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A REMARK ON JACQUET–LANGLANDS CORRESPONDENCE AND INVARIANT s

KAZUTOSHI KARIYAMA

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Abstract. Let F be a non-Archimedean local field, and let G be an inner form of $\mathrm{GL}_N(F)$ with $N \geq 1$. Let **JL** be the Jacquet–Langlands correspondence between $\mathrm{GL}_N(F)$ and G. In this paper, we compute the invariant s associated with the essentially square-integrable representation $\mathbf{JL}^{-1}(\rho)$ for a cuspidal representation ρ of G by using the recent results of Bushnell and Henniart, and we restate the second part of a theorem given by Deligne, Kazhdan, and Vignéras in terms of the invariant s. Moreover, by using the parametric degree, we present a proof of the first part of the theorem.

Introduction. Let F be a non-Archimedean local field, and let D be a central division F-algebra of dimension d^2 with $d \geq 1$. We fix positive integers N, m with $N = md$ and denote by G the group $\mathrm{GL}_m(D)$. For an element x of $F^\times$, we denote by $|x|_F$ the normalized absolute value of x.

The Jacquet–Langlands correspondence, denoted by **JL**, is a canonical bijection between the isomorphism classes of essentially square-integrable representations of $\mathrm{GL}_N(F)$ and G. The existence of **JL** was proved by Deligne, Kazhdan, and Vignéras [7] and Rogawski [9] for F of characteristic zero, and by Badulescu [1] for F of positive characteristic (see Theorem 2.7 for the definition of **JL**). In [7, Théorèm 2.B.b], the correspondence **JL** is described by using an invariant s. In fact, for a cuspidal representation ρ of G, the invariant $s = s(\rho)$ is defined as a positive integer k uniquely determined by the essentially square-integrable representation $\mathbf{JL}^{-1}(\rho)$ of $\mathrm{GL}_N(F)$.

In the present paper, we define the invariant $s(\pi)$ for an essentially square-integrable representation π of G. It is proved by Sécherre and Stevens [12], [13] that the representation π contains a *simple type* (J, λ), in the sense of [12], consisting of a compact open subgroup J of G and its irreducible smooth representation λ. The simple type is associated with a *simple stratum* $[\mathfrak{A}, n, 0, \beta]$, defined in [5], [10], consisting of a principal hereditary order $\mathfrak{A}$ of $A = \mathrm{M}_m(D)$, a positive integer n, and an element $\beta \in A$ that generates a subfield $E = F[\beta]$. The invariant is defined by

$$s(\pi) = d'/\ell\,,$$

where d' and ℓ are the positive integers determined by the simple type (J, λ). It turns out that $s(\pi)$ is a positive integer that does not depend on the choice of the simple type (J, λ)

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and depends only on the isomorphism class of π. Let B be the A-centralizer of β. Then, the invariant $s(\pi)$ is closely related to the *parametric degree* $\delta(\pi)$, introduced by Bushnell and Henniart [3], [4] as

$$s(\pi)\delta(\pi) = N/r\,,$$

where r is the period of the order $\mathfrak{B} = B \cap \mathfrak{A}$ for a simple type (J, λ) in G contained in π. Thus, by [13], π is cuspidal if and only if

$$s(\pi)\delta(\pi) = N$$

is satisfied. In particular, if $G = \mathrm{GL}_N(F)$, then we have $s(\pi) = 1$, so that π is cuspidal if and only if $\delta(\pi) = N$ is satisfied. This fact was obtained in [3]. By using these equalities, we obtain the following result, which is the main theorem of this paper.

THEOREM 0.1. *Let π be an essentially square-integrable representation of $\mathrm{GL}_N(F)$, and assume that $\pi' = \mathbf{JL}(\pi)$ is a cuspidal representation of G. Then, there exists a cuspidal representation ρ of $\mathrm{GL}_{N/s}(F)$, for $s = s(\pi')$ determined above, such that π is equivalent to a subquotient of the parabolically induced representation*

$$I_{\mathrm{GL}_{N/s}(F)^s}^{\mathrm{GL}_N(F)}(\rho \otimes \rho\nu \otimes \cdots \otimes \rho\nu^{s-1})\,,$$

where $\nu(g) = |\det(g)|_F$ for $g \in \mathrm{GL}_{N/s}(F)$.

