Tohoku Math. J.
68 (2016), 591–605

CONSTRUCTION OF SIGN-CHANGING SOLUTIONS FOR A SUBCRITICAL PROBLEM ON THE FOUR DIMENSIONAL HALF SPHERE

RABEH GHOUDI AND KAMAL OULD BOUH

(Received September 22, 2014, revised March 5, 2015)

Abstract. This paper is devoted to studying the nonlinear problem with subcritical exponent (S_ε) : $-\Delta_g u + 2u = K|u|^{2-\varepsilon}u$, in S^4_+, $\partial u/\partial \nu = 0$, on ∂S^4_+, where g is the standard metric of S^4_+ and K is a C^3 positive Morse function on $\overline{S^4_+}$. We construct some sign-changing solutions which blow up at two different critical points of K in interior. Furthermore, we construct sign-changing solutions of (S_ε) having two bubbles and blowing up at the same critical point of K.

1. Introduction. We consider the problem of prescribing the scalar curvature under minimal boundary conditions on the standard four dimensional half sphere. More precisely, let K be a C^3 positive Morse function on $\overline{S^4_+}$, we look for conditions on K to ensure the existence of solution for the problem

$$(S) \quad \begin{cases} L_g u := -\Delta_g u + 2u = K u^3, \quad u > 0 & \text{in } S^4_+ \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial S^4_+ , \end{cases}$$

where g is the standard metric of $S^n_+ = \{x \in \mathbb{R}^{n+1}/|x| = 1, x_{n+1} > 0\}$.

It is well known that there are topological obstructions of Kazdan-Warner type to solve (S) (see [5]) and so a natural question arises: under which conditions on K, (S) has a solution.

Regarding problem (S), Ben Ayed et al [3] proved that we have a balance phenomenon, that is, the self interaction of the functions failing the Palais-Smale and the interaction of two of those functions are of the same size, if we assume that $(\partial K/\partial \nu)(y) < 0$ at any critical point y of $K_1 = K_{|\partial S^4_+}$. Moreover, it is proved that this phenomenon appears also when the manifold is the three dimensional half sphere (see [6]).

Note that the embedding of $H^1(S^4_+)$ into $L^4(S^4_+)$ is noncompact. Hence, for the study of problem (S), it is interesting to approach it by a family of subcritical problems (S_ε)

$$(S_\varepsilon) \quad \begin{cases} -\Delta_g u + 2u = K|u|^{2-\varepsilon}u, & \text{in } S^4_+ \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial S^4_+ , \end{cases}$$

and we need to study the asymptotic behavior of the solutions (u_ε) as $\varepsilon \to 0$. Observe that, since $\varepsilon > 0$, problem (S_ε) has always a positive solution (u_ε). It is well known that, for the minimizing solutions, (u_ε) has to converge to a solution of (S) or to blow up at one point where the maximum of K or of K_1 attains.

2010 *Mathematics Subject Classification.* Primary 35J20; Secondary 35J60.
Key words and phrases. Critical points, Variational problem, Scalar curvature, Bubble-tower solutions.

For the other solutions (u_ε) (not minimizing solutions) and in the case of the three dimensional half sphere, Djadli et al [6] proved that (u_ε) can blow up at $\{x_1, \dots, x_p\}$ such that the points x_i's are different critical points of K_1 with $(\partial K/\partial \nu)(x_i) > 0$. Furthermore they proved that the x_i's are isolated simple blow ups (see [8] for the definition), which implies that, writing $u_\varepsilon = \sum_{i\le p} \alpha_i \delta_{(a_i,\lambda_i)} + v_\varepsilon$, we have that $|a_i - a_j| \ge c > 0$ for $i \ne j$ (the function $\delta_{(a,\lambda)}$ is defined in (1.1)). Hence, the tower bubble solutions do not exist. Moreover, In [4] (see also [3]), we proved that there are critical points at infinity (following the terminology of A. Bahri) for the functional associated to the problem (S). This implies the existence of solutions (u_ε) which blow up at $\{y_1, \dots, y_p\}$ such that the points y_i's are different critical points of K in S^4_+.

In this paper, we aim to construct some sign-changing solutions (u_ε) of (S_ε) which blow up at one or two different points in the interior.

Before stating the result, we need to introduce some notations. For $a \in \overline{S^4_+}$ and $\lambda > 0$, let

$$\delta_{(a,\lambda)}(x) = c_0 \frac{\lambda}{(\lambda^2 + 1 + (1-\lambda^2)\cos d(a,x))} \tag{1.1}$$

where d is the geodesic distance on $(\overline{S^4_+}, g)$ and c_0 is chosen so that $\delta_{(a,\lambda)}$ is the family of solutions of the following problem

$$-\Delta u + 2u = u^3, \quad u > 0, \quad \text{in } S^4 .$$

We denote by $P\delta_{(a,\lambda)}$ the projection of the function $\delta_{(a,\lambda)}$ defined by

$$-\Delta P\delta_{(a,\lambda)} + 2P\delta_{(a,\lambda)} = -\Delta \delta_{(a,\lambda)} + 2\delta_{(a,\lambda)}, \quad \text{in } S^4_+, \quad \frac{\partial P\delta_{(a,\lambda)}}{\partial \nu} = 0 \text{ on } \partial S^4_+ .$$

It is easy to obtain that $P\delta_{(a,\lambda)} = \delta_{(a,\lambda)}$ if $a \in \partial S^4_+$.

Let G be the Greens function of $L_g := -\Delta + 2Id$ on S^4_+ and H its regular part defined by

$$\begin{cases} G(x,y) = (1-\cos(d(x,y)))^{-1} + H(x,y), \\ L_g H = 0 \text{ in } S^4_+; \quad \partial G/\partial \nu = 0 \text{ on } \partial S^4_+ . \end{cases}$$

It is well known that H is a positive function and $H(x,x) \to +\infty$ as x goes to the boundary.

The space $H^1(S^4_+)$ is equipped with the norm $\|.\|$ and its corresponding inner product $\langle .,. \rangle$ defined by

$$\|u\|^2 = \int_{S^4_+} |\nabla u|^2 + 2\int_{S^4_+} u^2, \quad \text{and} \quad \langle u, v\rangle = \int_{S^4_+} \nabla u \nabla v + 2\int_{S^4_+} uv, \quad u, v \in H^1(S^4_+) .$$

Our first result deals with the construction of some sign-changing solutions (u_ε) of (S_ε) which blow up at two different points in the interior of S^4_+.

THEOREM 1.1. *Let y_1 and y_2 be nondegenerate critical points of K with $(-\Delta K(y_i)/3K(y_i) - 4H(y_i, y_i)) > 0$ for $i = 1, 2$. Then, there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, the problem (S_ε) has a solution (u_ε) of the form*

$$u_\varepsilon = \alpha_1 P\delta_{(x_1,\lambda_1)} - \alpha_2 P\delta_{(x_2,\lambda_2)} + v , \tag{1.2}$$

where $\alpha_i \to K(y_i)^{-1/2}$; $\|v\| \to 0$; $x_i \to y_i$, $\lambda_i \to +\infty$; $\lambda_1 = \gamma_1\lambda_2(1+o(1))$ *as* $\varepsilon \to 0$. *Here,* γ_1 *is a positive fixed constant.*

In the case of positive solutions, the blow up occur with comparable speeds. But for sign-changing solutions, Pistoia and Weth [10] constructed some solutions (u_ε) of analogous problem of (S_ε) with many bubbles ($u_\varepsilon = \sum_{i=1}^{q}(-1)^i P\delta_{a_i,\lambda_i}$, for $q \geq 2$) blowing up at the same point (bubble-tower solutions). This is a new phenomenon for sign-changing solutions compared with the positive one (see [7] and [9]). In our case, we prove that this phenomenon also appear for each $q \geq 2$. In fact, we prove that:

THEOREM 1.2. *Assume that* $\overline{y}$ *is a nondegenerate critical point of* K *satisfying* $(-\Delta K(\overline{y})/3K(\overline{y}) - 4H(\overline{y},\overline{y})) > 0$. *Then, there exists* $\varepsilon_0 > 0$ *such that for each* $\varepsilon \in (0,\varepsilon_0)$, *the problem* (S_ε) *has a solution* (u_ε) *of the form* (1.2) *where, as* $\varepsilon \to 0$,

$$\alpha_i \to K(\overline{y})^{-1/2};\ \|v\| \to 0;\ x_i \to \overline{y},\ \lambda_i \to +\infty;\ \textit{for } i \in \{1,2\} \tag{1.3}$$
$$\textit{and } \lambda_1 = \gamma_2\lambda_2^3(1+o(1)) \textit{ with } \quad \gamma_2 = 2\Big(-H(\overline{y},\overline{y}) - \tfrac{\Delta K(\overline{y})}{12K(\overline{y})}\Big)^{-1}.$$

The remaind of this paper is organized as follows. In Section 2, we recall some preliminaries. In Section 3, we give some careful expansions of gradient of the associated variational functional I_ε for $(\varepsilon > 0)$. While Sections 4 and 5 are devoted to the proof of Theorem 1.1 and Theorem 1.2 respectively.

