Tohoku Math. J.
69 (2017), 287–304

CONTIGUITY RELATIONS OF LAURICELLA'S F_D REVISITED

YOSHIAKI GOTO

(Received January 23, 2015, revised August 12, 2015)

Abstract. We study contiguity relations of Lauricella's hypergeometric function F_D, by using the twisted cohomology group and the intersection form. We derive contiguity relations from those in the twisted cohomology group and give the coefficients in these relations by the intersection numbers. Furthermore, we construct twisted cycles corresponding to a fundamental set of solutions to the system of differential equations satisfied by F_D, which are expressed as Laurent series. We also give the contiguity relations of these solutions.

1. Introduction. Lauricella's hypergeometric series F_D of m variables $x_1, \dots, x_m$ with complex parameters $a, b_1, \dots, b_m, c$ is defined by

$$F_D(a,b,c;x) = \sum_{n_1,\dots,n_m=0}^{\infty} \frac{(a, n_1+\cdots+n_m)(b_1,n_1)\cdots(b_m,n_m)}{(c, n_1+\cdots+n_m)n_1!\cdots n_m!} x_1^{n_1}\cdots x_m^{n_m},$$

where $x = (x_1, \dots, x_m)$, $b = (b_1, \dots, b_m)$, $c \notin \{0, -1, -2, \dots\}$, and $(a, n) = \Gamma(a+n)/\Gamma(a)$. This series converges in the domain $\{x \in \mathbb{C}^m \mid |x_i| < 1\ (1 \le i \le m)\}$. It is known that $F_D(a, b, c; x)$ admits an Euler-type integral representation:

$$(1)\qquad F_D(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_1^{\infty} t^{\sum_{i=1}^m b_i - c}(t-1)^{c-a-1} \prod_{i=1}^m (t-x_i)^{-b_i} dt.$$

The contiguity relations of Lauricella's F_D have been studied from several points of view. In the 1970s, W. Miller Jr. [6] gave the contiguity relations of F_D as a representation of a Lie algebra, and Aomoto [1] studied the contiguity relations of F_D and its generalization to the hypergeometric functions of type (k, n). In 1991, Sasaki [11] studied the contiguity relations in the framework of the Aomoto-Gel'fand system on the Grassmannian manifold. In 1989, an algorithmic method that used Gröbner bases to derive the contiguity relations was given by Takayama [12]. Recently, Ogawa, Takemura, and Takayama [7] have illustrated that the Pfaffian system and the contiguity relations for F_D combine to give a method to evaluate the normalizing constant of the hypergeometric distribution on the 2 by N contingency tables with given marginal sums. On the other hand, Matsumoto [5] recently proposed a method that utilizes the intersection numbers of twisted cohomology groups to derive Pfaffian systems. In this paper, we reconsider the problem of the contiguity relations of F_D, in order to produce

2010 *Mathematics Subject Classification.* Primary 33C65; Secondary 33C90.
Key words and phrases. Lauricella's F_D, contiguity relations, twisted (co)homology groups, contingency tables.
This work was supported by JSPS KAKENHI Grant Numbers 25220001, 14J01252.

formulas for application to statistics [7]. Matsumoto's method can be applied to derive the contiguity relations for our purpose, and further generalizations will be possible.

We derive the contiguity relations of F_D by considering the twisted cohomology groups associated with the integral representation (1). We regard the contiguity relations as those between the twisted cocycles. To obtain the coefficients in the contiguity relations, we use the intersection form of the twisted cohomology group. In the way, we are able to derive the contiguity relations for the basis given in [5], which was also used in [7]. An advantage of our method is that it makes it easy to systematically derive the contiguity relations for a given basis of the twisted cohomology group.

This paper is arranged as follows. In Sections 2, 3, and 4, we introduce our method for using the intersection form to derive the contiguity relations. By evaluating the intersection numbers, we obtain explicit forms for the contiguity relations. In Section 5, we introduce the system $E_D(a, b, c)$ of differential equations satisfied by $F_D(a, b, c; x)$, and we introduce the Laurent series solution $f^{(k)}(a, b, c; x)$ to $E_D(a, b, c)$ and construct a fundamental set of solutions. In Section 6, we construct the twisted cycle r_k corresponding to the solution $f^{(k)}(a, b, c; x)$. Since our contiguity relations are obtained from those in the twisted cohomology group, the integration on r_k gives the contiguity relations of $f^{(k)}$. In Section 7, we present an application of our formula, in which we evaluate the normalizing constant of the hypergeometric distribution on the 2 by $m + 1$ contingency tables; this is also explained in [7] in the context of statistics. We also explain how to apply our results when the parameters (a, b, c) are integers. This assumption is necessary for our applications to statistics. The discussion of twisted cycles in Section 6 as well as Theorem 3.4 are fully utilized to evaluate the normalizing constant with arbitrary marginal sums.

Although the contiguity relations of F_D have been studied by several authors, those of the other solutions $f^{(k)}$ that appear in applications to statistics have not been studied.

Acknowledgments. The author thanks Professor Nobuki Takayama for posing this problem and for his constant encouragement.

2. Twisted cohomology group and intersection pairing. We summarize some results in [2], [3], and [5] that will be used in this paper. We consider the twisted cohomology group for

$$T_x := \mathbb{C} - \{x_0, x_1, \dots, x_m, x_{m+1}\}$$

and the multivalued function

$$u_x(t) := \prod_{i=0}^{m+1} (t - x_i)^{\alpha_i} ,$$

where

$$x_0 := 0 , \quad x_{m+1} := 1 ,$$

(2) $\alpha_0 := -c + \sum_{j=1}^{m} b_j\,, \quad \alpha_k := -b_k \ (1 \le k \le m)\,, \quad \alpha_{m+1} := c - a\,, \quad \alpha_{m+2} := a\,.$

Except in Section 7, we assume the condition

(3) $$\alpha_k \notin \mathbb{Z} \quad (0 \le k \le m+2)\,.$$

We denote the vector space consisting of the smooth k-forms on T_x and that with compact support by $\mathcal{E}^k(T_x)$ and $\mathcal{E}^k_c(T_x)$, respectively. We set $\omega := d \log u_x$ and $\nabla_\omega := d + \omega\wedge$, where d is the exterior derivative with respect to the variable t (note that this is not with respect to $x_1, \dots, x_m$, which are regarded as parameters). The twisted cohomology group and that with compact support are defined as

$$H^1(T_x, \nabla_\omega) = \mathrm{Ker}(\nabla_\omega : \mathcal{E}^1(T_x) \to \mathcal{E}^2(T_x))/\nabla_\omega(\mathcal{E}^0(T_x))\,,$$
$$H^1_c(T_x, \nabla_\omega) = \mathrm{Ker}(\nabla_\omega : \mathcal{E}^1_c(T_x) \to \mathcal{E}^2_c(T_x))/\nabla_\omega(\mathcal{E}^0_c(T_x))\,,$$

respectively. The expression (1) means that the integral

$$\int_1^\infty u_x\varphi_0, \quad \varphi_0 := \frac{dt}{t-1}$$

represents $F_D(a, b, c; x)$ modulo Gamma factors. By [2], $H^1(T_x, \nabla_\omega)$ has $m+1$ dimensions, and there is a canonical isomorphism $\jmath : H^1(T_x, \nabla_\omega) \to H^1_c(T_x, \nabla_\omega)$; see also [5, Fact 6.1]. Hereafter, we identify $H^1_c(T_x, \nabla_\omega)$ with $H^1(T_x, \nabla_\omega)$.

