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# ON A CHARACTERIZATION OF UNBOUNDED HOMOGENEOUS DOMAINS WITH BOUNDARIES OF LIGHT CONE TYPE

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**Abstract.** We determine the automorphism groups of unbounded homogeneous domains with boundaries of light cone type. Furthermore we present a group-theoretic characterization of one of the domains. As a corollary we prove the non-existence of compact quotients of the homogeneous domain. We also give a counterexample of the characterization.

**Introduction.** We study some unbounded homogeneous domains, mainly concerned with a characterization of the domains by their automorphism groups. The group-theoretic characterization problem of complex manifolds is studied widely in complex analysis. This problem asks whether the automorphism group of a complex manifold determines only one biholomorphic equivalence class of complex manifolds under some conditions. Since there are many complex manifolds whose automorphism groups are trivial, this characterization problem does not make sense for such manifolds. Hence let us restrict our attention only to homogeneous complex manifolds, or in our case, homogeneous domains in the complex euclidean spaces.

By H. Cartan, it was shown that the automorphim groups of bounded domains have Lie group structures, and this result leads us to various studies of bounded homogeneous domains, e.g. normal j-algebra of automorphim groups (see [6]). Normal j-algebras determine bounded homogeneous domains with 1-1 correspondence, and therefore characterize bounded homogeneous domains in this category. It was shown by Dotti-Miatello that any irreducible homogeneous domain is determined by its automorphism group up to complex conjugates [3].

For unbounded homogeneous domains, in contrast to bounded domains, automorphim groups are, in general, not Lie groups, and we do not have a general theory of automorphim groups and the characterization theorem. Therefore any unbounded homogeneous domain is of interest, and some important cases are studied by Shimizu and Kodama [4], [5], Byun, Kodama and Shimizu [1], etc.

In this paper, we study other interesting unbounded homogeneous domains and give the group-theoretic characterization for one of the domains. Also we give a counterexample of the group-theoretic characterization for some domains. In order to describe our results, let us fix notations here. If $A_1, \dots, A_k$ are square matrices, $\mathrm{diag}[A_1, \dots, A_k]$ denotes the matrix with $A_1, \dots, A_k$ in the diagonal blocks and 0 in all other blocks. Let $\Omega$ be a complex manifold. An *automorphism* of $\Omega$ means a biholomorphic mapping of $\Omega$ onto itself. We denote by $\mathrm{Aut}(\Omega)$

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the group of all automorphisms of $\Omega$ equipped with the compact-open topology. $\Omega$ is called *homogeneous* if $\mathrm{Aut}(\Omega)$ acts transitively on $\Omega$. The purpose of our paper is that we would describe the automorphim group of the unbounded domain

$$D^{n,1} = \left\{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : -|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 > 0\right\},$$

and give the characterization theorem of $D^{n,1}$ by the automorphim group $\mathrm{Aut}(D^{n,1})$. $D^{n,1}$ is analogous to the de Sitter space

$$\left\{(x_1, \dots, x_n) \in \mathbb{R}^n : -x_1^2 + x_2^2 + \cdots + x_n^2 = 1\right\}.$$

The de Sitter space has a well-known property called the Calabi-Markus phenomenon, that is, any isometry subgroup which acts properly discontinuously on the de Sitter space is finite [2]. This phenomenon implies that the de Sitter space has no compact quotient. It is interesting whether similar results occur in other geometry. We will study subgroups of the automorphism group $\mathrm{Aut}(D^{n,1})$ which act properly discontinuously on $D^{n,1}$ and prove the non-existence of compact quotients of $D^{n,1}$. It is not the precise Calabi-Markus phenomenon, but a rigid phenomenon. For these purposes, we also need to consider the domain

$$C^{n,1} = \left\{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : -|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 < 0\right\},$$

the exterior of $D^{n,1}$ in $\mathbb{C}^{n+1}$. To describe the automorphism groups $\mathrm{Aut}(D^{n,1})$ and $\mathrm{Aut}(C^{n,1})$, put

$$GU(n,1) = \{A \in GL(n+1, \mathbb{C}) : A^* J A = \nu(A) J, \text{for some } \nu(A) \in \mathbb{R}_{>0}\},$$

where $J = \mathrm{diag}[-1, E_n]$. Consider $\mathbb{C}^*$ as a subgroup of $GU(n,1)$:

$$\mathbb{C}^* \simeq \left\{\mathrm{diag}[\alpha, \dots, \alpha] \in GL(n+1, \mathbb{C}), \alpha \in \mathbb{C}^*\right\} \subset GU(n,1).$$

Since $U(n,1) = \{A \in GL(n+1, \mathbb{C}) : A^* J A = J\} \subset GU(n,1)$ acts transitively on each level sets of $-|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 (\neq 0)$, and $\mathbb{C}^*$ acts on $D^{n,1}$ and $C^{n,1}$, $GU(n,1)$ is a subgroup of the automorphism groups of these two domains $D^{n,1}$ and $C^{n,1}$. It can be easily seen that $C^{n,1}$ and $D^{n,1}$ are homogeneous. Now we state our main results.

THEOREM 3.1. $\mathrm{Aut}(D^{n,1}) = GU(n,1)$ *for* $n > 1$.

We give the group-theoretic characterization theorem of $D^{n,1}$ in the class of complex manifolds containing Stein manifolds.

THEOREM 5.1. *Let $M$ be a connected complex manifold of dimension $n+1$ that is holomorphically separable and admits a smooth envelope of holomorphy. Assume that* $\mathrm{Aut}(M)$ *is isomorphic to* $\mathrm{Aut}(D^{n,1})$ *as topological groups. Then $M$ is biholomorphic to $D^{n,1}$.*

For the domain $C^{n,1}$, the characterization theorem was shown by Byun, Kodama and Shimizu [4] (see also the remark before Theorem 2.2).

Our paper organizes as follows. In Section 1, we will recall the notion of Reinhardt domains and Kodama-Shimizu's generalized standardization theorem, which is the key lemma for our theorem. Also we record some results, which will be used in the proof of Theorem 5.1 several times. To determine $\mathrm{Aut}(D^{n,1})$ we need an explicit form of the automorphism group

Aut($C^{n,1}$). In Section 2 we determine Aut($C^{n,1}$). We determine the automorphism groups of $D^{n,1}$ in Section 3. We will show the non-existence of compact quotients of $D^{n,1}$ in Section 4, using the Calabi-Markus phenomenon. In Section 5 we prove the characterization theorem of $D^{n,1}$ by its automorphism group Aut($D^{n,1}$). In Section 6, we construct a counterexample of the group-theoretic characterization of unbounded homogeneous domains.

THEOREM 6.1. *There exist unbounded homogeneous domains in* $\mathbb{C}^n$, $n \geq 5$ *which are not biholomorphically equivalent, while its automorphism groups are isomorphic as topological groups.*

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**1. Preliminary.** In order to establish terminology and notation, we recall some basic facts about Reinhardt domains, following Kodama and Shimizu [4], [5] for convenience.

Let $G$ be a Lie group and $\Omega$ a domain in $\mathbb{C}^n$. Consider a continuous group homomorphism $\rho : G \longrightarrow \mathrm{Aut}(\Omega)$. Then the mapping

$$G \times \Omega \ni (g, x) \longmapsto (\rho(g))(x) \in \Omega$$

is continuous, and in fact $C^\omega$. We say that $G$ acts on $\Omega$ as a Lie transformation group through $\rho$. Let $T^n = (U(1))^n$, the $n$-dimensional torus. $T^n$ acts as a holomorphic automorphism group on $\mathbb{C}^n$ in the following standard manner:

$$T^n \times \mathbb{C}^n \ni (\alpha, z) \longmapsto \alpha \cdot z := (\alpha_1 z_1, \dots, \alpha_n z_n) \in \mathbb{C}^n .$$

A *Reinhardt domain* $\Omega$ in $\mathbb{C}^n$ is, by definition, a domain which is stable under this standard action of $T^n$. Namely, there exists a continuous map $T^n \hookrightarrow \mathrm{Aut}(\Omega)$. We denote the image of $T^n$ of this inclusion map by $T(\Omega)$.

Let $f$ be a holomorphic function on a Reinhardt domain $\Omega$. Then $f$ can be expanded uniquely into a Laurent series

$$f(z) = \sum_{\nu \in \mathbb{Z}^n} a_\nu z^\nu ,$$

which converges absolutely and uniformly on any compact set in $\Omega$. Here $z^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n}$ for $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n$.

$(\mathbb{C}^*)^n$ acts holomorphically on $\mathbb{C}^n$ as follows:

$$(\mathbb{C}^*)^n \times \mathbb{C}^n \ni ((\alpha_1, \dots, \alpha_n), (z_1, \dots, z_n)) \longmapsto (\alpha_1 z_1, \dots, \alpha_n z_n) \in \mathbb{C}^n .$$

We denote by $\Pi(\mathbb{C}^n)$ the group of all automorphisms of $\mathbb{C}^n$ of this form. For a Reinhardt domain $\Omega$ in $\mathbb{C}^n$, we denote by $\Pi(\Omega)$ the subgroup of $\Pi(\mathbb{C}^n)$ consisting of all elements of $\Pi(\mathbb{C}^n)$ leaving $\Omega$ invariant. We need the following two lemmas to prove the characterization theorem.

LEMMA 1.1 ([4]). *Let $\Omega$ be a Reinhardt domain in $\mathbb{C}^n$ . Then $\Pi(\Omega)$ is the centralizer of $T(\Omega)$ in* $\mathrm{Aut}(\Omega)$.