The theorem implies that the invariant $s = s(\pi')$ is equal to the integer k associated with the essentially square-integrable representation $\pi = \mathbf{JL}^{-1}(\pi')$.

Consequently, we can restate the assertion of [7, Théorème B.2.b(2)] as a generalization of Theorem 0.1 as follows.

THEOREM 0.2. *Let π be an essentially square-integrable representation of $\mathrm{GL}_N(F)$, and let $\pi' = \mathbf{JL}(\pi)$. Then, there exist a positive integer r dividing m, a cuspidal representation ρ' of $\mathrm{GL}_{m/r}(D)$ and a cuspidal representation ρ of $\mathrm{GL}_{N/rs}(F)$ for $s = s(\rho')$, such that*

1. *π' is equivalent to a subquotient of the parabolically induced representation*

$$I_{\mathrm{GL}_{m/r}(D)^r}^{\mathrm{GL}_m(D)}(\rho' \otimes \rho'\nu_{\rho'} \otimes \cdots \otimes \rho'\nu_{\rho'}^{r-1})\,,$$

where $\nu_{\rho'}(g) = |\mathrm{Nrd}(g)|_F^s$ for $g \in \mathrm{GL}_{m/r}(D)$ and Nrd denotes the reduced norm map $\mathrm{GL}_{m/r}(D) \to F^\times$;

2. *π is equivalent to a subquotient of the parabolically induced representation*

$$I_{\mathrm{GL}_{N/rs}(F)^{rs}}^{\mathrm{GL}_N(F)}(\rho \otimes \rho\nu \otimes \cdots \otimes \rho\nu^{rs-1})\,,$$

where $\nu(g) = |\det(g)|_F$ for $g \in \mathrm{GL}_{N/rs}(F)$.

The remainder of the present paper is organized as follows. In Section 1, we recall the definition of simple type, as given in [10], [11], [12]. In Section 2, we prove Theorem 0.1. Moreover, by using the parametric degree, we present a proof of [7, Théorème B.2.b(1)] for the base field F of arbitrary characteristic.

1. Simple types. Hereafter, a *representation* of a totally disconnected, locally compact group means a smooth complex representation.

In this section, we recall the results of Sécherre [10], [11], [12].

Let F be a non-Archimedean local field, and let D be a central division F-algebra of dimension d^2, $d \geq 1$. Set $A = \mathrm{M}_m(D), m \geq 1$. Then, A is a simple central F-algebra of dimension N^2 with $N = md$. Set $G = A^\times$. For a finite field extension K/F, we denote by $\mathfrak{o}_K$ its ring of integers, by $\mathfrak{p}_K$ the maximal ideal of $\mathfrak{o}_K$, and by k_K the residue field of K.

Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in A, and let $\mathfrak{P}$ be the Jacobson radical of $\mathfrak{A}$. An integer e, also denoted by $e = e(\mathfrak{A}|\mathfrak{o}_D)$, is referred to as the $\mathfrak{o}_D$*-period* of $\mathfrak{A}$ if $\mathfrak{p}_D\mathfrak{A} = \mathfrak{P}^e$ is satisfied. Then, we define the compact open subgroups of G by

$$U(\mathfrak{A}) = U^0(\mathfrak{A}) = \mathfrak{A}^\times,\ U^k(\mathfrak{A}) = 1 + \mathfrak{P}^k,\ k \geq 1\,,$$

and write the G-normalizer of $\mathfrak{A}$ as $\mathfrak{K}(\mathfrak{A})$. The latter is an open, compact-mod-center subgroup of G. There exists a canonical homomorphism $\nu_{\mathfrak{A}} : \mathfrak{K}(\mathfrak{A}) \to \mathbb{Z}$ defined by $g\mathfrak{A} = \mathfrak{A}g = \mathfrak{P}^{\nu_{\mathfrak{A}}(g)},\ g \in \mathfrak{K}(\mathfrak{A})$. A hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in A is referred to as *principal* if there exists an element $x \in \mathfrak{K}(\mathfrak{A})$ such that $\mathfrak{P} = x\mathfrak{A} = \mathfrak{A}x$.

