

HOMOGENEOUS RICCI SOLITON HYPERSURFACES IN THE COMPLEX HYPERBOLIC SPACES

TAKAHIRO HASHINAGA, AKIRA KUBO AND HIROSHI TAMARU

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Abstract. A Lie hypersurface in the complex hyperbolic space is an orbit of a cohomogeneity one action without singular orbit. In this paper, we classify Ricci soliton Lie hypersurfaces in the complex hyperbolic spaces.

1. Introduction. Homogeneous submanifolds in Riemannian symmetric spaces of noncompact type G/K have provided a lot of interesting examples of submanifolds. We refer to [1, 3, 4, 5, 6, 8, 9, 10, 16] and references therein. Typical and the simplest examples are given by the nilpotent part N of the Iwasawa decomposition $G = KAN$. All orbits of N in G/K are isometrically congruent to each other (see [3, 10]), and they are known to be Ricci solitons ([12]). Recall that a complete Riemannian manifold (M, g) is called a *Ricci soliton* if

$$(1.1) \quad \text{Ric}_g = cg - \frac{1}{2}\mathcal{L}_X g$$

holds for some $c \in \mathbb{R}$ and some complete vector field X on M , where Ric_g denotes the Ricci operator of (M, g) and \mathcal{L}_X is the usual Lie derivative. The vector field X is called the *potential vector field*. A Ricci soliton is a natural generalization of an Einstein manifold.

In symmetric spaces of noncompact type, homogeneous hypersurfaces called Lie hypersurfaces are of particular interest. For an isometric action on a Riemannian manifold, maximal dimensional orbits are said to be *regular*, and other orbits *singular*. The codimension of a regular orbit is called the *cohomogeneity* of an action. A *Lie hypersurface* is an orbit of a cohomogeneity one action without singular orbit. The notion of Lie hypersurfaces has been introduced by Berndt ([1]). For symmetric spaces of noncompact type G/K , the orbits of a cohomogeneity one action without singular orbit form a Riemannian foliation ([2]). Furthermore, when G/K is irreducible, such actions have been completely classified up to orbit equivalence ([4]). Recall that two isometric actions on a Riemannian manifold are said to be *orbit equivalent* if there exists an isometry of the manifold mapping the orbits of one of these actions onto the orbits of the other action.

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In this paper, we focus on Lie hypersurfaces in the complex hyperbolic spaces

$$(1.2) \quad \mathbb{C}H^n = \mathrm{SU}(1, n)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))$$

with $n \geq 2$. Owing to [4], for rank one symmetric spaces of noncompact type, there exist exactly two cohomogeneity one actions without singular orbit up to orbit equivalence. One of the actions is given by N , the nilpotent part of the Iwasawa decomposition of the isometry group. The orbits of N , which are isometrically congruent to each other, are horospheres. The other action induces the so-called *solvable foliation*. In the case of the real hyperbolic space $\mathbb{R}H^n$, intrinsic geometry of the Lie hypersurfaces is well-known. In fact, the horosphere is flat, and the solvable foliation consists of a totally geodesic $\mathbb{R}H^{n-1}$ and its equidistant hypersurfaces. It is easy to see that all Lie hypersurfaces in $\mathbb{R}H^n$ have constant curvature. On the contrary, in the case of $\mathbb{C}H^n$ with $n \geq 2$, the situation is much more interesting and nontrivial. In this case, the solvable foliation consists of the homogeneous ruled minimal hypersurface and its equidistant hypersurfaces. The homogeneous ruled minimal hypersurface in $\mathbb{C}H^n$ is the ruled real hypersurface determined by a horocycle in a totally geodesic $\mathbb{R}H^2$ in $\mathbb{C}H^n$ ([14]). Note that this is isometric to a fan (we refer to [11]).

The purpose of this paper is to study and classify Ricci soliton Lie hypersurfaces in $\mathbb{C}H^n$. It is well-known that there exist no Einstein hypersurfaces in $\mathbb{C}H^n$ (see [15]). On the other hand, as we mentioned before, the horosphere is a Ricci soliton. Our main theorem classifies Ricci soliton Lie hypersurfaces in $\mathbb{C}H^n$. It would be interesting that the result is different in the cases $n = 2$ and $n > 2$.