LEMMA 1.2 (Generalized Standardization Theorem [5]). *Let $M$ be a connected complex manifold of dimension $n$ that is holomorphically separable and admits a smooth envelope of holomorphy, and let $K$ be a connected compact Lie group of rank $n$. Assume that an injective continuous group homomorphism $\rho$ of $K$ into* $\mathrm{Aut}(\Omega)$ *is given. Then there exists a biholomorphic map $F$ of $M$ onto a Reinhardt domain $\Omega$ in $\mathbb{C}^n$ such that*

$$F\rho(K)F^{-1} = U(n_1) \times \cdots \times U(n_s) \subset \mathrm{Aut}(\Omega)\,,$$

*where $\sum_{j=1}^{s} n_j = n$.*

We record some results, which will be used in the proof of Theorem 5.1 several times.

LEMMA 1.3. *Let $D$ and $D'$ be domains in $\mathbb{C}^n$. Then the automorphism groups of the domains $\mathbb{C} \times D$, $\mathbb{C}^* \times D'$ and $(\mathbb{C} \times D) \cup (\mathbb{C}^* \times D')$ are not Lie groups.*

PROOF. For any nowhere vanishing holomorphic function $u$ on $\mathbb{C}^n$,

$$f(z) = (u(z_1, \dots, z_n)z_0, z_1, \dots, z_n)$$

is an automorphism on each domain. Indeed, the inverse is given by

$$g(z) = (u(z_1, \dots, z_n)^{-1}z_0, z_1, \dots, z_n)\,.$$

Thus the automorphism groups of these domains have no Lie group structures. □

LEMMA 1.4. *Let $p, q, k$ be non-negative integers and $p + q \geq 2$. For $p + q > k$, any Lie group homomorphism*

$$\rho : SU(p, q) \longrightarrow GL(k, \mathbb{C})$$

*is trivial.*

PROOF. Put $n = p + q$. It is enough to show that the Lie algebra homomorphism

$$d\rho : \mathfrak{su}(p, q) \longrightarrow \mathfrak{gl}(k, \mathbb{C})$$

is trivial. Consider its complex linear extension

$$d\rho_{\mathbb{C}} : \mathfrak{su}(p, q) \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathfrak{gl}(k, \mathbb{C})\,.$$

Since $\mathfrak{su}(p, q) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{sl}(n, \mathbb{C})$ is a simple Lie algebra, $d\rho_{\mathbb{C}}$ is injective or trivial. On the other hand, $\dim_{\mathbb{C}} \mathfrak{su}(p, q) \otimes_{\mathbb{R}} \mathbb{C} = n^2 - 1 > k^2 = \dim_{\mathbb{C}} \mathfrak{gl}(k, \mathbb{C})$. Thus $d\rho_{\mathbb{C}}$ must be trivial, and so is $d\rho$. □

**2. The automorhpsim group of $C^{n,1}$.** In this section, we consider the automorphism group $\mathrm{Aut}(C^{n,1})$ of

$$C^{n,1} = \left\{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : -|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 < 0\right\}.$$

THEOREM 2.1. *For* $f = (f_0, f_1, f_2, \dots, f_n) \in \mathrm{Aut}(C^{n,1})$,

$$f_0(z_0, z_1, z_2, \dots, z_n) = c\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \dots, \frac{z_n}{z_0}\right) z_0 \text{ or } c\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \dots, \frac{z_n}{z_0}\right) z_0^{-1},$$

and

$$f_i(z_0, z_1, z_2, \dots, z_n) = f_0(z_0, z_1, z_2, \dots, z_n)\frac{\sum_{j=0}^n a_{ij} z_j}{\sum_{j=0}^n a_{0j} z_j}, \quad \text{for } i = 1, \dots, n,$$

*where* $c$ *is a nowhere vanishing holomorphic function on* $\mathbb{B}^n$*, and the matrix* $(a_{ij})_{0\le i,j\le n}$ *is an element of* $PU(n, 1)$.

PROOF. First we remark that $C^{n,1}$ is biholomorphic to the product domain $\mathbb{C}^* \times \mathbb{B}^n$. In fact, a biholomorphic map is given by

$$\Psi : C^{n,1} \ni (z_0, z_1, z_2, \dots, z_n) \mapsto \left(z_0, \frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right) \in \mathbb{C}^* \times \mathbb{B}^n .$$

Therefore, we consider the automorphism group of $\mathbb{C}^* \times \mathbb{B}^n$.

Let $(w_0, w_1, \dots, w_n)$ be a coordinate of $\mathbb{C}^* \times \mathbb{B}^n$, and

$$\gamma = (\gamma_0, \gamma_1, \dots, \gamma_n) \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{B}^n) .$$

Fix $(w_1, \dots, w_n) \in \mathbb{B}^n$. Then, by the definition, $\gamma_i(\cdot, w_1, \dots, w_n)$, for $i = 1, \dots, n$, are bounded holomorphic functions on $\mathbb{C}^*$. The Riemann removable singularities theorem implies that $\gamma_i(\cdot, w_1, \dots, w_n)$ extends to an entire function. By the Liouville theorem, $\gamma_i(\cdot, w_1, \dots, w_n)$ for $i = 1, \dots, n$ are constant functions. Hence $\gamma_i$, for $i = 1, \dots, n$, do not depend on $w_0$. In the same manner, we see that, for the inverse

$$\tau = (\tau_0, \tau_1, \dots, \tau_n) = \gamma^{-1} \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{B}^n)$$

of $\gamma$, $\tau_i$, for $i = 1, \dots, n$, are independent of $w_0$. It follows that

$$\overline{\gamma} := (\gamma_1, \gamma_2, \dots, \gamma_n) \in \mathrm{Aut}(\mathbb{B}^n) .$$

It is well-known (see [6]) that $\gamma \in \mathrm{Aut}(\mathbb{B}^n)$ is a linear fractional transformation of the form

$$\gamma_i(w_1, w_2, \dots, w_n) = \frac{a_{i0} + \sum_{j=1}^n a_{ij} w_j}{a_{00} + \sum_{j=1}^n a_{0j} w_j}, \quad i = 1, 2, \dots, n,$$

where the matrix $(a_{ij})_{0\le i,j\le n}$ is an element of $PU(n, 1)$.

Next we consider $\gamma_0$ of $\gamma$ and $\tau_0$ of $\tau$. By regarding $\overline{\gamma}$ with the standard action of $\mathrm{Aut}(\mathbb{B}^n)$ on $\mathbb{C}^* \times \mathbb{B}^n$, we obtain a biholomorphic map

$$\gamma \circ \overline{\gamma}^{-1}(w_0, w_1, w_2, \dots, w_n) = (\gamma_0(w_0, \overline{\gamma}^{-1}(w_1, w_2, \dots, w_n)), w_1, w_2, \dots, w_n)$$

on $\mathbb{C}^* \times \mathbb{B}^n$. Thus for fixed $(w_1, w_2, \dots, w_n) \in \mathbb{B}^n$, $\gamma_0$ is bijective on $\mathbb{C}^*$ with respect to $w_0$, and $\tau_0(w_0, \overline{\gamma}(w_1, w_2, \dots, w_n))$ is its inverse. Since $\mathrm{Aut}(\mathbb{C}^*) = \{cw^{\pm 1} : c \in \mathbb{C}^*\}$, we have $\gamma_0 = c(w_1, w_2, \dots, w_n)w_0$ or $c(w_1, w_2, \dots, w_n)w_0^{-1}$, where $c(w_1, w_2, \dots, w_n)$ is a nowhere vanishing holomorphic function on $\mathbb{B}^n$.

Since $\Psi^{-1}\mathrm{Aut}(\mathbb{C}^* \times \mathbb{B}^n)\Psi = \mathrm{Aut}(C^{n,1})$, we have shown the theorem. □

We remark that the group-theoretic characterization of the domain $\mathbb{C}^* \times \mathbb{B}^n$ is proved by J. Byun, A. Kodama and S. Shimizu [1], and in their paper more general domains are treated.

THEOREM 2.2 ([1]). *Let $M$ be a connected complex manifold of dimension $n+1$ that is holomorphically separable and admits a smooth envelope of holomorphy. Assume that* $\mathrm{Aut}(M)$ *is isomorphic to* $\mathrm{Aut}(C^{n,1})$ *as topological groups. Then $M$ is biholomorphic to $C^{n,1}$.*

**3. The automorphism Group of $D^{n,1}$.** In this section, we determine the automorphism group $\mathrm{Aut}(D^{n,1})$ of

$$D^{n,1} = \left\{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : -|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 > 0\right\},$$

the exterior of $C^{n,1}$. We assume $n > 1$. We show the following theorem using Theorem 2.1 in the preceding section.

THEOREM 3.1. $\mathrm{Aut}(D^{n,1}) = GU(n, 1)$ *for $n > 1$.*

PROOF. Let $f = (f_0, f_1, \dots, f_n) \in \mathrm{Aut}(D^{n,1})$. If $z'_0 \in \mathbb{C}$ is fixed, then each of the holomorphic functions $f_i(z'_0, z_1, \dots, z_n)$ for $i = 0, \dots, n$, on $D^{n,1} \cap \{z_0 = z'_0\}$ extends to a holomorphic function on $\mathbb{C}^{n+1} \cap \{z_0 = z'_0\}$ by Hartogs' extension theorem. Hence, when $z_0$ varies, we obtain an extended holomorphic map $\tilde{f} : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}$ i.e. $\tilde{f}|_{D^{n,1}} = f$. The same consideration for $f^{-1} \in \mathrm{Aut}(D^{n,1})$ shows that there exists a holomorphic map $g : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}$, such that $g|_{D^{n,1}} = f^{-1}$. Since $g \circ \tilde{f} = \mathrm{id}$ and $\tilde{f} \circ g = \mathrm{id}$ on $D^{n,1}$, the uniqueness of analytic continuation shows that $g \circ \tilde{f} = \mathrm{id}$ and $\tilde{f} \circ g = \mathrm{id}$ on $\mathbb{C}^{n+1}$. Hence we have $\tilde{f} \in \mathrm{Aut}(\mathbb{C}^{n+1})$, or $\mathrm{Aut}(D^{n,1}) \subset \mathrm{Aut}(\mathbb{C}^{n+1})$.

