(\cdot), q, [ns_\omega - n])$ atoms with $\operatorname{supp} a_j \subseteq 3Q_j$. Let $\{\lambda_j\}_{j\in\mathbb{N}}$ satisfy (6.1).

Write

$$\Big\|\mathcal{M}\Big(\sum_{j\in\mathbb{N}}\lambda_j a_j\Big)\Big\|_{L^{p(\cdot)}_\omega} \leq C\bigg(\Big\|\sum_{j\in\mathbb{N}}\lambda_j\chi_{3Q_j}\mathcal{M}(a_j)\Big\|_{L^{p(\cdot)}_\omega} + \Big\|\sum_{j\in\mathbb{N}}\lambda_j\chi_{\mathbb{R}^n\setminus 3Q_j}\mathcal{M}(a_j)\Big\|_{L^{p(\cdot)}_\omega}\bigg) = I + II\,.$$

We consider I. As $\Phi \in \mathcal{S}(\mathbb{R}^n)$, Φ has a radial majorant that is non-increasing, bounded and integrable. In view of [50, Chapter II, (16)], we have

$$\sup_{t>0}|\Phi_t * a_j(x)| \leq M(a_j)(x)\int_{\mathbb{R}^n}|\Phi(z)|dz \leq C\mathfrak{N}_N(\Phi)M(a_j)(x)\,, \quad \forall x \in 3Q_j$$

for some $N, C > 0$ independent of $j \in \mathbb{N}$, $\Phi \in \mathcal{S}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $t > 0$.

By taking supreme over those $\Phi \in \mathcal{S}(\mathbb{R}^n)$ with $\mathfrak{N}_N(\Phi) \leq 1$, we obtain

$$\mathcal{M}a_j(x) \leq CM(a_j)(x)\,, \quad \forall x \in 3Q_j \tag{6.6}$$

for some $C > 0$. Therefore, the θ-inequality gives

$$I \leq C\Big\|\sum_{j\in\mathbb{N}}|\lambda_j|M(a_j)\Big\|_{L^{p(\cdot)}_\omega} \leq C\bigg\|\Big(\sum_{j\in\mathbb{N}}\big(|\lambda_j|M(a_j)\big)^\theta\Big)^{1/\theta}\bigg\|_{L^{p(\cdot)}_\omega} = C\Big\|\sum_{j\in\mathbb{N}}\big(|\lambda_j|M(a_j)\big)^\theta\Big\|^{1/\theta}_{L^{p(\cdot)/\theta}_{\omega^\theta}}.$$

The boundedness of the Hardy-Littlewood maximal operator M on L^q yields

$$\|(M(a_j))^\theta\|_{L^{q/\theta}} = \|M(a_j)\|^\theta_{L^q} \leq C\|a_j\|^\theta_{L^q} \leq C\frac{|Q_j|^{\frac{\theta}{q}}}{\|\chi_{Q_j}\|_{L^{p(\cdot)/\theta}_{\omega^\theta}}}$$

for some $C > 0$.

Since $\frac{1}{\theta} \in \mathbb{S}_\omega$ and $q > \theta(\kappa_\omega^{1/\theta})'$, we apply Lemma 5.4 with $b_j = (M(a_j)\chi_{Q_j})^\theta$ and $A_j = \|\chi_{Q_j}\|^{-1}_{L^{p(\cdot)/\theta}_{\omega^\theta}}$ to obtain

$$I \leq C\Big\|\sum_{j\in\mathbb{N}}\big(|\lambda_j|M(a_j)\big)^\theta\Big\|^{1/\theta}_{L^{p(\cdot)/\theta}_{\omega^\theta}} \leq C\Big\|\sum_{j\in\mathbb{N}}\Big(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}}\Big)^\theta\chi_{3Q_j}\Big\|^{1/\theta}_{L^{p(\cdot)/\theta}_{\omega^\theta}}.$$

Furthermore, (5.12) and Theorem 3.1 yield

$$I \leq C\bigg\|\bigg(\sum_{j\in\mathbb{N}}\Big(\frac{|\lambda_j|^{\theta/2}}{\|\chi_{Q_j}\|^{\theta/2}_{L^{p(\cdot)}_\omega}}M\chi_{Q_j}\Big)^2\bigg)^{\frac{1}{2}}\bigg\|^{2/\theta}_{L^{2p(\cdot)/\theta}_{\omega^{\theta/2}}} \leq C\Big\|\sum_{j\in\mathbb{N}}\frac{|\lambda_j|^\theta}{\|\chi_{Q_j}\|^\theta_{L^{p(\cdot)}_\omega}}\chi_{Q_j}\Big\|^{1/\theta}_{L^{p(\cdot)/\theta}_{\omega^\theta}} \tag{6.7}$$

for some $C > 0$.