THEOREM. *A Lie hypersurface in $\mathbb{C}H^n$ is a Ricci soliton if and only if*

- (1) *it is isometrically congruent to a horosphere, or*
- (2) *$n = 2$ and it is isometrically congruent to the homogeneous ruled minimal hypersurface.*

The above theorem is relevant to the results by Cho and Kimura ([7]). Among others, for real hypersurfaces in non-flat complex space forms, they proved

- there do not exist compact Hopf hypersurfaces which are Ricci solitons,
- there do not exist ruled hypersurfaces which are gradient Ricci solitons.

Recall that a Ricci soliton is said to be *gradient* if the potential vector field can be expressed as the gradient of a smooth function. We also recall that a hypersurface in $\mathbb{C}H^n$ is said to be *Hopf* if the structure vector field $J\xi$ is an eigenvector of the shape operator, where J is the complex structure and ξ is a unit normal vector. Our theorem implies that the assumptions on these nonexistence results cannot be removed. Note that a horosphere is a Hopf hypersurface (see [1, 9]), which is a Ricci soliton, but not compact. When $n = 2$, the homogeneous ruled minimal hypersurface is a ruled hypersurface, which is a Ricci soliton, but not gradient (see [13]).

This paper is organized as follows. In Section 2, we recall algebraic Ricci solitons, which play important roles in studying homogeneous Ricci solitons. In Section 3, we recall some necessary results on Lie hypersurfaces in $\mathbb{C}H^n$. In Section 4 we prove the main theorem.

2. Ricci solitons and algebraic Ricci solitons. The notion of algebraic Ricci solitons has essentially been introduced by Lauret ([12, 13]). In this section, we recall a relationship between left-invariant Ricci solitons and algebraic Ricci solitons. Let G be a simply-connected Lie group, and g be a left-invariant metric on G . The *corresponding metric Lie algebra* of (G, g) is a pair $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$, where \mathfrak{g} is the Lie algebra of G and $\langle \cdot, \cdot \rangle$ is the inner product on \mathfrak{g} corresponding to g . We can study curvatures of (G, g) in term of the corresponding metric Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$.

First of all, we recall some curvature formulas for a metric Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$. Let $X, Y \in \mathfrak{g}$. The Levi-Civita connection ∇ of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is given by

$$(2.1) \quad 2\langle \nabla_X Y, Z \rangle = \langle [X, Y], Z \rangle + \langle [Z, X], Y \rangle + \langle X, [Z, Y] \rangle \quad (Z \in \mathfrak{g}).$$

Then the *Riemannian curvature* R of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is defined by

$$(2.2) \quad R(X, Y) := \nabla_{[X, Y]} - \nabla_X \nabla_Y + \nabla_Y \nabla_X,$$

and the *Ricci operator* Ric of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is defined by

$$(2.3) \quad \text{Ric}(X) := \sum R(E_i, X)E_i,$$

where $\{E_i\}$ is an orthonormal basis of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$.

We now recall the notion of algebraic Ricci solitons. For a Lie algebra \mathfrak{g} , let us denote the Lie algebra of derivations by

$$(2.4) \quad \text{Der}(\mathfrak{g}) := \{D \in \mathfrak{gl}(\mathfrak{g}) \mid D[X, Y] = [D(X), Y] + [X, D(Y)] \ (\forall X, Y \in \mathfrak{g})\}.$$

DEFINITION 2.1. A metric Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is called an *algebraic Ricci soliton* if the following holds:

$$(2.5) \quad \text{Ric} = c \cdot \text{id} + D,$$

for some $c \in \mathbb{R}$ and $D \in \text{Der}(\mathfrak{g})$.

A relationship between left-invariant Ricci solitons and algebraic Ricci solitons is given as follows. According to [13], if the corresponding metric Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is an algebraic Ricci soliton, then (G, g) is a Ricci soliton. The converse also holds if G is completely solvable. Recall that G is said to be *completely solvable* if G is solvable and the eigenvalues of any $\text{ad } X$ are all real.

THEOREM 2.2 ([13]). *Let (G, g) be a simply-connected completely solvable Lie group equipped with a left-invariant metric, and $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ be the corresponding metric Lie algebra. Then (G, g) is a Ricci soliton if and only if $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is an algebraic Ricci soliton.*

3. Lie hypersurfaces in the complex hyperbolic space. In this section, we recall Lie hypersurfaces in the complex hyperbolic space $\mathbb{C}H^n$ and their curvature properties. We refer to [1, 4, 9].