2. Preliminary results. First, let us introduce the general setting. For $\varepsilon > 0$, we define the functional

$$I_\varepsilon(u) = \frac{1}{2}\int_{S^4_+}|\nabla u|^2 + \int_{S^4_+}u^2 - \frac{1}{4-\varepsilon}\int_{S^4_+}K|u|^{4-\varepsilon}, \quad u \in H^1(S^4_+). \tag{2.1}$$

Note that if u is a positive critical point of I_ε, then u is a solution of (S_ε), and inversely.

Furthermore, we mention that it will be convenient to perform some stereographic projection in order to reduce our problem to $\mathbb{R}^4_+$. Let $D^{1,2}(\mathbb{R}^4_+)$ denote the completion of $C_c^\infty(\overline{\mathbb{R}^4_+})$ with respect to Dirichlet norm. The stereographic projection π_a through a point $a \in \partial S^4_+$ induces an isometry $\imath : H^1(S^4_+) \to D^{1,2}(\mathbb{R}^4_+)$ according to the following formula

$$(\imath v)(x) = \left(\frac{2}{1+|x|^2}\right) v(\pi_a^{-1}(x)), \qquad v \in H^1(S^4_+),\ x \in \mathbb{R}^4_+. \tag{2.2}$$

In particular, one can check that the following holds true, for every $v \in H^1(S^4_+)$

$$\int_{S^4_+}(|\nabla v|^2 + 2v^2) = \int_{\mathbb{R}^4_+}|\nabla(\imath v)|^2 \qquad \text{and} \quad \int_{S^4_+}|v|^4 = \int_{\mathbb{R}^4_+}|\imath v|^4.$$

In the sequel, we will identify the function K and its composition with the stereographic projection π_a. We will also identify a point b of S^4_+ and its image by π_a. Moreover, it is easy to see that, by (2.2) with π_{-a}, the function $\imath\delta_{(a,\lambda)}$ is equal to

$$\imath\delta_{(a,\lambda)} = c_0\frac{\lambda}{1+\lambda^2|x-a|^2}.$$

For sake of simplicity, we will write $\delta_{(a,\lambda)}$ instead of $\imath\delta_{(a,\lambda)}$. These facts will be assumed in the sequel.

LEMMA 2.1 ([3]). *For $a \in \partial S^4_+$, we have $(\partial\delta_{(a,\lambda)})/(\partial\nu) = 0$ and $\delta_{(a,\lambda)} = P\delta_{(a,\lambda)}$. For $a \in S^4_+$, we have*

$$P\delta_{(a,\lambda)} - \delta_{(a,\lambda)} = c_0 \frac{H(a,.)}{\lambda} + f_{(a,\lambda)}$$

such that $f_{(a,\lambda)}$ satisfies

$$|f_{(a,\lambda)}|_{L^\infty} \le \frac{c}{\lambda^3 d^4} \quad \text{and} \quad \lambda\frac{\partial f}{\partial\lambda} = O\Big(\frac{1}{\lambda^3 d^4}\Big),$$

where $d = d(a, \partial S^4_+)$.

PROOF. We will repeat the proof here to be self-contained.

Using a stereographic projection, we are led to prove the corresponding estimates on $\mathbb{R}^4_+$. We still denote by G and H the Greens function and its regular part of the Laplacian on $\mathbb{R}^4_+$ under Neumann boundary conditions. In this case, we have

$$\delta_{(a,\lambda)} = c_0 \frac{\lambda}{1+\lambda^2|x-a|^2} \text{ and } H(a,x) = \frac{1}{|\overline{a}-x|^2},$$

where $\overline{a}$ is the symmetric of a with respect to $\partial\mathbb{R}^4_+$.

Observe that, for $f_{(a,\lambda)} = P\delta_{(a,\lambda)} - \delta_{(a,\lambda)} - H(a,.)/\lambda$,

$$\Delta f_{(a,\lambda)} = 0 \text{ in } \mathbb{R}^4_+, \quad \frac{\partial f_{(a,\lambda)}}{\partial\nu} = \frac{\partial\delta}{\partial\nu} - \frac{1}{\lambda}\frac{\partial H}{\partial\nu} = O\Big(\frac{1}{\lambda^3 d^5}\Big).$$

Thus, using the Green formula, we derive

$$f_{(a,\lambda)}(x) = c\int_{\partial\mathbb{R}^4_+} G\Big(\frac{\partial\delta}{\partial\nu} - \frac{1}{\lambda}\frac{\partial H}{\partial\nu}\Big) \le \frac{c'}{\lambda^3 d_a^2}\int_{\partial\mathbb{R}^4_+} G\frac{1}{|a-x|^3},$$

where d_a is the distance of a to the boundary. G satisfies

$$\int_{\partial\mathbb{R}^4_+} G(x,y)\frac{1}{|a-x|^3} = O\Big(\frac{1}{d_a^2}\Big).$$

Let

$$E_{(x,\lambda)} = \Big\{w \in H^1(S^4_+)/\langle w,\varphi\rangle = 0 \quad \forall\varphi \in \mathrm{Span}\Big\{P\delta_i, \frac{\partial P\delta_i}{\partial\lambda_i}, \frac{\partial P\delta_i}{\partial x_i^j},\ i = 1,2;\ j \le 4\Big\}\Big\}.$$

Here, x_i^j denotes the j-th component of x_i. For sake of simplicity, we will write $P\delta_i$ instead of $P\delta_{(x_i,\lambda_i)}$ and therefore, for $u = \alpha_1 P\delta_{(x_1,\lambda_1)} - \alpha_2 P\delta_{(x_2,\lambda_2)} + v$ we can write $u = \alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v$. □

3. Expansions of the gradient of the functional I_ε. In this section, we collect some expansions of the gradient of the functional I_ε associated to the problem (S_ε) for $\varepsilon > 0$ which are needed in Section 4. We start by giving the following remark which is proved in [11] when S^4_+ is replaced by a bounded domain of $\mathbb{R}^3$.

REMARK 3.1. Let $\delta_{(a,\lambda)}$ be the function defined in (1.1). Assume that $\varepsilon\log\lambda$ is small enough. For $\varepsilon>0$, we have

$$\delta_{(a,\lambda)}^{-\varepsilon}(x)=1-\varepsilon\log\delta_{(a,\lambda)}+O(\varepsilon^2\log^2\lambda)\quad \text{in } S^4_+ .$$

Now, explicit computations, by Remark 3.1 and Lemma 2.1, yield the following propositions

PROPOSITION 3.2. *For $u=\alpha_1P\delta_{(x_1,\lambda_1)}-\alpha_2P\delta_{(x_2,\lambda_2)}+v$, with $v\in E_{x,\lambda}$, we have*

$$\langle\nabla I_\varepsilon,P\delta_i\rangle=(-1)^{i+1}\alpha_iS_4(1-\alpha_i^{2-\varepsilon}K(x_i))+O\Big(\varepsilon\log\lambda_i+\frac{1}{\lambda_2^2}+\varepsilon_{12}+\|v\|^2\Big),$$

where

$$S_4=64\int_{\mathbb{R}^4}\frac{dx}{(1+|x|^2)^4}.$$

PROOF. We have

$$\langle\nabla I_\varepsilon,h\rangle=\int_{S^4_+}\nabla u\nabla h+2\int_{S^4_+}uh-\int_{S^4_+}K|u|^{2-\varepsilon}uh. \tag{3.1}$$

A computation similar to the one performed in [1] shows that

$$||P\delta_i||^2=\int_{\mathbb{R}^4_+}|\nabla P\delta_i|^2=S_4+O\Big(\frac{1}{\lambda_i^2}\Big), \tag{3.2}$$

and

$$\int_{S^4_+}\nabla P\delta_1\nabla P\delta_2+2\int_{S^4_+}P\delta_1P\delta_2=\int_{\mathbb{R}^4_+}\nabla P\delta_1\nabla P\delta_2=\int_{\mathbb{R}^4_+}\delta_1^3P\delta_2=O(\varepsilon_{12}). \tag{3.3}$$

For the integral, we write

$$\int_{S^4_+}K|u|^{2-\varepsilon}uP\delta_1=\int_{S^4_+}K|\alpha_1P\delta_1-\alpha_2P\delta_2|^{2-\varepsilon}(\alpha_1P\delta_1-\alpha_2P\delta_2)P\delta_1+O\big(\varepsilon_{12}^2\log\varepsilon_{12}^{-1}+|v|^2\big). \tag{3.4}$$

We also write

$$\begin{aligned}\int_{S^4_+}K|\alpha_1P\delta_1-\alpha_2P\delta_2|^{2-\varepsilon}(\alpha_1P\delta_1-\alpha_2P\delta_2)P\delta_1&=\alpha_1^{3-\varepsilon}\int_{S^4_+}KP\delta_1^{4-\varepsilon}-\alpha_2^{3-\varepsilon}\int_{S^4_+}KP\delta_2^{3-\varepsilon}P\delta_1\\&-(3-\varepsilon)\alpha_1^{2-\varepsilon}\alpha_2\int_{S^4_+}KP\delta_1^{3-\varepsilon}P\delta_2+O\big(\varepsilon_{12}^2\log\varepsilon_{12}^{-1}\big).\end{aligned} \tag{3.5}$$