The intersection form I_c on the twisted cohomology groups is the pairing between $H^1(T_x, \nabla_\omega)$ and $H^1(T_x, \nabla_{-\omega})$, and it is defined as follows:

$$I_c(\psi, \psi') := \int_{T_x} \jmath(\psi) \wedge \psi'\,, \quad \psi \in H^1(T_x, \nabla_\omega),\ \psi' \in H^1(T_x, \nabla_{-\omega})\,.$$

We put

$$\varphi_{i,m+2} := \frac{dt}{t-x_i}\,, \quad \varphi_{i,j} := \varphi_{i,m+2} - \varphi_{j,m+2} = \frac{(x_i - x_j)dt}{(t-x_i)(t-x_j)},$$
$$\varphi_0 = \varphi_{m+1,m+2} = \frac{dt}{t-1}\,, \quad \varphi_k := \varphi_{m+1,k} = \frac{(1-x_k)dt}{(t-x_k)(t-1)}\,,$$

where $0 \le i, j \le m+1$ and $1 \le k \le m$. The intersection numbers among these 1-forms are evaluated in [3]; see also [5, Fact 6.2].

FACT 2.1 ([3]). *We have*

$$I_c(\varphi_{i,j}, \varphi_{p,q}) = 2\pi\sqrt{-1}\left(\frac{\delta_{i,p} - \delta_{i,q}}{\alpha_i} - \frac{\delta_{j,p} - \delta_{j,q}}{\alpha_j}\right)\,,$$

where $i, j, p, q \in \{0, 1, \dots, m+2\}$, and $\delta_{i,p}$ is the Kronecker delta. Thus, the intersection matrix $C(a, b, c) := \big(I_c(\varphi_i, \varphi_j)\big)_{i,j=0,\dots,m}$ is

$$C(a, b, c) = 2\pi\sqrt{-1}\left\{\frac{1}{\alpha_{m+1}}N + \mathrm{diag}\left(\frac{1}{\alpha_{m+2}}, \frac{1}{\alpha_1}, \dots, \frac{1}{\alpha_m}\right)\right\}\,,$$

where

$$N = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}.$$

Under assumption (3)*, we have*

$$\det\bigl(C(a,b,c)\bigr) = (2\pi\sqrt{-1})^{m+1} \frac{-\alpha_0}{\prod_{i=1}^{m+2} \alpha_i} \neq 0\,,$$

and hence $\varphi_0, \dots, \varphi_m$ *form a basis of* $H^1(T_x, \nabla_\omega)$.

3. Contiguity relations. In this section, we derive the contiguity relations by using the intersection form. We define two column vectors of size $m+1$:

$$F(a,b,c;x) := {}^t\!\left(F_D(a,b,c;x), \frac{x_1-1}{\alpha_1}\frac{\partial}{\partial x_1}F_D(a,b,c;x), \dots, \frac{x_m-1}{\alpha_m}\frac{\partial}{\partial x_m}F_D(a,b,c;x)\right),$$

$$\tilde{F}(a,b,c;x) := \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} F(a,b,c;x).$$

For $(v_0, \dots, v_m) \in \mathbb{C}^{m+1}$, we regard v_i as the i-th entry. For example, the 0-th entry of $F(a,b,c;x)$ is $F_D(a,b,c;x)$. By [5, Corollary 7.2], we have

$${}^t\!\left(\int_1^\infty u_x\varphi_0, \dots, \int_1^\infty u_x\varphi_m\right) = \tilde{F}(a,b,c;x)\,.$$

Our main theorem (Theorem 3.4) states the contiguity relations of the vector-valued function $F(a,b,c;x)$.

For $0 \le k \le m+1$, there exist $p_{il}^{(k)}(a,b,c;x)$'s such that

$$(t - x_k)\cdot\varphi_i = \sum_{l=0}^{m} p_{il}^{(k)}(a,b,c;x)\cdot\varphi_l \tag{4}$$

as elements in the twisted cohomology group $H^1(T_x, \nabla_\omega)$. We put $P_k(a,b,c,x) := \left(p_{ij}^{(k)}(a,b,c;x)\right)_{i,j}$ and $Q_k(a,b,c;x) := \left(I_c\left((t-x_k)\varphi_i, \varphi_j\right)\right)_{i,j}$. Because of

$$I_c\left((t-x_k)\varphi_i, \varphi_j\right) = \sum_{l=0}^{m} p_{il}^{(k)}(a,b,c;x)\cdot I_c(\varphi_l, \varphi_j)\,,$$

we obtain $Q_k(a,b,c;x) = P_k(a,b,c;x)C(a,b,c)$, that is,

$$P_k(a,b,c;x) = Q_k(a,b,c;x)C(a,b,c)^{-1}\,. \tag{5}$$

In the next section, we will show the following proposition.

PROPOSITION 3.1. *We have*

$$\begin{aligned}
&Q_k(a,b,c;x)\\
&= 2\pi\sqrt{-1}\biggl\{\frac{1-x_k}{\alpha_{m+1}}N+\frac{1}{\alpha_{m+2}}\mathrm{diag}\,(0,1-x_1,\ldots,1-x_m)\cdot N\cdot \mathrm{diag}\,(1,0,\ldots,0)\\
&\quad -\frac{1}{1-\alpha_{m+2}}\mathrm{diag}\,(1,0,\ldots,0)\cdot N\cdot\mathrm{diag}\,(0,1-x_1,\ldots,1-x_m)\\
&\quad +\mathrm{diag}\biggl(\frac{1-x_k}{\alpha_{m+2}}-\frac{1}{1-\alpha_{m+2}}\biggl(\frac{\sum_{p=1}^{m+1}\alpha_p x_p}{\alpha_{m+2}}+1\biggr),\frac{x_1-x_k}{\alpha_1},\ldots,\frac{x_m-x_k}{\alpha_m}\biggr)\biggr\}.
\end{aligned}$$

Note that

$$\mathrm{diag}(p_0,\ldots,p_m)\cdot N\cdot\mathrm{diag}(q_0,\ldots,q_m)=\begin{pmatrix} p_0q_0 & p_0q_1 & \cdots & p_0q_m\\ p_1q_0 & p_1q_1 & \cdots & p_1q_m\\ \vdots & \vdots & \vdots & \vdots\\ p_mq_0 & p_mq_1 & \cdots & p_mq_m\end{pmatrix}.$$

We give the contiguity relations by using the matrices P_k, Q_k, and C. Let e_k be the k-th unit vector in $\mathbb{C}^m$. For example, we have $b-e_1=(b_1-1,b_2,\ldots,b_m)$.

LEMMA 3.2.

$$\begin{aligned}
\tilde F(a-1,b,c;x)&=P_{m+1}(a,b,c;x)\tilde F(a,b,c;x)\,,\\
\tilde F(a-1,b,c-1;x)&=P_0(a,b,c;x)\tilde F(a,b,c;x)\,,\\
\tilde F(a-1,b-e_k,c-1;x)&=P_k(a,b,c;x)\tilde F(a,b,c;x)\quad (1\le k\le m)\,.
\end{aligned}$$

PROOF. Recall that $x_{m+1}=1$. We consider the integration of (4) on $(1,\infty)$. By (1), we have

$$\begin{aligned}
\int_1^\infty u_x\cdot(t-1)\varphi_0 &= \int_1^\infty t^{\sum_i b_i-c}(t-1)^{c-a}\prod_{i=1}^m (t-x_i)^{-b_i}dt\\
&=\frac{\Gamma(a-1)\Gamma(c-a+1)}{\Gamma(c)}F_D(a-1,b,c;x)\,,
\end{aligned}$$

which is the 0-th entry of $\tilde F(a-1,b,c;x)$. Then, the first equality follows. The other ones are shown in an analogous way. □

The following lemma is obvious.

LEMMA 3.3.

$$\begin{aligned}
\tilde F(a-1,b,c;x)&=P_{m+1}(a,b,c;x)\tilde F(a,b,c;x)\,,\\
\tilde F(a,b,c-1;x)&=P_0(a+1,b,c;x)P_{m+1}(a+1,b,c;x)^{-1}\tilde F(a,b,c;x)\,,\\
\tilde F(a,b-e_k,c;x)&=P_k(a+1,b,c+1;x)P_0(a+1,b,c+1;x)^{-1}\tilde F(a,b,c;x)\,.
\end{aligned}$$

We can reduce this lemma to the relations between the $F(a,b,c;x)$'s by the formulas $\Gamma(s+1)=s\cdot\Gamma(s)$ and (5).