DEFINITION 3.1. The triple $(\mathfrak{s}, \langle \cdot, \cdot \rangle, J)$ is called the *solvable model* of the complex hyperbolic space $\mathbb{C}H^n$ if

- (1) \mathfrak{s} is a Lie algebra, and there exists a basis $\{A_0, X_1, Y_1, \dots, X_{n-1}, Y_{n-1}, Z_0\}$ whose bracket products are given by

$$[A_0, X_i] = (1/2)X_i, [A_0, Y_i] = (1/2)Y_i, [A_0, Z_0] = Z_0, [X_i, Y_i] = Z_0,$$

and zero otherwise,

- (2) $\langle \cdot, \cdot \rangle$ is an inner product on \mathfrak{s} so that the above basis is orthonormal,
 (3) J is a complex structure on \mathfrak{s} given by

$$J(A_0) = Z_0, J(Z_0) = -A_0, J(X_i) = Y_i, J(Y_i) = -X_i.$$

We use \mathfrak{s} instead of $(\mathfrak{s}, \langle \cdot, \cdot \rangle, J)$ for simplicity, and denote by S the corresponding simply-connected solvable Lie group with the induced left-invariant Riemannian metric and complex structure. It is known that S can be identified with $\mathbb{C}H^n$ with holomorphic sectional curvature -1 . In fact, S coincides with the solvable part of the Iwasawa decomposition of $SU(1, n)$.

In the rest of the section, we recall some known results on Lie hypersurfaces in $\mathbb{C}H^n$. First of all, we recall the definition.

DEFINITION 3.2. A Lie hypersurface in $\mathbb{C}H^n$ is an orbit of a cohomogeneity one action without singular orbits.

It is known that there exist exactly two cohomogeneity one actions on $\mathbb{C}H^n$ without singular orbit up to orbit equivalence.

THEOREM 3.3 ([4]). An isometric action on $\mathbb{C}H^n$ is a cohomogeneity one action without singular orbit if and only if it is orbit equivalent to one of the actions of $S(\pi/2)$ or $S(0)$, where $S(\theta)$ is the connected Lie subgroup of S with Lie algebra

$$\mathfrak{s}(\theta) := \mathfrak{s} \ominus \mathbb{R}(\cos(\theta)X_1 + \sin(\theta)A_0).$$

Note that \ominus means the orthogonal complement with respect to $\langle \cdot, \cdot \rangle$. By studying the orbits of these actions, one obtains the classification of Lie hypersurfaces in $\mathbb{C}H^n$ up to isometric congruence.

THEOREM 3.4 ([1]). Every Lie hypersurface in $\mathbb{C}H^n$ is isometrically congruent to the orbit $S(\theta).o$ through the origin o for some $\theta \in [0, \pi/2]$.

Note that $S(\pi/2) = N$ and hence the orbit $S(\pi/2).o$ is a horosphere. The orbit $S(0).o$ is the homogeneous ruled minimal hypersurface. If $0 < \theta < \pi/2$, then the orbit $S(\theta).o$ is isometrically congruent to $S(0).p$ for some p , which is an equidistant hypersurface to $S(0).o$.

We now study the geometry of Lie hypersurfaces. The Lie hypersurface $S(\theta).o$ can be studied in terms of the metric Lie algebra $(\mathfrak{s}(\theta), \langle \cdot, \cdot \rangle)$, where this $\langle \cdot, \cdot \rangle$ denotes the restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{s}(\theta)$. First of all, by definition, $\mathfrak{s}(\theta)$ is a codimension one subalgebra of \mathfrak{s} , and it is solvable. Denote by

$$T := \cos(\theta)A_0 - \sin(\theta)X_1, \quad \mathfrak{v}_0 := \text{span}\{X_2, Y_2, \dots, X_{n-1}, Y_{n-1}\}.$$

Then we have the orthogonal decomposition

$$(3.1) \quad \mathfrak{s}(\theta) = \text{span}\{T\} \oplus \text{span}\{Y_1\} \oplus \mathfrak{v}_0 \oplus \text{span}\{Z_0\}.$$

One can directly see the following equations, which will be used hereafter.

LEMMA 3.5. *Let $V, W \in \mathfrak{v}_0$. Then we have*

- (1) $[T, Y_1] = (1/2) \cos(\theta)Y_1 - \sin(\theta)Z_0$,
- (2) $[T, V] = (1/2) \cos(\theta)V$,
- (3) $[T, Z_0] = \cos(\theta)Z_0$,
- (4) $[V, W] = \langle [V, W], Z_0 \rangle Z_0$, especially, $[X_k, Y_k] = Z_0$ for all $k = 2, \dots, n - 1$.

