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CALABI–YAU 3-FOLDS OF BORCEA–VOISIN TYPE AND ELLIPTIC FIBRATIONS

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Abstract. We consider Calabi–Yau 3-folds of Borcea–Voisin type, i.e. Calabi–Yau 3-folds obtained as crepant resolutions of a quotient $(S \times E)/(\alpha_S \times \alpha_E)$, where S is a K3 surface, E is an elliptic curve, $\alpha_S \in \mathrm{Aut}(S)$ and $\alpha_E \in \mathrm{Aut}(E)$ act on the period of S and E respectively with order $n = 2, 3, 4, 6$. The case $n = 2$ is very classical, the case $n = 3$ was recently studied by Rohde, the other cases are less known. First, we construct explicitly a crepant resolution, X, of $(S \times E)/(\alpha_S \times \alpha_E)$ and we compute its Hodge numbers; some pairs of Hodge numbers we found are new. Then, we discuss the presence of maximal automorphisms and of a point with maximal unipotent monodromy for the family of X. Finally, we describe the map $\mathcal{E}_n : X \to S/\alpha_S$ whose generic fiber is isomorphic to E.

1. Introduction. In the Nineties several constructions of Calabi–Yau 3-folds, and in particular of mirror pairs of Calabi–Yau 3-folds, were proposed. One of them, the so called Borcea–Voisin construction, was independently described by Borcea and by Voisin in [Bo], [V1] respectively. The main idea is to consider a quotient of a product of Calabi–Yau varieties of lower dimension. More precisely, one considers two pairs: (E, ι_E), where E is an elliptic curve and ι_E is its hyperelliptic involution, and (S, ι_S), where S is a K3 suface (i.e. a Calabi–Yau surface) and ι_S is an involution of S which does not preserve the period. The quotient $(S \times E)/(\iota_S \times \iota_E)$ is a singular 3-fold admitting a resolution which is a Calabi–Yau 3-fold. Both Borcea and Voisin explicitly constructed such a resolution and computed the Hodge numbers of the families of Calabi–Yau 3-folds obtained in such a way. The classification of the involutions ι_S which act on a K3 surface without fixing the period of S (and so of the K3 surfaces that can be used in the Borcea–Voisin construction) was given by Nikulin in [N]. Several generalizations of the Borcea–Voisin construction have been introduced in the last years (see e.g. [CH], [R1], [R2], [D2], [G]), essentially considering desingularizations of quotients $(Y_1 \times Y_2)/(\alpha_1 \times \alpha_2)$, where Y_i are Calabi–Yau varieties and $\alpha_i \in \mathrm{Aut}(Y_i)$. In order to obtain a Calabi–Yau variety one has to require that the automorphism α_i does not fix the period of Y_i, but $\alpha_1 \times \alpha_2$ fixes the wedge product of the periods of Y_1 and of Y_2.

Here we restrict our attention to Calabi–Yau 3-folds, and thus we can assume that $Y_1 =:$ S is a K3 surface and $Y_2 =: E$ is an elliptic curve. If we require that the order of α_1 and α_2 is the same, say n, then it has to be $n = 2, 3, 4, 6$ (for a more precise statement see Proposition 3.2). Hence, we consider Calabi–Yau 3-folds constructed as resolution of a quo-

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tient $(S \times E)/(\mathbb{Z}/n\mathbb{Z})$ for $n = 2, 3, 4, 6$ and we call them of Borcea–Voisin type. In case $n = 2$ one obtains the "classical" and well known Borcea–Voisin construction. A systematical analysis of the case $n = 3$ is presented in [R1] and [D2] and uses the classification of the non-symplectic automorphisms of K3 surfaces of order 3, described independently by Artebani and Sarti, [AS1], and by Taki, [T]. Sporadic examples of the case $n = 4$ are analyzed in [G], where some peculiar K3 surfaces with a non-symplectic automorphism of order 4 are constructed and the associated Calabi–Yau 3-folds are presented. The complete classification of the K3 surfaces with non-symplectic automorphisms of order 4 and 6 is still unknown, but a lot of it is understood, see [AS2] for $n = 4$ and [D1] for $n = 6$. Hence several families of Calabi–Yau 3-folds of Borcea–Voisin type obtained from quotients by automorphisms of order 4 and 6 can be described.

Given a quotient $(S \times E)/(\mathbb{Z}/n\mathbb{Z})$ as before, there could exist more then one crepant resolution. We construct explicitly one specific crepant resolution (see Sections 4.1, 5.1, 6.1, 7.2) and we call it of type X_n. The properties of the fixed loci of α_S^j, $j = 1, \dots, n-1$, on S determine the Hodge numbers of this 3-fold. We compute these for each admissible value of n (see Propositions 4.1, 5.1, 6.3, 7.3). Some of the Calabi–Yau 3-folds constructed have "new" Hodge numbers (here we refer to the database [J] of the known Calabi–Yau 3-folds).

The 3-folds X of type X_n admit an automorphism induced by $\alpha_S \times \mathrm{id}_E$. In certain cases (see Proposition 3.7) this automorphism is a maximal automorphism for the family, i.e. it deforms to an automorphism of the varieties which are deformations of X. Such an automorphism acts non-trivially on the period of X. In [R1], the maximal automorphisms which act non-trivially on the period are analyzed. In particular, it is proved that if a family of Calabi–Yau 3-folds admits a maximal automorphism which acts on the period as the multiplication by an n-th root of unity, $n \neq 2$, then the family does not admit a point with maximal unipotent monodromy. In [R1] the families of Calabi–Yau 3-folds of Borcea–Voisin type associated to $n = 3$ are considered and the ones with maximal automorphisms are classified, in order to construct families of Calabi–Yau 3-folds without maximal unipotent monodromy. Similarly, here we consider the family of Calabi–Yau 3-folds of Borcea–Voisin type associated to $n = 4, 6$ without maximal unipotent monodromy (see Remarks 6.4, 7.4 and Tables 2, 4, 5). In some of these cases we can moreover prove that the variation of Hodge structures of the family of Calabi–Yau 3-folds of type X_n is essentially the variation of Hodge structures of a family of curves (see Remarks 5.3, 6.5, and Example 7.9).

By construction, each variety X of type X_n is endowed with a map $\mathcal{E}_n : X \to S/\alpha_S$ whose generic fiber is an elliptic curve isomorphic to E. The study of this map is one of the main tools of this paper (see Propositions 4.4, 5.5, 6.8, 7.5): $\mathcal{E}_n$ is an elliptic fibration (with section) if and only if α_S^j does not fix isolated points for any $j = 1, \dots, n-1$. On the other hand, if α_S^j fixes some isolated point for a certain j, the fibers of $\mathcal{E}_n$ over the image of these points in S/α_S are the unique fibers which are not of Kodaira type, and they contain divisors. In any case a distinguished (rational) section is naturally given. In case S/α_S (i.e. the base of the fibration) is smooth, we give a Weierstrass equation for $\mathcal{E}_n$ (see Sections 4.4, 5.2.1, 5.2.2, 6.2.1, 6.2.2, 7.3.1, 7.3.2).

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2. Preliminary results.

2.1. Calabi–Yau d-folds and their automorphisms.

DEFINITION 2.1. A *Calabi–Yau d-fold* is a smooth compact Kähler variety X of dimension d such that the canonical bundle of X is trivial and $h^{i,0}(X) = 0$ if $i \neq 0, d$.

If X is a Calabi–Yau d-fold, $H^{d,0}(X) \simeq \mathbb{C}$ is generated by any non-trivial element in $H^{d,0}(X)$. We call *period* of X, denoted by ω_X, a chosen non-trivial element in $H^{d,0}(X)$, so $H^{d,0}(X) = \langle \omega_X \rangle$.

Let α be an automorphism of X. Then α^* acts on $H^d(X, \mathbb{C})$ preserving the Hodge structure. In particular, $\alpha^*(\omega_X) = \lambda_\alpha \omega_X$ for a certain $\lambda_\alpha \in \mathbb{C}^*$. If $\lambda_\alpha = 1$, we will say that α preserves the period of X.

The Calabi–Yau varieties of dimension 1 are the elliptic curves, the Calabi–Yau varieties of dimension 2 are the K3 surfaces.

2.1.1. *Automorphisms of elliptic curves.*

PROPOSITION 2.2 (see e.g. [ST]). *Let E be an elliptic curve (i.e. a Calabi–Yau variety of dimension 1). If α_E is an automorphism of E which does not preserve the period, then $\alpha^*(\omega_E) = \zeta_n \omega_E$ where $n = 2, 3, 4, 6$. If $n \neq 2$, then E is a rigid elliptic curve with complex multiplication. More precisely: if $n = 3, 6$, then $j(E) = 0$; if $n = 4$, then $j(E) = 1728$.*

We recall that any elliptic curve over $\mathbb{C}$ admits a Weierstrass equation of the form $v^2 = u^3 + au + b$. The hyperelliptic involution, denoted by ι, acts on these coordinates in the following way: $\iota : (u, v) \mapsto (u, -v)$. It acts as the multiplication by -1 on the period.

We will denote by E_{ζ_3} the elliptic curve with j-invariant equal to 0, and with E_i the one with j-invariant equal to 1728.

We recall that a Weierstrass equation for E_{ζ_3} is $v^2 = u^3 + 1$. We denote by $\alpha_E : E_{\zeta_3} \to E_{\zeta_3}$ the automorphism $\alpha_E : (u, v) \mapsto (\zeta_3 u, v)$ and by $\gamma_E := \alpha_E^2 \circ \iota_E$.

Similarly, a Weierstrass equation for E_i is $v^2 = u^3 + u$. We denote by $\alpha_E : E_i \to E_i$ the automorphism $\alpha_E : (u, v) \mapsto (-u, iv)$ and we observe that $\alpha_E^2 = \iota$.

2.1.2. *Automorphisms of K3 surfaces.*

DEFINITION 2.3. Let S be a K3 surface. An automorphism $\alpha_S \in \mathrm{Aut}(S)$ which preserves the period is called *symplectic*. An automorphism $\alpha_S \in \mathrm{Aut}(S)$ of finite order $n := |\alpha_S|$ is *purely non-symplectic* (of order n) if $\alpha_S^*(\omega_S) = \zeta_n \omega_S$, where ζ_n is a primitive n-th root of unity.

The choice of the period of a K3 surface determines a symplectic structure on S (this motivates the previous definition of symplectic automorphism).

PROPOSITION 2.4 ([K]). *Let S be a K3 surface and α_S be a purely non-symplectic automorphism of order n. Then $n \leq 66$ and if $n = p$ is a prime number, then $p \leq 19$. For every $p \leq 19$ there exists at least one K3 surface admitting a purely non-symplectic automorphism of order p.*

In the following we will be interested in purely non-symplectic automorphisms of K3 surfaces of order 2, 3, 4, 6. So we recall that there is a complete classification of the K3 surfaces admitting a purely non-symplectic automorphism of prime order ([AST]) and partial results on K3 surfaces admitting a purely non-symplectic automorphism of order 4 and 6 ([AS2] and [D1] respectively).