Next, we deal with II. For $x \in \mathbb{R}^n \backslash 3Q_j$, we use the vanishing moment condition satisfied by a_j to obtain

$$|(a_j * \Phi_t)(x)| \leq \int_{\mathbb{R}^n} \left| a_j(y) \left(\Phi_t(x-y) - \sum_{|\gamma| \leq d_\omega} \frac{(y - x_{Q_j})^\gamma}{\gamma!} \partial^\gamma \Phi_t(x - x_{Q_j}) \right) \right| dy .$$

By using the reminder terms of the Taylor expansion of Φ_t, we have

$$|(a_j * \Phi_t)(x)| \leq \int_{\mathbb{R}^n} |a_j(y)| \sum_{|\gamma| = d_\omega + 1} \left| \frac{(y - x_{Q_j})^\gamma}{\gamma!} \partial^\gamma \Phi_t(x - y + \theta(y - x_{Q_j})) \right| dy$$

for some $0 \leq \theta \leq 1$. Since $y \in Q_j$, we have $|(y - x_{Q_j})^\gamma| \leq |Q_j|^{\frac{d_\omega+1}{n}}$ for any $|\gamma| = d_\omega + 1$. Moreover, for any $y \in Q_j$,

$$|x - y + \theta(y - x_{Q_j})| \geq |x - x_{Q_j}| - (1-\theta)|y - x_{Q_j}| \geq \frac{1}{2}|x - x_{Q_j}|.$$

We obtain

$$|(a_j * \Phi_t)(x)| \leq C\mathfrak{N}_N(\Phi) t^{-(d_\omega+n+1)} |Q_j|^{\frac{d_\omega+1}{n}} (1 + t^{-1}|x - x_{Q_j}|)^{-L} \int_{3Q_j} |a_j(y)| dy$$

for some sufficient large $L > n + d_\omega + 1$ and some $C > 0$ independent of $t > 0$ and Φ. The Hölder inequality and the definition of a_j yield

$$\int_{3Q_j} |a_j(y)| dy \leq \|a_j\|_{L^q} \|\chi_{3Q_j}\|_{L^{q'}} \leq \frac{|Q_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}} \tag{6.8}$$

where q' is the conjugate of q. That is,

$$|(a_j * \Phi_t)(x)| \leq C\mathfrak{N}_N(\Phi) t^{-(d_\omega+n+1)} \frac{|Q_j|^{\frac{n+d_\omega+1}{n}}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}} (1 + t^{-1}|x - x_{Q_j}|)^{-L} .$$

As $L > n + d_\omega + 1$, by taking supremum over $t > 0$ and $\Phi \in \mathcal{S}(\mathbb{R}^n)$ with $\mathfrak{N}_N(\Phi) \leq 1$ on both sides of the above inequality, we obtain

$$\mathcal{M}a_j(x) \leq C \frac{|Q_j|^{\frac{n+d_\omega+1}{n}}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}} \frac{1}{|x - x_{Q_j}|^{n+d_\omega+1}} , \quad \forall x \in \mathbb{R}^n \backslash 3Q_j .$$

Furthermore, (4.7) yields

$$\mathcal{M}a_j(x) \leq C \frac{(M\chi_{Q_j}(x))^{\frac{n+d_\omega+1}{n}}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}} , \quad \forall x \in \mathbb{R}^n \backslash 3Q_j \tag{6.9}$$

for some $C > 0$ independent of the atoms $\{a_j\}$.

Write $\alpha = \frac{n+d_\omega+1}{n}$. Consequently,

$$II \leq C \left\| \left(\sum_{j \in \mathbb{N}} \frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}} (M\chi_{Q_j}(x))^\alpha \right)^{1/\alpha} \right\|^\alpha_{L^{\alpha p(\cdot)}_{\omega^{1/\alpha}}} .$$

Since

$$\alpha = \frac{n + d_\omega + 1}{n} \geq \frac{n + [ns_\omega - n] + 1}{n} > s_\omega\,,$$

the Fefferman-Stein vector-valued maximal inequality asserts that

$$II \leq C \left\| \left(\sum_{j\in\mathbb{N}} \frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}} \chi_{Q_j} \right)^{1/\alpha} \right\|^{\alpha}_{L^{\alpha p(\cdot)}_{\omega^{1/\alpha}}} = C \left\| \sum_{j\in\mathbb{N}} \frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}} \chi_{Q_j} \right\|_{L^{p(\cdot)}_\omega}$$

for some $C > 0$. Then, the θ-inequality gives

$$II \leq C \left\| \sum_{j\in\mathbb{N}} \frac{|\lambda_j|^\theta}{\|\chi_{Q_j}\|^\theta_{L^{p(\cdot)}_\omega}} \chi_{Q_j} \right\|^{1/\theta}_{L^{p(\cdot)/\theta}_{\omega^\theta}}. \tag{6.10}$$

In conclusion, (6.7) and (6.10) yield (6.2). □

Finally, Theorem 6.1 is a straightforward consequence of Theorems 5.2, 5.3, 6.2 and 6.3. Hence, the quasi-norms $\|\cdot\|_{H^{p(\cdot)}_\omega}$ and $\|\cdot\|_{\mathcal{H}^{p(\cdot)}_\omega}$ are mutually equivalent. When $\omega \equiv 1$, this result extends the Littlewood-Paley characterization for Hardy spaces with variable exponents in [38] because the exponent function considered in Theorem 6.1 is not necessarily to be log-Hölder continuous.

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