Finally in this section, we recall the formulas for Ricci curvatures of the Lie hypersurfaces. By direct calculations in terms of $(\mathfrak{s}(\theta), \langle, \rangle)$, one has the following.

PROPOSITION 3.6 ([9]). *The Ricci operator of $(\mathfrak{s}(\theta), \langle, \rangle)$ satisfies*

- (1) $\text{Ric}(T) = -(1/4)(2 + (2n - 1) \cos^2(\theta))T$,
- (2) $\text{Ric}(Y_1) = -(1/4)(2 + (2n - 3) \cos^2(\theta))Y_1 + (n/2) \sin(\theta) \cos(\theta)Z_0$,
- (3) $\text{Ric}(V) = -(1/4)(2 + (2n - 1) \cos^2(\theta))V$ for any $V \in \mathfrak{v}_0$,
- (4) $\text{Ric}(Z_0) = (n/2) \sin(\theta) \cos(\theta)Y_1 + (1/2)((n - 1) - 2n \cos^2(\theta))Z_0$.

4. Main result. In this section we classify Ricci soliton Lie hypersurfaces in $\mathbb{C}H^n$. First of all, we see that we have only to work on algebraic Ricci solitons.

LEMMA 4.1. *For any $\theta \in [0, \pi/2]$, the Lie hypersurface $S(\theta)$ is a Ricci soliton if and only if $(\mathfrak{s}(\theta), \langle, \rangle)$ is an algebraic Ricci soliton.*

PROOF. Let $\theta \in [0, \pi/2]$. Owing to Theorem 2.2, we have only to check that $\mathfrak{s}(\theta)$ is completely solvable. Take any $X \in \mathfrak{s}(\theta)$. One can write

$$(4.1) \quad X = a_1T + a_2Y_1 + V + a_3Z_0 \quad (\text{for } V \in \mathfrak{v}_0).$$

According to Lemma 3.5, we can show directly that

$$(4.2) \quad \begin{aligned} \text{ad } X(T) &= -(1/2)a_2 \cos(\theta)Y_1 - (1/2) \cos(\theta)V + (a_2 \sin(\theta) - a_3 \cos(\theta))Z_0, \\ \text{ad } X(Y_1) &= (1/2)a_1 \cos(\theta)Y_1 - a_1 \sin(\theta)Z_0, \\ \text{ad } X(W) &= (1/2)a_1 \cos(\theta)W + \langle [V, W], Z_0 \rangle Z_0, \\ \text{ad } X(Z_0) &= a_1 \cos(\theta)Z_0, \end{aligned}$$

where $W \in \mathfrak{v}_0$. Hence $\text{ad } X$ is represented by a triangular matrix with respect to the canonical basis

$$(4.3) \quad \{T, Y_1, X_2, Y_2, \dots, X_{n-1}, Y_{n-1}, Z_0\}$$

of $\mathfrak{s}(\theta)$. Therefore all eigenvalues of $\text{ad } X$ are real, which completes the proof. □

From now on, we shall study whether $(\mathfrak{s}(\theta), \langle, \rangle)$ is an algebraic Ricci soliton. We separate this into three cases, namely $\theta = \pi/2$, $\theta \in]0, \pi/2[$, and $\theta = 0$, and discuss them individually. We note that some of the propositions below have been known, and that some of the others can be proved by applying the structure theorem ([13, Theorem. 4.8]) for algebraic Ricci solitons. However, we here give proofs for all propositions in terms of the explicit formulas for the Ricci operators, since they are elementary and direct.

Firstly, we see that $(\mathfrak{s}(\pi/2), \langle, \rangle)$ is always an algebraic Ricci soliton. Note that $\mathfrak{s}(\pi/2)$ is the Heisenberg Lie algebra, and that \langle, \rangle is the standard inner product. It has been known that $(\mathfrak{s}(\pi/2), \langle, \rangle)$ is an algebraic Ricci soliton (see [12]).