PROPOSITION 2.5 (see e.g. [AST]). *Let α_S be a purely non-symplectic automorphism of order n. Let $\mathrm{Fix}_{\alpha_S}(S) := \{s \in S | \alpha_S(s) = s\}$ be the fixed locus of α_S. Then there are the following possibilities:*

(1) $\mathrm{Fix}_{\alpha_S}(S)$ *is empty, in this case $n = 2$;*
(2) $\mathrm{Fix}_{\alpha_S}(S)$ *is the disjoint union of two curves of genus 1, in this case $n = 2$;*
(3) $\mathrm{Fix}_{\alpha_S}(S) = C \coprod_{i=1}^{k-1} R_i \coprod_{j=1}^{h} P_j$ *where P_j are isolated fixed points, C and R_j are curves, C is the one with highest genus $g(C) \geq 0$ and R_j are rational curves.*

In the third case the fixed locus of α_S is determined by the triple $(g(C), k, h)$ and its Euler characteristic is $e(\mathrm{Fix}_{\alpha_S}(S)) = h + 2k - 2g(C)$. If $n = p$ is a prime number then for every prime $p \leq 19$ there exists a known finite list of admissible triples $(g(C), k, h)$ such that there exists at least a K3 surface admitting a non-symplectic automorphism of order p with fixed locus associated to one of these triples.

DEFINITION 2.6. Let α_S be a purely non-symplectic automorphism of order n on a K3 surface S. Let us denote by $H^2(S, \mathbb{C})_{\zeta_n^j}$ the eigenspace of the eigenvalue ζ_n^j for the action of α_S^* on $H^2(S, \mathbb{C})$. For every $i \in \mathbb{Z}/n\mathbb{Z}$ of order n, the dimension $\dim(H^2(S, \mathbb{C})_{\zeta_n^i})$ does not depend on i and will be denoted by m. We will denote by $r := \dim(H^2(S, \mathbb{C})^{\alpha_S})$.

PROPOSITION 2.7. *Let S be a K3 surface admitting a purely non-symplectic automorphism α_S of order n. The numbers r and m are uniquely determined by the Euler characteristics of the fixed loci of α_S^j for $j = 1, \ldots, n-1$.*

The dimension of the family of K3 surfaces S admitting a purely non symplectic automorphism α_S of order n with prescribed Euler characteristics of the fixed loci of α_S^j, $j = 1, \ldots, n-1$, is $m - 1$ if $n \neq 2$ and is $m - 2$ if $n = 2$.

PROOF. The first statement follows immediately by the Lefschetz fixed points formula (see (7.1)), the second one by [DK, Section 11]. □

We observe that if α_S acts as ζ_n on the period of S, then $\dim(H^{1,1}(S)_{\zeta_n}) = m - 1$ if $n \neq 2$ and $\dim(H^{1,1}(S)_{\zeta_n}) = m - 2$ if $n = 2$.

REMARK 2.8. A deformation of the pair (S, α_S) is a deformation of the K3 surface S in the family of the K3 surfaces S_t admitting a purely non-symplectic automorphism α_t which deforms α_S. The topological properties of the fixed loci of α_t^j, $j = 1, \dots, n-1$ coincide with the ones of α_S^j. Similarly, the action of α_t^* on $H^2(S_t, \mathbb{Z})$ coincides with the one of α_S^* on $H^2(S, \mathbb{Z})$. This determines the family of the deformation of the pair (S, α_S) and the dimension of this family is in fact $m - 1$ if $n \neq 2$ and $m - 2$ if $n = 2$.

2.1.3. *Maximal automorphisms of Calabi–Yau 3-folds.*

DEFINITION 2.9. Let $\mathcal{X} \to B$ be a family of Calabi–Yau d-folds X_t. A *maximal automorphism* of such a family of Calabi–Yau d-folds is an automorphism $\alpha_{\bar{t}}$ of a smooth fiber $X_{\bar{t}}$ which extends to the local universal deformation of $X_{\bar{t}}$.

Let B be a polydisc and let $\mathcal{X} \to B$ be a local family of Calabi–Yau 3-folds. Let us fix $\bar{t} \in B$. For a generic $t \in B$, $H^3(X_t, \mathbb{Q}) \simeq H^3(X_{\bar{t}}, \mathbb{Q})$. If $\alpha_{\bar{t}}$ is an automorphism of $X_{\bar{t}}$, then $\alpha_{\bar{t}}^*$ acts on $H^3(X_t, \mathbb{Q})$ for any t. If $\alpha_{\bar{t}}$ is a maximal automorphism for the family, the action of $\alpha_{\bar{t}}^*$ on $H^3(X_t, \mathbb{Q})$ is induced by the automorphism α_t of X_t. In particular, the action of α_t^* preserves the Hodge structure of $H^3(X_t, \mathbb{Q})$, i.e. $\alpha_t^* = \alpha_{\bar{t}}^*$ is compatible with the variation of the Hodge structures of $X_{\bar{t}}$. Thus, the action of α_t^* on ω_{X_t} does not depend on t, i.e. there exists a non-zero complex number λ such that $\alpha_t^*(\omega_{X_t}) = \lambda(\omega_{X_t})$ for the generic $t \in B$.

PROPOSITION 2.10 ([R1, Theorem 8]). *Let $X_{\bar{t}}$ be a Calabi–Yau 3-fold and let $\alpha_{\bar{t}} \in$ $\mathrm{Aut}(X_{\bar{t}})$ be a maximal automorphism of the family of $X_{\bar{t}}$. If $\alpha_{\bar{t}}$ acts on the period of $X_{\bar{t}}$ with finite order then either it acts trivially or with one of the following orders: 2,3,4,6 (i.e. the value λ associated to $\alpha_{\bar{t}}$ is one of the followings: $1, -1, \zeta_3^k, k = 1, 2, \pm i, \zeta_6^h, h = 1, 5$).*

PROPOSITION 2.11 ([R1, Theorem 7]). *Let $X_{\bar{t}}$ be a Calabi–Yau 3-fold and let $\alpha_{\bar{t}} \in$ $\mathrm{Aut}(X_{\bar{t}})$ be a maximal automorphism of the family of $X_{\bar{t}}$. If $\alpha_{\bar{t}}$ acts with order 3, 4 or 6 on the period, then the family of $X_{\bar{t}}$ does not admit a point with maximal unipotent monodromy.*

The previous Proposition is used in several papers (see e.g. [R1], [GvG], [G]) to construct explicit examples of Calabi–Yau 3-folds without maximal unipotent monodromy. We observe that there exist also families of Calabi–Yau 3-folds without maximal unipotent monodromy which do not admit a maximal automorphism acting on the period as described in Proposition 2.11, see [CvS].

2.2. Elliptic fibrations on 3-folds. Let Y be a 3-fold. We now define the notion of elliptic fibration. Since several 3-folds we construct in the following do not admit an elliptic fibration, but have a natural map whose generic fiber is an elliptic curve, we also give a less restrictive definition (the one of almost elliptic fibration) which is useful to describe our situation.

DEFINITION 2.12. Let Y be a 3-fold and R be a surface. We will say that a surjective map with connected fibers $\mathcal{E} : Y \to R$ is an *elliptic fibration* if:

(1) the generic fiber of $\mathcal{E}$ is a smooth genus one curve;

(2) a section of $\mathcal{E}$ is given, i.e. there exists a map $s : R \to Y$ such that $\mathcal{E} \circ s = \mathrm{id}$;
(3) all the fibers of $\mathcal{E}$ are of dimension 1.

DEFINITION 2.13. Let Y be a 3-fold and R be a (possibly singular) surface. We will say that a surjective map with connected fibers $\pi : Y \to R$ is an *almost elliptic fibration* if:

(1) the generic fiber of π is a smooth genus one curve;
(2) a rational section is given, i.e. there exists a rational map $s : R \dashrightarrow Y$ such that $\pi \circ s = \mathrm{id}$ on the domain of s.

REMARK 2.14. We observe that if $\pi : Y \to R$ is an almost elliptic fibration, then we do not require that all the fibers have dimension 1, and in fact we accept the presence of divisors contained in a fiber.

2.3. Hirzebruch surfaces. In the following we will construct elliptic fibrations on Calabi–Yau 3-folds whose bases are one of the following surfaces: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_n$, the Hirzebruch surface. For this reason we recall some results on the Hirzebruch surfaces.

Let $\mathbb{F}_n$ denote the Hirzebruch surface $\mathbb{F}_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-n) \oplus \mathcal{O}_{\mathbb{P}^1})$, $n \in \mathbb{N}$. These surfaces are toric varieties, whose fan has four edges ([F])

$$v_s = (1, 0)\,, \qquad v_t = (-1, n)\,,$$
$$v_y = (0, -1)\,, \quad v_z = (0, 1)\,,$$

and four maximal cones

$$\mathrm{Cone}(v_s, v_z)\,, \quad \mathrm{Cone}(v_z, v_t)\,,$$
$$\mathrm{Cone}(v_t, v_y)\,, \quad \mathrm{Cone}(v_y, v_s)\,.$$

We can describe $\mathbb{F}_n$ also as a quotient space ([CLS, §5]): it is the quotient of $\mathbb{C}^4_{(s,t,y,z)} \setminus \{s = t = 0,\ y = z = 0\}$ by the action of $\mathbb{C}^* \times \mathbb{C}^*$

$$(\lambda, \mu)(s, t, y, z) = (\lambda s, \lambda t, \lambda^n \mu y, \mu z)$$

and so we can use $(s : t : y : z)$ as global homogeneous toric coordinates on $\mathbb{F}_n$.
From the fan we can also see that the Picard group of $\mathbb{F}_n$ is generated by the four divisors D_s, D_t, D_y and D_z, with the relations

$$D_s \equiv D_t\,, \qquad D_y \equiv nD_t + D_z\,,$$

and so

$$\mathrm{Pic}\,\mathbb{F}_n = \mathbb{Z} \cdot D_t \oplus \mathbb{Z} \cdot D_z\,.$$

The intersection properties of these divisors are $D_t^2 = 0$, $D_z^2 = -n$ and $D_t D_z = 1$. We observe that $D_y^2 = n$ and so we call D_y the positive curve, and D_z the negative curve.
We recall $K_{\mathbb{F}_n} = -(n+2)D_t - 2D_z$.

REMARK 2.15. Every Hirzebruch surface admits an automorphism of order d, for every $d \in \mathbb{N}$, whose quotient is another Hirzebruch surface. Indeed let us consider

$$\alpha : (s : t : y : z) \longmapsto (s : t : \zeta_d y : z)\,,$$

where ζ_d denotes a d-th primitive root of unity. Then α is an automorphism of order d on $\mathbb{F}_n$, whose fixed locus consists of the two disjoint rational curves $y = 0$ and $z = 0$ respectively.

The quotient of $\mathbb{F}_n$ by the action of this automorphism is another Hirzebruch surface:

$$\begin{array}{cccc} q: & \mathbb{F}_n & \longrightarrow & \mathbb{F}_{dn} \\ & (s:t:y:z) & \longmapsto & (s:t:y^d:z^d)\,. \end{array}$$

In particular, $\mathbb{F}_{2n}$ is the quotient of $\mathbb{F}_n$ by the involution $(s:t:y:z) \longmapsto (s:t:-y:z)$.