From now on, we write $f$ for $\tilde{f}$. Now we know that $f|_{C^{n,1}} \in \mathrm{Aut}(C^{n,1})$. By Theorem 2.1 of the preceding section,

$$f_0(z_0, z_1, z_2, \dots, z_n) = c\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \dots, \frac{z_n}{z_0}\right) z_0^{\pm 1},$$

and

$$f_i(z_0, z_1, z_2, \dots, z_n) = f_0(z_0, z_1, z_2, \dots, z_n)\gamma_i\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \dots, \frac{z_n}{z_0}\right),$$

for $i = 1, \dots, n$, on $C^{n,1}$, where $c$ is a nowhere vanishing holomorphic function on $\mathbb{B}^n$ and

$$\gamma_i(w_1, \dots, w_n) = \frac{a_{i0} + \sum_{j=1}^n a_{ij} w_j}{a_{00} + \sum_{j=1}^n a_{0j} w_j}.$$

If

$$f_0(z_0, z_1, z_2, \dots, z_n) = c\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \dots, \frac{z_n}{z_0}\right) z_0^{-1},$$

considering the Taylor expansion of $c$ at the origin in $\mathbb{C}$, we see that $f_0$ is not holomorphic at $z_0 = 0$, which contradicts to the fact that $f_0$ is an entire holomorphic function. Thus

$$f_0(z_0, z_1, z_2, \dots, z_n) = c\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \dots, \frac{z_n}{z_0}\right) z_0.$$

Furthermore, by the Taylor expansion of $c$ at the origin, we see that $c$ is a polynomial function of degree 1, since $f_0$ has no singular point in $\mathbb{C}^{n+1}$, and therefore, $f_0$ is a linear function $\sum_{j=0}^n c_j z_j$. Then the entire functions $f_i$ $(i = 1, \dots, n)$ are

$$\begin{aligned} f_i(z_0, z_1, z_2, \dots, z_n) &= \gamma_i\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \dots, \frac{z_n}{z_0}\right) f_0(z_0, z_1, z_2, \dots, z_n) \\ &= \frac{a_{i0} + \sum_{j=1}^n a_{ij}\frac{z_j}{z_0}}{a_{00} + \sum_{j=1}^n a_{0j}\frac{z_j}{z_0}}\left(\sum_{j=0}^n c_j z_j\right) \\ &= \frac{\left(\sum_{j=0}^n a_{ij} z_j\right)\left(\sum_{j=0}^n c_j z_j\right)}{\sum_{j=0}^n a_{0j} z_j}. \end{aligned}$$

Then $\sum_{j=0}^n c_j z_j$ must be divided by $\sum_{j=0}^n a_{0j} z_j$ since $f_i$ has no singular point in $\mathbb{C}^{n+1}$. We now write

$$\sum_{j=0}^n c_j z_j = C\sum_{j=0}^n a_{0j} z_j\,,$$

where $C$ is a non-zero constant. Consequently,

$$f(z_0, \dots, z_n) = \left(C\sum_{j=0}^n a_{0j} z_j, C\sum_{j=0}^n a_{1j} z_j, \dots, C\sum_{j=0}^n a_{nj} z_j\right).$$

Thus we have shown the theorem. □

**4. The non-existence of compact quotients of $D^{n,1}$.** In this section, we prove the following theorem:

THEOREM 4.1. *$D^{n,1}$ has no compact quotients by a discrete subgroup of* $\mathrm{Aut}(D^{n,1})$ *acting properly discontinuously.*

We remark that $C^{n,1}$ has compact quotients since $\mathbb{B}^n$ and $\mathbb{C}^*$ have compact quotients. Recall the following result called the Calabi–Markus phenomenon:

LEMMA 4.2 (Calabi–Markus [2], Wolf [7]). *Let $\Gamma$ be a subgroup of $O(q+1, p)$ acting properly discontinuously on*

$$\left\{(x_1, \dots, x_p, x_{p+1}, \dots, x_{p+q+1}) \in \mathbb{R}^{p+q+1} : -x_1^2 - \dots - x_p^2 + x_{p+1}^2 + \dots + x_{p+q+1}^2 = 1\right\},$$

*where $1 < p \le q$. Then $\Gamma$ is finite.*

PROOF. From Theorem 3.1, we know that $\mathrm{Aut}(D^{n,1}) = GU(n,1) = \mathbb{R}_{>0} \times U(n,1)$, which acts on the complex euclidean space as linear transformations. We regard $\mathbb{R}_{>0} \times U(n,1)$ as a subgroup of $\mathbb{R}_{>0} \times O(2n,2)$.

Suppose that there exists a discrete subgroup

$$\Gamma = \{f_m\}_{m=1}^\infty \subset \mathbb{R}_{>0} \times O(2n,2)$$

such that $\Gamma$ acts properly discontinuously on $D^{n,1}$ and that the quotient $D^{n,1}/\Gamma$ is compact. By Selberg's lemma, we may assume without loss of generality that $\Gamma$ is torsion free. Set $f_m = (r_m, T_m)$, where $r_m \in \mathbb{R}_{>0}$ and $T_m \in O(2n, 2)$. It is clear that $\Gamma$ is not included in $O(2n, 2)$ by Lemma 4.2. We consider two cases.

First we consider the case where there exists the minimum of the set $\{r_m \mid 1 < r_m\}$. We denote the minimum by $R$:

$$R = \min\{r_m \mid 1 < r_m\}\,.$$

Then we see that, for any $r_m$, there exists an integer $l$ such that $r_m = R^l$. Therefore we can write

$$\Gamma = \left\{f_{l,k} = (R^l, T_{l,k})\right\}_{l\in\mathbb{Z},k\in\mathbb{N}}$$

by changing the indices. Put $\Gamma_0 = \{f_{0,k}\}$, a subgroup of $O(2n, 2)$. By Lemma 4.2, it follows that $\Gamma_0$ is a finite group. Since $\Gamma_0$ is torsion free, $\Gamma_0 = \{\mathrm{id}\}$. Therefore, $\Gamma$ is the group generated by the element $(R, T) \in \Gamma$. Hence we see that $D^{n,1}/\Gamma$ is not compact.

Next we consider the case where there does not exist the minimum of the set $\{r_m \mid 1 < r_m\}$. Let $R'$ be the infimum of the set $\{r_m \mid 1 < r_m\}$:

$$R' = \inf\{r_m \mid 1 < r_m\}\,.$$

Then, for any $\epsilon > 0$, by arranging the indices of the elements of $\Gamma$, we can take an infinite distinct sequence

$$R' + \epsilon > r_1 > r_2 > r_3 > \cdots > r_m > \cdots > R'\,.$$

Let

$$\Pi = \{z_0 = 0\} \subset \mathbb{C}^{n+1}$$

and

$$K = \left\{z_0 = 0, 1 \le |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 \le (R' + \epsilon)^2 + 1\right\} \subset \mathbb{C}^{n+1}\,.$$

It is clear that $K$ is compact in $D^{n,1}$. Let $\gamma_m = (r_m, T_m)$. We can easily see that $\gamma_m(\Pi) \cap \Pi$ contains a nontrivial linear subspace by the dimension formula of linear map. Then there exist $v_m \in \gamma_m(\Pi) \cap \Pi$ and $w_m \in \Pi$ such that $v_m = \gamma_m(w_m)$ and that $|w_m| = 1$. Note that $w_m \in K$. We see that $|v_m| = r_m|w_m| = r_m \le R' + \epsilon$, since $v_m \in \Pi$, and thus $v_m \in K$. We obtain that $\gamma_m(K) \cap K \ne \emptyset$ for any $m \ge 1$. However this is a contradiction since $\Gamma$ acts on $D^{n,1}$ properly discontinuously. The proof is complete. □

**5. The characterization of $D^{n,1}$ by its automorphism group.** Now we prove the following characterization theorem.

THEOREM 5.1. *Let $M$ be a connected complex manifold of dimension $n + 1$ that is holomorphically separable and admits a smooth envelope of holomorphy. Assume that* $\mathrm{Aut}(M)$ *is isomorphic to* $\mathrm{Aut}(D^{n,1}) = GU(n, 1)$ *as topological groups. Then $M$ is biholomorphic to $D^{n,1}$.*

PROOF. Denote by $\rho_0 : GU(n,1) \longrightarrow \mathrm{Aut}(M)$ a topological group isomorphism. Let us consider $U(1) \times U(n)$ as a matrix subgroup of $GU(n,1)$ in the natural way, and identify $U(n)$ with $\{1\} \times U(n)$. Then, by Theorem 1.2, there is a biholomorphic map $F$ from $M$ onto a Reinhardt domain $\Omega$ in $\mathbb{C}^{n+1}$ such that

$$F\rho_0(U(1) \times U(n))F^{-1} = U(n_1) \times \cdots \times U(n_s) \subset \mathrm{Aut}(\Omega)\,,$$

where $\sum_{j=1}^{s} n_j = n+1$. Then, after a permutation of coordinates if we need, we may assume $F\rho_0(U(1) \times U(n))F^{-1} = U(1) \times U(n)$. We define an isomorphism

$$\rho : GU(n,1) \longrightarrow \mathrm{Aut}(\Omega)$$

by $\rho(g) := F \circ \rho_0(g) \circ F^{-1}$. We will prove that $\Omega$ is biholomorphic to $D^{n,1}$.