PROPOSITION 4.2. *Let $n \geq 2$. Then $(\mathfrak{s}(\pi/2), \langle, \rangle)$ is an algebraic Ricci soliton.*

PROOF. We consider the basis $\{X_1, Y_1, \dots, X_{n-1}, Y_{n-1}, Z_0\}$ of $\mathfrak{s}(\pi/2)$. Denote by Ric the matrix expression of the Ricci operator of $\mathfrak{s}(\pi/2)$ with respect to this basis. Then Proposition 3.6 yields that

$$(4.4) \quad Ric = \frac{1}{2} \begin{pmatrix} -1 & & & & & \\ & \ddots & & & & \\ & & -1 & & & \\ & & & \ddots & & \\ & & & & n-1 & \\ & & & & & \ddots \end{pmatrix}.$$

In order to show that it is an algebraic Ricci soliton, we define

$$(4.5) \quad c := -\frac{n+1}{2}, \quad D := \frac{n}{2} \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 2 \end{pmatrix}.$$

Then one can show directly that $Ric = c \cdot id + D$ and $D \in Der(\mathfrak{s}(\pi/2))$. □

Secondly, we consider the case $\theta \in]0, \pi/2[$. In this case, $(\mathfrak{s}(\theta), \langle, \rangle)$ cannot be an algebraic Ricci soliton. The proof starts with the following lemma for a derivation of $\mathfrak{s}(\theta)$.

LEMMA 4.3. *Let $n \geq 2$. If $\theta \in [0, \pi/2[$, then $\langle D(Z_0), Y_1 \rangle = 0$ holds for any $D \in Der(\mathfrak{s}(\theta))$.*

PROOF. Take any $D \in Der(\mathfrak{s}(\theta))$. By the definition of a derivation, one has

$$(4.6) \quad \langle D[T, Z_0], Y_1 \rangle = \langle [D(T), Z_0], Y_1 \rangle + \langle [T, D(Z_0)], Y_1 \rangle.$$

We calculate the both sides of this equation. By Lemma 3.5, the left-hand side satisfies

$$(4.7) \quad \langle D[T, Z_0], Y_1 \rangle = \langle D(\cos(\theta)Z_0), Y_1 \rangle = \cos(\theta)\langle D(Z_0), Y_1 \rangle.$$

Next we calculate the right-hand side. It also follows from Lemma 3.5 that

$$(4.8) \quad [\mathfrak{s}(\theta), Z_0] \subset \text{span}\{Z_0\}.$$

Then the first term of the right-hand side of (4.6) is

$$(4.9) \quad \langle [D(T), Z_0], Y_1 \rangle = 0.$$

In order to calculate the second term, take the canonical orthonormal basis

$$(4.10) \quad \{E_i\} = \{T, Y_1, X_2, Y_2, \dots, X_{n-1}, Y_{n-1}, Z_0\}$$

of $\mathfrak{s}(\theta)$. Lemma 3.5 yields that

$$(4.11) \quad \langle [T, E_i], Y_1 \rangle = \begin{cases} (1/2) \cos(\theta) & (E_i = Y_1), \\ 0 & (\text{otherwise}). \end{cases}$$

Then the second term of the right-hand side of (4.6) satisfies

$$(4.12) \quad \begin{aligned} \langle [T, D(Z_0)], Y_1 \rangle &= \langle [T, \sum \langle D(Z_0), E_i \rangle E_i], Y_1 \rangle \\ &= \sum \langle D(Z_0), E_i \rangle \langle [T, E_i], Y_1 \rangle \\ &= (1/2) \cos(\theta) \langle D(Z_0), Y_1 \rangle. \end{aligned}$$

Altogether, we obtain

$$(4.13) \quad \cos(\theta) \langle D(Z_0), Y_1 \rangle = 0 + (1/2) \cos(\theta) \langle D(Z_0), Y_1 \rangle.$$

Since $\theta \in [0, \pi/2[$, this completes the proof. □

PROPOSITION 4.4. *Let $n \geq 2$. If $\theta \in]0, \pi/2[$, then $(\mathfrak{s}(\theta), \langle, \rangle)$ is not an algebraic Ricci soliton.*

PROOF. We show this by contradiction. Assume that $(\mathfrak{s}(\theta), \langle, \rangle)$ is an algebraic Ricci soliton. By definition, there exist $c \in \mathbb{R}$ and $D \in \text{Der}(\mathfrak{s}(\theta))$ such that

$$(4.14) \quad \text{Ric} = c \cdot \text{id} + D.$$

One thus has

$$(4.15) \quad \langle \text{Ric}(Z_0), Y_1 \rangle = c \langle Z_0, Y_1 \rangle + \langle D(Z_0), Y_1 \rangle.$$