3. Calabi–Yau 3-folds of Borcea–Voisin type. Here we describe the main construction of this paper and we summarize some of our main results (see Propositions 3.4, 3.7, 3.9).

At least in the cases $n = 2$ and $n = 3$ the construction is well known, and in particular it was introduced in case $n = 2$ by Borcea and by Voisin, see [Bo] and [V1] respectively. In case $n = 3$ it is extensively studied by Rohde in [R1]. Several generalizations of such construction are proposed, see for example in [D2] and [G]. Here we describe one of them.

First we recall an essential result on the existence of certain crepant resolutions. Let Z be a 3-fold with trivial canonical bundle and let G be a finite group of automorphisms of Z which preserves the period. This implies that for every $g \in G$, the action of g^* on the tangent space at a fixed point is represented by a diagonal matrix in SL(3). Under this condition the following holds:

PROPOSITION 3.1 ([Y], [Ba]). *The singular 3-fold Z/G admits a crepant resolution. If a crepant resoltuion of Z/G is a Calabi–Yau 3-fold, then every crepant resolution of Z/G is a Calabi–Yau and the Hodge numbers of every crepant resolution do not depend on the specific resolution that we are considering.*

PROPOSITION 3.2. *Let S be a K3 surface admitting a purely non-symplectic automorphism α_S of order n such that $\alpha_S^*(\omega_S) = \zeta_n\omega_S$. Let E be an elliptic curve admitting an automorphism α_E such that $\alpha_E^*(\omega_E) = \zeta_n\omega_E$. Then $n = 2, 3, 4, 6$ and $(S \times E)/(\alpha_S \times \alpha_E^{n-1})$ is a singular variety which admits a desingularization which is a Calabi–Yau 3-fold.*

PROOF. The condition on n follows by Proposition 2.2.
The 3-fold $S \times E$ has trivial canonical bundle and a generator of $H^{3,0}(S \times E, \mathbb{C})$ is $\omega_S \wedge \omega_E$. By construction $\alpha_S \times \alpha_E^{n-1}$ preserves the period, hence there exists a crepant resolution of $(S \times E)/(\alpha_S \times \alpha_E^{n-1})$, i.e. a resolution with trivial canonical bundle, by Proposition 3.1. Since $H^{2,0}(S \times E) = \langle\omega_S\rangle$ and $H^{1,0}(S \times E) = \langle\omega_E\rangle$ are not preserved by $\alpha_S \times \alpha_E^{n-1}$, and since $h^{i,0}$ are birational invariant for any i, the Hodge numbers $h^{1,0}$ and $h^{2,0}$ of any resolution of $(S \times E)/(\alpha_S \times \alpha_E^{n-1})$ are trivial. Hence there exists a resolution of $(S \times E)/(\alpha_S \times \alpha_E^{n-1})$ which is a Calabi–Yau 3-fold. □

DEFINITION 3.3. Any crepant resolution of $(S \times E)/(\alpha_S \times \alpha_E^{n-1})$ will be called a *Calabi–Yau 3-fold of Borcea–Voisin type (associated to $(S, \alpha_S, E, \alpha_E)$).*

PROPOSITION 3.4. *The Hodge numbers of any Calabi–Yau 3-fold of Borcea–Voisin* *type associated to* $(S, \alpha_S, E, \alpha_E)$ *depend only on the topological properties of the fixed loci of* α_S^j*, for* $j = 1, \dots, n-1$.

This Proposition follows immediately by the computations of the Hodge numbers of X done in Propositions 4.1, 5.1, 6.3, 7.3 for $n = 2, 3, 4, 6$ respectively. Anyway it is based on a general idea:

Let $\widetilde{(S \times E)/(\alpha_S \times \alpha_E^{n-1})}$ be a crepant resolution of $(S \times E)/(\alpha_S \times \alpha_E^{n-1})$. Since $\widetilde{(S \times E)/(\alpha_S \times \alpha_E^{n-1})}$ is a Calabi–Yau 3-fold, $h^{0,0} = h^{3,0} = 1$ and $h^{1,0} = h^{2,0} = 0$. The numbers $h^{1,1}$ and $h^{2,1}$ depend on the action of $\alpha_S \times \alpha_E^{n-1}$ on $S \times E$. They are the sum of two contributions: one comes from the desingularization of the singular locus of $(S \times E)/(\alpha_S \times \alpha_E^{n-1})$, the other comes from the cohomology of $S \times E$ which is invariant for $\alpha_S \times \alpha_E^{n-1}$. Since the fixed loci of α_E^j, $j = 1, \dots, n-1$ are uniquely determined by n, the fixed loci of $(\alpha_S \times \alpha_E^{n-1})^j$ depend only the properties of the fixed loci of α_S^j. The part of the cohomology which comes from the cohomology of $S \times E$ can be computed in this general setting:

(3.1)

$$\begin{aligned}
H^{1,1}(S \times E)^{\alpha_S \times \alpha_E^{n-1}} &= (H^{0,0}(S) \otimes H^{1,1}(E))^{\alpha_S \times \alpha_E^{n-1}} \oplus (H^{1,1}(S) \otimes H^{0,0}(E))^{\alpha_S \times \alpha_E^{n-1}} \\
&= (H^{0,0}(S) \otimes H^{1,1}(E)) \oplus \left(H^{1,1}(S)^{\alpha_S} \otimes H^{0,0}(E)\right); \\
H^{2,1}(S \times E)^{\alpha_S \times \alpha_E^{n-1}} &= (H^{2,0}(S) \otimes H^{0,1}(E))^{\alpha_S \times \alpha_E^{n-1}} \oplus (H^{1,1}(S) \otimes H^{1,0}(E))^{\alpha_S \times \alpha_E^{n-1}} \\
&= \left(H^{2,0}(S)_{\zeta_n^{n-1}} \otimes H^{0,1}(E)\right) \oplus \left(H^{1,1}(S)_{\zeta_n} \otimes H^{1,0}(E)\right).
\end{aligned}$$

With the notation introduced in Definition 2.6, the dimension of these spaces are $1 + r$ and $m - 1$ respectively. By Proposition 2.7, r and m depend only on the properties of the fixed loci of α_S^j.

DEFINITION 3.5. Let us consider the automorphism $\alpha_S \times \mathrm{id}_E \in \mathrm{Aut}(S \times E)$. It clearly commutes with $\alpha_S \times \alpha_E^{n-1}$ and so descends to an automorphism α_X of any Calabi–Yau 3-fold X which desingularizes $(S \times E)/(\alpha_S \times \alpha_E^{n-1})$.

3.1. Borcea–Voisin maximal families. By choosing a K3 surface S with a non-symplectic automorphism α_S and an elliptic curve E with an automorphism α_E as in Propostion 3.2, we produce a Calabi–Yau 3-fold, X, of Borcea–Voisin type associated to $(S, \alpha_S, E, \alpha_E)$. We now consider the family of Calabi–Yau 3-folds which deform X. By the Tian–Todorov theorem, the dimension of such a family is $h^{2,1}(X)$. In general not all the members of this family are of Borcea–Voisin type.

Let us now consider a deformation of the pair (S, α_S) to a pair (S_t, α_t). It induces a deformation of the quadruple $(S, \alpha_S, E, \alpha_E)$ which induces a deformation of X to X_t, which is a crepant resolution of $(S_t \times E)/(\alpha_t \times \alpha_E^{n-1})$. Similarly each deformation of the pair (E, α_E) induces a deformation of X.

The deformations of $(S, \alpha_S, E, \alpha_E)$ are the deformations of $(S \times E, \alpha_S \times \alpha_E^{n-1})$ induced by the deformations of the pair (S, α_S) and by the deformations of (E, α_E).

DEFINITION 3.6. Let X be a Calabi–Yau 3-fold of Borcea–Voisin type associated to $(S, \alpha_S, E, \alpha_E)$. The family of X is a *Borcea–Voisin maximal family* if the generic deformation of X is induced by a deformation of $(S, \alpha_S, E, \alpha_E)$.

PROPOSITION 3.7. *Let $n = |\alpha_S|$ and let $\mathcal{F}_X$ be the family of X.*

(1) *If $h^{2,1}(X) = m - 1$, then $\mathcal{F}_X$ is Borcea–Voisin maximal.*
(2) *If $\mathcal{F}_X$ is Borcea–Voisin maximal, then α_X is a maximal automorphism and $\alpha_X(\omega_X) = \zeta_n \omega_X$.*
(3) *If $\mathcal{F}_X$ is Borcea–Voisin maximal and $n = 3, 4, 6$, then $\mathcal{F}_X$ does not admit maximal unipotent monodromy.*
(4) *If $\mathcal{F}_X$ is Borcea–Voisin maximal and $n = 3, 4, 6$, then the variation of the Hodge structures of $\mathcal{F}_X$ depends only on the variation of the Hodge structures of the family of S.*

PROOF. The dimension of the family of K3 surfaces S admitting the purely non-symplectic automorphism α_S of order n is $m - 1$ if $n \neq 2$ and $m - 2$ if $n = 2$ (see Proposition 2.7). The dimension of the family of elliptic curves E admitting the automorphism α_E as considered is 1 if $n = 2$, and 0 otherwise. Let us consider the deformations of the quadruple $(S, \alpha_S, E, \alpha_E)$. They are given by the deformations of (S, α_S) plus the deformations of (E, α_E). So the dimension of the deformations space of $(S, \alpha_S, E, \alpha_E)$ is $m-1 = (m-2)+1$ if $n = 2$ and $m - 1 = m - 1 + 0$ otherwise.

The dimension of the family $\mathcal{F}_X$ is $h^{2,1}(X)$. So, if $h^{2,1}(X) = m - 1$, the generic deformation of X is induced by a deformation of $(S, \alpha_S, E, \alpha_E)$ and so $\mathcal{F}_X$ is a Borcea–Voisin maximal family. Then the automorphism α_X is maximal (since it is defined on every deformation of $(S \times E)/(\alpha_S \times \alpha_E^{n-1})$). By the Proposition 2.11, if $\mathcal{F}_X$ is Borcea–Voisin maximal and $n = 3, 4, 6$, then $\mathcal{F}_X$ does not admit maximal unipotent monodromy. If $n = 3, 4, 6$, then E is a rigid curve. Moreover, if $\mathcal{F}_X$ is a Borcea–Voisin maximal family, $H^{3,0}(X) \oplus H^{2,1}(X) = \big(H^{2,0}(S) \oplus H^{1,1}(S)_{\zeta_n}\big) \otimes H^{1,0}(E) = \big(H^{2,0}(S) \oplus H^{1,1}(S)_{\zeta_n}\big) \otimes \mathbb{C}$ hence the variation of the Hodge structures of $\mathcal{F}_X$ depends only on the variation of the Hodge structures of S. □

Under certain hypothesis, if S is a K3 surface with a purely non-symplectic automorphism, the variation of the Hodge structures of S is essentially the variation of the Hodge structures of a family of curves. So, if we are in case (4) of the previous proposition and moreover the variation of the Hodge structures of S depends only on the variation of the Hodge structures of a family of curves, then the variation of the Hodge structures of X is essentially the variation of the Hodge structures of a family of curves. In particular the Picard–Fuchs equation of X is the Picard–Fuchs equation of a family of curves. Examples of this phenomenon are given in [GvG], [G], in Remarks 5.3, 6.4, and in Section 7.5.