Put

$$T_{1,n} = \{\mathrm{diag}[u_1, u_2 E_n] \in GL(n+1, \mathbb{C}) : u_1, u_2 \in U(1)\} \subset GU(n,1)\,.$$

Since $T_{1,n}$ is the center of the group $U(1) \times U(n)$, we have $\rho(T_{1,n}) = T_{1,n} \subset \mathrm{Aut}(\Omega)$. Consider $\mathbb{C}^*$ as a subgroup of $GU(n,1)$. So $\mathbb{C}^*$ represents center of $GU(n,1)$. Since $\rho(\mathbb{C}^*)$ is commutative with $T^{n+1}$, Lemma 1.1 tells us that $\rho(\mathbb{C}^*) \subset \Pi(\Omega)$, that is, $\rho(\mathbb{C}^*)$ is represented by diagonal matrices. Furthermore, $\rho(\mathbb{C}^*)$ commutes with $\rho(U(1) \times U(n)) = U(1) \times U(n)$, so that we have

$$\rho\big(e^{2\pi i(s+it)}\big) = \mathrm{diag}\big[e^{2\pi i\{a_1 s + (b_1 + ic_1)t\}}, e^{2\pi i\{a_2 s + (b_2 + ic_2)t\}} E_n\big] \in \rho(\mathbb{C}^*)\,,$$

where $s, t \in \mathbb{R}$, $a_1, a_2 \in \mathbb{Z}$, $b_1, b_2, c_1, c_2 \in \mathbb{R}$. Since $\rho$ is injective, $a_1, a_2$ are relatively prime and $(c_1, c_2) \neq (0,0)$. To consider the actions of $\rho(\mathbb{C}^*)$ and $U(1) \times U(n)$ on $\Omega$ together, we put

$$G(U(1) \times U(n)) = \big\{e^{-2\pi t}\mathrm{diag}[u_0, U] \in GU(n,1) : t \in \mathbb{R}, u_0 \in U(1), U \in U(n)\big\}\,.$$

Then we have

$$\begin{aligned} G &:= \rho(G(U(1) \times U(n))) \\ &= \big\{\mathrm{diag}[e^{-2\pi c_1 t} u_0, e^{-2\pi c_2 t} U] \in GL(n+1, \mathbb{C}) : t \in \mathbb{R}, u_0 \in U(1), U \in U(n)\big\}\,. \end{aligned}$$

Note that $G$ is the centralizer of $T_{1,n} = \rho(T_{1,n})$ in $\mathrm{Aut}(\Omega)$.

Let $f = (f_0, f_1, \dots, f_n) \in \mathrm{Aut}(\Omega) \setminus G$ and consider Laurent expansions of its components:

$$f_0(z_0, \dots, z_n) = \sum_{\nu \in \mathbb{Z}^{n+1}} a_\nu^{(0)} z^\nu\,, \tag{1}$$

$$f_i(z_0, \dots, z_n) = \sum_{\nu \in \mathbb{Z}^{n+1}} a_\nu^{(i)} z^\nu, \quad 1 \le i \le n\,. \tag{2}$$

If $f$ is a linear map of the form

$$\begin{pmatrix} a^{(0)}_{(1,0,\dots,0)} & 0 & \cdots & 0 \\ 0 & a^{(1)}_{(0,1,0,\dots,0)} & \cdots & a^{(1)}_{(0,\dots,0,1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a^{(n)}_{(0,1,0,\dots,0)} & \cdots & a^{(n)}_{(0,\dots,0,1)} \end{pmatrix} \in GL(n+1,\mathbb{C}) ,$$

then $f$ commutes with $\rho(T_{1,n})$, which contradicts $f \notin G$. Thus for any $f \in \mathrm{Aut}(\Omega) \setminus G$, there exists $\nu \in \mathbb{Z}^{n+1}(\neq (1,0,\dots,0))$ such that $a^{(0)}_\nu \neq 0$ in (1), or there exists $\nu \in \mathbb{Z}^{n+1}(\neq (0,1,0,\dots,0),\dots,(0,0,\dots,0,1))$ such that $a^{(i)}_\nu \neq 0$ in (2) for some $1 \leq i \leq n$.

REMARK 5.2. We remark here that, in (1) and (2), there are no negative degree terms of $z_1,\dots,z_n$, since $\Omega \cup \{z_i = 0\} \neq \emptyset$ for $1 \leq i \leq n$ by the $U(n)$-action on $\Omega$, and since the Laurent expansion of a holomorphic function on $\Omega$ are globally defined on $\Omega$. Write $\nu = (\nu_0,\nu') = (\nu_0,\nu_1,\dots,\nu_n)$ and $|\nu'| = \nu_1 + \cdots + \nu_n$. Let us consider $\nu' \in \mathbb{Z}^n_{\geq 0}$ and put

$$\sum_\nu{}' = \sum_{\nu_0 \in \mathbb{Z},\, \nu' \in \mathbb{Z}^n_{\geq 0}}$$

from now on.

CLAIM 5.3. *$a_1 = \pm 1$, $c_1 c_2 \neq 0$, and $\lambda := c_2/c_1 = a_2/a_1 \in \mathbb{Z} \setminus \{0\}$.*

PROOF. To prove the claim, we divide three cases.

Case (i): $c_1 c_2 \neq 0$.

Since $\mathbb{C}^*$ is the center of $GU(n,1)$, it follows that, for any $f \in \mathrm{Aut}(\Omega) \setminus G$,

$$f \circ \rho(e^{2\pi i(s+it)}) = \rho(e^{2\pi i(s+it)}) \circ f .$$

By (1) and (2), this equation means

$$\begin{aligned} e^{2\pi i\{a_1 s+(b_1+ic_1)t\}} \sum_\nu{}' a^{(0)}_\nu z^\nu &= \sum_\nu{}' a^{(0)}_\nu (e^{2\pi i\{a_1 s+(b_1+ic_1)t\}} z_0)^{\nu^{(0)}_0} (e^{2\pi i\{a_2 s+(b_2+ic_2)t\}} z')^{\nu'} \\ &= \sum_\nu{}' a^{(0)}_\nu e^{2\pi i\{a_1 s+(b_1+ic_1)t\}\nu^{(0)}_0} e^{2\pi i\{a_2 s+(b_2+ic_2)t\}|\nu'|} z^\nu \end{aligned}$$

and

$$\begin{aligned} e^{2\pi i\{a_2 s+(b_2+ic_2)t\}} \sum_\nu{}' a^{(i)}_\nu z^\nu &= \sum_\nu{}' a^{(i)}_\nu (e^{2\pi i\{a_1 s+(b_1+ic_1)t\}} z_0)^{\nu^{(i)}_0} (e^{2\pi i\{a_2 s+(b_2+ic_2)t\}} z')^{\nu'} \\ &= \sum_\nu{}' a^{(i)}_\nu e^{2\pi i\{a_1 s+(b_1+ic_1)t\}\nu^{(i)}_0} e^{2\pi i\{a_2 s+(b_2+ic_2)t\}|\nu'|} z^\nu , \end{aligned}$$

for $1 \leq i \leq n$. Thus for each $\nu \in \mathbb{Z}^{n+1}$, we have

$$e^{2\pi i\{a_1 s+(b_1+ic_1)t\}} a^{(0)}_\nu = e^{2\pi i\{a_1 s+(b_1+ic_1)t\}\nu^{(0)}_0} e^{2\pi i\{a_2 s+(b_2+ic_2)t\}|\nu'|} a^{(0)}_\nu ,$$

and

$$e^{2\pi i\{a_2 s+(b_2+ic_2)t\}} a^{(i)}_\nu = e^{2\pi i\{a_1 s+(b_1+ic_1)t\}\nu^{(i)}_0} e^{2\pi i\{a_2 s+(b_2+ic_2)t\}|\nu'|} a^{(i)}_\nu ,$$

for $1 \le i \le n$. Therefore, if $a_\nu^{(0)} \ne 0$ for $\nu = \big(\nu_0^{(0)}, \nu'\big) = \big(\nu_0^{(0)}, \nu_1^{(0)}, \dots, \nu_n^{(0)}\big)$, we have

$$\text{(3)} \qquad \begin{cases} a_1\big(\nu_0^{(0)} - 1\big) + a_2\big(\nu_1^{(0)} + \cdots + \nu_n^{(0)}\big) = 0\,, \\ c_1\big(\nu_0^{(0)} - 1\big) + c_2\big(\nu_1^{(0)} + \cdots + \nu_n^{(0)}\big) = 0\,. \end{cases}$$

Similarly, if $a_\nu^{(i)} \ne 0$ for $\nu = \big(\nu_0^{(i)}, \nu'\big) = \big(\nu_0^{(i)}, \nu_1^{(i)}, \dots, \nu_n^{(i)}\big)$, we have

$$\text{(4)} \qquad \begin{cases} a_1\nu_0^{(i)} + a_2\big(\nu_1^{(i)} + \cdots + \nu_n^{(i)} - 1\big) = 0\,, \\ c_1\nu_0^{(i)} + c_2\big(\nu_1^{(i)} + \cdots + \nu_n^{(i)} - 1\big) = 0\,, \end{cases}$$

for $1 \le i \le n$.