DEFINITION 1.1. A *stratum* in A is a 4-tuple $[\mathfrak{A}, n, m, \beta]$ consisting of a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$, two integers m, n such that $0 \leq m < n$ and an element $\beta \in \mathfrak{P}^{-n}$.

Let $[\mathfrak{A}, n, m, \beta]$ be a stratum in A and denote by E the F-subalgebra $F[\beta]$ of A generated by β. This stratum is referred to as *pure* if E is a field, $\mathfrak{A}$ is E-pure, that is, $E^\times \subset \mathfrak{K}(\mathfrak{A})$, and $\nu_{\mathfrak{A}}(\beta) = -n$.

Let $[\mathfrak{A}, n, m, \beta]$ be a pure stratum, let $E = F[\beta]$ and let B be the A-centralizer of β. Write $B = C_A(E)$. For each $k \in \mathbb{Z}$, we set $\mathfrak{n}_k(\beta, \mathfrak{A}) = \{x \in \mathfrak{A} : \beta x - x\beta \in \mathfrak{P}^k\}$. Set

$$k_0(\beta, \mathfrak{A}) = \min\{k \in \mathbb{Z} : k \geq \nu_{\mathfrak{A}}(\beta),\ \mathfrak{n}_{k+1}(\beta, \mathfrak{A}) \subset \mathfrak{A} \cap B + \mathfrak{P}\}\,.$$

DEFINITION 1.2. A stratum $[\mathfrak{A}, n, m, \beta]$ in A is referred to as *simple* if it is pure and $m \leq -k_0(\beta, \mathfrak{A}) - 1$.

Hereafter, we assume that $[\mathfrak{A}, n, 0, \beta]$ is a simple stratum in A. Then, the stratum $[\mathfrak{A}, n, 0, \beta]$ gives rise to a pair

$$\mathfrak{H}(\beta, \mathfrak{A}) \subset \mathfrak{J}(\beta, \mathfrak{A})$$

of $\mathfrak{o}_F$-orders in A. We have the standard filtration subgroups of unit groups

$$H^k(\beta, \mathfrak{A}) = \mathfrak{H}(\beta, \mathfrak{A}) \cap U^k(\mathfrak{A})\,,$$
$$J^k(\beta, \mathfrak{A}) = \mathfrak{J}(\beta, \mathfrak{A}) \cap U^k(\mathfrak{A})\,,$$

for $k \in \mathbb{Z}, k \geq 0$. In particular, we write $J = J(\beta, \mathfrak{A}) = J^0(\beta, \mathfrak{A})$.

DEFINITION 1.3 ([8, §0.6], [11, §2.5.1]). A simple type of *level zero* in G is a pair (U, τ) satisfying

1. $U = U(\mathfrak{A})$ for a principal hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ of A with $r = e(\mathfrak{A}|\mathfrak{o}_D)$;

2. τ is an irreducible representation of $U = U(\mathfrak{A})$, trivial on $U^1(\mathfrak{A})$ and inflated from a representation $\sigma_0^{\otimes r}$ of the quotient group $U(\mathfrak{A})/U^1(\mathfrak{A})$ that is isomorphic to $\mathrm{GL}_s(k_D)^r$ with $rs = m$, where σ_0 is a cuspidal representation of $\mathrm{GL}_s(k_D)$. Hereinafter, we write $\tau = \sigma_0^{\otimes r}$.