THEOREM 3.4 (Contiguity relations). *We have*

$$
\begin{aligned}
F(a-1,b,c;x) &= D_a(a,b,c;x)F(a,b,c;x)\,,\\
F(a,b,c-1;x) &= D_c(a,b,c;x)F(a,b,c;x)\,,\\
F(a,b-e_k,c;x) &= D_k(a,b,c;x)F(a,b,c;x) \quad (1\le k\le m)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
D_a(a,b,c;x) &:= \frac{a-1}{c-a}\cdot Q_{m+1}(a,b,c;x)\cdot C(a,b,c)^{-1}\,,\\
D_c(a,b,c;x) &:= \frac{c-a-1}{c-1}\cdot Q_0(a+1,b,c;x)\cdot Q_{m+1}(a+1,b,c;x)^{-1}\,,\\
D_k(a,b,c;x) &:= Q_k(a+1,b,c+1;x)\cdot Q_0(a+1,b,c+1;x)^{-1}\,.
\end{aligned}
$$

The explicit forms for $C(a,b,c)$ and $Q_k(a,b,c;x)$ are given in Fact 2.1 and Proposition 3.1, respectively.

EXAMPLE 3.5. If $m=2$, the matrices $C(a,b,c)$ and $Q_k(a,b,c;x)$ are as follows:

$$
C(a,b,c;x) = 2\pi\sqrt{-1}\left\{\frac{1}{\alpha_3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}+\begin{pmatrix}\frac{1}{\alpha_4}&0&0\\0&\frac{1}{\alpha_1}&0\\0&0&\frac{1}{\alpha_2}\end{pmatrix}\right\},
$$

$$
\begin{aligned}
&Q_k(a,b,c;x)\\
&=2\pi\sqrt{-1}\left\{\frac{1-x_k}{\alpha_3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}+\begin{pmatrix}\frac{1-x_k}{\alpha_4}-\frac{1}{1-\alpha_4}\cdot\frac{\alpha_1x_1+\alpha_2x_2+\alpha_3+\alpha_4}{\alpha_4} & -\frac{1-x_1}{1-\alpha_4} & -\frac{1-x_2}{1-\alpha_4}\\ \frac{1-x_1}{\alpha_4} & \frac{x_1-x_k}{\alpha_1} & 0\\ \frac{1-x_2}{\alpha_4} & 0 & \frac{x_2-x_k}{\alpha_2}\end{pmatrix}\right\}.
\end{aligned}
$$

The first equality in Theorem 3.4 is written as

$$
\begin{aligned}
&\begin{pmatrix} F_D(a-1,b_1,b_2,c;x_1,x_2)\\ \frac{1-x_1}{b_1}\cdot\frac{\partial}{\partial x_1}F_D(a-1,b_1,b_2,c;x_1,x_2)\\ \frac{1-x_2}{b_2}\cdot\frac{\partial}{\partial x_2}F_D(a-1,b_1,b_2,c;x_1,x_2)\end{pmatrix}\\
&=\begin{pmatrix}\frac{-b_1x_1-b_2x_2+c-a}{c-a} & \frac{b_1x_1}{c-a} & \frac{b_2x_2}{c-a}\\ \frac{(a-1)(1-x_1)}{c-a} & \frac{(a-1)(x_1-1)}{c-a} & 0\\ \frac{(a-1)(1-x_2)}{c-a} & 0 & \frac{(a-1)(x_2-1)}{c-a}\end{pmatrix}\begin{pmatrix} F_D(a,b_1,b_2,c;x_1,x_2)\\ \frac{1-x_1}{b_1}\cdot\frac{\partial}{\partial x_1}F_D(a,b_1,b_2,c;x_1,x_2)\\ \frac{1-x_2}{b_2}\cdot\frac{\partial}{\partial x_2}F_D(a,b_1,b_2,c;x_1,x_2)\end{pmatrix}.
\end{aligned}
$$

The 3×3 matrix on the right-hand side is equal to $D_a(a,b,c;x)=\frac{a-1}{c-a}\cdot Q_3(a,b,c;x)\cdot C(a,b,c)^{-1}$.

REMARK 3.6. The determinant of $Q_k(a, b, c; x)$ is as follows:

$$\det\bigl(Q_k(a,b,c;x)\bigr) = (2\pi\sqrt{-1})^{m+1} \cdot \frac{\alpha_0(1+\delta_{k,0}\alpha_0)}{\prod_{\substack{j=1\\ j\neq k}}^{m+2} \alpha_j \cdot (\alpha_{m+2}-1)} \cdot \prod_{\substack{j=0\\ j\neq k}}^{m+1} (x_j - x_k)\,.$$

4. Proof of Proposition 3.1. In this section, we evaluate the intersection numbers that are the entries of $Q_k(a, b, c; x)$, by using Fact 2.1.

We denote $\varphi \sim \psi$, if φ is ∇_ω-cohomologous to ψ, that is,

$$\begin{aligned}\varphi \sim \psi &\Longleftrightarrow \varphi = \psi + \nabla_\omega f \quad \text{for some } f \in \mathcal{E}^0(T_x)\,,\\ &\Longleftrightarrow \varphi \text{ and } \psi \text{ give the same element in } H^1(T_x, \nabla_\omega)\,.\end{aligned}$$

LEMMA 4.1.

$$dt \sim -\frac{1}{1-\alpha_{m+2}} \sum_{p=1}^{m+1} \alpha_p x_p \varphi_{p,m+2}\,.$$

PROOF. This lemma follows from

$$\begin{aligned}0 \sim \nabla_\omega(t) &= dt + \sum_{p=0}^{m+1} \alpha_p \frac{t}{t-x_p} dt = dt + \sum_{p=0}^{m+1} \alpha_p \frac{t - x_p + x_p}{t-x_p} dt\\ &= \Bigl(1 + \sum_{p=0}^{m+1} \alpha_p\Bigr) dt + \sum_{p=0}^{m+1} \alpha_p x_p \frac{dt}{t-x_p} = (1-\alpha_{m+2})dt + \sum_{p=1}^{m+1} \alpha_p x_p \varphi_{p,m+2}\,.\end{aligned}$$

Here, we use $x_0 = 0$ and $\sum_{p=0}^{m+2} \alpha_p = 0$. □

Then, we have

$$\begin{aligned}(t-x_k)\cdot\varphi_{l,m+2} &= \frac{t-x_k}{t-x_l} dt = \frac{t - x_l + x_l - x_k}{t-x_l} dt\\ &\sim (x_l - x_k)\varphi_{l,m+2} - \frac{1}{1-\alpha_{m+2}} \sum_{p=1}^{m+1} \alpha_p x_p \varphi_{p,m+2}\,.\end{aligned}$$

Fact 2.1 and a straightforward calculation show the following lemma.