Expanding of K around x_1 and x_2, we get

$$\int_{S^4_+}KP\delta_i^{4-\varepsilon}=\int_{\mathbb{R}^4_+}KP\delta_i^{4-\varepsilon}=K(x_i)S_4+O\Big(\varepsilon\log\lambda_i+\frac{1}{\lambda_i^2}\Big), \tag{3.6}$$

$$\int_{S^4_+}KP\delta_2^{3-\varepsilon}P\delta_1=\int_{\mathbb{R}^4_+}KP\delta_2^{3-\varepsilon}P\delta_1=O\left(\varepsilon\log\lambda_2+\varepsilon_{12}\right), \tag{3.7}$$

$$\int_{S^4_+}KP\delta_1^{3-\varepsilon}P\delta_2=\int_{\mathbb{R}^4_+}KP\delta_1^{3-\varepsilon}P\delta_2=O\left(\varepsilon\log\lambda_1+\varepsilon_{12}\right). \tag{3.8}$$

Combining (3.1)–(3.8), we easily derive our proposition. □

PROPOSITION 3.3. *For $u = \alpha_1 P\delta_{(x_1,\lambda_1)} - \alpha_2 P\delta_{(x_2,\lambda_2)} + v$, with $v \in E_{x,\lambda}$, we have the following expansion:*

$$\Big\langle \nabla I_\varepsilon(u), \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\Big\rangle = (-1)^{i+1}\Big[\alpha_j \frac{c_2}{2}\Big(\alpha_i^{2-\varepsilon}K(x_i) + \alpha_j^{2-\varepsilon}K(x_j) - 1\Big)\Big(\lambda_i \frac{\partial \varepsilon_{12}}{\partial \lambda_i} - 2\frac{H(x_1,x_2)}{\lambda_1\lambda_2}\Big)$$
$$+ \alpha_i^{3-\varepsilon}\frac{\varepsilon S_4 K(x_i)}{4} + \alpha_i^{3-\varepsilon}\frac{c_2}{12}\frac{\Delta K(x_i)}{\lambda_i^2} + \alpha_i c_2 \frac{H(x_i,x_i)}{\lambda_i^2}(2\alpha_i^{2-\varepsilon}K(x_i) - 1)\Big]$$
$$+ O\Big(\varepsilon^2 \log\lambda_i + \frac{\varepsilon \log \lambda_i}{\lambda_i^2} + \frac{1}{\lambda_i^3} + \|v\|^2\Big)$$
$$+ O\Big(\varepsilon\varepsilon_{12}(\log \varepsilon_{12}^{-1})^{\frac12} + \varepsilon_{12}^2 \log \varepsilon_{12}^{-1} + \varepsilon_{12}(\log\varepsilon_{12}^{-1})^{\frac12}\Big(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\Big)\Big),$$

where

$$c_1 = 64\int_{\mathbb{R}^4_+} \frac{x_4(|x|^2-1)}{(1+|x|^2)^5}dx, \quad c_2 = 64\int_{\mathbb{R}^4}\frac{dx}{(1+|x|^2)^3}.$$

PROOF. Observe that (see [1])

$$\int_{\mathbb{R}^4_+} \nabla P\delta_i \nabla\Big(\lambda_i \frac{\partial P\delta_i}{\partial\lambda_i}\Big) = -c_2 \frac{H(x_1,x_2)}{\lambda_i^2} + O\Big(\frac{1}{\lambda_i^3}\Big) \tag{3.9}$$

$$\int_{\mathbb{R}^4_+} \nabla P\delta_j \nabla\Big(\lambda_i \frac{\partial\delta_i}{\partial\lambda_i}\Big) = \int_{\mathbb{R}^4_+}\delta_j^3 \lambda_i \frac{\partial\delta_i}{\partial\lambda_i} = \frac{c_2}{2}\Big(\lambda_i\frac{\partial\varepsilon_{12}}{\partial\lambda_i} - 2\frac{H(x_1,x_2)}{\lambda_1\lambda_2}\Big) + O\big(\varepsilon_{12}^2 \log(\varepsilon_{12}^{-1})\big). \tag{3.10}$$

For the other part

$$\int_{\mathbb{R}^4_+} KP\delta_i^{3-\varepsilon}\lambda_i \frac{\partial P\delta_i}{\partial\lambda_i} = -\frac{c_2}{\lambda_i^2}\frac{\Delta K(x_i)}{12} - \frac{S_4\varepsilon}{4}K(x_i) - 2c_2 K(x_i)\frac{H(x_i,x_i)}{\lambda_i^2} + O\Big(\varepsilon^2\log\lambda_i + \frac{1}{\lambda_i^3} + \frac{\varepsilon\log\lambda_i}{\lambda_i^2}\Big), \tag{3.11}$$

$$\int_{\mathbb{R}^4_+} KP\delta_j^{3-\varepsilon}\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i} = K(x_j)\frac{c_2}{2}\Big(\lambda_i\frac{\partial\varepsilon_{12}}{\partial\lambda_i} - 2\frac{H(x_1,x_2)}{\lambda_1\lambda_2}\Big) + O\Big(\varepsilon\varepsilon_{12}(\log(\varepsilon_{12}^{-1}))^{\frac12} + \frac{1}{\lambda_i^3}\Big) + O\big(\varepsilon_{12}^2\log(\varepsilon_{12}^{-1})\big), \tag{3.12}$$

$$(3-\varepsilon)\int_{\mathbb{R}^4_+} KP\delta_i^{2-\varepsilon}P\delta_j\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i} = K(x_i)\frac{c_2}{2}\Big(\lambda_i\frac{\partial\varepsilon_{12}}{\partial\lambda_i} - 2\frac{H(x_1,x_2)}{\lambda_1\lambda_2}\Big) + O\big(\varepsilon\varepsilon_{12}(\log(\varepsilon_{12}^{-1}))^{\frac12}\big) + O\Big(\varepsilon_{12}^2\log(\varepsilon_{12}^{-1}) + \frac{\varepsilon_{12}}{\lambda_i}(\log(\varepsilon_{12}^{-1}))^{\frac12}\Big). \tag{3.13}$$

Combining (3.1) and (3.9)–(3.13), we derive our proposition. □

PROPOSITION 3.4. *For $u = \alpha_1 P\delta_{(x_1,\lambda_1)} - \alpha_2 P\delta_{(x_2,\lambda_2)} + v$, with $v \in E_{x,\lambda}$, we have*

$$\Big\langle \nabla I_\varepsilon(u), \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial x_i}\Big\rangle = \frac{(-1)^{i+1}}{\lambda_i}\Big[\alpha_i c_2\Big(1 - \sum \alpha_j^{2-\varepsilon}K(x_j)\Big)\Big(\frac{\partial\varepsilon_{12}}{\partial x_i} + \frac{2}{\lambda_1\lambda_2}\frac{\partial H(x_1,x_2)}{\partial x_i}\Big)$$

$$- \alpha_i^{3-\varepsilon} c_3 \nabla K(x_i) + \Big(1 - \alpha_i^{2-\varepsilon} K(x_i)\Big) \frac{\alpha_i c_2}{\lambda_i^2} \frac{\partial H(x_i, x_i)}{\partial x_i} \Big]$$
$$+ O\Big(\frac{\varepsilon \log \lambda_i}{\lambda_i} |\nabla K(x_i)| + \frac{1}{\lambda_i^2} + \|v\|^2 + \lambda_j |x_1 - x_2| \varepsilon_{12}^{5/2}\Big)$$
$$+ O\bigg(\varepsilon \varepsilon_{12} (\log \varepsilon_{12}^{-1})^{\frac{1}{2}} + \varepsilon_{12}^2 \log \varepsilon_{12}^{-1} + \varepsilon_{12} (\log \varepsilon_{12}^{-1})^{\frac{1}{2}} \Big(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\Big)\bigg),$$

where

$$c_3 = 16 \int_{\mathbb{R}^4} \frac{x_4^2}{(1+|x|^2)^5} dx, \quad c_4 = 132 \int_{\mathbb{R}^4_+} \frac{x_4}{(1+|x|^2)^5} dx.$$

PROOF. An easy computation shows

$$\int_{\mathbb{R}^4_+} \nabla P\delta_i \nabla \Big(\frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial x_i}\Big) = \frac{c_2}{\lambda_i^3} \frac{\partial H(x_i, x_i)}{\partial x_i} + \Big(\frac{1}{\lambda_i^4}\Big), \tag{3.14}$$

$$\int_{\mathbb{R}^4_+} \nabla P\delta_j \nabla \left(\frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial x_i}\right) = \int_{\mathbb{R}^4_+} \delta_j^3 \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial x_i} \tag{3.15}$$
$$= \frac{c_2}{\lambda_i} \Big(\frac{\partial \varepsilon_{12}}{\partial x_i} + \frac{2}{\lambda_1 \lambda_2} \frac{\partial H(x_1, x_2)}{\partial x_i}\Big) + O\bigg(\varepsilon_{12}^2 \log(\varepsilon_{12}^{-1}) + \varepsilon_{12}^{\frac{5}{2}} \lambda_i |x_1 - x_2|\bigg).$$