We refer to a simple type $(U(\mathfrak{A}), \tau)$ of level zero in G as *associated with* the null simple stratum $[\mathfrak{A}, 0, 0, 0]$ in A (cf. [11, Remark 4.1]).

A finite set $\mathcal{C}(\mathfrak{A}, 0, \beta)$ of simple characters of the group $H^1(\beta, \mathfrak{A})$ was defined in [10, §3.3] (cf. [13, §2]).

Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum in A, let $E = F[\beta]$ and let $B = C_A(E)$. Then, we have $B \simeq \mathrm{M}_{m'}(D')$, where D' is a central division algebra of dimension d'^2 over E.

PROPOSITION 1.4 ([11, §2.2]). *Let* $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$.

1. *There exists a unique irreducible representation* η_θ *of* $J^1(\beta, \mathfrak{A})$ *such that* $\eta_\theta | H^1(\beta, \mathfrak{A})$ *is equal to* θ.
2. *There exists an irreducible representation* κ *of* $J = J^0(\beta, \mathfrak{A})$ *such that*
   (a) $\kappa | J^1 \simeq \eta_\theta$,;
   (b) κ *is intertwined by every element of* $B^\times$.

Following [4, §2.5], we refer to a representation κ of J as in Proposition 1.4 (2) as a *wide extension* of η_θ.

DEFINITION 1.5. A simple type of *positive level* in G is a pair (J, λ), given as follows:

1. there exists a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in A such that $J = J^0(\beta, \mathfrak{A})$ and that if $E = F[\beta]$, $B = C_A(E) \simeq \mathrm{M}_{m'}(D')$ and $\mathfrak{B} = \mathfrak{A} \cap B$, then $\mathfrak{B}$ is an $\mathfrak{o}_E$-order in B with $r = e(\mathfrak{B}|\mathfrak{o}_{D'})$;
2. there exist a simple character $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ and a simple type $(U(\mathfrak{B}), \tau)$ of level zero in $B^\times$ such that λ is a representation of J of the form

$$\lambda = \kappa \otimes \tau\,,$$

   where
   (a) κ is a wide extension of η_θ;
   (b) $\tau = \sigma_0^{\otimes r}$ is regarded as the inflation of a representation of $J/J^1 \simeq U(\mathfrak{B})/U^1(\mathfrak{B}) \simeq \mathrm{GL}_s(k_{D'})^r$ as in Definition 1.3.

Denote by Z the center of the group $G = \mathrm{GL}_m(D)$. A representation π of G is referred to as *cuspidal* if π is irreducible and has a nonzero coefficient that is compactly supported modulo Z. A representation π of G is referred to as *essentially square-integrable* if π is irreducible and there exists a character χ of G such that $\chi \otimes \pi$ is unitary and has a nonzero coefficient which is square-integrable over G/Z.

THEOREM 1.6 ([13, Corollaire 5.20]). *Let* π *be an irreducible representation of* G *that contains a simple type* (J, λ) *in* G *associated with a simple stratum* $[\mathfrak{A}, n, 0, \beta]$ *in* A. *Set* $E = F[\beta]$, $B = C_A(\beta) \simeq \mathrm{M}_{m'}(D')$ *and* $\mathfrak{B} = \mathfrak{A} \cap B$. *Then,* π *is cuspidal if and only if* $\mathfrak{B}$ *is a maximal order in* B*, that is,* $e(\mathfrak{B}|\mathfrak{o}_{D'}) = 1$.

2. Jacquet–Langlands correspondence and invariant s. We first recall the definition of the parametric degree of an essentially square-integrable representation π of $G = \mathrm{GL}_m(D)$.