Suppose $a_\nu^{(0)} \ne 0$ for some $\nu = \big(\nu_0^{(0)}, \nu_1^{(0)}, \dots, \nu_n^{(0)}\big) \ne (1, 0, \dots, 0)$. Then, by (3) and the assumption $c_1c_2 \ne 0$, it follows that $\nu_0^{(0)} - 1 \ne 0$ and $\nu_1^{(0)} + \cdots + \nu_n^{(0)} \ne 0$. Hence $c_2/c_1 \in \mathbb{Q}$ by (3). On the other hand, if $a_\nu^{(i)} \ne 0$ for some $1 \le i \le n$ and $\nu = \big(\nu_0^{(i)}, \nu_1^{(i)}, \dots, \nu_n^{(i)}\big) \ne (0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 0, 1)$, then $\nu_0^{(i)} \ne 0$ and $\nu_1^{(i)} + \cdots + \nu_n^{(i)} - 1 \ne 0$ by (4) and the assumption $c_1c_2 \ne 0$. In this case, we also obtain $c_2/c_1 \in \mathbb{Q}$ by (4). Note that $a_1, a_2$ are relatively prime. Consequently, we have

$$\lambda := a_2/a_1 = c_2/c_1 \in \mathbb{Q}$$

by (3) or (4).

We now prove that $\lambda$ is an integer. For the purpose, we assume $\lambda \notin \mathbb{Z}$, that is, $a_1 \ne \pm 1$. First we consider the case $\lambda < 0$. Since $\nu_1^{(i)} + \cdots + \nu_n^{(i)} \ge 0$ for $0 \le i \le n$, we have $\nu_0^{(0)} \ge 1$ and $\nu_0^{(i)} \ge 0$ by (3) and (4) . Furthermore, the Laurent expansions of the components of $f \in \mathrm{Aut}(\Omega)$ are

$$\text{(5)} \qquad f_0(z_0, \dots, z_n) = \sum_{k=0}^{\infty} \sideset{}{'}\sum_{|\nu'|=k|a_1|} a_{\nu'}^{(0)} z_0^{1+k|a_2|} (z')^{\nu'}$$

and

$$\text{(6)} \qquad f_i(z_0, \dots, z_n) = \sum_{k=0}^{\infty} \sideset{}{'}\sum_{|\nu'|=1+k|a_1|} a_{\nu'}^{(i)} z_0^{k|a_2|} (z')^{\nu'}$$

for $1 \le i \le n$. Here we have written $a_{\nu'}^{(0)} = a_{(1+k|a_2|,\nu')}^{(0)}$ and $a_{\nu'}^{(i)} = a_{(k|a_2|,\nu')}^{(i)}$, and so as from now on. We focus on the first degree terms of the Laurent expansions. It follows from (5) and (6) that the first degree terms of the Laurent expansions of the components of the composite $f \circ h$ are the composites of the first degree terms of Laurent expansions of the components of $f$ and $h$, where $h \in \mathrm{Aut}(\Omega)$. We put

$$\text{(7)} \qquad Pf(z) := \left( a_{(1,0,\dots,0)}^{(0)} z_0, \sideset{}{'}\sum_{|\nu'|=1} a_{\nu'}^{(1)} (z')^{\nu'}, \dots, \sideset{}{'}\sum_{|\nu'|=1} a_{\nu'}^{(n)} (z')^{\nu'} \right).$$

Then as a matrix we can write

$$Pf = \begin{pmatrix} a^{(0)}_{(1,0,\dots,0)} & 0 & \cdots & 0 \\ 0 & a^{(1)}_{(0,1,0,\dots,0)} & \cdots & a^{(1)}_{(0,\dots,0,1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a^{(n)}_{(0,1,0,\dots,0)} & \cdots & a^{(n)}_{(0,\dots,0,1)} \end{pmatrix},$$

which belongs to $GL(n+1, \mathbb{C})$ since $f$ is an automorphism. Hence we have a representation of $GU(n, 1)$ given by

$$GU(n, 1) \ni g \longmapsto Pf \in GL(n+1, \mathbb{C}),$$

where $f = \rho(g)$. The restriction of this representation to the simple Lie group $SU(n, 1)$ is nontrivial since $\rho(U(1) \times U(n)) = U(1) \times U(n)$. However this contradicts Lemma 1.4. Thus it does not occur that $\lambda$ is a negative non-integer.

Next we consider the case $\lambda > 0$ and $\lambda \notin \mathbb{Z}$. Then $\nu^{(0)}_0 \leq 1$ and $\nu^{(i)}_0 \leq 0$ by (3) and (4) since $\nu^{(i)}_1 + \dots + \nu^{(i)}_n \geq 0$ for $0 \leq i \leq n$. Furthermore, the Laurent expansions of components of $f$ are

$$\begin{aligned} f_0(z_0, \dots, z_n) &= \sum_{k=0}^{\infty} \sum_{|\nu'|=k|a_1|}{}' a^{(0)}_{\nu'} z_0^{1-k|a_2|} (z')^{\nu'} \\ &= a^{(0)}_{(1,0,\dots,0)} z_0 + \sum_{|\nu'|=|a_1|}{}' a^{(0)}_{(1-|a_2|,\nu')} z_0^{1-|a_2|} (z')^{\nu'} \\ &\qquad + \sum_{|\nu'|=2|a_1|}{}' a^{(0)}_{(1-2|a_2|,\nu')} z_0^{1-2|a_2|} (z')^{\nu'} + \cdots, \end{aligned}$$

and

$$\begin{aligned} f_i(z_0, \dots, z_n) &= \sum_{k=0}^{\infty} \sum_{|\nu'|=1+k|a_1|}{}' a^{(i)}_{\nu'} z_0^{-k|a_2|} (z')^{\nu'} \\ &= \sum_{|\nu'|=1}{}' a^{(i)}_{(0,\nu')} (z')^{\nu'} + \sum_{|\nu'|=1+|a_1|}{}' a^{(i)}_{(-|a_2|,\nu')} z_0^{-|a_2|} (z')^{\nu'} \\ &\qquad + \sum_{|\nu'|=1+2|a_1|}{}' a^{(0)}_{(-2|a_2|,\nu')} z_0^{-2|a_2|} (z')^{\nu'} + \cdots \end{aligned}$$

for $1 \leq i \leq n$. We claim that $a^{(0)}_{(1,0,\dots,0)} \neq 0$. Indeed, if $a^{(0)}_{(1,0,\dots,0)} = 0$, then $f(z_0, 0, \dots, 0) = (0, \dots, 0) \in \mathbb{C}^{n+1}$. This contradicts that $f$ is an automorphism. Take another $h \in \mathrm{Aut}(\Omega) \setminus G$ and put Laurent expansions of its components

$$h_0(z_0, \dots, z_n) = \sum_{k=0}^{\infty} \sum_{|\nu'|=k|a_1|}{}' b^{(0)}_{\nu'} z_0^{1-k|a_2|} (z')^{\nu'}$$

$$h_i(z_0,\dots,z_n)=\sum_{k=0}^{\infty}\ \sum_{|\nu'|=1+k|a_1|}{}' \ b^{(i)}_{\nu'} z_0^{-k|a_2|}(z')^{\nu'}$$

for $1\le i\le n$. We have $b^{(0)}_{(1,0,\dots,0)}\neq 0$ as above. We mention the first degree terms of the Laurent expansions of the components of $f\circ h$. For the first component,

$$f_0(h_0,\dots,h_n)=a^{(0)}_{(1,0,\dots,0)}h_0+\sum_{k=1}^{\infty}\ \sum_{|\nu'|=k|a_1|}{}' \ a^{(0)}_{\nu'}h_0^{1-k|a_2|}(h')^{\nu'}.$$

Then, for $k>0$,

$$\begin{aligned}h_0(z)^{1-k|a_2|}&=\Bigl(\sum_{l=0}^{\infty}\ \sum_{|\nu'|=l|a_1|}{}' b^{(0)}_{\nu'}z_0^{1-l|a_2|}(z')^{\nu'}\Bigr)^{1-k|a_2|}=z_0^{1-k|a_2|}\Bigl(\sum_{l=0}^{\infty}\ \sum_{|\nu'|=l|a_1|}{}' b^{(0)}_{\nu'}z_0^{-l|a_2|}(z')^{\nu'}\Bigr)^{1-k|a_2|}\\&=(b^{(0)}_{0_n}z_0)^{1-k|a_2|}\Bigl(1-\frac{1-k|a_2|}{b^{(0)}_{0_n}}z_0^{-|a_2|}\sum_{|\nu'|=|a_1|}{}' b^{(0)}_{\nu'}(z')^{\nu'}+\cdots\Bigr).\end{aligned}$$

Thus $h_0(z)^{1-k|a_2|}$ has the maximum degree of $z_0$ at most $1-k|a_2|<1$ and has the minimum degree of $z'$ at least $|a_1|>1$ in its Laurent expansion. For $|\nu'|=k|a_1|$ and $k>0$, $(h')^{\nu'}$ has the maximum degree of $z_0$ at most $-|a_2|<0$ and the first degree terms of $z'$ are with coefficients of negative degree terms of $z_0$ in its Laurent expansion. Hence the first degree term of the Laurent expansion of $f_0(h_0,\dots,h_n)$ is $a^{(0)}_{(1,0,\dots,0)}b^{(0)}_{(1,0,\dots,0)}z_0$.

Similarly, we consider

$$f_i(h_0,\dots,h_n)=\sum_{|\nu'|=1}{}' \ a^{(i)}_{\nu'}(h')^{\nu'}+\sum_{k=1}^{\infty}\ \sum_{|\nu'|=1+k|a_1|}{}' \ a^{(i)}_{\nu'}h_0^{-k|a_2|}(h')^{\nu'}$$

for $1\le i\le n$. Then, for $k>0$,

$$h_0^{-k|a_2|}=(b^{(0)}_{0_n}z_0)^{-k|a_2|}\Bigl(1-\frac{-k|a_2|}{b^{(0)}_{0_n}}z_0^{-|a_2|}\sum_{|\nu'|=|a_1|}{}' b^{(0)}_{\nu'}(z')^{\nu'}+\cdots\Bigr).$$

Thus $h_0^{-k|a_2|}$ has the maximum degree of $z_0$ at most $-k|a_2|<0$ and has the minimum degree of $z'$ at least $|a_1|>1$ in its Laurent expansion. For $|\nu'|=1+k|a_1|$ and $k>0$, $(h')^{\nu'}$ has the maximum degree of $z_0$ at most $-|a_2|<0$ and the first degree terms of $z'$ are with coefficients of negative degree terms of $z_0$ in its Laurent expansion. Hence the first degree terms of the Laurent expansion of $f_i(h_0,\dots,h_n)$ are

$$\sum_{j=1}^{n}\sum_{|\nu'|=1}{}' \ a^{(i)}_{\nu_j}b^{(j)}_{\nu'}(z')^{\nu'},$$

where $\nu_j=(0,\dots,0,1_j,0,\dots,0)$, that is, the $j$-th component is 1 and the others are 0.