LEMMA 4.2.

$$\begin{aligned}&I_c\left((t-x_k)\varphi_{l,m+2}, \varphi_j\right)\\ &= \begin{cases} 2\pi\sqrt{-1}\Bigl((x_l - x_k)\left(\frac{\delta_{l,m+1}}{\alpha_{m+1}} + \frac{1}{\alpha_{m+2}}\right) - \frac{1}{1-\alpha_{m+2}}\left(\frac{\sum_{p=1}^{m+1}\alpha_p x_p}{\alpha_{m+2}} + 1\right)\Bigr) & (j=0)\,,\\ 2\pi\sqrt{-1}\left((x_l - x_k)\frac{\delta_{l,m+1}-\delta_{l,j}}{\alpha_l} - \frac{1-x_j}{1-\alpha_{m+2}}\right) & (1 \le j \le m)\,.\end{cases}\end{aligned}$$

PROOF OF PROPOSITION 3.1. Let $Q_k(i,j)$ be the (i,j) entry of $Q_k(a,b,c;x)$, that is, $Q_k(i,j) = I_c\big((t-x_k)\cdot\varphi_i,\varphi_j\big)$. For $1 \le i,j \le m$, we have

$$\begin{aligned}
Q_k(0,0) &= I_c\big((t-x_k)\cdot\varphi_{m+1,m+2},\varphi_0\big)\\
&= 2\pi\sqrt{-1}\left(\frac{1-x_k}{\alpha_{m+1}} + \frac{1-x_k}{\alpha_{m+2}} - \frac{1}{1-\alpha_{m+2}}\left(\frac{\sum_{p=1}^{m+1}\alpha_p x_p}{\alpha_{m+2}} + 1\right)\right),\\
Q_k(0,j) &= I_c\big((t-x_k)\cdot\varphi_{m+1,m+2},\varphi_j\big)\\
&= 2\pi\sqrt{-1}\left(\frac{1-x_k}{\alpha_{m+1}} - \frac{1-x_j}{1-\alpha_{m+2}}\right),\\
Q_k(i,0) &= I_c\big((t-x_k)\cdot\varphi_{m+1,m+2},\varphi_0\big) - I_c\big((t-x_k)\cdot\varphi_{i,m+2},\varphi_0\big)\\
&= 2\pi\sqrt{-1}\left(\frac{1-x_k}{\alpha_{m+1}} + \frac{1-x_i}{\alpha_{m+2}}\right),\\
Q_k(i,j) &= I_c\big((t-x_k)\cdot\varphi_{m+1,m+2},\varphi_j\big) - I_c\big((t-x_k)\cdot\varphi_{i,m+2},\varphi_j\big)\\
&= 2\pi\sqrt{-1}\left(\frac{1-x_k}{\alpha_{m+1}} + \frac{x_i-x_k}{\alpha_i}\delta_{i,j}\right),
\end{aligned}$$

by Lemma 4.2. These equalities imply Proposition 3.1. □

5. Differential equations and solutions. Lauricella's $F_D(a,b,c;x)$ satisfies the differential equations

$$\begin{aligned}
&[\theta_i(\theta+c-1) - x_i(\theta+a)(\theta_i+b_i)]\,f(x) = 0 \quad (1 \le i \le m),\\
&\big[(x_i-x_j)\partial_i\partial_j - b_j\partial_i + b_i\partial_j\big]\,f(x) = 0 \quad (1 \le i < j \le m),
\end{aligned}$$

where $\partial_i := \frac{\partial}{\partial x_i}$, $\theta_i := x_i\partial_i$, and $\theta := \sum_{j=1}^m \theta_j$. The system generated by them is called Lauricella's hypergeometric system $E_D(a,b,c)$ of differential equations. It is known that the A-hypergeometric system associated with the matrix $A(\Delta_1 \times \Delta_m)$ can be transformed into the system $E_D(a,b,c)$, and combinatorial methods for constructing a fundamental set of solutions to the A-hypergeometric system are known [4], [10]. Thus, we can use the general method for constructing series solutions to A-hypergeometric systems to obtain a fundamental set of solutions to $E_D(a,b,c)$ with generic parameters (a,b,c).

FACT 5.1 ([4, Section 3.3], [10, Section 1.5]). *For $1 \le k \le m$, we put*

$$\begin{aligned}
f^{(k)}(a,b,c;x) := &\prod_{l=1}^{k-1} x_l^{-b_l} \cdot x_k^{\sum_{l=1}^{k-1} b_l - c + 1}\\
&\cdot \sum_{n_1,\dots,n_m=0}^{\infty} \frac{1}{\Gamma^{(k)}_{n_1,\dots,n_m}(a,b,c)} \cdot \prod_{l=1}^{k-1}\left(\frac{x_k}{x_l}\right)^{n_l} \cdot x_k^{n_k} \cdot \prod_{l=k+1}^{m}\left(\frac{x_l}{x_k}\right)^{n_l},
\end{aligned}$$

where

$$\Gamma^{(k)}_{n_1,\dots,n_m}(a,b,c)$$

$$:= \Gamma(c-a-n_k) \cdot \prod_{\substack{1\le l\le m \\ l\neq k}} \Gamma(1-b_l-n_l) \cdot \prod_{l=1}^{m} \Gamma(1+n_l)$$
$$\cdot \Gamma\left(-\sum_{l=1}^{k} b_l + c - \sum_{l=1}^{k} n_l + \sum_{l=k+1}^{m} n_l\right) \cdot \Gamma\left(2+\sum_{l=1}^{k-1} b_l - c + \sum_{l=1}^{k} n_l - \sum_{l=k+1}^{m} n_l\right).$$

Then, each $f^{(k)}(a,b,c;x)$ is a solution to $E_D(a,b,c)$. Moreover, the set of $F_D(a,b,c;x)$ and $f^{(k)}(a,b,c;x)$ $(1\le k\le m)$ is a set of fundamental solutions to $E_D(a,b,c)$.

6. Twisted cycles corresponding to solutions. We consider the twisted homology group $H_1(T_x, u_x)$ on T_x that is associated with the multivalued function $u_x(t)$. For the definition of the twisted homology groups, refer to [2] and [5]. By [2], $H_1(T_x, u_x)$ has $m+1$ dimensions. If $(a,b,c;x)$ are generic, then the local solution space Sol_x of $E_D(a,b,c)$ around x can be identified with the twisted homology group $H_1(T_x,u_x)$ by the integration of $u_x\varphi_0$; see [5, Proposition 4.1]. Thus, there exists a twisted cycle that corresponds to the series solution $f^{(k)}(a,b,c;x)$. In this section, we construct such a cycle explicitly.

Let ε and ξ be real numbers satisfying

$$0<\varepsilon<\frac{1}{2}, \quad \xi<\min\left\{\varepsilon, \frac{1}{1+\varepsilon}\right\}.$$

We construct the twisted cycle r_k in T_x with x belonging to a small neighborhood of

$$x^{(k)} := (\xi, \xi^2, \dots, \xi^{k-1}, e^{-\pi\sqrt{-1}}\xi^k, \xi^{k+1}, \dots, \xi^m).$$

Once we construct the twisted cycle in $T_{x^{(k)}}$, this cycle is uniquely continued to the twisted cycle in each T_x. Thus, we may assume $x = x^{(k)}$. We put

$$S_x := \mathbb{C} - \left\{\frac{x_k}{x_m}, \dots, \frac{x_k}{x_{k+1}}, \frac{x_k}{x_{k-1}}, \dots, \frac{x_k}{x_1}, x_k, 0, 1\right\},$$
$$v_x(s) := \prod_{l=1}^{k-1}\left(s-\frac{x_k}{x_l}\right)^{\alpha_l} \cdot (s-x_k)^{\alpha_{m+1}-1} \cdot \prod_{l=k+1}^{m}\left(1-\frac{x_l}{x_k}s\right)^{\alpha_l}$$
$$\cdot s^{\alpha_{m+2}} \cdot (1-s)^{\alpha_k+1}$$
$$= \prod_{l=1}^{k-1}\left(1-\frac{x_k}{x_l}\frac{1}{s}\right)^{\alpha_l} \cdot \left(1-x_k\frac{1}{s}\right)^{\alpha_{m+1}-1} \cdot \prod_{l=k+1}^{m}\left(1-\frac{x_l}{x_k}s\right)^{\alpha_l}$$
$$\cdot s^{\sum_{l=1}^{k-1}\alpha_l+\alpha_{m+1}+\alpha_{m+2}-1} \cdot (1-s)^{\alpha_k+1}.$$