PROPOSITION 2.1. *Let π be an essentially square-integrable representation of $G = \mathrm{GL}_m(D)$. Then, there exist a positive integer r dividing m and a cuspidal representation ρ' of $G_0 = \mathrm{GL}_{m/r}(D)$ such that the cuspidal support of π consists of unramified twists of ρ'. The integer r is uniquely determined by the representation π.*

PROOF. The first assertion follows directly from [4, A.1.1, Proposition], and the second one follows from [5, (7.3.11)]. In fact, it is proved by [6, (6.3.7), (6.3.11)]. □

In the situation of Proposition 2.1, let M be a Levi subgroup of G that is isomorphic to $G_0^r = G_0 \times \cdots \times G_0$. Then, the inertial (equivalence) class of π is represented by the cuspidal pair $(M, (\rho')^{\otimes r})$. We write $[M, (\rho')^{\otimes r}]_G$ for the inertial class.

COROLLARY 2.2. *Let π be an essentially square-integrable representation of G that has the inertial class $[M, (\rho')^{\otimes r}]_G$, and let (J, λ) be a simple type in G contained in π. Then, there exists a maximal simple type (J_0, λ_0), with $\lambda_0 = \kappa_0 \otimes \sigma_0$, in $G_0 = \mathrm{GL}_{m/r}(D)$ contained in ρ' such that $\lambda = \kappa \otimes \sigma_0^{\otimes r}$ for some wide extension κ of J.*

PROOF. This follows from [13, Theorem 5.23] (cf. [4, (A1.3.1)]). □

Assume that π is an essentially square-integrable representation of G that has the inertial class $[M, (\rho')^{\otimes r}]_G$ and contains a simple type (J, λ) in G associated with a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $A = \mathrm{M}_m(D)$. Then, from Corollary 2.2, we have $\lambda = \kappa \otimes \sigma_0^{\otimes r}$. Set $E = F[\beta]$, $B = C_A(E)$ and $\mathfrak{B} = B \cap \mathfrak{A}$. Then, we have $B \simeq \mathrm{M}_{m'}(D')$, where D' is a central division algebra of dimension d'^2 over E, and we have $J/J^1 \simeq U(\mathfrak{B})/U^1(\mathfrak{B}) \simeq \mathrm{GL}_s(k_{D'})^r$. Let ℓ be the number of $\mathrm{Gal}(k_{D'}/k_E)$-orbits of the representation σ_0 of $\mathrm{GL}_s(k_{D'})$ (cf. Definition 1.5).

DEFINITION 2.3. Let the notation and assumptions be as above. The *parametric degree*, denoted by $\delta(\pi)$, of the representation π is defined by

$$\delta(\pi) = s\ell[E : F].$$

From Corollary 2.2, the parametric degree $\delta(\pi)$ in the definition coincides with that defined in [4, 2.6, 2.8], that is,

$$\delta(\pi) = \delta(\rho') = \delta_0(\lambda_0),$$

where $\lambda_0 = \kappa_0 \otimes \sigma_0$ is as in Corollary 2.2. Thus, by [4, 2.7, Proposition], the parametric degree $\delta(\pi)$ does not depend on the choice of the simple type (J, λ) in G contained in π.

The parametric degree can be expressed in another form as follows.

PROPOSITION 2.4. *Let π be an essentially square-integrable representation of G that contains a simple type (J, λ) in G with $\lambda = \kappa \otimes \sigma_0^{\otimes r}$, as above. Then, we have*

$$\delta(\pi) = N\ell/rd'.$$

PROOF. This follows immediately from the equalities $rsd' = m'd' = N/[E : F]$. $\square$

We define another invariant for such a representation π of G.

DEFINITION 2.5. In the situation of Proposition 2.4, we define the quantity $s(\pi)$ by

$$s(\pi) = d'/\ell .$$

By the definition of the positive integer ℓ, $s(\pi)$ is a positive integer that divides d' and so d, because $d' = d/\gcd(d, [E : F])$ by [16, Proposition 1]. From Proposition 2.4, we obtain

$$s(\pi)\delta(\pi) = N/r . \tag{1}$$

The integer r and the parametric degree $\delta(\pi)$ do not depend on the choice of the simple type (J, λ) in G contained in π as was seen above. Thus, from Eq. (1), $s(\pi)$ is well defined.