REMARK 3.8. Let us assume that the family of X is a Borcea–Voisin maximal family. We recall that X is a desingularization of $(S \times E)/(\alpha_S \times \alpha_E^{n-1})$ where S lies in the family of the K3 surfaces admitting a non-symplectic automorphism α_S with certain properties. Let us assume that every K3 surface in the family of S admits an automorphism σ which commutes with α_S. Then $\sigma \times \mathrm{id}_E \in \mathrm{Aut}(S \times E)$ induces a maximal automorphism of the family of X (which is in general not equal to α_X).

PROPOSITION 3.9. *Let X be a Calabi–Yau 3-fold of Borcea–Voisin type and $\mathcal{F}_X$ its family. If $\alpha_S^j \in \mathrm{Aut}(S)$ does not fix curves of positive genus for every $j = 1, \dots, n-1$, then $\mathcal{F}_X$ is a Borcea–Voisin maximal family.*

There exists at least one Borcea–Voisin maximal family for every $n = 2, 3, 4, 6$.

PROOF. The first statement follows by Proposition 3.7 and by the computations of the Hodge numbers of X in Propositions 4.1, 5.1, 6.3, 7.3 for $n = 2, 3, 4, 6$ respectively. It is known that there exist purely non-symplectic automorphisms α_S of order 2, 3 and 4 such that α_S^j does not fix curves of positive genus for every $j = 1, \dots, n-1$ (see [N], [AS1],[AS2] and [G] respectively; for $n = 4$ see also Table 2, 4 and Remark 6.4). In case $n = 6$ we construct an explicit example in Section 7.5. □

3.2. Fibrations on Calabi–Yau 3-folds of Borcea–Voisin type. In the next sections we will construct explicitly one crepant resolution X of $(S \times E)/(\alpha_S \times \alpha_E^{n-1})$ and so a particular Calabi–Yau 3-fold of Borcea–Voisin type, called of type X_n. In the following proposition we summarize some geometric properties of these 3-folds.

PROPOSITION 3.10. *Let $\mathcal{G}_n : X \to E/\alpha_E^{n-1} \simeq \mathbb{P}^1$ and $\mathcal{E}_n : X \to S/\alpha_S$ be the maps induced on X by $(S \times E)/(\alpha_S \times \alpha_E^{n-1}) \to E/\alpha_E^{n-1}$ and $(S \times E)/(\alpha_S \times \alpha_E^{n-1}) \to S/\alpha_S$ respectively. Let $g : E \to E/\alpha_E^{n-1}$ and $q : S \to S/\alpha_S$ be the quotient maps.*

The map $\mathcal{G}_n$ is an isotrivial fibration in K3 surfaces. The fiber $\mathcal{G}_n^{-1}(g(P))$ is reducible if and only if $P \in E$ is a point with non-trivial stabilizer for the action of α_E on E and the fixed locus of α_S is non empty.

The map $\mathcal{E}_n$ is an almost elliptic fibration. More precisely:

- *the fiber $\mathcal{E}_n^{-1}(q(Q))$ is isomorphic to E if and only if $Q \in S$ has trivial stabilizer for the action of α_S on S;*
- *the fiber $\mathcal{E}_n^{-1}(q(Q))$ is singular of dimension 1 if and only if $Q \in S$ has a non-trivial stabilizer for the action of α_S on S, but Q is not an isolated fixed point for α_S^j for any $j = 1, \dots, n-1$; in this case $\mathcal{E}_n^{-1}(q(Q))$ is of Kodaira type;*
- *the fiber $\mathcal{E}_n^{-1}(q(Q))$ contains a divisor if and only if $Q \in S$ is an isolated fixed point for α_S^j for at least one $j \in \{1, \dots, n-1\}$.*

PROOF. The fibers of $\mathcal{G}_n$ are clearly equidimensional and the smooth ones are isomorphic to S. The ones which are not smooth contain divisors which come from the resolution of $(S \times E)/(\alpha_S \times \alpha_E^{n-1})$ and hence project to points in E/α^{n-1} which are branch points.

The map $\mathcal{E}_n$ will be considered in the Propositions 4.4, 5.5, 6.8, 7.5. □

We observe that $\mathcal{E}_n$ is an elliptic fibration if and only if α_S^j does not fix isolated points for every $j \in \{1, \dots, n-1\}$. For example, this surely happens if $n = 2$.

4. The "original" Borcea–Voisin construction: order 2. In this section we assume $|\alpha_S| = |\alpha_E| = 2$. We present the explicit construction of a crepant resolution, of type X_2, of $(S \times E)/(\alpha_S \times \alpha_E)$ following [V1, Section 1].

4.1. The construction of Calabi–Yau 3-folds of type X_2. Let S be a K3 surface with a non-symplectic involution ι_S. These K3 surfaces are classified in [N]. The fixed locus of ι_S on S is either empty or consists of N (disjoint) curves, and ι_S linearizes near the fixed locus to the matrix diag$(-1, 1)$. Let E be an elliptic curve and ι_E its hyperelliptic involution. The fixed locus of ι_E consists of 4 points, P_1, P_2, P_3 and P_4, and clearly the local action of ι_E near the fixed points is -1. Thus, the fixed locus of $\iota := \iota_S \times \iota_E$ on $S \times E$ consists of $4N$ disjoint curves and the local action near the fixed locus linearizes to diag$(-1, 1, -1)$. The singularities of $(S \times E)/\iota$ are the images of the curves in $S \times E$ fixed by ι and a crepant resolution of such singularities can be obtained blowing up each of these curves. More precisely the following diagram commutes:

$$\begin{array}{ccccc} \iota \circlearrowright & S\times E & \stackrel{\beta}{\leftarrow} & \widetilde{S\times E} & \circlearrowleft \widetilde{\iota} \\ & \downarrow & & \downarrow & \\ & (S\times E)/\iota & \leftarrow & \widetilde{(S\times E)}/\widetilde{\iota} & \simeq X \end{array} \tag{4.1}$$

where $\beta : \widetilde{S \times E} \to S \times E$ is the blow up of $S \times E$ in the fixed locus $\mathrm{Fix}_\iota(S \times E)$, $\widetilde{\iota}$ is the involution induced on $S \times E$ by ι and the vertical arrows are the quotient maps. Thus, a desingularization of $(S \times E)/\iota$ is constructed blowing up the fixed locus $\mathrm{Fix}_\iota(S \times E)$ and then considering the quotient by the induced automorphism.

In order to compute the Hodge numbers of the Calabi–Yau 3-fold we first observe that

$$\begin{aligned} H^i(\widetilde{S\times E}) &= H^i(S\times E), \qquad i = 0, 1, \\ H^2(\widetilde{S\times E}) &= H^2(S\times E) \bigoplus \oplus_{i=1}^{4N} H^0(D_i), \\ H^3(\widetilde{S\times E}) &= H^3(S\times E) \bigoplus \oplus_{i=1}^{4N} H^1(D_i), \qquad \text{(cf. [V1])}, \end{aligned}$$

where D_i is the exceptional divisor over the fixed curve C_i blowed up, and is isomorphic to a $\mathbb{P}^1$-bundle over C_i.

Since $\widetilde{S \times E}/\widetilde{\iota}$ is a smooth quotient of $\widetilde{S \times E}$, the cohomology groups of X coincide with the invariant part of the cohomology groups of $\widetilde{S \times E}$ under $\widetilde{\iota}$. We notice that the exceptional divisors over the fixed locus of ι are clearly invariant under $\widetilde{\iota}$ in $\widetilde{S \times E}$.

PROPOSITION 4.1 (cf. [V1]). *Let S be a K3 surface admitting a non-symplectic involution ι_S fixing N curves and let $N' = \sum_{C_i \in \mathrm{Fix}_{\iota_S}(S)} g(C_i)$. Let E be an elliptic curve and ι_E its hyperelliptic involution. The Hodge numbers of any crepant resolution of $(S\times E)/(\iota_S \times \iota_E)$, and in particular the ones of X, are*

$$h^{0,0} = h^{3,0} = 1, \quad h^{1,0} = h^{2,0} = 0, \quad h^{1,1} = 1 + r + 4N, \quad h^{2,1} = m - 1 + 4N'.$$

Equivalently

$$\begin{aligned} h^{1,1} &= 11 + 5N - N' = 5 + 3r - 2a, \\ h^{2,1} &= 11 + 5N' - N = 65 - 3r - 2a, \end{aligned}$$

where a is defined by the following property: $(H^2(S,\mathbb{Z})^{\iota_S})^\vee/(H^2(S,\mathbb{Z})^{\iota_S}) \simeq (\mathbb{Z}/2\mathbb{Z})^a$.

The proposition follows immediately by the construction of X of type X_2 given before, by (3.1) and by the following known relations among (r, a) and (N, N'): $N = \frac{1}{2}(r - a + 2)$, $N' = \frac{1}{2}(22 - r - a)$, cf. [V1].

REMARK 4.2. If $N' = 0$, the automorphism ι_X induced on X by $\iota_S \times \mathrm{id}_E$ is a maximal automorphism, by Proposition 3.7.

REMARK 4.3. If $N' = 0$ any automorphism σ_S commuting with ι_S induces a maximal automorphism of X by Remark 3.8. In [GS] it is proved that, if ι_S is such that $N' = 0$, then S admits at least one symplectic involution, σ_S. So, if $N' = 0$, we obtain at least one maximal automorphism of X (in fact an involution) preserving the period of X. We observe that there exists a crepant resolution of the quotient of X by such an automorphism which is a Calabi–Yau 3-fold, and it is again of Borcea–Voisin type. However it is not in general a deformation of X.

4.2. The elliptic fibration. We now consider the map $\mathcal{E}_2 : X \to S/\iota_S$ (cf. Proposition 3.10), which turns out to be an elliptic fibration on X, a Calabi–Yau of type X_2, and whose analogues will be considered in the following sections.

PROPOSITION 4.4. *Let* $\mathcal{E}_2 : X \to S/\iota_S$ *be the natural map induced on* X *by* $(S \times E)/(\iota_S \times \iota_E) \to S/\iota_S$.

The map $\mathcal{E}_2 : X \to S/\iota_S$ *is an isotrivial elliptic fibration whose general fiber is isomorphic to* E. *Let us consider the quotient map* $q : S \to S/\iota_S$. *Let us assume that* S *is generic in the family of K3 surfaces with the non-symplectic involution* ι_S *(i.e.* $\rho(S) = r$*). The fiber* F_P *of* $\mathcal{E}_2$ *over* $P \in S/\iota_S$ *is singular if and only if* P *is in the branch locus of* $q : S \to S/\iota_S$ *and in this case* F_P *is of type* I_0^*. *The Mordell–Weil group of this fibration is generically trivial.*

PROOF. Since ι_S does not fix isolated points, the quotient S/ι_S is smooth. The generic fiber of the map $\mathcal{E}_2 : X \to S/\iota_S$ is an elliptic curve isomorphic to E by construction. The discriminant locus of $\mathcal{E}_2$ is the branch locus of $q : S \to S/\iota_S$ and so is isomorphic to $\mathrm{Fix}_{\iota_S}(S)$. Any of its components is a copy of a curve C fixed by ι_S. By the construction of X we introduce four $\mathbb{P}^1$-bundles over C for each curve C, which are in fact the blow up of $C \times P_i \subset S \times E$, $i = 1, 2, 3, 4$. Thus, if we consider the fiber F_P over a point P of $q(C)$ we find the strict transform of E, which is a rational curve, and 4 rational curves which are the fibers over the point P of the four $\mathbb{P}^1$-bundles over $C \simeq q(C)$. Hence we find exactly a fiber of type I_0^*. □

REMARK 4.5. The automorphism α_X defined in Definition 3.5 is the hyperelliptic involution on the elliptic fibration $\mathcal{E}_2 : X \to S/\iota_S$.