The last equality holds when $0<s<1$. We define the twisted cycle $\tilde{r}_k$ that gives an element in $H_1(S_x, v_x)$. We put $\lambda_j := e^{2\pi\sqrt{-1}\alpha_j}$ and

$$\tilde{r}_k := \frac{1}{\prod_{l=1}^{k-1}\lambda_l \cdot \lambda_{m+1}\lambda_{m+2}-1} C_0 \otimes v_x + [\varepsilon, 1-\varepsilon]\otimes v_x - \frac{1}{\lambda_k - 1} C_1 \otimes v_x.$$

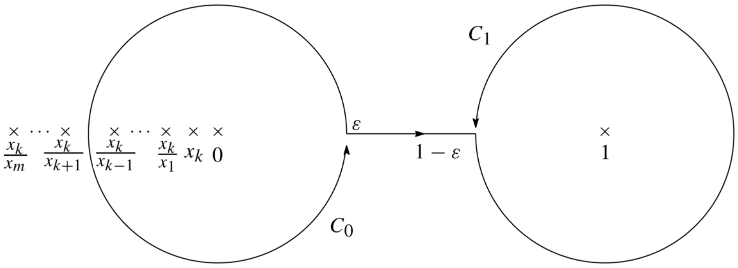

FIGURE 1. $\tilde{r}_k$.

Here, C_0 (resp. C_1) is the circle of center 0 (resp. 1) and radius ε with starting point ε (resp. $1-\varepsilon$), which turns in the counterclockwise direction, and the branch of v_x is obtained by the analytic continuation along C_0 (resp. C_1). Let us verify that $\tilde{r}_k$ is a twisted cycle. Let D_i be the disk whose boundary is C_i $(i = 0,\ 1)$. Since

$$\left|\frac{x_k}{x_{k-1}}\right| = \xi < \varepsilon < 1 < \frac{1}{\xi} = \left|\frac{x_k}{x_{k+1}}\right| ,$$

we have

$$D_0 \cap (\mathbb{C} - S_x) = \left\{\frac{x_k}{x_{k-1}}, \dots, \frac{x_k}{x_1}, x_k, 0\right\} , \quad D_1 \cap (\mathbb{C} - S_x) = \{1\} ;$$

see Figure 1. Then, the difference between the branches of v_x at the ending and starting points of the circle C_0 (resp. C_1) is $\prod_{l=1}^{k-1} \lambda_l \cdot \lambda_{m+1}\lambda_{m+2}$ (resp. λ_k), which implies that $\tilde{r}_k$ is a twisted cycle (cf. [2, Example 2.1]).

LEMMA 6.1.

$$(6)\quad \int_{\tilde{r}_k} v_x \frac{ds}{s(1-s)} = \Gamma(c-a) \cdot \prod_{l=1}^{m} \Gamma(1-b_l) \cdot \Gamma\left(\sum_{l=1}^{k-1} b_l - c\right) \cdot \Gamma\left(1 - \sum_{l=1}^{k-1} b_l + c\right)$$
$$\cdot \sum_{n_1,\dots,n_m=0}^{\infty} \frac{1}{\Gamma^{(k)}_{n_1,\dots,n_m}(a,b,c)} \cdot \prod_{l=1}^{k-1} \left(\frac{x_k}{x_l}\right)^{n_l} \cdot x_k^{n_k} \cdot \prod_{l=k+1}^{m} \left(\frac{x_l}{x_k}\right)^{n_l} .$$

PROOF. Note that if s belongs to $C_0 \cup [\varepsilon, 1-\varepsilon] \cup C_1$, it satisfies $\varepsilon < |s| < 1+\varepsilon$. Since

$$\left|\frac{x_k}{x_l}\frac{1}{s}\right| < \xi^{k-l} \cdot \frac{1}{\varepsilon} < 1 \quad (1 \le l \le k-1) ,$$
$$\left|x_k \frac{1}{s}\right| < \xi^{k} \cdot \frac{1}{\varepsilon} < 1 ,$$
$$\left|\frac{x_l}{x_k} s\right| < \xi^{l-k} \cdot (1+\varepsilon) < 1 \quad (k+1 \le l \le m) ,$$

the following power series expansions are uniformly and absolutely convergent on $C_0 \cup [\varepsilon, 1-\varepsilon] \cup C_1$:

$$\left(1-\frac{x_k}{x_l}\frac{1}{s}\right)^{\alpha_l} = \sum_{n_l=0}^{\infty} \frac{(-\alpha_l, n_l)}{n_l!}\left(\frac{x_k}{x_l}\frac{1}{s}\right)^{n_l} \quad (1 \le l \le k-1),$$

$$\left(1-x_k\frac{1}{s}\right)^{\alpha_{m+1}-1} = \sum_{n_k=0}^{\infty} \frac{(1-\alpha_{m+1}, n_k)}{n_k!}\left(x_k\frac{1}{s}\right)^{n_k},$$

$$\left(1-\frac{x_l}{x_k}s\right)^{\alpha_l} = \sum_{n_l=0}^{\infty} \frac{(-\alpha_l, n_l)}{n_l!}\left(\frac{x_l}{x_k}s\right)^{n_l} \quad (k+1 \le l \le m).$$

We replace the power functions on the left-hand side of (6) by these expansions, and exchange the sum and the integral. Then, the coefficient of $\prod_{l=1}^{k-1}\left(\frac{x_k}{x_l}\right)^{n_l} \cdot x_k^{n_k} \cdot \prod_{l=k+1}^{m}\left(\frac{x_l}{x_k}\right)^{n_l}$ is

$$\frac{(1-\alpha_{m+1}, n_k)}{n_k!}\prod_{l\neq k}\frac{(-\alpha_l, n_l)}{n_l!} \cdot \int_{\tilde{r}_k} s^{\sum_{l=1}^{k-1}\alpha_l+\alpha_{m+1}+\alpha_{m+2}-1-\sum_{l=1}^{k}n_l+\sum_{l=k+1}^{m}n_l} \cdot (1-s)^{\alpha_k+1}\frac{ds}{s(1-s)}. \tag{7}$$

By the construction of $\tilde{r}_k$, the twisted cycle $\tilde{r}_k$ of this integral can be identified with the usual regularization of the open interval (0, 1) loaded with the multivalued function

$$s^{\sum_{l=1}^{k-1}\alpha_l+\alpha_{m+1}+\alpha_{m+2}-1-\sum_{l=1}^{k}n_l+\sum_{l=k+1}^{m}n_l} \cdot (1-s)^{\alpha_k+1}$$

on $\mathbb{C} - \{0, 1\}$. Hence the integral in (7) is equal to

$$\frac{\Gamma\left(\sum_{l\le k-1}\alpha_l+\alpha_{m+1}+\alpha_{m+2}-1-\sum_{l\le k}n_l+\sum_{l\ge k+1}n_l\right)\Gamma(\alpha_k+1)}{\Gamma\left(\sum_{l\le k}\alpha_l+\alpha_{m+1}+\alpha_{m+2}-\sum_{l\le k}n_l+\sum_{l\ge k+1}n_l\right)}.$$

By (2) and $(a, n) = \Gamma(a+n)/\Gamma(a)$, (7) is equal to

$$\frac{\Gamma(1-c+a+n_k)}{\Gamma(1-c+a)}\prod_{l\neq k}\frac{\Gamma(b_l+n_l)}{\Gamma(b_l)} \cdot \frac{\Gamma\left(-\sum_{l\le k-1}b_l+c-1-\sum_{l\le k}n_l+\sum_{l\ge k+1}n_l\right)\Gamma(1-b_k)}{\Gamma\left(-\sum_{l\le k}b_l+c-\sum_{l\le k}n_l+\sum_{l\ge k+1}n_l\right)}\prod_{l=1}^{m}\frac{1}{\Gamma(1+n_l)}.$$

By using $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, we obtain, for example,

$$\begin{aligned}\frac{\Gamma(b_l+n_l)}{\Gamma(b_l)} &= \frac{1}{\Gamma(b_l)} \cdot \frac{\pi}{\Gamma(1-b_l-n_l)\sin\pi(b_l+n_l)} \\ &= (-1)^{n_l}\frac{1}{\Gamma(1-b_l-n_l)}\frac{\pi}{\Gamma(b_l)\sin\pi b_l} = (-1)^{n_l}\frac{\Gamma(1-b_l)}{\Gamma(1-b_l-n_l)}.\end{aligned}$$

In an analogous way, other Gamma functions with n_l's in the numerator can be moved to the denominator. Thus, we obtain the lemma. □

We will construct a twisted cycle standing for the series solution $f^{(k)}(a,b,c;x)$ by the bijection

$$\iota : S_x \to T_x\,; \quad s \mapsto t = \frac{x_k}{s}\,.$$

Let r_k be the twisted cycle defined as $r_k := \iota_*(\tilde{r}_k)$, which gives an element in $H_1(T_x, u_x)$.