PROPOSITION 2.6. *Let π be an essentially square-integrable representation of G. Then, π is cuspidal if and only if*

$$s(\pi)\delta(\pi) = N .$$

In particular, if G is equal to $\mathrm{GL}_N(F)$, then π is cuspidal if and only if $\delta(\pi) = N$.

PROOF. By Theorem 1.6, the first assertion follows immediately from Eq. (1). If $G = \mathrm{GL}_N(F)$, then we have $s(\pi) = d'/\ell = 1$ and so $\delta(\pi) = N$. $\square$

The last assertion in Proposition 2.6 is already obtained in [3]. We denote by $\mathcal{A}^{(2)}(G)$ the set of isomorphism classes of essentially square-integrable representations of G. In particular, write $H = \mathrm{GL}_N(F)$ with $N = md$.

THEOREM 2.7 ([7], [9], [1]). *There exists a unique bijection*

$$\mathbf{JL} : \mathcal{A}^{(2)}(H) \to \mathcal{A}^{(2)}(G)$$

such that, for $\pi \in \mathcal{A}^{(2)}(H)$, we have

$$\operatorname{tr} \pi(g) = (-1)^{N-m} \operatorname{tr} \mathbf{JL}(\pi)(g') ,$$

where $g \in H$ and $g' \in G$ are elliptic regular elements that have the same characteristic polynomial over F.

We refer to the map **JL** as the *Jacquet–Langlands correspondence* between H and G. By using Proposition 2.6, we can give a condition for $\mathbf{JL}(\pi)$ to be cuspidal, which is different from that of [7, Théorème B.2.b(1)], as follows.

THEOREM 2.8. *Let $\pi \in \mathcal{A}^{(2)}(H)$, and set $\pi' = \mathbf{JL}(\pi) \in \mathcal{A}^{(2)}(G)$. Assume that π contains a simple type (J, λ) in H associated with a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $A = \mathrm{M}_N(F)$. Set $E = F[\beta]$, $B = C_A(E)$ and $\mathfrak{B} = \mathfrak{A} \cap B$. Then, π' is cuspidal if and only if*

$$s(\pi') = e(\mathfrak{B}|\mathfrak{o}_E) .$$

PROOF. Assume that π' is cuspidal. Then, from Proposition 2.6, we obtain $s(\pi')\delta(\pi') = N$. Since **JL** preserves the parametric degree by [4, §2.8, Corollary 1], we thus obtain

$$\delta(\pi) = \delta(\mathbf{JL}(\pi)) = \delta(\pi') = N/s(\pi')\,.$$

While, since $s(\pi) = 1$ is satisfied for $H = \mathrm{GL}_N(F)$ as in the proof of Proposition 2.6, we have

$$\delta(\pi) = N/r\,,$$

where $r = e(\mathfrak{B}|\mathfrak{o}_E)$. Hence, we obtain

$$s(\pi') = r = e(\mathfrak{B}|\mathfrak{o}_E)\,.$$

Conversely, if $s(\pi') = e(\mathfrak{B}|\mathfrak{o}_E)$ is satisfied, we obtain similarly

$$N/s(\pi') = N/r = \delta(\pi) = \delta(\pi')\,,$$

and, again from Proposition 2.6, π' is cuspidal. □

In view of the result of [15], Theorem 0.1 follows from Theorem 2.8. The proof of Theorem 0.1 is complete.

A proof of [7, Théorème B.2.b(1)] for the base field F of arbitrary characteristic was given by Lemma 2.4 and comments after the proof in [2]. However, by using the results of [4], we give an alternate proof of the theorem.