Consequently, $P(f\circ h)=Pf\circ Ph$, where $Pf$ is defined as (7). Then the same argument as that in the previous case ($\lambda<0$) shows that this is a contradiction. Indeed, $Pf\in GL(n+$

$1, \mathbb{C})$ since $f$ is an automorphism, so that we have a representation of $GU(n,1)$ by

$$GU(n,1) \ni g \longmapsto Pf \in GL(n+1,\mathbb{C}),$$

where $f = \rho(g)$. Therefore this contradicts Lemma 1.4, since this representation is nontrivial on $SU(n,1)$ by $\rho(U(1)\times U(n)) = U(1)\times U(n)$. Thus it does not occur that $\lambda$ is a positive non-integer.

Hence we have $\lambda = c_2/c_1 = a_2/a_1 \in \mathbb{Z}\setminus\{0\}$ and $a_1 = \pm 1$.

Case (ii) : $c_1 \neq 0, c_2 = 0$.
In this case, $\Omega \subset \mathbb{C}^{n+1}$ can be written of the form $(\mathbb{C}\times D)\cup(\mathbb{C}^*\times D')$, where $D$ and $D'$ are open sets in $\mathbb{C}^n$. Indeed, $\Omega = (\Omega\cap\{z_0 = 0\})\cup(\Omega\cap\{z_0\neq 0\})$. Then $\{0\}\times D := \Omega\cap\{z_0 = 0\}\subset\Omega$ implies $\mathbb{C}\times D\subset\Omega$ by $\rho(\mathbb{C}^*)$- and $T^{n+1}$-actions on $\Omega$. On the other hand, $\Omega\cap\{z_0\neq 0\} = \mathbb{C}^*\times D'$ for some open set $D'\subset\mathbb{C}^n$ by $\rho(\mathbb{C}^*)$- and $T^{n+1}$-actions. Thus $\Omega = (\mathbb{C}\times D)\cup(\mathbb{C}^*\times D')$. Then, by Lemma 1.3, $\mathrm{Aut}(\Omega)$ has no Lie group structure, and this contradicts the assumption $\mathrm{Aut}(\Omega) = GU(n,1)$.

Case (iii) : $c_1 = 0$ and $c_2 \neq 0$.
As in the previous case, $\Omega\subset\mathbb{C}^{n+1}$ can be written of the form $(D''\times\mathbb{C}^n)\cup(D'''\times(\mathbb{C}^n\setminus\{0\}))$ by $\rho(\mathbb{C}^*)$- and $T^{n+1}$-actions on $\Omega$, where $D''$ and $D'''$ are open sets in $\mathbb{C}$. Then, for a similar reason as for the proof of Lemma 1.3, $\mathrm{Aut}(\Omega)$ has no Lie group structure, and this contradicts our assumption. □

REMARK 5.4. Since $\lambda\in\mathbb{Z}\setminus\{0\}$, the Laurent expansions of its components of $f\in \mathrm{Aut}(\Omega)$ are

$$f_0(z_0,\dots,z_n) = \sum_{k=0}^{\infty}\sum_{|\nu'|=k}{}' a^{(0)}_{\nu'} z_0^{1-k\lambda}(z')^{\nu'},$$

and

$$f_i(z_0,\dots,z_n) = a^{(i)}_{(\lambda,0,\dots,0)} z_0^{\lambda} + \sum_{k=0}^{\infty}\sum_{|\nu'|=1+k}{}' a^{(i)}_{\nu'} z_0^{-k\lambda}(z')^{\nu'}$$

for $1\le i\le n$.

Since $G = \rho(G(U(1)\times U(n)))$ acts as linear transformations on $\Omega\subset\mathbb{C}^{n+1}$, it preserves the boundary $\partial\Omega$ of $\Omega$. We now study the action of $G$ on $\partial\Omega$. The $G$-orbits of points in $\mathbb{C}^{n+1}$ consist of four types as follows:

(i) If $p = (p_0, p_1,\dots,p_n)\in\mathbb{C}^*\times(\mathbb{C}^n\setminus\{0\})$, then

$$G\cdot p = \left\{(z_0,\dots,z_n)\in\mathbb{C}^{n+1}\setminus\{0\} : -a|z_0|^{2\lambda} + |z_1|^2+\cdots+|z_n|^2 = 0\right\}, \tag{8}$$

where $a := (|p_1|^2+\cdots+|p_n|^2)/|p_0|^{2\lambda} > 0$ and $\lambda\in\mathbb{Z}\setminus\{0\}$ by Claim 5.3.

(ii) If $p' = (0, p'_1,\dots,p'_n)\in\mathbb{C}^{n+1}\setminus\{0\}$, then

$$G\cdot p' = \{0\}\times(\mathbb{C}^n\setminus\{0\}). \tag{9}$$

(iii) If $p'' = (p''_0, 0, \ldots, 0) \in \mathbb{C}^{n+1} \setminus \{0\}$, then

$$G \cdot p'' = \mathbb{C}^* \times \{0\}\,. \tag{10}$$

(iv) If $p''' = (0, \ldots, 0) \in \mathbb{C}^{n+1}$, then

$$G \cdot p''' = \{0\} \subset \mathbb{C}^{n+1}\,. \tag{11}$$

We show that $\partial\Omega \cap (\mathbb{C}^* \times (\mathbb{C}^n \setminus \{0\})) \neq \emptyset$.

CLAIM 5.5. *$\Omega \cap (\mathbb{C}^* \times (\mathbb{C}^n \setminus \{0\}))$ is a proper subset of $\mathbb{C}^* \times (\mathbb{C}^n \setminus \{0\})$.*

PROOF. If $\Omega \cap (\mathbb{C}^* \times (\mathbb{C}^n \setminus \{0\})) = \mathbb{C}^* \times (\mathbb{C}^n \setminus \{0\})$, then $\Omega$ equals one of the following domains by the $G$-actions of type (9) and (10) above:

$$\mathbb{C}^{n+1},\ \mathbb{C}^{n+1} \setminus \{0\},\ \mathbb{C}^* \times (\mathbb{C}^n \setminus \{0\}),\ \mathbb{C} \times (\mathbb{C}^n \setminus \{0\}) \text{ or } \mathbb{C}^* \times \mathbb{C}^n\,.$$

However these can not occur since all automorphism groups of these domains are not Lie groups by Lemma 1.3. This contradicts that $\mathrm{Aut}(\Omega) = GU(n, 1)$. □

By Claim 5.5, $\partial\Omega \cap (\mathbb{C}^* \times (\mathbb{C}^n \setminus \{0_n\})) \neq \emptyset$. Thus we can take a point

$$p = (p_0, \ldots, p_n) \in \partial\Omega \cap (\mathbb{C}^* \times (\mathbb{C}^n \setminus \{0_n\}))\,.$$

Let

$$a = \big(|p_1|^2 + \cdots + |p_n|^2\big)/|p_0|^{2\lambda} > 0\,,$$
$$A_{a,\lambda} = \big\{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : -a|z_0|^{2\lambda} + |z_1|^2 + \cdots + |z_n|^2 = 0\big\}\,.$$

Note that

$$\partial\Omega \supset A_{a,\lambda}\,.$$

If $\lambda > 0$, then $\Omega$ is included in

$$D^+_{a,\lambda} = \big\{|z_1|^2 + \cdots + |z_n|^2 > a|z_0|^{2\lambda}\big\}$$

or

$$C^+_{a,\lambda} = \big\{|z_1|^2 + \cdots + |z_n|^2 < a|z_0|^{2\lambda}\big\}\,.$$

If $\lambda < 0$, then $\Omega$ is included in

$$D^-_{a,\lambda} = \big\{(|z_1|^2 + \cdots + |z_n|^2)|z_0|^{-2\lambda} > a\big\}$$

or

$$C^-_{a,\lambda} = \big\{(|z_1|^2 + \cdots + |z_n|^2)|z_0|^{-2\lambda} < a\big\}\,.$$

CLAIM 5.6. *If $\Omega = D^+_{a,\lambda}$, then $\lambda = 1$ and $\Omega$ is biholomorphic to $D^{n,1}$.*

PROOF. If $\lambda \neq 1$, then by Remark 5.4, for $f \in \mathrm{Aut}(\Omega)$, the Laurent expansions of its components are

$$f_0(z_0, \dots, z_n) = \sum_{k=0}^{\infty} {\sum_{|\nu'|=k}}' a_{\nu'}^{(0)} z_0^{1-k\lambda} (z')^{\nu'} ,$$

and

$$f_i(z_0, \dots, z_n) = a_{(\lambda,0,\dots,0)}^{(i)} z_0^{\lambda} + \sum_{k=0}^{\infty} {\sum_{|\nu'|=1+k}}' a_{\nu'}^{(i)} z_0^{-k\lambda} (z')^{\nu'} ,$$

for $1 \leq i \leq n$. Since $D_{a,\lambda}^{+} \cap \{z_0 = 0\} \neq \emptyset$, it follows that the negative degrees of $z_0$ do not arise in the Laurent expansions. Therefore

$$f_0(z_0, \dots, z_n) = a_{(1,0,\dots,0)}^{(0)} z_0 ,$$

and

$$f_i(z_0, \dots, z_n) = a_{(\lambda,0,\dots,0)}^{(i)} z_0^{\lambda} + {\sum_{|\nu'|=1}}' a_{\nu'}^{(i)} (z')^{\nu'} ,$$

for $1 \leq i \leq n$. Consider

$$Pf(z) = (a_{(1,0,\dots,0)}^{(0)} z_0, {\sum_{|\nu'|=1}}' a_{\nu'}^{(1)} (z')^{\nu'}, \dots, {\sum_{|\nu'|=1}}' a_{\nu'}^{(n)} (z')^{\nu'}) .$$

Then $Pf$ gives a representation of $GU(n, 1)$ by

$$P\rho : GU(n, 1) \ni g \longmapsto P(\rho(g)) \in GL(n + 1, \mathbb{C}) ,$$

as in the proof of Claim 5.3, and we showed that this can not occur by Lemma 1.4. Thus $\lambda = 1$ and $\Omega$ is biholomorphic to $D^{n,1}$. □

We will show that Claim 5.6 is the only case that a domain has the automorphism group isomorphic to $GU(n, 1)$.