4.3. Non-symplectic automorphisms of order 2 on K3 surfaces. Every pair (S, ι_S), where S is a K3 surface admitting a non-symplectic involution ι_S, can be described in one of the following ways:

(1) S is the minimal resolution of the double cover of $\mathbb{P}^2$ branched over a (possibly singular) sextic and ι_S is induced on S by the cover involution;
(2) S is an elliptic fibration with section and ι_S is induced by the hyperelliptic involution on each smooth fiber;
(3) S is the minimal resolution of the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over a (possibly singular) curve of bi-degree (4, 4) and ι_S is induced on S by the cover involution.

In the second case, and more precisely if S is generic among the K3 surfaces admitting an elliptic fibration with section, S/ι_S is the Hirzebruch surface $\mathbb{F}_4$. Indeed, S can be embedded in $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1})$ with a Weierstrass equation $y^2z = x^3 + f_8(s:t)xz^2 + f_{12}(s:t)z^3$, where $f_8(s:t) \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(8))$ and $f_{12}(s:t) \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(12))$. The quotient by the hyperelliptic involution corresponds to the projection of the K3 surface S from the constant section $(x:y:z) = (0:1:0)$ on the surface $y = 0$. This defines a $2:1$ covering of $\mathbb{F}_4$ ($y = 0$ is a surface isomorphic to $\mathbb{F}_4$), branched along $(x^3 + f_8(s:t)xz^2 + f_{12}(s:t)z^3)z = 0$. So the branch divisor in $\mathbb{F}_4$ is $12D_t + 4D_z$ and is the disjoint union of the curve D_z, which corresponds to the section of the elliptic fibration on S, and of the curve $12D_t + 3D_z$ which is the image of the trisection passing through the points of order 2 of the elliptic fibration on S. The first is a rational curve, the latter has genus 10. In particular the surface S admits the following equation $w^2 = (x^3 + f_8(s:t)xz^2 + f_{12}(s:t)z^3)z$, where $(s:t:x:z)$ are the coordinates of $\mathbb{F}_4$ introduced in Section 2.3 (more precisely, x was y with the notation of the Section 2.3).

Hence a (possibly singular) model of S has one of the following equations:

(1) $w^2 = f_6(x_0:x_1:x_2)$, where $(x_0:x_1:x_2)$ are coordinates of $\mathbb{P}^2$;
(2) $w^2 = (x^3 + f_8(s:t)xz^2 + f_{12}(s:t)z^3)z$, where $(s:t:x:z)$ are coordinates of $\mathbb{F}_4$;
(3) $w^2 = f_{4,4}((x_0:x_1),(y_0:y_1))$, where $((x_0:x_1),(y_0:y_1))$ are coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$;

and in all these cases ι_S acts trivially on all the coordinates but w and changing the sign of w.

4.4. Equations. Let us now assume that E has the following equation $v^2 = u^3 + au + b$ and S is the double cover of $\mathbb{P}^2$ branched along a sextic $V(f_6(x_0:x_1:x_2))$. An equation for a singular model of X_2 is

$$Y^2 = X^3 + af_6^2(x_0:x_1:x_2)X + bf_6^3(x_0:x_1:x_2) \tag{4.2}$$

where the functions $Y := vw^3$, $X := uw^2$ are invariant for $\iota_S \times \iota_E : ((w,(x_0:x_1:x_2)),(v,u)) \mapsto ((-w,(x_0:x_1:x_2)),(-v,u))$.

Similarly, if S is a double cover of $\mathbb{F}_{4(s:t:x:z)}$, a Weierstrass equation for the elliptic fibration $\mathcal{E}_2 : X \to \mathbb{F}_4$ is

$$\begin{aligned} Y^2 = \; & X^3 + a(x^3 + f_8(s:t)xz^2 + f_{12}(s:t)z^3)^2z^2X \\ & + b(x^3 + f_8(s:t)xz^2 + f_{12}(s:t)z^3)^3z^3 \,. \end{aligned} \tag{4.3}$$

If S is the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along the curve $V(f_{4,4}((x_0:x_1),(y_0:y_1)))$, an equation for a singular model of X_2 is

$$Y^2 = X^3 + af_{4,4}^2((x_0 : x_1), (y_0 : y_1))X + bf_{4,4}^3((x_0 : x_1), (y_0 : y_1)) . \tag{4.4}$$

If $V(f_6(x_0 : x_1 : x_2))$ and $V(f_{4,4}((x_0 : x_1), (y_0 : y_1)))$ are smooth, then (4.2), (4.3), (4.4) are Weierstrass equations for the elliptic fibrations $\mathcal{E}_2 : X_2 \to \mathbb{P}^2$, $\mathcal{E}_2 : X_2 \to \mathbb{F}_4$ $\mathcal{E}_2 : X_2 \to \mathbb{P}^1 \times \mathbb{P}^1$ described in Proposition 4.4 respectively and the results of such a proposition can be directly checked on the equation.

5. Quotient of order 3.

5.1. The construction of Calabi–Yau 3-folds of type X_3. Let S be a K3 surface admitting a non-symplectic automorphism α_S of order 3. Such K3 surfaces are classified in [AS1], [T]. The fixed locus of α_S consists of n isolated points and of k curves and the linearization of α_S near the fixed locus is $\mathrm{diag}(\zeta_3^2, \zeta_3^2)$ and $\mathrm{diag}(\zeta_3, 1)$ respectively.

Let E_{ζ_3} and its automorphism of order 3, α_E, be as in Section 2.1.1. The automorphism α_E fixes 3 points on E_{ζ_3} and its local action near the fixed locus is the multiplication by ζ_3. Let α be the automorphism $\alpha_S \times \alpha_E^2$ of $S \times E$. The fixed locus of α on $S \times E$ consists of $3n$ points and $3k$ curves, and the linearization of α near the fixed locus is $\mathrm{diag}(\zeta_3^2, \zeta_3^2, \zeta_3^2)$ and $\mathrm{diag}(\zeta_3, 1, \zeta_3^2)$ respectively. As in the case of the involutions one can construct a desingularization of $(S \times E)/\alpha$ by blowing up $S \times E$ in the fixed locus of α to obtain a variety $\widetilde{S \times E}$ such that the induced automorphism $\widetilde{\alpha}$ fixes only divisors, and then considering the quotient $(\widetilde{S \times E})/\widetilde{\alpha}$, but in this case one has to contract some divisors on $(\widetilde{S \times E})/\widetilde{\alpha}$ in order to obtain a Calabi–Yau 3-fold. One finds that it suffices to blow up once the fixed points, introducing a copy of $\mathbb{P}^2$ as exceptional divisor. The situation is a little bit more complicated in the case of fixed curves: one has to blow up the fixed curve C, introducing a divisor D_1 which is a $\mathbb{P}^1$-bundle over C, and the induced automorphism fixes two disjoint sections (copies of C) on D_1. Hence, one has to blow up these 2 copies of C introducing two other exceptional divisors D_2, D_3, which are again $\mathbb{P}^1$-bundles over C. The induced automorphism fixes only divisors and then the quotient by it is smooth. The image of the divisor D_1 under the quotient has to be contracted to obtain a smooth Calabi–Yau 3-fold.

Globally, we introduce $3n$ exceptional divisors isomorphic to $\mathbb{P}^2$ over the $3n$ fixed points, and $6k$ exceptional divisors which are $\mathbb{P}^1$-bundles over C_i, two for each fixed curve C_i. A more detailed construction of the crepant resolution of $(S \times E)/\alpha$ of type X_3 can be found in [CH] and [R1], see also [S].

Examples of Calabi–Yau 3-folds constructed in this way are studied in [R1] and [GvG].

PROPOSITION 5.1. *Let S be a K3 surface admitting a non-symplectic automorphism α_S of order 3, fixing n isolated points and k curves. Let us denote by C the curve of highest genus fixed by α_S and by $g(C)$ its genus. Let E_{ζ_3} be the elliptic curve with Weierstrass form $v^2 = u^3 + 1$ and $\alpha_E : (v, u) \to (v, \zeta_3 u)$. Then any crepant resolution of $(S \times E)/(\alpha_S \times \alpha_E^2)$ is a Calabi–Yau 3-fold with Hodge numbers*

$$h^{0,0} = h^{3,0} = 1 , \qquad h^{1,0} = h^{2,0} = 0 ,$$
$$h^{1,1} = r + 1 + 3n + 6k , \quad h^{2,1} = m - 1 + 6g(C) .$$

Equivalently

$$h^{1,1} = 7 + 4r - 3a \quad \textit{and} \quad h^{2,1} = 43 - 2r - 3a$$

where a is defined by the following property: $(H^2(S,\mathbb{Z})^{\alpha_S})^\vee/(H^2(S,\mathbb{Z})^{\alpha_S}) \simeq (\mathbb{Z}/3\mathbb{Z})^a$. *In particular, the Euler characteristic is* $2(h^{1,1} - h^{2,1}) = -72 + 12r$.

PROOF. The proposition follows immediately by the construction of the Calabi–Yau of type X_3, by the fact that $r + 2m = 22 = \dim(H^2(S,\mathbb{C}))$ and by the following relations among the pair (r, a) and the invariant $(g(C), k, n)$ describing the fixed locus $\mathrm{Fix}_{\alpha_S}(S)$ of α_S on S: $g(C) = (22 - r - 2a)/4$, $k = (6 + r - 2a)/4$, $n = r/2 - 1$ ([AS1, Section 2]). □

Let X be a Calabi–Yau of type X_3.

REMARK 5.2. If $g(C) = 0$, then $h^{2,1}(X) = m - 1$. So, by Proposition 3.7 the automorphism α_X is a maximal automorphism of the family of Calabi–Yau 3-folds X, and this family does not admit maximal unipotent monodromy. For this reason, the families of Calabi–Yau 3-folds constructed in this way are analyzed both in [R1] and [GvG].

REMARK 5.3. More precisely, in [GvG] it is proved that every K3 surface S such that $g(C) = 0$ and $k \geq 3$ is birational to $(D \times E_{\zeta_3})/(\mathbb{Z}/3\mathbb{Z})$ where D is an appropriate curve. Hence the Calabi–Yau 3-fold X is birational to $(D \times E_{\zeta_3} \times E_{\zeta_3})/(\mathbb{Z}/3\mathbb{Z})^2$. Using that the curve E_{ζ_3} is rigid, one proves that the variation of the Hodge structures of X depends only on the variation of the Hodge structures of the curve D, see [GvG]. We will see in the following that this result generalizes to other Calabi–Yau's of type X_n.