For the other part

$$\int_{\mathbb{R}^4_+} K P\delta_i^{3-\varepsilon} \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial x_i} = c_3 \frac{\nabla K(x_i)}{\lambda_i} + O\Big(\frac{1}{\lambda_i^2} + \varepsilon^2 \log^2 \lambda_i\Big), \tag{3.16}$$

$$\int_{\mathbb{R}^4_+} K P\delta_j^{3-\varepsilon} \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial x_i} = K(x_j) \frac{c_2}{\lambda_i} \Big(\frac{\partial \varepsilon_{12}}{\partial x_i} + \frac{2}{\lambda_1 \lambda_2} \frac{\partial H(x_1, x_2)}{\partial x_i}\Big) + O\Big(\varepsilon_{12}^{\frac{5}{2}} \lambda_i |x_1 - x_2|\Big) \tag{3.17}$$
$$+ O\bigg(\varepsilon_{12}^2 \log(\varepsilon_{12}^{-1}) + \frac{1}{\lambda_i} \varepsilon_{12} \Big(\log(\varepsilon_{12}^{-1})\Big)^{\frac{1}{2}}\bigg),$$

$$(3-\varepsilon) \int_{\mathbb{R}^4_+} K P\delta_i^{2-\varepsilon} P\delta_j \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial x_i} = K(x_i) \frac{c_2}{\lambda_i} \Big(\frac{\partial \varepsilon_{12}}{\partial x_i} + \frac{2}{\lambda_1 \lambda_2} \frac{\partial H(x_1, x_2)}{\partial x_i}\Big) + O\Big(\varepsilon_{12}^{\frac{5}{2}} \lambda_2 |x_1 - x_2|\Big)$$
$$+ O\left(\varepsilon_{12}^2 \log(\varepsilon_{12}^{-1}) + \frac{1}{\lambda_i} \varepsilon_{12} \left(\log(\varepsilon_{12}^{-1})\right)^{\frac{1}{2}}\right). \tag{3.18}$$

Using (3.1), (3.14)–(3.18), we have our proposition. □

4. Proof of Theorem 1.1. Let

$$M_{\varepsilon,1} = \bigg\{ m = (\alpha, \lambda, x_1, x_2, v) \in \mathbb{R}^2 \times (\mathbb{R}_+^*)^2 \times S_+^4 \times S_+^4 \times H^1(S_+^4) : v \in E_{(x,\lambda)}, \|v\| < \nu_0;$$
$$\left| \frac{\alpha_i^2 K(x_i)}{\alpha_j^2 K(x_j)} - 1 \right| < \nu_0, \lambda_i > \frac{1}{\nu_0}, \varepsilon \log \lambda_i < \nu_0, \forall i;\ c_0 < \frac{\lambda_1}{\lambda_2} < c_0^{-1}; |x_1 - x_2| > d_0 \bigg\},$$

where σ, c_0, d_0 are some suitable positive constants, ν_0 is a small positive constant. Let us define the function by

$$\Psi_{\varepsilon,1} : M_{\varepsilon,1} \to \mathbb{R}; \quad m = (\alpha, \lambda, x, v) \mapsto I_\varepsilon\big(\alpha_1 P\delta_{(x_1,\lambda_1)} - \alpha_2 P\delta_{(x_2,\lambda_2)} + v\big)\,. \tag{4.1}$$

As in [2], using the Euler-Lagrange's coefficients, we easily get the following proposition.

PROPOSITION 4.1. *Let $m = (\alpha, \lambda, x, v) \in M_{\varepsilon,1}$. m is a critical point of $\Psi_{\varepsilon,1}$ if and only if $u = \alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v$ is a critical point of I_ε, i.e. if and only if there exists $\big(A, B, C\big) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \big(\mathbb{R}^4\big)^2$ such that the following holds:*

$$(E_{\alpha_i})\ \frac{\partial \Psi_{\varepsilon,1}}{\partial \alpha_i} = 0, \ \forall i = 1, 2 \tag{4.2}$$

$$(E_{\lambda_i})\ \frac{\partial \Psi_{\varepsilon,1}}{\partial \lambda_i} = B_i\Big\langle \frac{\partial^2 P\delta_i}{\partial \lambda_i^2}, v\Big\rangle + \sum_{j=1}^{4} C_{ij}\Big\langle \frac{\partial^2 P\delta_i}{\partial x_i^j \partial \lambda_i}, v\Big\rangle, \ \forall i = 1, 2 \tag{4.3}$$

$$(E_{x_i})\ \frac{\partial \Psi_{\varepsilon,1}}{\partial x_i} = B_i\Big\langle \frac{\partial^2 P\delta_i}{\partial \lambda_i \partial x_i}, v\Big\rangle + \sum_{j=1}^{4} C_{ij}\Big\langle \frac{\partial^2 P\delta_i}{\partial x_i^j \partial x_i}, v\Big\rangle, \ \forall i = 1, 2 \tag{4.4}$$

$$(E_v)\ \frac{\partial \Psi_{\varepsilon,1}}{\partial v} = \sum_{i=1,2}\Big(A_i P\delta_i + B_i \frac{\partial P\delta_i}{\partial \lambda_i} + \sum_{j=1}^{4} C_{ij}\frac{\partial P\delta_i}{\partial x_i^j}\Big)\,. \tag{4.5}$$

The results of Theorem 1.1 will be obtained through a careful analysis of (4.2)–(4.5) on $M_{\varepsilon,1}$. As usual in this type of problems, we first deal with the v-part of u, in order to show that it is negligible with respect to the concentration phenomenon. The study of (E_v) yields:

PROPOSITION 4.2. *There exists a smooth map which to any $(\varepsilon, \alpha, \lambda, x)$ such that $(\alpha, \lambda, x, 0)$ in $M_{\varepsilon,1}$ associates $\overline{v} \in E_{(x,\lambda)}$ such that $\|v\| < \nu_0$ and (E_v) is satisfied for some $\big(A, B, C\big) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \big(\mathbb{R}^4\big)^2$. Such a $\overline{v}$ is unique, minimizes $\Psi_{\varepsilon,1}(\alpha, \lambda, x, v)$ with respect to v in $\{v \in E_{(x,\lambda)}/\|v\| < \nu_0\}$, and we have the following estimate*

$$\|\overline{v}\| = O\bigg(\varepsilon + \sum\Big(\frac{|\nabla K(x_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\Big) + \varepsilon_{12}(\log \varepsilon_{12}^{-1})^{1/2}\bigg)\,. \tag{4.6}$$

PROOF. Expanding I_ε with respect to $v \in E_{(x,\lambda)}$, we obtain

$$I_\varepsilon(\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v\big) = c(\alpha, x, \lambda) + \frac{1}{2}Q(v, v) - f(v) + R(v)\,, \tag{4.7}$$

where $Q(.,.)$ is a quadratic form positive definite, $f(.)$ is a linear form and $R(v)$ satisfies $R(v) = o(\|v\|^2)$, $R'(v) = o(\|v\|)$ and $R''(v) = o(1)$.

Since $Q(v, v)$ is positive definite, we derive that the following problem

$$\min\{I_\varepsilon(\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v),\ v \in E_{(x,\lambda)} \text{ and } \|v\| < \nu_0\} \tag{4.8}$$

is achieved by a unique function $\overline{v}$ which satisfies $\|\overline{v}\| \leq c\|f\|$. Now, following [3] we get the estimate (4.6). Since $\overline{v}$ is orthogonal to the functions $\{P\delta_i, \partial P\delta_i/\partial\lambda_i, \partial P\delta_i/\partial x_i^j, i \leq 2,\ j \leq 4\}$, there exist A, B and C such that

$$\frac{\partial \Psi_{\varepsilon,1}}{\partial v}(\alpha,\lambda,x,\overline{v}) = \nabla I_\varepsilon(\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \overline{v})$$

$$= \sum_{i=1,2}\Big(A_i P\delta_i + B_i \frac{\partial P\delta_i}{\partial \lambda_i} + \sum_{j=1}^{4} C_{ij}\frac{\partial P\delta_i}{\partial x_i^j}\Big). \tag{4.9}$$