THEOREM 6.2.

$$\int_{r_k} u_x\varphi_0 = \Gamma(c-a)\cdot\prod_{l=1}^{m}\Gamma(1-b_l)\cdot\Gamma\Big(\sum_{l=1}^{k-1}b_l - c\Big)\cdot\Gamma\Big(1-\sum_{l=1}^{k-1}b_l + c\Big)$$
$$\cdot\; e^{\pi\sqrt{-1}(\sum_{l=1}^{k-1}b_l - c + a)}\cdot f^{(k)}(a,b,c;x)\,.$$

PROOF. Note that $\arg(x_k) = -\pi$. We have

$$u_x(\iota(s))\cdot\iota^*\varphi_0$$
$$= \Big(\frac{x_k}{s}\Big)^{\alpha_0}\cdot\Big(\frac{x_k}{s}-1\Big)^{\alpha_{m+1}}\cdot\prod_{l=1}^{k-1}\Big(\frac{x_k}{s}-x_l\Big)^{\alpha_l}\cdot\Big(\frac{x_k}{s}-x_k\Big)^{\alpha_k}\cdot\prod_{l=k+1}^{m}\Big(\frac{x_k}{s}-x_l\Big)^{\alpha_l}$$
$$\cdot\frac{-x_k ds}{s^2(\frac{x_k}{s}-1)}$$
$$= -\prod_{l=1}^{k-1}x_l^{\alpha_l}\cdot x_k^{\alpha_0+\sum_{l=k}^{m}\alpha_l+1}\cdot s^{-\alpha_0-\sum_{l=k}^{m}\alpha_l-1}\cdot\Big(\frac{x_k}{s}-1\Big)^{\alpha_{m+1}-1}$$
$$\cdot\prod_{l=1}^{k-1}\Big(\frac{x_k}{x_l s}-1\Big)^{\alpha_l}\cdot(1-s)^{\alpha_k+1}\cdot\prod_{l=k+1}^{m}\Big(1-\frac{x_l s}{x_k}\Big)^{\alpha_l}\cdot\frac{ds}{s(1-s)}$$
$$= e^{-\pi\sqrt{-1}(\sum_{l=1}^{k-1}\alpha_m+\alpha_{m+1})}\cdot\prod_{l=1}^{k-1}x_l^{\alpha_l}\cdot x_k^{\alpha_0+\sum_{l=k}^{m}\alpha_l+1}\cdot v_x(s)\frac{ds}{s(1-s)}\,.$$

Here, we use $-\alpha_0-\sum_{l=k}^{m}\alpha_m = \sum_{l=1}^{k-1}\alpha_l+\alpha_{m+1}+\alpha_{m+2}$. By Lemma 6.1 and the relations

$$\alpha_l = -b_l \quad (1\le l\le k-1)\,,$$
$$\alpha_0+\sum_{l=k}^{m}\alpha_l+1 = -\sum_{l=1}^{k-1}\alpha_l-\alpha_{m+1}-\alpha_{m+2}+1 = \sum_{l=1}^{k-1}b_l-c+1\,,$$
$$\sum_{l=1}^{k-1}\alpha_m+\alpha_{m+1} = -\sum_{l=1}^{k-1}b_l+c-a\,,$$

we obtain the identity of the theorem. □

By replacing the cycle $(1,\infty)$ in Section 3 with r_k, we can obtain the contiguity relations of $f^{(k)}$. We put

$$F^{(k)}(a,b,c;x) := {}^t\!\left(f^{(k)}(a,b,c;x), \frac{x_1-1}{-b_1}\frac{\partial}{\partial x_1}f^{(k)}(a,b,c;x), \dots, \frac{x_m-1}{-b_m}\frac{\partial}{\partial x_m}f^{(k)}(a,b,c;x)\right).$$

By Theorem 6.2 and [5], we have

$$\begin{aligned}\tilde{F}^{(k)}(a,b,c;x) &:= {}^t\!\left(\int_{r_k} u_x\varphi_0, \dots, \int_{r_k} u_x\varphi_m\right)\\ &= \Gamma(c-a)\cdot\prod_{l=1}^{m}\Gamma(1-b_l)\cdot\Gamma\left(\sum_{l=1}^{k-1}b_l-c\right)\cdot\Gamma\left(1-\sum_{l=1}^{k-1}b_l+c\right)\\ &\quad\cdot e^{\pi\sqrt{-1}(\sum_{l=1}^{k-1}b_l-c+a)}\cdot F^{(k)}(a,b,c;x).\end{aligned}$$

It is clear that Lemmas 3.2 and 3.3 hold even if $\tilde{F}$ is replaced by $\tilde{F}^{(k)}$. Therefore, we obtain the following corollary.

COROLLARY 6.3.

$$\begin{aligned}F^{(k)}(a-1,b,c;x) &= D_a^{(k)}(a,b,c;x)F^{(k)}(a,b,c;x),\\ F^{(k)}(a,b,c-1;x) &= D_c^{(k)}(a,b,c;x)F^{(k)}(a,b,c;x),\\ F^{(k)}(a,b-e_l,c;x) &= D_l^{(k)}(a,b,c;x)F^{(k)}(a,b,c;x) \quad (1\le l\le m),\end{aligned}$$

where

$$\begin{aligned}D_a^{(k)}(a,b,c;x) &:= \frac{1}{a-c}\cdot Q_{m+1}(a,b,c;x)\cdot C(a,b,c)^{-1},\\ D_c^{(k)}(a,b,c;x) &:= (c-a-1)\cdot Q_0(a+1,b,c;x)\cdot Q_{m+1}(a+1,b,c;x)^{-1},\\ D_l^{(k)}(a,b,c;x) &:= \frac{1}{1-b_l}\cdot Q_l(a+1,b,c+1;x)\cdot Q_0(a+1,b,c+1;x)^{-1}.\end{aligned}$$

In fact, $D_\bullet^{(k)}$ is independent of k.

7. Application—Normalizing constant for $2\times(m+1)$ contingency tables. Contiguity relations of F_D and $f^{(k)}$ are applied to the numerical evaluation of the normalizing constant of the hypergeometric distribution of the $2\times(m+1)$ contingency tables with fixed marginal sums. In this section, we explain how our results are applied.