REMARK 5.4. If $g(C) = 0$ and $k \geq 3$, then S admits also a symplectic automorphism σ of order 3 which commutes with α_S, as proved in [GS]. So we are in the assumption of Remark 3.8 and thus both $\alpha_S \times \mathrm{id}_E \in \mathrm{Aut}(S \times E)$ and $\sigma \times \mathrm{id} \in \mathrm{Aut}(S \times E)$ induce maximal automorphisms of the family of X; the first one does not preserve the period, the second one, denoted by σ_X, preserves the period. In particular there exists a crepant resolution, Y, of X/σ_X which is again a Calabi–Yau 3-fold. Since S/σ is birational to a K3 surface admitting a non-symplectic automorphism of order 3, Y is birational to a Calabi–Yau 3-fold of type X_3.

5.2. (Almost) elliptic fibrations. Let α_S be a non-symplectic automorphism of S of order 3 which fixes n isolated points and k curves. Let X be the Calabi–Yau 3-fold of type X_3 associated to S. Let us consider the map $\mathcal{E}_3 : X \to S/\alpha_S$ induced on X by $(S \times E)/(\alpha_S \times \alpha_E^2) \to S/\alpha_S$. Moreover we consider the quotient map $q_S : S \to S/\alpha_S$ and we recall that S/α_S has exactly n singular points, the image of the n isolated fixed points on S. We will assume (S, α_S) to be generic in the family of K3 surfaces S with the non-symplectic automorphism of order 3 α_S (i.e. $\rho(S) = r$).

PROPOSITION 5.5. *The map* $\mathcal{E}_3 : X \to S/\alpha_S$ *is an almost elliptic fibration whose general fiber is isomorphic to* E_{ζ_3} *and it is an elliptic fibration if and only if* $n = 0$. *The fiber* F_P *of* $\mathcal{E}_3$ *over* $P \in S/\alpha_S$ *is of dimension 1 if and only if* P *is a smooth point and is singular if and only if* P *is in the branch locus of* $q : S \to S/\alpha_S$. *In particular:*

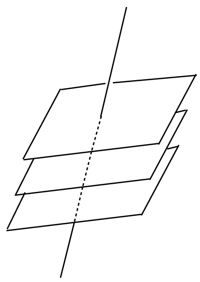


FIGURE 1. The fiber over a singular point of S/α_S.

- *if P is a singular point of S/α_S, the fiber F_P consists of a rational curve which meets three disjoint copies of $\mathbb{P}^2$, each in 1 point (see Figure 1);*
- *if P is a smooth point of S/α_S in the branch locus of q, then F_P is a fiber of type IV^*.*

The Mordell–Weil group of this fibration is generically equal to $\mathbb{Z}/3\mathbb{Z}$.

PROOF. Again the proof follows directly by the construction of X. Let us denote by R_i the 3 fixed points of α_E on E. We denote by $P' \in S$ a point such that $q(P') = P$. If $P' \in S$ is an isolated fixed point of α_S (i.e., P is a singular point of S/α_S), the points $P' \times R_i$, are isolated fixed points of $\alpha_S \times \alpha_E^2$. On each of them we introduce a $\mathbb{P}^2$. Moreover, the image of the strict transform of $P' \times E_{\zeta_3}$ is a rational curve which is a component of the fiber F_P over P.

Similarly, one constructs the reducible fiber over P if P' lies on a curve C' fixed by α_S. In this case P is a smooth point of S/α_S. In order to construct X, we introduce six $\mathbb{P}^1$-bundles over C', 2 on each curve $C' \times R_j$, $j = 1, 2, 3$. Considering the fiber of these $\mathbb{P}^1$-bundles over $P' \times R_j$ we obtain 3 copies of A_2 (where A_2 is a configuration of 2 rational curves meeting in a point). Moreover, there is a component of the fiber F_P, which consists of the image of the strict transform of $P' \times E_{\zeta_3}$. It meets in one point one of the two rational curves of each configuration of type A_2. Hence we obtain a configuration of (-2)-curves which corresponds to a fiber of Kodaira type IV^*.

The (rational) sections are the image of the strict transform of the divisors $S \times R_j$. It is immediate to show that they are sections if S/α_S is smooth and rational sections otherwise. □

REMARK 5.6. The automorphism α_X defined in Definition 3.5 is induced by the complex multiplication of order 3 on each smooth fiber of the fibration.

We now give a Weierstrass equation of $\mathcal{E}_3$ in case S/α_S is smooth. This implies that α_S has no isolated fixed points, i.e. $n = 0$. By Proposition 5.5, this condition is equivalent to the condition that $\mathcal{E}_3$ is an elliptic fibration.

By [AS1], [T] there exist exactly two families of K3 surfaces admitting a non-symplectic automorphism of order 3 which do not fix isolated points. Each of these families is 9 dimensional. They are characterized by the number of fixed curves of α_S on S. If α_S fixes exactly one curve on S, then $S/\alpha_S \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and if α_S fixes exactly 2 curves on S, then $S/\alpha_S \simeq \mathbb{F}_6$.

5.2.1. *Weierstrass equation of $\mathcal{E}_3$ if $S/\alpha_S \simeq \mathbb{P}^1 \times \mathbb{P}^1$.* Let S be a K3 surface admitting a non-symplectic automorphism of order 3 whose fixed locus consists of exactly one curve. In this case the fixed curve has genus 4 and an equation for S is ([AS1, Proposition 4.7])

$$(5.1)\qquad \begin{cases} f_2(x_0 : x_1 : x_2 : x_3) & = 0 \\ f_3(x_0 : x_1 : x_2 : x_3) + x_4^3 & = 0 \end{cases}$$

where the polynomials $f_n(x_0 : x_1 : x_2 : x_3)$ are generic homogeneous polynomials of degree n. In this case α_S is induced by the projectivity $(x_0 : x_1 : x_2 : x_3 : x_4) \to (x_0 : x_1 : x_2 : x_3 : \zeta_3 x_4)$. It is now clear that S is a $3:1$ cover of a quadric, $V(f_2(x_0 : x_1 : x_2 : x_3)) \subset \mathbb{P}^3$, branched along the curve $V(f_2(x_0 : x_1 : x_2 : x_3)) \cap V(f_3(x_0 : x_1 : x_2 : x_3)) \subset \mathbb{P}^3$. More intrinsically, S is the triple cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along a curve of bidegree $(3, 3)$, i.e. S admits an equation of type $w^3 = f_{3,3}((x_0 : x_1), (y_0 : y_1))$ and we can assume $\alpha_S \colon (w, ((x_0 : x_1), (y_0 : y_1))) \mapsto (\zeta_3 w, ((x_0 : x_1), (y_0 : y_1)))$.

So a Weierstrass equation for the elliptic fibration described in Proposition 5.5 is

$$Y^2 = X^3 + f_{3,3}^4((x_0 : x_1), (y_0 : y_1)),$$

where the functions $Y := vw^6$, $X := uw^4$ are invariant for $\alpha_S \times \alpha_E^2$. We can directly check that the reducible fibers of this elliptic fibration are of type IV^* and that there are 3 sections: the one at infinity and the sections $(X, Y) \mapsto (0, \pm f_{3,3}^2((x_0 : x_1), (y_0 : y_1)))$ which have order 3.

5.2.2. *Weierstrass equation of $\mathcal{E}_3$ if $S/\alpha_S \simeq \mathbb{F}_6$.* Let S be a K3 surface admitting a non-symplectic automorphism of order 3 whose fixed locus consists of exactly 2 curves. In [AS1, Proposition 4.2] it is proved that S admits an isotrivial elliptic fibration whose equation is $y^2 = x^3 + f_{12}(t)$ where $f_{12}(t)$ is a polynomial of degree 12 without multiple roots. The automorphism α_S in this case is $(x, y, t) \mapsto (\zeta_3 x, y, t)$. The fixed curves are the section of this elliptic fibration (which is of course rational) and the bisection $y^2 = f_{12}(t)$. Using the homogeneous coordinates in $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1})$, the surface S has equation $y^2 z = x^3 + f_{12}(s : t) z^3$ and so we see that the projection from the constant section $(x : y : z) = (1 : 0 : 0)$ defines a $3:1$ cover of the surface $x = 0$, which is the Hirzebruch surface $\mathbb{F}_6$. Using the coordinates $(s : t : y : z)$ on $\mathbb{F}_6$ introduced in Section 2.3, the branch locus is given by the two disjoint curves $z = 0$, i.e. the negative curve of $\mathbb{F}_6$, which corresponds to the section of the elliptic fibration, and $y^2 = f_{12}(s : t) z^2$ which corresponds to the bisection. Observe that S admits a birational model S' in $\mathcal{O}_{\mathbb{F}_6}(4D_t + D_z)$, whose equation is $w^3 = (y^2 - f_{12}(s : t) z^2) z$. The birational morphism

$$\begin{array}{ccc} S & \longrightarrow & S' \\ (s, t, x, y, z) & \longmapsto & (w, (s : t : y : z)) = (x, (s : t : y : z)) \end{array}$$

is compatible with the non-symplectic automorphism α_S, in the following sense: via this birational morphism, α_S induces on S' the covering transformation $\alpha_{S'} : (w, (s : t : y :$

$z)) \longmapsto (\zeta_3 w, (s:t:y:z))$ which is non-symplectic.
The functions $Y := vw^6$ and $X := uw^4$ are then invariant for $\alpha_{S'} \times \alpha_E^2$ and satisfy

$$Y^2 = X^3 + (y^2 z - f_{12}(s:t)z^3)^4\,,$$

which is a Weierstrass equation for the elliptic fibration $\mathcal{E}_3 : X_3 \to S/\alpha_S \simeq \mathbb{F}_6$ in $\mathbb{P}(\mathcal{O}_{\mathbb{F}_6}(-2K_{\mathbb{F}_6}) \oplus \mathcal{O}_{\mathbb{F}_6}(-3K_{\mathbb{F}_6}) \oplus \mathcal{O}_{\mathbb{F}_6})$.

6. Quotient of order 4.

6.1. The construction of Calabi–Yau 3-folds of type X_4. Let E_i be the elliptic curve admitting an automorphism, α_E, of order 4 such that $\alpha_E^*(\omega_E) = i\omega_E$. We recall that α_E^2 is the hyperelliptic involution and thus fixes 4 points. Among these points, two are switched by α_E and two are fixed. We denote by P_i, $i = 1, 2$ the points such that $\alpha_E(P_i) = P_i$ and by Q_j, $j = 1, 2$ the points such that $\alpha_E^2(Q_j) = Q_j$ and $\alpha_E(Q_1) = Q_2$.

Let S be a K3 surface admitting a purely non-symplectic automorphism of order 4, α_S. Such K3 surfaces are not completely classified, but a lot of them are studied and listed in [AS2]. The fixed locus of α_S consists of points and curves. Of course $\mathrm{Fix}_{\alpha_S}(S) \subset \mathrm{Fix}_{\alpha_S^2}(S)$. Since α_S^2 is a non-symplectic involution of S and thus fixes only curves, all the points fixed by α_S lie on curves fixed by α_S^2.