Let us first consider the case $\partial\Omega = A_{a,\lambda}$, and we derive contradictions if $\Omega = C_{a,\lambda}^{+}$, $D_{a,\lambda}^{-}$ or $C_{a,\lambda}^{-}$.

CLAIM 5.7. *$\mathrm{Aut}(C_{a,\lambda}^{+})$ and $\mathrm{Aut}(D_{a,\lambda}^{-})$ are not Lie groups, so $\Omega \neq C_{a,\lambda}^{+}, D_{a,\lambda}^{-}$.*

PROOF. Indeed, $C_{a,\lambda}^{+}$ is biholomorphic to $\mathbb{C}^* \times \mathbb{B}^n$, and $D_{a,\lambda}^{-}$ is biholomorphic to $\mathbb{C}^* \times (\mathbb{C}^n \setminus \overline{\mathbb{B}^n})$. The automorphism groups of these domains are not Lie groups by Lemma 1.3. □

CLAIM 5.8. *$\Omega \neq C_{a,\lambda}^{-}$.*

PROOF. Suppose $\Omega = C_{a,\lambda}^{-}$. By remark 5.4, for $f \in \mathrm{Aut}(\Omega)$, the Laurent expansions of its components are

$$f_0(z_0, \dots, z_n) = \sum_{k=0}^{\infty} {\sum_{|\nu'|=k}}' a_{\nu'}^{(0)} z_0^{1-k\lambda} (z')^{\nu'} ,$$

and

$$f_i(z_0,\dots,z_n)=a^{(i)}_{(\lambda,0,\dots,0)}z_0^{\lambda}+\sum_{k=0}^{\infty}\sum_{|\nu'|=1+k}{}' a^{(i)}_{\nu'}z_0^{-k\lambda}(z')^{\nu'}$$

for $1\le i\le n$. Since $C^-_{a,\lambda}\cap\{z_0=0\}\neq\emptyset$, the negative degrees of $z_0$ do not arise in the Laurent expansions. Therefore

$$f_0(z_0,\dots,z_n)=\sum_{k=0}^{\infty}\sum_{|\nu'|=k}{}' a^{(0)}_{\nu'}z_0^{1-k\lambda}(z')^{\nu'},$$

and

$$f_i(z_0,\dots,z_n)=\sum_{k=0}^{\infty}\sum_{|\nu'|=1+k}{}' a^{(i)}_{\nu'}z_0^{-k\lambda}(z')^{\nu'}$$

for $1\le i\le n$. Consider

$$Pf(z)=(a^{(0)}_{(1,0,\dots,0)}z_0,\ \sum_{|\nu'|=1}{}' a^{(1)}_{\nu'}(z')^{\nu'},\dots,\ \sum_{|\nu'|=1}{}' a^{(n)}_{\nu'}(z')^{\nu'}).$$

Then $Pf$ gives a representation of $GU(n,1)$ by

$$P\rho: GU(n,1)\ni g\longmapsto P(\rho(g))\in GL(n+1,\mathbb{C}),$$

as in the proof of Claim 5.3, and we showed that this can not occur by Lemma 1.4. Thus $\Omega\neq C^-_{a,\lambda}$. □

Let us consider the case $\partial\Omega\neq A_{a,\lambda}$.

Case (I) : $(\partial\Omega\setminus A_{a,\lambda})\cap(\mathbb{C}^*\times(\mathbb{C}^n\setminus\{0\}))=\emptyset$.

In this case, $\partial\Omega$ is the union of $A_{a,\lambda}$ and some of the following sets

(12) $$\{0\}\times(\mathbb{C}^n\setminus\{0\}),\ \mathbb{C}^*\times\{0\}\ \text{or}\ \{0\}\subset\mathbb{C}^{n+1},$$

by the $G$-actions on the boundary of type (9), (10) and (11). If $\Omega\subset D^-_{a,\lambda}$, then the sets of (12) can not be included in the boundary of $\Omega$. Thus we must consider only the cases $\Omega\subsetneq D^+_{a,\lambda}$, $C^+_{a,\lambda}$ or $C^-_{a,\lambda}$.

Case (I-i) : $\Omega\subsetneq D^+_{a,\lambda}$.

In this case, $\mathbb{C}^*\times\{0\}$ can not be a subset of the boundary of $\Omega$, and $\{0\}\in A_{a,1}$. Thus

$$\partial\Omega=A_{a,\lambda}\cup(\{0\}\times\mathbb{C}^n),$$
$$\Omega=D^+_{a,\lambda}\setminus(\{0\}\times\mathbb{C}^n).$$

Then, $\Omega$ is biholomorphic to $\mathbb{C}^*\times(\mathbb{C}^n\setminus\overline{\mathbb{B}^n})$ and $\mathrm{Aut}(\mathbb{C}^*\times(\mathbb{C}^n\setminus\overline{\mathbb{B}^n}))$ does not have a Lie group structure. This contradicts the assumption that $\mathrm{Aut}(\Omega)=GU(n,1)$. Thus this case does not occur.

Case (I-ii) : $\Omega\subsetneq C^+_{a,\lambda}$.

In this case, $\{0\}\times(\mathbb{C}^n\setminus\{0\})$ can not be a subset of the boundary of $\Omega$, and $\{0\}\in A_{a,1}$. Thus

$$\partial\Omega=A_{a,\lambda}\cup(\mathbb{C}\times\{0\}),$$

$$\Omega = C^+_{a,\lambda} \setminus (\mathbb{C} \times \{0\}) .$$

Then, $\Omega$ is biholomorphic to $\mathbb{C}^* \times (\mathbb{B}^n \setminus \{0\})$ and $\mathrm{Aut}(\mathbb{C}^* \times (\mathbb{B}^n \setminus \{0\}))$ does not have a Lie group structure. This contradicts the assumption that $\mathrm{Aut}(\Omega) = GU(n, 1)$, and this case does not occur.

Case (I-iii) : $\Omega \subsetneq C^-_{a,\lambda}$.

In this case, $\Omega$ coincides with one of the followings:

$$\begin{aligned} C_1 &= C^-_{a,\lambda} \setminus (\{0\} \times \mathbb{C}^n) \cup (\mathbb{C} \times \{0\}) , \\ C_2 &= C^-_{a,\lambda} \setminus (\{0\} \times \mathbb{C}^n) , \\ C_3 &= C^-_{a,\lambda} \setminus (\mathbb{C} \times \{0\}) , \\ C_4 &= C^-_{a,\lambda} \setminus \{0\} . \end{aligned}$$

Then $C_1$ is biholomorphic to $\mathbb{C}^* \times (\mathbb{B}^n \setminus \{0\})$, and $C_2$ is biholomorphic to $\mathbb{C}^* \times \mathbb{B}^n$. The automorphism groups of these domains are not Lie groups. This contradicts the assumption. The proof of Claim 5.8 also leads that $\Omega \neq C_3, C_4$ since $C_3 \cap \{z_0 = 0\} \neq \emptyset$ and $C_4 \cap \{z_0 = 0\} \neq \emptyset$. Thus this case does not occur.

Case (II) : $(\partial\Omega \setminus A_{a,\lambda}) \cap (\mathbb{C}^* \times (\mathbb{C}^n \setminus \{0_n\})) \neq \emptyset$.

In this case, we can take a point $p' = (p'_0, \dots, p'_n) \in (\partial\Omega \setminus A_{a,\lambda}) \cap (\mathbb{C}^* \times (\mathbb{C}^n \setminus \{0_n\}))$. Put

$$\begin{aligned} b &= \big(|p'_1|^2 + \cdots + |p'_n|^2\big)/|p'_0|^{2\lambda} > 0 , \\ B_{b,\lambda} &= \big\{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : -b|z_0|^{2\lambda} + |z_1|^2 + \cdots + |z_n|^2 = 0\big\} . \end{aligned}$$

We may assume $a > b$ without loss of generality.

Case (II-i) : $\partial\Omega = A_{a,\lambda} \cup B_{b,\lambda}$.