The proposition follows. □

Proof of Theorem 1.1. Once $\overline{v}$ is defined by Proposition 4.2, we estimate the corresponding numbers A, B, C by taking the scalar product in $H^1(S^4_+)$ of (E_v) with $P\delta_1$, $P\delta_2$, $\partial P\delta_1/\partial\lambda_1$, $\partial P\delta_2/\partial\lambda_2$, $\partial P\delta_1/\partial x_1$ and $\partial P\delta_2/\partial x_2$ respectively. Thus we get a quasi-diagonal system whose coefficients are given by

$$\int_{\mathbb{R}^4_+}|\nabla P\delta_i|^2 = S_4 + O\Big(\frac{1}{\lambda_i^2}\Big);\quad \int_{\mathbb{R}^4_+}\nabla P\delta_i\nabla P\delta_j = O\Big(\frac{1}{\lambda_i\lambda_j}\Big);\quad \int_{\mathbb{R}^4_+}\nabla P\delta_i\nabla\frac{\partial P\delta_i}{\partial\lambda_i} = O\Big(\frac{1}{\lambda_i^3}\Big),$$

$$\int_{\mathbb{R}^4_+}\nabla P\delta_i\nabla\frac{\partial P\delta_j}{\partial\lambda_j} = O\Big(\frac{1}{\lambda_i\lambda_j^2}\Big); \int_{\mathbb{R}^4_+}\Big|\nabla\frac{\partial P\delta_i}{\partial\lambda_i}\Big|^2 = \frac{\Gamma_1}{\lambda_i^2} + O\Big(\frac{1}{\lambda_i^3}\Big); \int_{\mathbb{R}^4_+}\nabla\frac{\partial P\delta_i}{\partial\lambda_i}\nabla\frac{\partial P\delta_j}{\partial\lambda_j} = O\Big(\frac{1}{\lambda_i^2\lambda_j^2}\Big),$$

$$\int_{\mathbb{R}^4_+}\nabla\frac{\partial P\delta_i}{\partial\lambda_i}\nabla\frac{\partial P\delta_i}{\partial x_i} = O\Big(\frac{1}{\lambda_i^3}\Big); \int_{\mathbb{R}^4_+}\Big|\nabla\frac{\partial P\delta_i}{\partial x_i}\Big|^2 = \Gamma_2\lambda_i^2 + O\Big(\frac{1}{\lambda_i}\Big); \int_{\mathbb{R}^4_+}\nabla P\delta_i\nabla\frac{\partial P\delta_i}{\partial x_i} = O\Big(\frac{1}{\lambda_i^2}\Big),$$

$$\int_{\mathbb{R}^4_+}\nabla P\delta_i\nabla\frac{\partial P\delta_j}{\partial x_j} = O\Big(\frac{1}{\lambda_i}\Big);\quad \int_{\mathbb{R}^4_+}\nabla\frac{\partial P\delta_i}{\partial x_i}\nabla\frac{\partial P\delta_j}{\partial x_j} = O\Big(\frac{1}{\lambda_i}\Big),$$

with Γ_1, Γ_2 are positive constants and where we have used the fact that $|x_1 - x_2| \geq c > 0$. The other hand side is given by

$$(-1)^i\frac{\partial\Psi_{\varepsilon,1}}{\partial\alpha_i} = \Big\langle\frac{\partial\Psi_{\varepsilon,1}}{\partial v}, P\delta_i\Big\rangle;\ \frac{(-1)^i}{\alpha_i}\frac{\partial\Psi_{\varepsilon,1}}{\partial\lambda_i} = \Big\langle\frac{\partial\Psi_{\varepsilon,1}}{\partial v}, \frac{\partial P\delta_i}{\partial\lambda_i}\Big\rangle;\ \frac{(-1)^i}{\alpha_i}\frac{\partial\Psi_{\varepsilon,1}}{\partial x_i} = \Big\langle\frac{\partial\Psi_{\varepsilon,1}}{\partial v}, \frac{\partial P\delta_i}{\partial x_i}\Big\rangle. \tag{4.10}$$

Using Proposition 3.2, some computations yield

$$\frac{\partial\Psi_{\varepsilon,1}}{\partial\alpha_i} = -2S_4\beta_i + V_{\alpha_i}(\varepsilon,\alpha,\lambda,x)\,, \tag{4.11}$$

with $\beta = (\beta_1,\beta_2)$ where $\beta_i = \alpha_i - 1/K(y_i)^{\frac{1}{2}}$ and V_{α_i} is a smooth function which satisfies

$$V_{\alpha_i} = O\Big(\beta_i^2 + \varepsilon\log\lambda_i + \frac{1}{\lambda_i^2} + |x_i - y_i|^2\Big). \tag{4.12}$$

In the same way, using Proposition 3.3, we get

$$\frac{\partial\Psi_{\varepsilon,1}}{\partial\lambda_i} = \frac{1}{K(y_i)}\Big(\frac{\varepsilon K(x_i)S_4}{4\lambda_i} + \frac{c_2}{4}\Big(\frac{\Delta K(x_i)}{3K(x_i)} + 4H(x_i,x_i)\Big)\frac{1}{\lambda_i^3}\Big)$$

$$+ \frac{c_2}{2(K(y_1)K(y_2))^{1/2}}\frac{1}{\lambda_i}\frac{G(x_1,x_2)}{\lambda_1\lambda_2} + V_{\lambda_i}(\varepsilon,\alpha,\lambda,x)\,, \tag{4.13}$$

where c_2 and c_3 are defined in Proposition 3.3 and V_{λ_i} is a smooth function satisfying

$$(4.14)\quad V_{\lambda_i} = O\Big[\frac{1}{\lambda_i}\Big(\frac{1}{\lambda_i^3}+\frac{|x_i-y_i|^2}{\lambda_i^2}+\varepsilon^2\log\lambda_i+\frac{\varepsilon\log\lambda_i}{\lambda_i^2}\Big)+(|\beta|+|x_i-y_i|^2)\Big(\frac{\varepsilon}{\lambda_i}+\frac{1}{\lambda_i^3}\Big)\Big].$$

Lastly, using Proposition 3.4, we have

$$(4.15)\qquad \frac{\partial\Psi_{\varepsilon,1}}{\partial x_i} = -c_3\nabla K(x_i) + V_{x_i}(\varepsilon,\alpha,\lambda,x),$$

where V_{x_i} is a smooth function such that

$$(4.16)\qquad V_{x_i} = O\Big(\frac{1}{\lambda_i} + (|\beta| + \varepsilon\log\lambda_i + |x_i-y_i|^2)|x_i-y_i|\Big).$$

Notice that these estimates imply

$$\frac{\partial\Psi_{\varepsilon,1}}{\partial\alpha_i} = O\Big(|\beta|+\varepsilon\log\lambda_i+\frac{1}{\lambda_i^2}+|x_i-y_i|^2\Big),\ \frac{\partial\Psi_{\varepsilon,1}}{\partial\lambda_i} = O\Big(\frac{1}{\lambda_i^3}+\frac{\varepsilon}{\lambda_i}\Big),\ \frac{\partial\Psi_{\varepsilon,1}}{\partial x_i} = O\Big(|x_i-y_i|+\frac{1}{\lambda_i}\Big).$$

The solution of the system in A, B and C shows that

$$A_i = O\Big(|\beta|+\varepsilon\log\lambda_i+\frac{1}{\lambda_i^2}+|x_i-y_i|^2\Big),\quad B_i = O\Big(\frac{1}{\lambda_i}+\varepsilon\lambda_i\Big),\quad C_i = O\Big(\frac{|x_i-y_i|}{\lambda_i^2}+\frac{1}{\lambda_i^3}\Big).$$

This allows us to evaluate the right hand side in the equations (E_{λ_i}) and (E_{x_i}), namely

$$(4.17)\qquad B_i\Big\langle\frac{\partial^2 P\delta_i}{\partial\lambda_i^2},\overline{v}\Big\rangle+\sum_{j=1}^{4}C_{ij}\Big\langle\frac{\partial^2 P\delta_i}{\partial x_i^j\partial\lambda_i},\overline{v}\Big\rangle = O\Big(\Big(\frac{1}{\lambda_i^3}+\frac{\varepsilon}{\lambda_i}+\frac{|y_i-x_i|}{\lambda_i^2}\Big)\|\overline{v}\|\Big),$$

$$(4.18)\qquad B_i\Big\langle\frac{\partial^2 P\delta_i}{\partial\lambda_i\partial x_i},\overline{v}\Big\rangle+\sum_{j=1}^{4}C_{ij}\Big\langle\frac{\partial^2 P\delta_i}{\partial x_i^j\partial x_i},\overline{v}\Big\rangle = O\Big(\Big(\frac{1}{\lambda_i}+\varepsilon\lambda_i+|x_i-y_i|\Big)\|\overline{v}\|\Big),$$

where we have used the following estimates

$$\Big\|\frac{\partial^2 P\delta_i}{\partial\lambda_i^2}\Big\| = O\Big(\frac{1}{\lambda_i^2}\Big);\quad \Big\|\frac{\partial^2 P\delta_i}{\partial x_i\partial\lambda_i}\Big\| = O(1);\quad \Big\|\frac{\partial^2 P\delta_i}{\partial x_i^2}\Big\| = O(\lambda_i^2).$$

Now, we consider a point $(y_1, y_2) \in S^4_+ \times S^4_+$ such that y_1 and y_2 are nondegenerate critical points of K. We set

$$\frac{1}{\lambda_i} = \varepsilon^{\frac{1}{2}}\Lambda_i(1+\zeta_i);\quad x_i = y_i+\xi_i,$$

where $\zeta_i \in \mathbb{R}$, $\xi_i \in \mathbb{R}^4$ are assumed to be small and for $i, j \in 1, 2$, $\Lambda_i = \Lambda_i(y_i)$ verifies

$$c_2\Lambda_i^2\Big(\frac{\Delta K(y_i)}{3K(y_i)}+4H(y_i,y_i)\Big)+S_4K(y_i)+c_2\Lambda_i\Lambda_j\Big(\frac{K(y_i)}{K(y_j)}\Big)^{1/2}G(y_i,y_j) = 0.$$

With these changes of variables and using (4.11), (E_{α_i}) is equivalent to

$$(4.19)\qquad \beta_i = V_{\alpha_i}(\varepsilon,\beta,\zeta,\xi) = O(\beta^2+\varepsilon|\log\varepsilon|+|\xi|^2).$$