PROPOSITION 2.9 ([7, Théorème B.2.b(1)]). *Let $\pi \in \mathcal{A}^{(2)}(H)$, and set $\pi' = \mathbf{JL}(\pi) \in \mathcal{A}^{(2)}(G)$. Assume that the representation π has a cuspidal support $\{\rho, \rho\nu, \ldots, \rho\nu^{k-1}\}$ for some positive integer k. Then, π' is cuspidal if and only if $N = \mathrm{lcm}(d, N/k)$.*

PROOF. Let (J, λ) be a simple type in G contained in π' that is associated with a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $A = \mathrm{M}_m(D)$. Set $E = F[\beta]$, $B = C_A(E)$ and $\mathfrak{B} = \mathfrak{A} \cap B$. Then, we have $B \simeq \mathrm{M}_{m'}(D')$, for a central division E-algebra D' of dimension d'^2, as before. Assume that π' is cuspidal. Then, from Theorem 2.8, we have $k = s(\pi')$. We first prove

$$\gcd(m, s(\pi')) = 1\,. \tag{2}$$

From [16, Proposition 1], we obtain

$$m' = \gcd(m, N/[E : F]) = \gcd(m, m'd')\,,$$

which implies that m/m' is an integer and $\gcd(m/m', d') = 1$. Since the invariant $s(\pi')$ divides d', we thus obtain

$$\gcd(m/m', s(\pi')) = 1\,,$$

and so

$$\gcd(m, s(\pi')) = \gcd(m'(m/m'), s(\pi')) = \gcd(m', s(\pi'))\,.$$

Hence, for Eq. (2), it is enough to show that $\gcd(m', s(\pi')) = 1$. By the assumption, (J, λ) is the maximal simple type in G with $\lambda = \kappa \otimes \sigma$. Let ρ' be a cuspidal representation of $\mathrm{GL}_{m'}(D')$ that contains the maximal simple type $(U(\mathfrak{B}), \sigma)$. Then, we have

$$\delta(\rho') = m'\ell\,,$$

where ℓ is the number of $\mathrm{Gal}(k_{D'}/k_E)$-orbits of the representation σ of $U(\mathfrak{B})/U^1(\mathfrak{B}) \simeq \mathrm{GL}_{m'}(k_{D'})$. Thus, applying [4, 2.4, Remark 2] to the representation ρ', we obtain

$$\gcd(N/[E:F]\delta(\rho'), m') = 1\,.$$

By assumption, we have $r = e(\mathfrak{B}|\mathfrak{o}_{D'}) = 1$. Since we have $\delta(\pi') = m'\ell[E:F] = \delta(\rho')[E:F]$ by definition, we thus obtain

$$1 = \gcd(N/[E:F]\delta(\rho'), m') = \gcd(N/\delta(\pi'), m') = \gcd(s(\pi'), m')$$

by Eq. (1). Hence, Eq. (2) holds. Write $k = s(\pi')$ as above. Then, we obtain $km = \mathrm{lcm}(k, m)$. Thus, we obtain

$$\begin{aligned} N = md = (d/k)(km) &= (d/k)\mathrm{lcm}(k, m) \\ &= \mathrm{lcm}(k(d/k), m(d/k)) = \mathrm{lcm}(d, N/k)\,, \end{aligned}$$

which proves the "only if" part of the proposition.

Conversely, assume that $N = \mathrm{lcm}(d, N/k)$. Then, from $N = md$, we obtain $k|d$ and $\gcd(m, k) = 1$. Again from Eq. (1), we obtain

$$N/k = \delta(\pi) = \delta(\pi') = N/rs(\pi')\,,$$

as in the proof of Theorem 2.8. Hence, we have

$$\gcd(m, rs(\pi')) = 1\,.$$

Since r divides m, we obtain

$$1 = \gcd(m, rs(\pi')) = r\gcd(m/r, s(\pi'))\,,$$

which implies that $r = e(\mathfrak{B}|\mathfrak{o}_{D'}) = 1$. Hence, by Theorem 1.6, π' is cuspidal. □

By Proposition 2.9, we obtain the following result.