Since $\Omega$ is connected, it coincides with

$$C^+_{a,\lambda} \cap D^+_{b,\lambda} = \big\{b|z_0|^{2\lambda} < |z_1|^2 + \cdots + |z_n|^2 < a|z_0|^{2\lambda}\big\} ,$$

if $\lambda > 0$, or

$$C^-_{a,\lambda} \cap D^-_{b,\lambda} = \big\{b < (|z_1|^2 + \cdots + |z_n|^2)|z_0|^{-2\lambda} < a\big\} ,$$

if $\lambda < 0$. These domains are biholomorphic to $\mathbb{C}^* \times \mathbb{B}^n(a, b)$, where

$$\mathbb{B}^n(a, b) = \big\{(z_1, \dots, z_n) \in \mathbb{C}^n : b < |z_1|^2 + \cdots + |z_n|^2 < a\big\} .$$

Then $\mathrm{Aut}(\mathbb{C}^* \times \mathbb{B}^n(a, b))$ does not have a Lie group structure by Lemma 1.3, and this contradicts the assumption that $\mathrm{Aut}(\Omega) = GU(n, 1)$. Thus this case does not occur.

Case (II-ii) : $\partial\Omega \neq A_{a,\lambda} \cup B_{b,\lambda}$.

Suppose $(\partial\Omega \setminus (A_{a,\lambda} \cup B_{b,\lambda})) \cap (\mathbb{C}^* \times \mathbb{C}^n \setminus \{0_n\}) \neq \emptyset$, then we can take

$$p'' = (p''_0, \dots, p''_n) \in (\partial\Omega \setminus (A_{a,\lambda} \cup B_{b,\lambda})) \cap (\mathbb{C}^* \times (\mathbb{C}^n \setminus \{0_n\})) .$$

Then put

$$\begin{aligned} c &= \big(|p''_1|^2 + \cdots + |p''_n|^2\big)/|p''_0|^{2\lambda} , \\ C_{c,\lambda} &= \big\{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : -c|z_0|^{2\lambda} + |z_1|^2 + \cdots + |z_n|^2 = 0\big\} . \end{aligned}$$

We have $A_{a,\lambda} \cup B_{b,\lambda} \cup C_{c,\lambda} \subset \partial\Omega$. However $\Omega$ is connected. Thus this is impossible, and therefore this case does not occur. Let us consider the remaining case:

$$(\partial\Omega \setminus (A_{a,\lambda} \cup B_{b,\lambda})) \cap (\mathbb{C}^* \times (\mathbb{C}^n \setminus \{0_n\})) = \emptyset .$$

However, $\mathbb{C}^* \times \{0_n\}$, $\{0\} \times (\mathbb{C}^n \setminus \{0_n\})$ and $\{0\} \in \mathbb{C}^{n+1}$ can not be subsets of the boundary of $\Omega$ since $\Omega \subset C^+_{a,\lambda} \cap D^+_{b,\lambda}$ or $\Omega \subset C^-_{a,\lambda} \cap D^-_{b,\lambda}$. Thus this case does not occur either.

We have shown that $\partial\Omega = A_{a,1}$ and $\Omega = D^+_{a,1}$ which is biholomorphic to $D^{n,1}$. □

## 6. A counterexample of the group-theoretic characterization.

THEOREM 6.1. *There exist unbounded homogeneous domains in $\mathbb{C}^n, n \geq 5$ which are not biholomorphically equivalent, while their automorphism groups are isomorphic as topological groups.*

PROOF. Suppose $p, q > 1$ and $p \neq q$. We put $n = p + q$. Let

$$D^{p,q} = \left\{(z_1, \ldots, z_p, w_1, \ldots, w_q) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_p|^2 - |w_1|^2 - \cdots - |w_q|^2 > 0\right\},$$
$$C^{p,q} = \left\{(z_1, \ldots, z_p, w_1, \ldots, w_q) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_p|^2 - |w_1|^2 - \cdots - |w_q|^2 < 0\right\}.$$

It is easy to see that $D^{p,q}$ and $C^{p,q}$ are homogeneous since both $\mathrm{Aut}(D^{p,q})$ and $\mathrm{Aut}(C^{p,q})$ contain a subgroup $GU(p, q)$, where

$$GU(p, q) = \left\{A \in GL(n, \mathbb{C}) : A^* J_{p,q} A = \nu(A) J_{p,q}, \text{for some } \nu(A) \in \mathbb{R}_{>0}\right\},$$

and $J_{p,q} = \mathrm{diag}[E_p, -E_q]$. Since $p \neq q$, $D^{p,q}$ and $C^{p,q}$ are not biholomorphically equivalent. Indeed, $D^{p,q}$ is homeomorphic to the product manifold $\mathbb{R}^{2q+1} \times S^{2p-1}$, but $C^{p,q}$ is homeomorphic to the product manifold $\mathbb{R}^{2p+1} \times S^{2q-1}$.

We will show that $\mathrm{Aut}(D^{p,q})$ is isomorphic to $\mathrm{Aut}(C^{p,q})$ as topological groups. As the proof of Theorem 3.1, we take $f = (f_1, \ldots, f_n) \in \mathrm{Aut}(D^{p,q})$. If $(w'_1, \ldots, w'_q) \in \mathbb{C}^q$ is fixed, then the holomorphic functions $f_i(\ldots, w'_1, \ldots, w'_q)$, for $i = 1, \ldots, n$, on $D^{p,q} \cap \{w_1 = w'_1, \ldots, w_q = w'_q\}$ extend continuously to the holomorphic functions on $\mathbb{C}^n \cap \{w_1 = w'_1, \ldots, w_q = w'_q\}$ by Hartogs' theorem, since $p > 1$. Hence, when $w_1, \ldots, w_q$ vary, we obtain an extended holomorphic map $\tilde{f} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that $\tilde{f}|_{D^{p,q}} = f \in \mathrm{Aut}(D^{p,q})$. The same consideration for $f^{-1} \in \mathrm{Aut}(D^{p,q})$ shows that there exists a holomorphic map $g : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}$ such that $g|_{D^{p,q}} = f^{-1}$. Since $g \circ \tilde{f} = \mathrm{id}$ and $\tilde{f} \circ g = \mathrm{id}$ on $D^{p,q}$, the uniqueness of analytic continuation shows that $g \circ \tilde{f} = \mathrm{id}$ and $\tilde{f} \circ g = \mathrm{id}$ on $\mathbb{C}^n$. Hence $\tilde{f} \in \mathrm{Aut}(\mathbb{C}^n)$. Now we see that $\tilde{f}|_{C^{p,q}} \in \mathrm{Aut}(C^{p,q})$ and therefore we have a group homomorphism

$$\phi : \mathrm{Aut}(D^{p,q}) \longrightarrow \mathrm{Aut}(C^{p,q}), \quad f \longmapsto \tilde{f}|_{C^{p,q}} .$$

In the same manner, we have

$$\psi : \mathrm{Aut}(C^{p,q}) \longrightarrow \mathrm{Aut}(D^{p,q}), \quad h \longmapsto \tilde{h}|_{C^{p,q}} ,$$

by Hartogs' theorem since $q > 1$. It is clear that $\phi \circ \psi = \mathrm{id}$ on $\mathrm{Aut}(C^{p,q})$ and $\psi \circ \phi = \mathrm{id}$ on $\mathrm{Aut}(D^{p,q})$. Thus we obtain $\mathrm{Aut}(D^{p,q}) \simeq \mathrm{Aut}(C^{p,q})$ as groups.

We will show that $\phi$ is continuous. Take a sequence $\{f^{(n)}\}_{n=1}^{\infty} \in \mathrm{Aut}(D^{p,q})$, which converges to $f \in \mathrm{Aut}(D^{p,q})$ uniformly on any compact subset in $D^{p,q}$. Then $\phi(f^{(n)}) = \widetilde{f^{(n)}}|_{C^{p,q}}$ and $\phi(f) = \widetilde{f}|_{C^{p,q}}$ as the above notation. Let $K$ be any compact subset in $C^{p,q}$. We can take $p$ and $q$-dimensional balls $\Delta^p \subset \mathbb{C}^p$ and $\Delta^q \subset \mathbb{C}^q$ centered at the origins in $\mathbb{C}^p$ and $\mathbb{C}^q$, respectively, such that $\Delta^p \times \Delta^q$ contains $K$ and $\partial\Delta^p \times \partial\Delta^q$ is included in $D^{p,q}$. By the maximal principle, we have

$$\begin{aligned}\sup_K |\widetilde{f_i^{(n)}}(z) - \tilde{f}_i(z)| &\le \sup_{\partial\Delta^p \times \partial\Delta^q} |\widetilde{f_i^{(n)}}(z) - \tilde{f}_i(z)| \\ &= \sup_{\partial\Delta^p \times \partial\Delta^q} |f_i^{(n)}(z) - f_i(z)|\end{aligned}$$

for $i = 1, \dots, n$. Since $\partial\Delta^p \times \partial\Delta^q$ is a compact subset in $D^{p,q}$, the right-hand side above converges to 0, and $\widetilde{f^{(n)}}$ converges to $\tilde{f}$ on the compact set $K$. We have shown that $\phi(f^{(n)})$ converges to $\phi(f)$ on any compact subset in $C^{p,q}$. In the same manner, we can prove that $\psi$ is continuos. Thus we obtain $\mathrm{Aut}(D^{p,q}) \simeq \mathrm{Aut}(C^{p,q})$ as topological groups. The proof is complete. □

We remark on the automorphism groups of $D^{p,q}$ and the characterization problem by automorphism groups. We have not yet obtained an explicit description of the automorphism groups $\mathrm{Aut}(D^{p,q})$ for $p, q > 1$. We only expect that $\mathrm{Aut}(D^{p,q}) = GU(p,q)$.

The difference between $D^{n,1}$ and $D^{p,q}$ for $p, q > 1$ is that the exterior of $D^{n,1}$ is holomorphically convex domain, but that of $D^{p,q}$ is not. It is known that some holomorphically convex homogeneous Reinhardt domains are characterized by their automorphism groups with some additional conditions (see [1] and [4]). We may study the group-theoretic characterization problem for holomorphically convex homogeneous Reinhardt domains, or for homogeneous Reinhardt domains with holomorphically convex exterior domains.

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