We calculate the both sides of this equation. By Proposition 3.6, we have

$$(4.16) \quad \langle \text{Ric}(Z_0), Y_1 \rangle = (n/2) \sin(\theta) \cos(\theta).$$

On the other hand, Lemma 4.3 yields that

$$(4.17) \quad c \langle Z_0, Y_1 \rangle + \langle D(Z_0), Y_1 \rangle = 0.$$

Since $\theta \in]0, \pi/2[$, this is a contradiction. □

Lastly, we study the case $\theta = 0$. First of all, we consider the case $n > 2$, and show that $(\mathfrak{s}(0), \langle, \rangle)$ is not an algebraic Ricci soliton. As in the previous case, we start with the next lemma on a derivation. Note that $\mathfrak{v}_0 \neq \{0\}$ if $n > 2$.

LEMMA 4.5. *Assume that $n > 2$, and let $D \in \text{Der}(\mathfrak{s}(0))$. Then we have*

- (1) $\langle D(A_0), A_0 \rangle = 0$, and
- (2) $\langle D(X_k), X_k \rangle + \langle D(Y_k), Y_k \rangle = \langle D(Z_0), Z_0 \rangle$ for $k = 2, \dots, n - 1$.

PROOF. Take any $D \in \text{Der}(\mathfrak{s}(0))$. The proof is similar to that of Lemma 4.3. We use the canonical orthonormal basis of $\mathfrak{s}(0)$,

$$(4.18) \quad \{E_i\} = \{A_0, Y_1, X_2, Y_2, \dots, X_{n-1}, Y_{n-1}, Z_0\}.$$

We show (1). By the definition of a derivation, one has

$$(4.19) \quad \langle D[A_0, Z_0], Z_0 \rangle = \langle [D(A_0), Z_0], Z_0 \rangle + \langle [A_0, D(Z_0)], Z_0 \rangle.$$

The left-hand side of (4.19) satisfies

$$(4.20) \quad \langle D[A_0, Z_0], Z_0 \rangle = \langle D(Z_0), Z_0 \rangle .$$

In order to calculate the right-hand side, one needs

$$(4.21) \quad \langle [E_i, Z_0], Z_0 \rangle = \begin{cases} 1 & (E_i = A_0), \\ 0 & (\text{otherwise}). \end{cases}$$

Hence we have

$$(4.22) \quad \langle [D(A_0), Z_0], Z_0 \rangle = \sum \langle D(A_0), E_i \rangle \langle [E_i, Z_0], Z_0 \rangle = \langle D(A_0), A_0 \rangle .$$

Similarly, one can see that

$$(4.23) \quad \langle [A_0, E_i], Z_0 \rangle = \begin{cases} 1 & (E_i = Z_0), \\ 0 & (\text{otherwise}). \end{cases}$$

This yields that

$$(4.24) \quad \langle [A_0, D(Z_0)], Z_0 \rangle = \sum \langle D(Z_0), E_i \rangle \langle [A_0, E_i], Z_0 \rangle = \langle D(Z_0), Z_0 \rangle .$$

Altogether, we have

$$(4.25) \quad \langle D(Z_0), Z_0 \rangle = \langle D(A_0), A_0 \rangle + \langle D(Z_0), Z_0 \rangle .$$

This completes the proof of the first assertion.

We show (2). Take any $k = 2, \dots, n - 1$. We start from

$$(4.26) \quad \langle D[X_k, Y_k], Z_0 \rangle = \langle [D(X_k), Y_k], Z_0 \rangle + \langle [X_k, D(Y_k)], Z_0 \rangle .$$

One can calculate each term as follows:

$$(4.27) \quad \begin{aligned} \langle D[X_k, Y_k], Z_0 \rangle &= \langle D(Z_0), Z_0 \rangle , \\ \langle [D(X_k), Y_k], Z_0 \rangle &= \sum \langle D(X_k), E_i \rangle \langle [E_i, Y_k], Z_0 \rangle = \langle D(X_k), X_k \rangle , \\ \langle [X_k, D(Y_k)], Z_0 \rangle &= \sum \langle D(Y_k), E_i \rangle \langle [X_k, E_i], Z_0 \rangle = \langle D(Y_k), Y_k \rangle . \end{aligned}$$