We consider the $2\times(m+1)$ contingency table

$$u = \begin{pmatrix} u_{10} & u_{11} & \cdots & u_{1m}\\ u_{20} & u_{21} & \cdots & u_{2m}\end{pmatrix} \in M_{2,m+1}(\mathbb{Z}_{\ge 0})$$

with row sums β_1 and β_2 and columns sums $\gamma_0,\dots,\gamma_m$. We put $t := \beta_1+\beta_2 = \sum_{i=0}^m \gamma_i$. We use the multi-index notation

$$p^u = \prod_{i=1}^{2}\prod_{j=0}^{m} p_{ij}^{u_{ij}}, \quad u! = \prod_{i=1}^{2}\prod_{j=0}^{m} u_{ij}!,$$

where p is the $2 \times (m+1)$ matrix variable. The polynomial

$$Z(\beta, \gamma; p) = t! \sum_u \frac{p^u}{u!}$$

is called the normalizing constant, where the sum is taken over all contingency tables u with marginal sums $\beta = (\beta_1, \beta_2)$ and $\gamma = (\gamma_0, \dots, \gamma_m)$. It is a fundamental problem in statistics to evaluate $Z(\beta, \gamma; p)$ numerically, where $\beta_i, \gamma_j \in \mathbb{Z}_{\geq 0}$ and $p_{ij} \in \mathbb{Q}_{\geq 0}$.

The normalizing constant Z can be expressed by F_D or $f^{(k)}$. To explain this, we will first define some notation. We put

$$\mathcal{B}_0 := \left\{ (\beta_1, \beta_2, \gamma_0, \dots, \gamma_m) \in (\mathbb{Z}_{>0})^{m+3} \,\middle|\, \beta_1 + \beta_2 = \sum_{i=0}^{m} \gamma_i,\ \beta_1 - \gamma_0 \leq 0 \right\},$$

$$\mathcal{B}_k := \left\{ (\beta_1, \beta_2, \gamma_0, \dots, \gamma_m) \in (\mathbb{Z}_{>0})^{m+3} \,\middle|\, \beta_1 + \beta_2 = \sum_{i=0}^{m} \gamma_i,\ \beta_1 - \sum_{i=0}^{k-1} \gamma_i > 0,\ \beta_1 - \sum_{i=0}^{k} \gamma_i \leq 0 \right\},$$

where $1 \leq k \leq m$. Then, $\{(\beta_1, \beta_2, \gamma_0, \dots, \gamma_m) \in (\mathbb{Z}_{>0})^{m+3} \mid \beta_1 + \beta_2 = \sum_{i=0}^{m} \gamma_i\}$ is the disjoint union of $\mathcal{B}_0, \dots, \mathcal{B}_m$. We also put

$$\ell_1 := \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \end{pmatrix}, \quad \ell_2 := \begin{pmatrix} -1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & 0 & \cdots & 0 \end{pmatrix}, \dots,$$

$$\ell_m := \begin{pmatrix} -1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & -1 \end{pmatrix},$$

$$u_0 := \begin{pmatrix} \beta_1 & 0 & 0 & \cdots & 0 \\ \gamma_0 - \beta_1 & \gamma_1 & \gamma_2 & \cdots & \gamma_m \end{pmatrix},$$

$$u_k := \begin{pmatrix} \gamma_0 & \cdots & \gamma_{k-1} & \beta_1 - \sum_{i=0}^{k-1} \gamma_i & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \sum_{i=0}^{k} \gamma_i - \beta_1 & \gamma_{k+1} & \cdots & \gamma_m \end{pmatrix} \quad (1 \leq k \leq m),$$

$$x_i := p^{\ell_i} = \frac{p_{1i} p_{20}}{p_{10} p_{2i}}.$$

If $(\beta_1, \beta_2, \gamma_0, \dots, \gamma_m) \in \mathcal{B}_k$, then all of the entries of u_k are non-negative integers, and hence it is one of the contingency tables with marginal sums β and γ. By straightforward calculation, we can prove the following lemma.

LEMMA 7.1. (1) *If $(\beta, \gamma) \in \mathcal{B}_0$, then*

$$Z(\beta, \gamma; p) = \frac{t!}{u_0!} \cdot p^{u_0} \cdot F_D(-\beta_1, (-\gamma_1, \dots, -\gamma_m), \gamma_0 - \beta_1 + 1; x_1, \dots, x_m).$$

(2) *If $(\beta, \gamma) \in \mathcal{B}_k$ with $1 \leq k \leq m$, then*

$$Z(\beta, \gamma; p) = t! \cdot p^{u_0} \cdot f^{(k)}(-\beta_1, (-\gamma_1, \dots, -\gamma_m), \gamma_0 - \beta_1 + 1; x_1, \dots, x_m).$$

In [7], our contiguity relations are applied for the difference holonomic gradient method, which evaluates the numerical value of the column vector $F(a, b, c; x)$ or that of $F^{(k)}(a, b, c; x)$, with $a, b_i \in \mathbb{Z}_{<0}$. For example, it follows from the below discussion of contiguity relations for integer parameters a, b, c that we can easily evaluate the numerical value of $F(a, b, c; x)$ from that of $F(-1, b, c; x)$ by using the matrix D_a in the contiguity relation. Note that

$$F_D(-1, b, c; x) = 1 - \sum_{i=1}^{m} \frac{b_i}{c} x_i .$$

For details of the difference holonomic gradient method, see [7].

We now consider the case in which the parameters are integers. Since $\beta_1, \beta_2, \gamma_0, \ldots, \gamma_m$ are integers, the parameters $(a, b, c) = (-\beta_1, (-\gamma_1, \ldots, -\gamma_m), \gamma_0 - \beta_1 + 1)$ do not satisfy the condition (3). For the above application, we need to give the contiguity relations that are valid even when the parameters are integers.

PROPOSITION 7.2. (1) *If $(\beta, \gamma) \in \mathcal{B}_0$, then the relation*

$$F(a-1, b, c; x) = \frac{a-1}{c-a} \cdot P_{m+1}(a, b, c; x) \cdot F(a, b, c; x)$$

holds when the generic parameter vector is specialized to an integral point $(a, b, c) \to (-\beta_1, (-\gamma_1, \ldots, -\gamma_m), \gamma_0 - \beta_1 + 1)$.

(2) *When $(\beta, \gamma) \in \mathcal{B}_k$ with $1 \le k \le m$, we consider the relation*

$$F^{(k)}(a-1, b, c-1; x) = -P_0(a, b, c; x) \cdot F^{(k)}(a, b, c; x).$$

If $(\beta_1+1, \beta_2-1, \gamma_0, \ldots, \gamma_m) \in \mathcal{B}_k$, then this relation holds when the generic parameter vector is specialized to an integral point $(a, b, c) \to (-\beta_1, (-\gamma_1, \ldots, -\gamma_m), \gamma_0 - \beta_1 + 1)$.

We put

$$\Gamma^{(0)}_{n_1,\ldots,n_m}(a, b, c) := \Gamma\left(1 - a - \sum_{l=1}^{m} n_l\right) \cdot \Gamma\left(c + \sum_{l=1}^{m} n_l\right) \cdot \prod_{l=1}^{m} \Gamma(1 - b_l - n_l) \cdot \prod_{l=1}^{m} \Gamma(1 + n_l) .$$

To prove this proposition, we will use the following lemma.

LEMMA 7.3. (1) *Let $\tilde{a}, \tilde{b}_1, \ldots, \tilde{b}_m, \tilde{c}$ be integers, and assume $\tilde{c} > 0$. Then, there exists $\tilde{x} \in \mathbb{C}^m$ such that the power series*

$$\sum_{n_1,\ldots,n_m=0}^{\infty} \frac{1}{\Gamma^{(0)}_{n_1,\ldots,n_m}(a, b, c)} \prod_{l=1}^{m} x_l^{n_l}$$

as a function in $(a, b, c; x)$ is holomorphic on a small neighborhood of $(\tilde{a}, \tilde{b}, \tilde{c}; \tilde{x})$. In particular, if $x \in \mathbb{C}^m$ belongs to a small neighborhood of $\tilde{x}$, then this series has a limit as $(a, b, c) \to (\tilde{a}, \tilde{b}, \tilde{c})$.