COROLLARY 2.10 (cf. [14, Sec. 2]). *Let the notation and assumptions be as in Theorem* 0.2. *Then, the invariant $s(\rho')$ satisfies the following conditions:*

1. *$s(\rho')$ divides d;*
2. $\gcd(m/r, s(\rho')) = 1$.

PROOF. Since ρ' is a cuspidal representation of $\mathrm{GL}_{m/r}(D)$, $\mathbf{JL}^{-1}(\rho')$ is an essentially square-integrable representation of $\mathrm{GL}_{N/r}(F)$. Thus, by replacing m, N and k by m/r, N/r and $s(\rho')$, respectively, by Proposition 2.9, we obtain

$$N/r = \mathrm{lcm}(d, N/rs(\rho'))\,,$$

which is written by $r = \mathrm{lcm}(d, n/k)$ in [7, Théorème 2.B.b(2)]. Thus, the corollary is proved similarly as Proposition 2.9. □

REFERENCES

[1] A. BADULESCU, Correspondence de Jacquet–Langlands pour les corps locaux de caractéristique non nulle, Ann. Sci. École Norm. Sup. 35 (2002), 695–747.

[2] A. BADULESCU, Jacquet–Langlands et unitarisabilité, J. Inst. Math. Jussieu 6 (2007), 349–379.

[3] C. J. BUSHNELL AND G. HENNIART, Local Jacquet–Langlands correspondence and parametric degrees, Manuscripta Math. 114 (2004), 1–7.

[4] C. J. BUSHNELL AND G. HENNIART, The essentially tame Jacquet–Langlands correspondence for inner forms of $\mathrm{GL}(n)$, Pure and Appl. Math. Q. 7 (2011), no. 3, 469–538.

[5] C. J. BUSHNELL AND P. C. KUTZKO, The admissible dual of $\mathrm{GL}(N)$ via compact open subgroups, Ann. Math. Stud. 129, Princeton University Press, 1993.

[6] W. CASSELMAN, Introduction of the theory of admmisible representations of p-adic reductive groups, Unpublished manuscript, 1974.

[7] P. DELIGNE, D. KAZHDAN AND M.-F. VIGNÉRAS, Représentations des algèbres centrales simples p-adiques, in Représentations des groups réductifs sur un corps local, 33–117, Travaux en Cours, Hermann, Paris, 1984.

[8] G. GRABITZ, A. Z. SILBERGER AND E.-W. ZINK, Level zero types and Hecke algebras for local central simple algebras, J. Number Theory 91 (2001), 92–125.

[9] J. ROGAWSKI, Representations of $GL(n)$ and division algebras over a local field, Duke Math. J. 50 (1983), 161–196.

[10] V. SÉCHERRE, Représentations lisses de $\mathrm{GL}(m, D)$, I: caractères simples, Bull. Soc. Math. France 132 (2004), 327–396.

[11] V. SÉCHERRE, Représentations lisses de $\mathrm{GL}(m, D)$, II: β-extensions, Compositio Math. 141 (2005), 1531–1550.

[12] V. SÉCHERRE, Représentations lisses de $\mathrm{GL}(m, D)$, III: types simples, Ann. Sci. École Norm. Sup. 38 (2005), 951–977.

[13] V. SÉCHERRE AND S. STEVENS, Représentations lisse de $\mathrm{GL}(m, D)$, IV: Représentations supercuspidales, J. Inst. Math. Jussieu 7 (2008), 527–574.

[14] M. TADIĆ, Induced representations of $\mathrm{GL}(n, A)$ for p-adic division algebra A, J. Reine Angew. Math. 405 (1990), 48–77.

[15] A. ZELEVINSKY, Induced representations and reductive p-adic groups II, Ann. Sci. École Norm. Sup. 13 (1980), 165–210.

[16] E.-W. ZINK, More on embedding of local fields in simple algebras, J. Number Theory 77 (1989), 51–61.

ONOMICHI CITY UNIVERSITY
ONOMICHI 722–8506
JAPAN

E-mail address: kazutoshi.kariyama@gmail.com