This completes the proof of the second assertion. □

PROPOSITION 4.6. *Let $n > 2$. Then $(\mathfrak{s}(0), \langle, \rangle)$ is not an algebraic Ricci soliton.*

PROOF. We show this by contradiction. Assume that $(\mathfrak{s}(0), \langle, \rangle)$ is an algebraic Ricci soliton. By definition, there exist $c \in \mathbb{R}$ and $D \in \text{Der}(\mathfrak{s}(0))$ such that

$$(4.28) \quad \text{Ric} = c \cdot \text{id} + D .$$

Then, for any $E \in \mathfrak{s}(0)$ with $\langle E, E \rangle = 1$, one has

$$(4.29) \quad \langle \text{Ric}(E), E \rangle = \langle cE + D(E), E \rangle = c + \langle D(E), E \rangle .$$

Let $V \in \mathfrak{v}_0$ be a unit vector, and we substitute A_0, V , and Z_0 for E . Then, Proposition 3.6 yields that

$$(4.30) \quad \begin{aligned} c + \langle D(A_0), A_0 \rangle &= \langle \text{Ric}(A_0), A_0 \rangle = -(2n + 1)/4 , \\ c + \langle D(V), V \rangle &= \langle \text{Ric}(V), V \rangle = -(2n + 1)/4 , \\ c + \langle D(Z_0), Z_0 \rangle &= \langle \text{Ric}(Z_0), Z_0 \rangle = -(n + 1)/2 . \end{aligned}$$

These equations yield that

$$(4.31) \quad \langle D(A_0), A_0 \rangle = \langle D(V), V \rangle \neq \langle D(Z_0), Z_0 \rangle.$$

Lemma 4.5 (1) yields that

$$(4.32) \quad 0 = \langle D(A_0), A_0 \rangle = \langle D(V), V \rangle.$$

Since $n > 2$, we have $X_2, Y_2 \in \mathfrak{v}_0$, which can be substituted for V . Thus, owing to Lemma 4.5 (2), we have

$$(4.33) \quad \langle D(Z_0), Z_0 \rangle = \langle D(X_2), X_2 \rangle + \langle D(Y_2), Y_2 \rangle = 0.$$

This is a contradiction. □

If $n = 2$, however, $(\mathfrak{s}(0), \langle, \rangle)$ is indeed an algebraic Ricci soliton. Note that $\dim \mathfrak{s}(0) = 3$ in this case, and three-dimensional algebraic Ricci solitons on solvable Lie algebras have been classified by Lauret ([13]). Hence, the following proposition has been essentially known.

PROPOSITION 4.7. *Let $n = 2$. Then $(\mathfrak{s}(0), \langle, \rangle)$ is an algebraic Ricci soliton.*

PROOF. Denote by Ric the matrix expression of the Ricci operator with respect to the basis $\{A_0, Y_1, Z_0\}$ of $\mathfrak{s}(0)$. Then, Proposition 3.6 yields that

$$(4.34) \quad \text{Ric} = \begin{pmatrix} -5/4 & & \\ & -3/4 & \\ & & -3/2 \end{pmatrix}.$$

Let us define

$$(4.35) \quad c := -\frac{5}{4}, \quad D := \begin{pmatrix} 0 & & \\ & 1/2 & \\ & & -1/4 \end{pmatrix}.$$

Then we can show directly that $\text{Ric} = c \cdot \text{id} + D$, and $D \in \text{Der}(\mathfrak{s}(0))$. □

In consequence of the arguments above we conclude the following theorem.

THEOREM 4.8. *The Lie hypersurfaces $S(\theta)$ in $\mathbb{C}H^n$ is a Ricci soliton if and only if*

- (1) $\theta = \pi/2$, or
- (2) $n = 2$ and $\theta = 0$.

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NATIONAL INSTITUTE OF TECHNOLOGY
KITAKYUSHU COLLEGE
5–2–1, SHII, KOKURAMINAMIKU
KITAKYUSHU, FUKUOKA 802–0985
JAPAN

E-mail address: hashinaga@kct.ac.jp

DEPARTMENT OF MATHEMATICS
HIROSHIMA UNIVERSITY
HIGASHI-HIROSHIMA 739–8526
JAPAN

E-mail address: tamaru@math.sci.hiroshima-u.ac.jp

FACULTY OF ECONOMIC SCIENCES
HIROSHIMA SHUDO UNIVERSITY
HIROSHIMA 731–3195
JAPAN

E-mail address: akubo@shubo-u.ac.jp