(2) *For $1 \le k \le m$, let $\tilde{a}, \tilde{b}_1, \ldots, \tilde{b}_m, \tilde{c}$ be integers that satisfy*

$$-\sum_{l \le k} \tilde{b}_l + \tilde{c} > 0, \quad 2 + \sum_{l \le k-1} \tilde{b}_l - \tilde{c} > 0.$$

Then, there exists $\tilde{x} \in \mathbb{C}^m$ such that the Laurent series

$$\sum_{n_1,\ldots,n_m=0}^{\infty} \frac{1}{\Gamma^{(k)}_{n_1,\ldots,n_m}(a,b,c)} \cdot \prod_{l \le k-1} \left(\frac{x_k}{x_l}\right)^{n_l} \cdot x_k^{n_k} \cdot \prod_{l \ge k+1} \left(\frac{x_l}{x_k}\right)^{n_l}$$

as a function in $(a, b, c; x)$ is holomorphic on a small neighborhood of $(\tilde{a}, \tilde{b}, \tilde{c}; \tilde{x})$. In particular, if $x \in \mathbb{C}^m$ belongs to a small neighborhood of $\tilde{x}$, then this series has a limit as $(a, b, c) \to (\tilde{a}, \tilde{b}, \tilde{c})$.

Further, we can differentiate these series term by term, and the partial derivatives of them also have limits as $(a, b, c) \to (\tilde{a}, \tilde{b}, \tilde{c})$.

We can show this lemma in a way that is analogous to that used for [9, Lemma 1]; see also [8, pp. 18–21]. Although, in [9], the parameter vector (a, b, c) belongs to a neighborhood of a generic point, an analogous estimation of Gamma functions can be done in our case.

SKETCH OF PROOF. Let $0 \le j \le m$. First, we can show that there exist $C, \rho_1, \ldots, \rho_m > 0$ such that the inequality

$$\left| \frac{1}{\Gamma^{(j)}_{n_1,\ldots,n_m}(a,b,c)} \right| \le C \rho_1^{n_1} \cdots \rho_m^{n_m}$$

holds on a small neighborhood of $(\tilde{a}, \tilde{b}, \tilde{c})$. Next, we put

$$\rho := \max\{\rho_1, \ldots, \rho_m, 2\}, \quad \tilde{x}_i := \frac{1}{\rho^{2i}},$$

and $\tilde{x} := (\tilde{x}_1, \ldots, \tilde{x}_m)$. We can show that there exists $0 < \eta < 1$ such that the series in the lemma has the form

$$\sum_{n_1,\ldots,n_m=0}^{\infty} \left(\prod_{l=1}^{m} \eta^{n_l} \right)$$

for a majorant on a small neighborhood of $(\tilde{a}, \tilde{b}, \tilde{c}; \tilde{x})$. Therefore, the series is uniformly and absolutely convergent, and it defines a holomorphic function. □

PROOF OF PROPOSITION 7.2. If $(a, b, c) = (-\beta_1, (-\gamma_1, \ldots, -\gamma_m), \gamma_0 - \beta_1 + 1)$, then the α_i's are expressed as follows:

$$\alpha_0 = \beta_1 - \sum_{l=0}^{m} \gamma_l - 1 = -\beta_2 - 1, \quad \alpha_k = \gamma_k \ (1 \le k \le m),$$
$$\alpha_{m+1} = \gamma_0 + 1, \quad \alpha_{m+2} = -\beta_1.$$

Since these values and $1-\alpha_{m+2}=\beta_1+1$ are not zero, it follows from Fact 2.1, Proposition 3.1, and Remark 3.6 that both of the matrices $Q_k(a,b,c;x)$ and $C(a,b,c)$ are well-defined and invertible.

(1) The definition of $F_D(a,b,c;x)$ can be expressed by the Gamma function:

$$F_D(a,b,c;x)=\Gamma(1-a)\cdot\Gamma(c)\cdot\prod_{l=1}^{m}\Gamma(1-b_l)\cdot\sum_{n_1,\dots,n_m=0}^{\infty}\frac{1}{\Gamma^{(0)}_{n_1,\dots,n_m}(a,b,c)}\prod_{l=1}^{m}x_l^{n_l}.$$

By $(\beta,\gamma)\in\mathcal{B}_0$, we have $c=\gamma_0-\beta_1+1>0$. Then we can apply Lemma 7.3 (1) to $F(a,b,c;x)$. Note that $a-1=-\beta_1-1\neq 0$, and $c-a=\gamma_0+1\neq 0$.

(2) Let σ be 0 or 1. $(\beta_1+\sigma,\beta_2-\sigma,\gamma_0,\dots,\gamma_m)\in\mathcal{B}_k$ implies

$$-\sum_{l=1}^{k}b_l+(c-\sigma)=-(\beta_1+\sigma)+\sum_{l=0}^{k}\gamma_l+1>0\,,$$

$$2+\sum_{l=1}^{k-1}b_l-(c-\sigma)=(\beta_1+\sigma)-\sum_{l=0}^{k-1}\gamma_l+1>0\,.$$

Then we can take the limit of $F^{(k)}(a,b,c;x)$ as $(a,b,c)\to(-\beta_1,(-\gamma_1,\dots,-\gamma_m),\gamma_0-\beta_1+1)$ by Lemma 7.3 (2).

By the identity theorem for holomorphic functions, it is sufficient to prove the proposition on a small neighborhood of some $x\in\mathbb{C}^m$. Therefore, the proof is completed. □

REFERENCES

[1] K. AOMOTO, Les équations aux différences linéaires et les intégrales des fonctions multiformes, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 22 (1975), no. 3, 271–297.

[2] K. AOMOTO AND M. KITA, translated by K. IOHARA, Theory of Hypergeometric Functions, Springer-Verlag, New York, 2011.

[3] K. CHO AND K. MATSUMOTO, Intersection theory for twisted cohomologies and twisted Riemann's period relations I, Nagoya Math. J. 139 (1995), 67–86.

[4] I. M. GEL'FAND, A. V. ZELEVINSKY AND M. M. KAPRANOV, Hypergeometric functions and toral manifolds, Funct. Anal. Appl. 23 (1989), no. 2, 94–106.

[5] K. MATSUMOTO, Monodromy and Pfaffian of Lauricella's F_D in terms of the intersection forms of twisted (co)homology groups, Kyushu J. Math. 67 (2013), no. 2, 367–387.

[6] W. MILLER JR., Lie theory and Lauricella functions F_D, J. Mathematical Phys. 13 (1972), 1393–1399.

[7] M. OGAWA, A. TAKEMURA AND N. TAKAYAMA, An application of A-hypergeometric equations to conditional maximal likelihood estimation, in preparation.

[8] K. OHARA AND N. TAKAYAMA, Holonomic rank of $\mathcal{A}$-hypergeometric differential-difference equations, in: Computational Analysis of Hypergeometric Differential Equations Research Report of Grant-in-Aid for Scientific Research (KAKENHI) 15340045, 2007, pp. 28–49 (in Japanese). http://www.math.kobe-u.ac.jp/HOME/taka/2007/kaken-hg-all.pdf.

[9] K. OHARA AND N. TAKAYAMA, Holonomic rank of $\mathcal{A}$-hypergeometric differential-difference equations, J. Pure Appl. Algebra 213 (2009), no. 8, 1536–1544.

[10] M. SAITO, B. STURMFELS AND N. TAKAYAMA, Gröbner Deformations of Hypergeometric Differential Equations, Springer-Verlag, Berlin, 2000.

[11] T. SASAKI, Contiguity relations of Aomoto-Gel'fand hypergeometric functions and applications to Appell's system F_3 and Goursat's system ${}_3F_2$, SIAM J. Math. Anal. 22 (1991), no. 3, 821–846.
[12] N. TAKAYAMA, Gröbner basis and the problem of contiguous relations, Japan J. Appl. Math. 6 (1989), no. 1, 147–160.

GENERAL EDUCATION
OTARU UNIVERSITY OF COMMERCE
OTARU, HOKKAIDO, 047–8501
JAPAN

E-mail address: goto@res.otaru-uc.ac.jp