Now, using (4.13), we show by an easy computation

$$\frac{\varepsilon K(y_i+\xi_i)S_4}{4\lambda_i}+\frac{c_2}{4}\Big(\frac{\Delta K(y_i+\xi_i)}{3K(y_i+\xi_i)}+4H(y_i+\xi_i,y_i+\xi_i)\Big)\frac{1}{\lambda_i^3}$$

$$
\begin{aligned}
&+ \frac{c_2}{2(K(y_j))^{1/2}}\frac{1}{\lambda_i}\frac{G(y_1+\xi_1, y_2+\xi_2)}{\lambda_1\lambda_2}\\
&= K(y_i)\frac{\varepsilon^{3/2}S_4}{4}\Lambda_i(1+\zeta_i) + \frac{c_2}{4}\varepsilon^{3/2}\Lambda_i^3(1+3\zeta_i)\Big(\frac{\Delta K(y_i)}{3K(y_i)} + 4H(y_i, y_i)\\
&+ \Big(\frac{1}{3K(y_i)}\nabla\Delta K(y_i) + 8\frac{\partial H}{\partial x_i}(y_i, y_i)\Big)\xi_i\Big) + c_2\varepsilon^{3/2}\Lambda_i^2\Lambda_j\frac{(1+2\zeta_i)(1+\zeta_j)}{K(y_j)^{1/2}}G(y_i, y_j)\\
&+ \frac{\varepsilon^{3/2}c_2\Lambda_i^2\Lambda_j}{K(y_j)^{1/2}}\frac{\partial G(y_i, y_j)}{\partial x_1}\xi_i + \frac{c_2\varepsilon^{3/2}\Lambda_i^2\Lambda_j}{K(y_j)^{1/2}}\frac{\partial G(y_i, y_j)}{\partial x_2}\xi_j + O\big(\varepsilon^{3/2}(\zeta_i^2 + |\xi_i|^2)\big)\\
&= \varepsilon^{3/2}\Big[\frac{\Lambda_i^3 c_2}{2}\Big(\frac{\Delta K(y_i)}{3K(y_i)} + 4H(y_i, y_i)\Big) + \frac{c_2\Lambda_i^2\Lambda_j}{K(y_j)^{1/2}}G(y_i, y_j)\Big]\zeta_i + \frac{\varepsilon^{3/2}c_2\Lambda_i^2\Lambda_j}{K(y_j)^{1/2}}G(y_i, y_j)\zeta_j\\
&+ \varepsilon^{3/2}\Big[\frac{\Lambda_i^3 c_2}{4}\Big(\frac{1}{3K(y_i)}\nabla(\Delta K)(y_i) + 8\frac{\partial H}{\partial x_i}(y_i, y_i)\Big) + \frac{c_2\Lambda_i^2\Lambda_j}{K(y_j)^{1/2}}\frac{\partial G(y_i, y_j)}{\partial x_1}\Big]\xi_i\\
&+ \frac{c_2\varepsilon^{3/2}\Lambda_i^2\Lambda_j}{K(y_j)^{1/2}}\frac{\partial G(y_i, y_j)}{\partial x_2}\xi_j + O\big(\varepsilon^{3/2}(\zeta_i^2 + |\xi_i|^2)\big).
\end{aligned}
$$

This implies that (E_{λ_i}) is equivalent, on account of (4.14) and (5.9), to

$$
\begin{aligned}
&\Big[\frac{\Lambda_i^3 c_2}{2}\Big(\frac{\Delta K(y_i)}{3K(y_i)} + 4H(y_i, y_i)\Big) + \frac{c_2\Lambda_i^2\Lambda_j}{K(y_j)^{1/2}}G(y_i, y_j)\Big]\zeta_i + \frac{c_2\Lambda_i^2\Lambda_j}{K(y_j)^{1/2}}G(y_i, y_j)\zeta_j\\
&+ \Big[\frac{\Lambda_i^3 c_2}{4}\Big(\frac{1}{3K(y_i)}\nabla(\Delta K)(y_i) + 8\frac{\partial H}{\partial x_i}(y_i, y_i)\Big) + \frac{c_2\Lambda_i^2\Lambda_j}{K(y_j)^{1/2}}\frac{\partial G(y_i, y_j)}{\partial x_1}\Big]\xi_i\\
&+ \frac{c_2\Lambda_i^2\Lambda_j}{K(y_j)^{1/2}}\frac{\partial G(y_i, y_j)}{\partial x_2}\xi_j = V_{\lambda_i}(\varepsilon, \beta, \zeta, \xi) = O(|\beta|^2 + \zeta_2^2 + |\xi|^2 + \varepsilon^{1/2}). \qquad (4.20)
\end{aligned}
$$

Lastly, using (4.15), (4.16) and (5.11), we see that (E_{x_i}) is equivalent to

$$
D^2K(y_i)\xi_i = V_{x_i}(\varepsilon, \beta, \zeta, \xi) = O(\varepsilon^{1/2} + |\beta|^2 + |\zeta|^2 + |\xi|^2). \qquad (4.21)
$$

We remark that V_{α_i}, V_{λ_i} and V_{x_i} are smooth functions. This system may be written as

$$
\begin{cases} \beta = V(\varepsilon, \beta, \zeta, \xi), \\ L(\zeta, \xi) = W(\varepsilon, \beta, \zeta, \xi), \end{cases} \qquad (4.22)
$$

where L is a fixed linear operator on $\mathbb{R}^{10}$ defined by (5.15) and (4.21) and V, W are smooth functions satisfying

$$
\begin{cases} V(\varepsilon, \beta, \zeta, \xi) = O(\varepsilon^{1/2} + |\beta|^2 + |\xi|^2), \\ W(\varepsilon, \beta, \zeta, \xi) = O(\varepsilon^{1/2} + |\beta|^2 + |\zeta|^2 + |\xi|^2). \end{cases}
$$

Moreover, a simple computation shows that the determinant of L is not equal to zero. Hence L is invertible, and Brouwer's fixed point theorem shows that (4.22) has a solution $(\beta^\varepsilon, \zeta^\varepsilon, \xi^\varepsilon)$ for ε small enough, such that

$$
|\beta^\varepsilon| = O(\varepsilon^{1/2}); \quad |\zeta^\varepsilon| = O(\varepsilon^{1/2}); \quad |\xi^\varepsilon| = O(\varepsilon^{1/2}).
$$

Hence, we have constructed $m^\varepsilon = (\alpha_1^\varepsilon, \alpha_2^\varepsilon, \lambda_1^\varepsilon, \lambda_2^\varepsilon, x_1^\varepsilon, x_2^\varepsilon)$ such that $u_\varepsilon := \alpha_1^\varepsilon P\delta_{(x_1^\varepsilon,\lambda_1^\varepsilon)} - \alpha_2^\varepsilon P\delta_{(x_2^\varepsilon,\lambda_2^\varepsilon)} + \overline{v_\varepsilon}$, satisfies (4.2)–(4.6). Therefore, by Proposition 4.1, u_ε is a critical point of I_ε, i.e., u_ε satisfies

$$-\Delta u_\varepsilon + 2u_\varepsilon = K|u_\varepsilon|^{2-\varepsilon}u_\varepsilon \quad \text{in } S_+^4, \quad \partial u_\varepsilon/\partial\nu = 0 \text{ on } \partial S_+^4 . \tag{4.23}$$

Hence, the proof of Theorem 1.1 is thereby completed. □

5. Proof of Theorem 1.2. As in the proof of Theorem 1.1, we introduce the set

$$M_{\varepsilon,2} = \Big\{ m = (\alpha, \lambda, x_1, x_2, v) \in \mathbb{R}^2 \times (\mathbb{R}_+^*)^2 \times S_+^4 \times S_+^4 \times H^1(S_+^4) : \Big| \frac{\alpha_i^2 K(x_i)}{\alpha_j^2 K(x_j)} - 1 \Big| < \nu_0 ,$$
$$\lambda_i > \frac{1}{\nu_0}, \varepsilon \log\lambda_i < \nu_0,\ d_0 < \frac{\lambda_1}{\lambda_2^3} < \frac{1}{d_0},\ \lambda_1|x_1 - x_2| < d_0',\ v \in E_{(x,\lambda)},\ \|v\| < \nu_0 \Big\},$$

where d_0 and d_0' are suitable positive constants, ν_0 is a small positive constant. Let us define the functional

$$\Psi_{\varepsilon,2} : M_{\varepsilon,2} \to \mathbb{R}; \quad m = (\alpha, \lambda, x, v) \mapsto I_\varepsilon\big(\alpha_1 P\delta_{(x_1,\lambda_1)} - \alpha_2 P\delta_{(x_2,\lambda_2)} + v\big) . \tag{5.1}$$