The automorphism α_S can act in three different ways on each curve fixed by α_S^2. We call:

(1) curves of first type: the curves which are fixed by α_S (and thus of course also by α_S^2), we denote by K_i these curves and by k their number;
(2) curves of second type: the curves which are fixed by α_S^2 and are invariant by α_S, we denote by B_i these curves and by b their number;
(3) curves of third type: the curves fixed by α_S^2 and sent to another curve by α_S, we denote by (A_i', A_i'') the pairs of these curves, assuming that $\alpha_S(A_i') = A_i''$, and we denote by a the number of these pairs.

The properties of the locus with non-trivial stabilizer for the action of α_S on S are described by the numbers $(N, g(D), a, b, k, n_1, n_2)$ defined here:

DEFINITION 6.1. The automorphism α_S^2 fixes N curves. We will denote by D the curve of highest genus fixed by α_S^2 and by $g(D)$ it genus. The automorphism α_S fixes k curves and $n_1 + n_2$ isolated points; n_2 of them lie on D. The isolated fixed points of α_S lie on the b curves (of second type) which are fixed by α_S^2 and are invariant for α_S. The automorphism α_S switches a pairs of curves (of third type) fixed by α_S^2.

REMARK 6.2. The isolated fixed points of α_S lie on curves of second type and α_S is an involution of each of these curves. Clearly $N = k + b + 2a$.

Let $\alpha := (\alpha_S \times \alpha_E^3) \in \mathrm{Aut}(S \times E_i)$. In order to construct a crepant resolution of the quotient $(S \times E_i)/\alpha$ we make two following quotients: first we consider the quotient $(S \times E_i)/\alpha^2$ (it is of the type described in Section 4.1) which admits a crepant resolution X';

then we consider the crepant resolution of the quotient X'/α' where α' is the automorphism induced by α on X'.

Let $\beta_1 : \widetilde{S \times E_i} \to S \times E_i$ be the blow up of $S \times E_i$ in the fixed locus of α^2, i.e. in $4N$ curves. The automorphism $\alpha \in \mathrm{Aut}(S \times E_i)$ induces an automorphism $\widetilde{\alpha}$ on $\widetilde{S \times E_i}$, indeed outside of the exceptional locus the 3-folds $S \times E_i$ and $\widetilde{S \times E_i}$ are isomorphic and thus $\widetilde{\alpha}$ is identified with α. We claim that there is a global automorphism $\widetilde{\alpha}$ which extends this action: to prove this it suffices to consider the action in a neighborhood of the exceptional locus.

Since the local action of α^2 near the fixed locus is represented by the diagonal matrix $\mathrm{diag}(-1, 1, -1)$, locally we are blowing up $\mathbb{C}^3_{(x,y,z)}$ in $x = z = 0$. Hence, locally the blow up is identified with $\widetilde{\mathbb{C}^3_{(x,y,z)}} := V(bx = az) \subset \mathbb{C}^3_{(x,y,z)} \times \mathbb{P}^1_{(a:b)}$. On the exceptional locus the action of $\widetilde{\alpha}$ depends on the type of curve that we blow up. Let us denote by D_1, D_2, (D_3', D_3'') the exceptional divisors over a curve of first type, over a curve of second type, over a pair of curves of third type respectively. The automorphism α linearizes as $(i, 1, -i)$ near the curves of the first type: locally we are considering a copy of $\mathbb{C}^3_{(x,y,z)}$ with the action $(x, y, z) \to (ix, y, -iz)$. This induces the map $((x, y, z); (a : b)) \to ((ix, y, -iz); (a : -b))$ on $\widetilde{\mathbb{C}^3_{(x,y,z)}}$. So, $\widetilde{\alpha}$ preserves the exceptional divisor D_1 and acts as $((x, y, z); (a : b)) \to ((ix, y, -iz); (a : -b))$ in the neighborhood of D_1 which can be identified with $V(bx = az) \subset \mathbb{C}_{(x:y:z)} \times \mathbb{P}^1_{(a:b)}$. Similarly, we could have argued that since α^* preserves the tangent directions, $\widetilde{\alpha}$ is well defined in a neighborhood of D_1.

The automorphism α linearizes as $\mathrm{diag}(-i, -1, -i)$ near the fixed points on the curves of the second type: locally we are considering a copy of $\mathbb{C}^3_{(x,y,z)}$ with the action $(x, y, z) \to (-ix, -y, -iz)$. This induces the map $((x, y, z); (a : b)) \to ((-ix, -y, -iz); (a : b))$ on $\widetilde{\mathbb{C}^3_{(x,y,z)}}$. Similarly, we could have argued that since α^* preserves the tangent directions, $\widetilde{\alpha}$ is well defined in a neighborhood of D_2.

The automorphism α switches pairs of curves of the third type, so $\widetilde{\alpha}$ switches the divisors D_3' and D_3''.

So, globally α induces an automorphism $\widetilde{\alpha}$ on $\widetilde{S \times E_i}$.

Let us denote by X' the smooth 3-fold $\widetilde{S \times E_i}/\widetilde{\alpha}^2$. We observe that X' is the crepant resolution of $(S \times E_i)/\alpha^2$ constructed in Section 4.1. Since X' is a resolution of quotient of $\widetilde{S \times E_i}$ by $\widetilde{\alpha}^2$, $\widetilde{\alpha}$ induces an automorphism (of order 2) on X'. We call it α'. It has order 2 and preserves the period of X' because α preserves the period of $S \times E_i$. The local action of α' near the fixed locus can be linearized to $\mathrm{diag}(-1, 1, -1)$. Thus, in order to construct a crepant resolution of X'/α' it suffices to blow up X' in the fixed locus of α' and then to consider the quotient by the automorphism induced by α' on the blow up. We denote by X the crepant resolution obtained in this way and we will say that it is of type X_4. We have the following commutative diagram:

$$
\begin{array}{ccccc}
S\times E_i & \xleftarrow{\beta_1} & \widetilde{S\times E_i} & & \\
\downarrow & & \downarrow{\scriptstyle \pi_1} & & \\
S\times E_i/\alpha^2 & \longleftarrow & \widetilde{S\times E_i}/\tilde{\alpha}^2 = X' & \xleftarrow{\beta_2} & \widetilde{X'} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \pi_2} \\
S\times E_i/\alpha & \longleftarrow & X'/\alpha' & \longleftarrow & \widetilde{X'}/\tilde{\alpha'} = X
\end{array}
\tag{6.1}
$$

where β_1 is the blow up of $S\times E_i$ in $\mathrm{Fix}_{\alpha^2}(S\times E)$, β_2 is the blow up of X' in $\mathrm{Fix}_{\alpha'}(X')$ and all the vertical arrows are the quotient maps and are $2:1$.

PROPOSITION 6.3. *Let S be a K3 surface with a purely non-symplectic automorphism α_S, such that $\alpha_S^*(\omega_S) = i\omega_S$. Let us denote by $m := \dim(H^2(S,\mathbb{Z})_i) = \dim(H^2(S,\mathbb{Z})_{-i})$ and by $r := \dim(H^2(S,\mathbb{Z})^{\alpha_S})$.*
Let X be a crepant resolution of $(S\times E_i)/(\alpha_S\times\alpha_E^3)$.

If α_S^2 fixes two elliptic curves, then the Hodge numbers of X are $h^{j,0}=1$, if $j=0,3$, $h^{j,0}=0$, if $j=1,2$, $h^{1,1}=19+r=25$, $h^{2,1}=m-1+8=13$.

If the fixed locus of α_S^2 does not consist of two elliptic curves, let us denote by D the curve of highest genus fixed by α_S^2. There are the following two possibilities: 1) D is of the first type; 2) D is of the second type.

The Hodge numbers of X are:

$$
\begin{array}{lll}
h^{j,0}=1 \text{ if } j=0,3, & h^{j,0}=0 \text{ if } j=1,2, & \\
h^{1,1}=1+r+7k+3b+2(n_1+n_2)+4a, & h^{2,1}=m-1+7g(D), & \text{in case 1)}\\
h^{1,1}=1+r+7k+3b+2(n_1+n_2)+4a, & h^{2,1}=m+2g(D)-n_2/2, & \text{in case 2)}.
\end{array}
$$

Equivalently,

$$
\begin{array}{lll}
h^{1,1}=22+17k+5a-10g(D), & h^{2,1}=4-k-a+8g(D), & \text{in case 1)}\\
h^{1,1}=\frac{102-7n_2-2g(D)}{4}+17k+5a, & h^{2,1}=\frac{18-n_2+10g(D)}{4}-k-a, & \text{in case 2)}.
\end{array}
$$

In particular $\chi(X)=36+36k-36g(D)+12a$ in case 1), $\chi(X)=42-3n_2-6g(D)+36k+12a$ in case 2) and $\chi(X)=24$ if α_S^2 fixes 2 elliptic curves.

PROOF. We make the proof in case α_S^2 does not fix two curves of genus 1. The remaining case is analogous.

By Proposition 3.1, the Hodge numbers of a crepant resolution of $(S\times E_i)/\alpha$ do not depend on the crepant resolution. Hence we compute them for the resolution described by Diagram (6.1). Let us denote by D_{tot} the divisor which is the sum of all the divisors introduced by the crepant resolution of $(S\times E_i)/\alpha$. It is the disjoint union of a certain number of $\mathbb{P}^1$-bundles over curves. We apply [V2, Théorèm 7.3.1], see also [G, Proposition 7], obtaining

$$
\begin{array}{rcll}
H^j(X) & = & H^j(S\times E_i), & j=0,1\,;\\
H^j(X) & = & H^j(S\times E_i)^{\alpha}\oplus H^{j-2}(D_{tot}), & j=2,3\,.
\end{array}
\tag{6.2}
$$

We already computed $(H^2(S\times E_i))^\alpha$ and $(H^3(S\times E_i))^\alpha$, see (3.1), obtaining $\dim((H^2(S\times E_i))^\alpha)=r+1$ and $\dim((H^3(S\times E_i))^\alpha)=\dim(H^{3,0}(S\times E_i))+\dim((H^{2,1}(S\times E_i))^\alpha)+\dim((H^{1,2}(S\times E_i))^\alpha)+\dim((H^{0,3}(S\times E_i)))=1+(m-1)+(m-1)+1=2m$. For $j=2$, $\dim(H^0(D_{tot}))$ is the number of disjoint divisors introduced by the resolution of the singular quotient. For $j=3$, we need to compute $\dim(H^1(D_{tot}))$ and since D_{tot} is a disjoint union of certain $\mathbb{P}^1$-bundles over certain curves, $\dim(H^1(D_{tot}))$ is the sum of the genera of the basis of these $\mathbb{P}^1$-bundles. So, in order to find the Hodge numbers of X we have to compute the number of divisors introduced by the resolution of $(S\times E_i)/\alpha$ and to consider the basis of the $\mathbb{P}^1$-bundles introduced by this resolution.

Step 1: Description of the resolution of the singularities.