Let $m = (\alpha, \lambda, x, v) \in M_{\varepsilon,2}$. m is a critical point of $\Psi_{\varepsilon,2}$ if and only if $u = \alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v$ is a critical point of I_ε, i.e. if and only if there exists $(A, B, C) \in \mathbb{R}^2 \times \mathbb{R}^2 \times (\mathbb{R}^4)^2$ such that the system (4.2)–(4.5) holds. Once $\overline{v}$ is defined by Proposition 4.2, we estimate the corresponding numbers A, B and C by taking the scalar product in $H^1(S_+^4)$ of (E_v) with $P\delta_i$,$\partial P\delta_i/\partial\lambda_i$ and $\partial P\delta_i/\partial x_i$ respectively. Thus we get a quasi-diagonal system whose coefficients are given by (we remark that in this region ε_{12} is of the order of λ_2^{-2})

$$\int_{\mathbb{R}_+^4} \nabla P\delta_i \nabla P\delta_j = S_4\delta_{ij} + O\Big(\frac{1}{\lambda_2^2}\Big); \qquad \int_{\mathbb{R}_+^4} \nabla P\delta_i \nabla \frac{\partial P\delta_j}{\partial\lambda_j} = O\Big(\text{if } (i=j) \frac{1}{\lambda_i^3};\ \text{if } (i \neq j) \frac{1}{\lambda_j\lambda_2^2} \Big),$$
$$\int_{\mathbb{R}_+^4} \nabla P\delta_i \nabla \frac{\partial P\delta_j}{\partial x_j} = O\Big(\text{if } (i=j) \frac{1}{\lambda_i^2};\ \text{if } (i \neq j) \frac{1}{\lambda_1} \Big); \qquad \int_{\mathbb{R}_+^4} \Big|\nabla \frac{\partial P\delta_i}{\partial\lambda_i}\Big|^2 = \frac{\Gamma_1}{\lambda_i^2} + O\Big(\frac{1}{\lambda_i^3}\Big),$$
$$\int_{\mathbb{R}_+^4} \nabla \frac{\partial P\delta_1}{\partial\lambda_1} \nabla \frac{\partial P\delta_2}{\partial\lambda_2} = O\Big(\frac{1}{\lambda_1\lambda_2^3}\Big), \int_{\mathbb{R}_+^4} \nabla \frac{\partial P\delta_1}{\partial\lambda_1} \nabla \frac{\partial P\delta_2}{\partial x_2} = O\Big(\frac{1}{\lambda_1^2}\Big), \int_{\mathbb{R}_+^4} \nabla \frac{\partial P\delta_2}{\partial\lambda_2} \nabla \frac{\partial P\delta_1}{\partial x_1} = O\Big(\frac{1}{\lambda_2^4}\Big),$$
$$\int_{\mathbb{R}_+^4} \nabla \frac{\partial P\delta_i}{\partial\lambda_i} \nabla \frac{\partial P\delta_i}{\partial x_i} = O\Big(\frac{1}{\lambda_i^3}\Big), \int_{\mathbb{R}_+^4} \nabla \frac{\partial P\delta_i}{\partial x_i} \nabla \frac{\partial P\delta_j}{\partial x_j} = \Gamma_2\lambda_i^2\delta_{ij} + O\Big(\text{if } (i=j) \frac{1}{\lambda_i};\ \text{if } (i \neq j) 1 \Big).$$

On the other hand side, $\Psi_{\varepsilon,2}$ satisfies (4.10). By Proposition 3.2, (4.11) and (4.12) are satisfied with λ_2 instead of λ. In the same way we get

$$\frac{\partial\Psi_{\varepsilon,2}}{\partial\lambda_1} = \frac{1}{K(\overline{y})}\Big(\frac{S_4}{4}\frac{\varepsilon}{\lambda_1} - c_2\frac{\lambda_2}{\lambda_1^2} \Big) + V_{\lambda_1}(\varepsilon, \alpha, \lambda, x) , \tag{5.2}$$

$$\frac{\partial\Psi_{\varepsilon,2}}{\partial\lambda_2} = \frac{1}{K(\overline{y})}\Big(\frac{S_4}{4}\frac{\varepsilon}{\lambda_2} + \frac{c_2}{4}\Big(\frac{\Delta K(x_2)}{3K(x_2)} + 4H(x_2, x_2)\Big)\frac{1}{\lambda_2^3} + \frac{c_2}{\lambda_1} \Big) + V_{\lambda_2}(\varepsilon, \alpha, \lambda, x) , \tag{5.3}$$

where V_{λ_i} is a smooth function verifying

(5.4)
$$V_{\lambda_i} = O\left\{\frac{1}{\lambda_i}\left(\varepsilon^2\log\lambda_i + \frac{\varepsilon\log\lambda_i}{\lambda_i^2} + (|\beta|+\varepsilon+|x_i-\overline{y}|^2)\left(\varepsilon+\frac{1}{\lambda_2^2}\right) + \sum_{q=1,2}\frac{|x_q-\overline{y}|^2}{\lambda_q^2} + \frac{1}{\lambda_2^3}\right)\right\}.$$

Lastly, we have

$$\frac{\partial\Psi_{\varepsilon,2}}{\partial x_i} = -\frac{c_3}{K(\overline{y})^2}\nabla K(x_i) + V_{x_i}(\varepsilon,\alpha,\lambda,x)\,, \tag{5.5}$$

where V_{x_i} is a smooth function verifying

$$V_{x_i} = O\left(\varepsilon^2\lambda_i\log\lambda_i + \frac{1}{\lambda_i} + \lambda_i\sum_{q=1,2}\frac{|x_q-\overline{y}|^2}{\lambda_q^2} + (|\beta|+\varepsilon+|x_i-\overline{y}|^2)|x_i-\overline{y}|\right). \tag{5.6}$$

Notice that these estimates imply

$$\frac{\partial\Psi_{\varepsilon,2}}{\partial\alpha_i} = O\left(|\beta| + \frac{1}{\lambda_2^2} + \varepsilon\log\lambda_2 + |x_i-\overline{y}|^2\right);\quad \frac{\partial\Psi_{\varepsilon,2}}{\partial\lambda_i} = O\left(\frac{1}{\lambda_i\lambda_2^2} + \frac{\varepsilon}{\lambda_i}\right),$$
$$\frac{\partial\Psi_{\varepsilon,2}}{\partial x_i} = O\left(\frac{1}{\lambda_i} + \varepsilon^2\lambda_i\log\lambda_i + |x_i-\overline{y}| + \lambda_i\sum_{q=1,2}\frac{|x_q-\overline{y}|^2}{\lambda_q^2}\right).$$

The solution of the system in A, B and C shows that

$$\begin{cases} A_i = O\left(|\beta| + \frac{1}{\lambda_2^2} + \varepsilon\log\lambda_2 + |x_i-\overline{y}|^2\right), \quad B_i = O\left(\frac{\lambda_i}{\lambda_2^2} + \varepsilon\lambda_i\right), \\ C_1 = O\left(\frac{1}{\lambda_1^3} + \frac{\varepsilon^2\log\lambda_1}{\lambda_1} + \frac{|x_1-\overline{y}|}{\lambda_1^2} + \frac{|x_2-\overline{y}|^2}{\lambda_1\lambda_2^2}\right);\ C_2 = O\left(\frac{1}{\lambda_2^3} + \frac{\varepsilon^2\log\lambda_2}{\lambda_2} + \frac{|x_2-\overline{y}|}{\lambda_2^2}\right). \end{cases} \tag{5.7}$$

This makes us able to evaluate the right hand side in the equations (E_{λ_i}) and (E_{x_i}), namely as in the proof of Theorem 1.1, we get

$$B_1\left\langle\frac{\partial^2 P\delta_1}{\partial\lambda_1^2},\overline{v}\right\rangle + \sum_{j=1}^{4} C_{1j}\left\langle\frac{\partial^2 P\delta_1}{\partial x_1^j\partial\lambda_1},\overline{v}\right\rangle = O\left(\left(\frac{\varepsilon}{\lambda_1} + \frac{|x_1-\overline{y}|}{\lambda_1^2} + \frac{1}{\lambda_1\lambda_2^2}\right)\|\overline{v}\|\right), \tag{5.8}$$

$$B_2\left\langle\frac{\partial^2 P\delta_2}{\partial\lambda_1^2},\overline{v}\right\rangle + \sum_{j=1}^{4} C_{2j}\left\langle\frac{\partial^2 P\delta_2}{\partial x_2^j\partial\lambda_2},\overline{v}\right\rangle = O\left(\left(\frac{1}{\lambda_2^2} + \frac{\varepsilon}{\lambda_2} + \frac{|\overline{y}-x_2|}{\lambda_2^2}\right)\|\overline{v}\|\right), \tag{5.9}$$

$$B_1\left\langle\frac{\partial^2 P\delta_1}{\partial\lambda_1\partial x_1},\overline{v}\right\rangle + \sum_{j=1}^{4} C_{1j}\left\langle\frac{\partial^2 P\delta_1}{\partial x_1^j\partial x_1},\overline{v}\right\rangle = O(\lambda_2\|\overline{v}\|)\,, \tag{5.10}$$

$$B_2\left\langle\frac{\partial^2 P\delta_2}{\partial\lambda_2\partial x_2},\overline{v}\right\rangle + \sum_{j=1}^{4} C_{2j}\left\langle\frac{\partial^2 P\delta_2}{\partial x_2^j\partial x_2},\overline{v}\right\rangle = O(\|\overline{v}\|)\,. \tag{5.11}$$

Now, we consider a point $\overline{y}$ in S^4_+ such that $\overline{y}$ is a nondegenerate critical point of K. We set

$$\frac{1}{\lambda_2} = \left(\frac{S_4}{4c_2}\right)^{1/2}\Lambda(\overline{y})(1+\zeta_2)\varepsilon^{1/2};\quad \frac{\lambda_2^3}{\lambda_1} = \frac{1}{\Lambda(\overline{y})^2}(1+\zeta_1),\qquad x_i = \overline{y}+\xi_i\,,$$

where $\zeta_i\in\mathbb{R}$ and $\xi_i\in\mathbb{R}^4$ are assumed to be small and

$$\Lambda(\overline{x}) := \overline{\Lambda} = \sqrt{2}\Big(-H(\overline{y},\overline{y}) - \frac{\Delta K(\overline{y})}{12K(\overline{y})}\Big)^{-1/2}.$$

With these changes of variables, (E_{α_i}) is equivalent to (4.19). Now, using (5.2), we show by an easy computation that

$$\begin{aligned}\frac{S_4}{4}\varepsilon - c_2\frac{\lambda_2}{\lambda_1} &= \frac{S_4}{4}\varepsilon - \frac{c_2}{\overline{\Lambda}^2}\frac{1}{\lambda_2^2}(1+\zeta_1) = \frac{S_4}{4}\varepsilon - \frac{S_4}{4}(1+\zeta_1)(1+\zeta_2)^2\varepsilon \\ &= -\frac{S_4}{4}(\zeta_1+2\zeta_2)\varepsilon + O\big(\varepsilon(\zeta_1^2+\zeta_2^2)\big).\end{aligned}$$

Thus, (E_{λ_1}) is equivalent, on account of (5.4) and (5.8), to

$$\zeta_1 + 2\zeta_2 = V_{\lambda_1}(\varepsilon,\beta,\zeta,\xi) = O\big(\varepsilon|\log\varepsilon| + |\beta| + |\zeta|^2 + |\xi|^2\big). \tag{5.12}$$

Using (5.3), we have

$$\begin{aligned}&\frac{S_4}{4}\varepsilon + \frac{c_2}{4}\Big(4H(\overline{y}+\xi_2,\overline{y}+\xi_2) + \frac{\Delta K(\overline{y}+\xi_2)}{3K(\overline{y}+\xi_2)}\Big)\frac{1}{\lambda_2^2} + c_2\frac{\lambda_2}{\lambda_1} \\ &= \frac{S_4}{4}\varepsilon + \frac{S_4}{4}\overline{\Lambda}^2(1+\zeta_2)^2\varepsilon\Big(H(\overline{y},\overline{y}) + \frac{\Delta K(\overline{y})}{12K(\overline{y})} + 2\frac{\partial H}{\partial a}(\overline{y},\overline{y})\xi_2 + \frac{\nabla\Delta K(\overline{y})\xi_2}{12K(\overline{y})} + O(|\xi_2|^2)\Big) \\ &\quad + \frac{\varepsilon S_4}{4}(1+\zeta_1)(1+\zeta_2)^2 \\ &= \frac{S_4}{4}\varepsilon\Big(\zeta_1 - 2\zeta_2 + \frac{\overline{\Lambda}^2}{12K(\overline{y})}\nabla\Delta K(\overline{y})\xi_2 + 2\overline{\Lambda}^2\frac{\partial H}{\partial a}(\overline{y},\overline{y})\xi_2\Big) + O\big(\varepsilon(|\zeta|^2+|\xi_2|^2)\big).\end{aligned}$$

This implies that (E_{λ_2}) is equivalent, on account of (5.4) and (5.8), to

$$\zeta_1 - 2\zeta_2 + \overline{\Lambda}^2\Big(\frac{1}{12K(\overline{y})}\nabla\Delta K(\overline{y}) + 2\frac{\partial H}{\partial a}(\overline{y},\overline{y})\Big)\xi_2 = V_{\lambda_2}(\varepsilon,\beta,\zeta,\xi), \tag{5.13}$$

where $V_{\lambda_2}(\varepsilon,\beta,\zeta,\xi) = O(\varepsilon^{1/2} + |\beta| + |\zeta|^2 + |\xi|^2)$. Using (5.5), (5.6) and (5.10), we see that (E_{x_i}) is equivalent to

$$D^2K(\overline{y})\xi_i = V_{x_i}(\varepsilon,\beta,\zeta,\xi) = O\big((\varepsilon|\ln\varepsilon|)^{1/2} + |\beta|^2 + |\xi|^2\big). \tag{5.14}$$

We remark that V_{α_i}, V_{λ_i} and V_{x_i} are smooth functions. This system may be written as

$$\begin{cases}\beta = V(\varepsilon,\beta,\zeta,\xi), \\ L_2(\zeta,\xi) = W_2(\varepsilon,\beta,\zeta,\xi),\end{cases} \tag{5.15}$$

where L_2 is a fixed linear operator of $\mathbb{R}^{10}$ defined by

$$L_2(\zeta,\xi) = \Big(\zeta_1 + 2\zeta_2;\, \zeta_1 - 2\zeta_2 + \overline{\Lambda}^2\Big(\frac{1}{12K(\overline{y})}\nabla\Delta K(\overline{y}) + \frac{\partial H(\overline{y},\overline{y})}{\partial a}\Big)\xi_2;\, D^2K(\overline{y})\xi_1;\, D^2K(\overline{y})\xi_2\Big),$$

and V, W_2 are smooth functions satisfying

$$\begin{cases}V(\varepsilon,\beta,\zeta,\xi) = O(\varepsilon|\log\varepsilon| + |\beta|^2 + |\xi|^2), \\ W_2(\varepsilon,\beta,\zeta,\xi) = O(\varepsilon^{1/2} + |\beta|^2 + |\zeta|^2 + |\xi|^2).\end{cases} \tag{5.16}$$

$\overline{y}$ is a nondegenerate critical point of K by assumption, L_2 is invertible, and Brouwer's fixed point theorem shows that (5.15) has a solution $(\beta^\varepsilon, \zeta^\varepsilon, \xi^\varepsilon)$ for ε small enough, such that

$$|\beta^\varepsilon| = O(\varepsilon|\log\varepsilon|), \quad |\zeta^\varepsilon| = O(\varepsilon^{1/2}), \quad |\xi^\varepsilon| = O\big((\varepsilon|\ln\varepsilon|)^{1/2}\big).$$

By construction, the corresponding $u_\varepsilon \in H^1(S^4_+)$ is a critical point of I_ε, i.e. u_ε satisfies (S_ε). The proof of Theorem 1.2 is thereby completed.

Acknowledgment. The second author gratefully acknowledges the Deanship of Scientific Research at Taibah University on material and moral support in the financing of this research project.

REFERENCES

[1] A. BAHRI, An invarient for Yamabe-type flows with applications to scalar curvature problems in high dimension, A celebration of J. F. Nash Jr., Duke Math. J. 81 (1996), 323–466.

[2] A. BAHRI, Y. Y. LI AND O. REY, On a variational problem with lack of compactness: The topological effect of the critical points at infinity, Calc. Var. Partial Differential Equations 3 (1995), 67–94.

[3] M. BEN AYED, K. EL MEHDI AND M. OULD AHMEDOU, The scalar curvature problem on the four dimensional half sphere, Calc. Var. 22 (2005), 465–482.

[4] M. BEN AYED, R. GHOUDI AND K. OULD BOUH, Existence of conformal metrics with prescribed scalar curvature on the four dimensional half sphere, NoDEA Nonlinear Differential Equations Appl. 19 (2012), 629–662.

[5] G. BIANCHI AND X. B. PAN, Yamabe equations on half-spaces, Nonlinear Anal. 37 (1999), 161–186.

[6] Z. DJADLI, A. MALCHIODI AND M. OULD AHMEDOU, Prescribing the scalar and the boundary mean curvature on the three dimensional half sphere, J. Geom. Anal. 13 (2003), 233–267.

[7] R. GHOUDI, Blowing up of sign-changing solutions to an elliptic subcritical equation, J. Partial Differ. Equ. 25 (2012), no. 4, 368–388.

[8] Y. Y. LI, Prescribing scalar curvature on S^n and related topics, Part I, J. Differental Equations 120 (1995), 319–410; Part II, existence and compactness, Comm. Pure Appl. Math. 49 (1996), 437–477.

[9] M. MUSSO AND A. PISTOIA, Tower of bubbles for almost critical problems in general domains, J. Math. Pures Appl. 93 (2010), no. 1–140.

[10] A. PISTOIA AND T. WETH, Sign-changing bubble-tower solutions in a slightly subcritical semilinear Dirichlet problem, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (2007), 325–340.

[11] O. REY, The topological impact of critical points at infinity in a variational problem with lack of compactness: the dimension 3, Adv. Differential Equations 4 (1999), 581–616.

UNIVERSITÉ DE GABÈS
FACULTÉ DES SCIENCES
CITÉ EL RIADH, GABÈS
TUNISIA

E-mail address: ghoudi.rabeh@yahoo.fr

DEPARTMENT OF MATHEMATICS
COLLEGE OF SCIENCES
TAIBAH UNIVERSITY
P.O.BOX: 30002
ALMADINAH ALMUNAWWARAH
KINGDOM OF SAUDI ARABIA

E-mail address: hbouh@taibahu.edu.sa
: kamal_bouh@yahoo.fr

http://www.math.tohoku.ac.jp/tmj/Esubmit.html#article