Let K be a **curve of the first type** on $S\times E_i$, i.e. a fixed curve for α. Then it is a fixed curve for α^2 and so it is in the locus blown up by β_1. This introduces a divisor D_1 in $\widetilde{S\times E_i}$, which is the projectivization of the normal bundle $\mathcal{N}_{K/(S\times E_i)}$ and is a $\mathbb{P}^1$-bundle over K. Since locally near K the action of α is $\mathrm{diag}(i,1,-1)$, the action restricted to the normal bundle has two different eigenvalues, i and -1. This splits the normal bundle in two eigenspaces which can not be interchanged by α. Each of these two eigenspaces determines a section of the exceptional divisor D_1, which is a fixed curve for the action of $\widetilde{\alpha}$ on $D_1\subset\widetilde{S\times E_i}$. The automorphism $\widetilde{\alpha}^2$ acts as the identity on D_1, so $\pi_1: D_1\to\pi_1(D_1)$ is a $1:1$ map and $\pi_1(D_1)$ is isomorphic to D_1. The action of α' on $\pi_1(D_1)$ coincides with the action of $\widetilde{\alpha}$ on D_1 and hence α' fixes two curves on $\pi_1(D_1)$, which are sections of a $\mathbb{P}^1$-bundle over K, so they are two copies of K. Hence β_2 blows up two copies of K introducing 2 other divisors D_1' and D_1'' which are $\mathbb{P}^1$-bundles over K. The action of $\widetilde{\alpha'}$ on D_1' and D_1'' is the identity, so $\pi_2(D_1')\simeq D_1'$ and $\pi_2(D_1'')\simeq D_1''$. Hence there are 3 divisors $(\pi_2(\beta_2^{-1}(\pi_1(D_1))),\pi_2(D_1'),\pi_2(D_1''))$ which are $\mathbb{P}^1$-bundles over K and which are mapped by $\beta_1\pi_1^{-1}\beta_2\pi_2^{-1}$ to the curve K. We observe that $\pi_2(\beta_2^{-1}(\pi_1(D_1)))$ intersects both $\pi_2(D_1')$ and $\pi_2(D_1'')$ in a curve isomorphic to K and that $\pi_2(D_1')\cap\pi_2(D_1'')=\emptyset$.

Let B be a **curve of the second type** on $S\times E_i$, i.e., it is fixed by α^2 and α restricts to an involution on B with n_B fixed points. The blow up β_1 introduces a divisor D_2 over B which is a $\mathbb{P}^1$-bundle over B. The local action of $\widetilde{\alpha}$ on D_2 was computed before (in local coordinates over $\widetilde{\mathbb{C}^3_{(x,y,z)}}$): $\widetilde{\alpha}$ acts on the basis of the $\mathbb{P}^1$-bundle D_2 as α acts on B. For each point $V\in B$ such that $\alpha(V)=V$, $\widetilde{\alpha}$ fixes all the fiber of D_2 over V. There are n_B fibers of D_2 which are fixed by $\widetilde{\alpha}$, i.e. n_B rational curves in D_2 which are fixed by $\widetilde{\alpha}$. Since $\widetilde{\alpha}^2$ is the identity on D_2, $D_2\simeq\pi_1(D_2)$ and α' fixes n_B rational curves on $\pi_1(D_2)$. For each of these curves β_2 introduces an exceptional divisor which is preserved by π_2. In order to resolve the singularities of X'/α' we introduce n_B divisors which are $\mathbb{P}^1$-bundles over $\mathbb{P}^1$. Moreover, since α is an involution on B, α is non-trivial on the basis of $\beta_2^{-1}(\pi_1(D_2))$, and thus $\pi_2(\beta_2^{-1}(\pi_1(D_2)))$ is a $\mathbb{P}^1$-bundle over B/α. Hence, in order to resolve the singularities of $(S\times E_i)/\alpha$ over the image of B, we introduced the divisor $\pi_2(\beta_2^{-1}(\pi_1(D_2)))$ which is a $\mathbb{P}^1$-bundle over B/α and n_B divisors which are $\mathbb{P}^1$-bundles over $\mathbb{P}^1$.

Let (A',A'') be a **pair of curves of the third type** on $S\times E_i$, i.e. both A' and A''

are fixed by α^2 and $\alpha(A') = A''$. Since α interchanges A' and A'', $A' \simeq A''$. The blow up β_1 introduces two divisors D_3' and D_3'' over A' and A'' respectively and both of them are $\mathbb{P}^1$-bundles over $A' \simeq A''$. These divisors are interchanged by $\widetilde{\alpha}$ and fixed by $\widetilde{\alpha}^2$. So the two divisors $\beta_2^{-1}(\pi_1(D_3')) \subset X'$ and $\beta_2^{-1}(\pi_1(D_3'')) \subset X'$ are isomorphic to D_3' and D_3'' and they are interchanged by α'. Hence $\pi_2(\beta_2^{-1}(\pi_1(D_3'))) = \pi_2(\beta_2^{-1}(\pi_1(D_3'')))$ is a divisor on X, isomorphic to D_3' and so it is a $\mathbb{P}^1$-bundle over $A' \simeq A''$. Hence in order to resolve the singularities of $(S \times E_i)/\alpha$ over the image of the pair (A', A'') we introduced one divisor, $\pi_2(\beta_2^{-1}(\pi_1(D_3')))$, which is a $\mathbb{P}^1$-bundle over A'.

Step 2: Computation of the exceptional divisors and of the Hodge numbers of X.

In order to compute the Hodge numbers of X, we need to count the number of curves of each type that appear on $S \times E_i$:

(1) for each curve $K \subset S$ which is of the first type for α_S, the curves $K \times P_1 \subset (S \times E_i)$ and $K \times P_2 \subset (S \times E_i)$ are of the first type for α and the pair of curves $(K \times Q_1, K \times Q_2)$ is a pair of curves of the third type for α;

(2) for each curve $B_i \subset S$ which is of the second type for α_S, the curves $B \times P_1 \subset (S \times E_i)$ and $B \times P_2 \subset (S \times E_i)$ are of the second type for α and there are n_B points fixed by α on each of them. The pair $(B \times Q_1, B \times Q_2) \subset (S \times E_i)$ is a pair of curves of the third type for α;

(3) for each pair of curves $(A', A'') \subset S$ which is a pair of curves of the third type for α_S, the pairs $(A' \times P_1, A'' \times P_1)$, $(A' \times P_2, A'' \times P_2)$, $(A' \times Q_1, A'' \times Q_2)$, $(A' \times Q_2, A'' \times Q_1)$ are of the third type for α.

So the locus with non-trivial stabilizer for the action of α on $S \times E_i$ is the following: there are $2k$ curves of the first type; $2b$ curves of the second type and $k + b + 4a$ pairs of curves of the third type.

Let us denote by n_{B_i} for $i = 1, \dots, b$ the number of points on B_i which are fixed by α_S. For each curve B_i of the second type, both $B_i \times P_1$ and $B_i \times P_2$ contain n_{B_i} points fixed by α. Moreover $\sum_{i=1}^{b} n_{B_i} = n_1 + n_2$ is the number of the isolated fixed points for α_S on S. So the number of the isolated fixed points of α on $S \times E_i$ is $2(n_1 + n_2) = 2\sum_{i=1}^{b} n_{B_i}$.

The number of exceptional divisors of X which arise by the desingularization of $(S \times E_i)/\alpha$ is $3(2k) + 2b + 2(\sum_{i=1}^{b} n_{B_i}) + k + b + 4a = 7k + 3b + 4a + 2(n_1 + n_2)$. Hence

$$h^{1,1}(X) = 1 + r + 7k + 3b + 4a + 2(n_1 + n_2)\,.$$

We recall that D is the curve with highest genus fixed by α_S^2. If D is of the first type, then $D \times P_1$ and $D \times P_2$ are curves of the first type and $(D \times Q_1, D \times Q_2)$ is a pair of curves of third type, so the resolution of $(S \times E_i)/\alpha$ introduces 7 divisors which are $\mathbb{P}^1$-bundles over D, hence

$$h^{2,1}(X) = m - 1 + 7g(D)\,.$$

If D is of the second type, then $D \times P_1$ and $D \times P_2$ are curves of the second type and $(D \times Q_1, D \times Q_2)$ is a pair of curves of third type, so the resolution of $(S \times E_i)/\alpha$ introduces 2 divisors which are $\mathbb{P}^1$-bundles over D/α, and 1 divisor which is a $\mathbb{P}^1$-bundle over D, hence

$h^{2,1}(X) = m - 1 + 2g(D/\alpha) + g(D)$. By Riemann–Hurwitz, $g(D/\alpha) = \frac{1}{2}(g(D) + 1 - n_2/2)$, so

$$h^{2,1}(X) = m - 1 + 2g(D) + 1 - n_2/2 = m + 2g(D) - n_2/2\,.$$

Step 3: the Hodge numbers depend on $(k, a, g(D), n_2)$.

By [AS2, Theorem 1.1] and [AS2, Proposition 1], one obtains the following relations:

$$r = \frac{1}{2}(12+k+2a+b-g(D)+4h), \quad m = \frac{1}{2}(12-k-2a-b+g(D)), \quad n_1+n_2 = 2h+4$$

where $h = \sum_{C \subseteq \mathrm{Fix}\alpha_S(S)}(1 - g(C))$. Moreover we observe that if D is of the first type $h = (k-1) + (1 - g(D)) = k - g(D)$, $n_2 = 0$, $n_1 = 2h + 4$, $b = n_1/2$; if D is of the second type, $h = k$, $n_1 + n_2 = 2h + 4$, $b = n_1/2 + 1$. □

REMARK 6.4. If $g(D) = 0$, then $h^{2,1}(X) = m - 1$. So, by Proposition 3.7, if D is rational α_X is a maximal automorphism of the family of Calabi–Yau 3-folds of X, and this family does not admit maximal unipotent monodromy. Some of these families were already constructed in [G] (the ones corresponding to lines 12 - 13 - 15 - 16 - 18 of Table 2).

Any other automorphism on S commuting with α_S induces a maximal automorphism of X. In particular if X admits an isotrivial elliptic fibration with fibers isomorphic to E_i (this surely happens for some S, for example in many cases constructed in [G]), then S admits a symplectic involution (the translation by the 2-torsion section of this elliptic fibration) and thus X admits a maximal automorphism of order 2 preserving the period.

REMARK 6.5. In [G] it is proved that some K3 surfaces admitting a non-symplectic automorphism of order 4 such that D is a rational curve (with the notation of Proposition 6.3), are birational to the quotient $(C \times E_i)/(\mathbb{Z}/4\mathbb{Z})$ where C is a certain curve of positive genus. As a consequence one obtains that X is birational to $(C \times E_i \times E_i)/(\mathbb{Z}/4\mathbb{Z})$ and thus the variation of the Hodge structures of X depends only on the variation of the Hodge structures of the curve C, see also Remark 5.3. Here we can extend this result to each K3 surface admitting a non-symplectic automorphism α_S of order 4 such that D is a rational curve and α_S is the cover automorphism of a $4:1$ map $S \to \mathbb{P}^2$. Indeed, if S is a $4:1$ cover of $\mathbb{P}^2$ the branch locus consists of a plane quartic curve Q, which is fixed by α_S. Since D is rational, Q is quite singular (it could also be reducible) and in particular it admits at least either one node or a cusp, say in the point $P \in Q$. The pencil of lines through P in $\mathbb{P}^2$ induces a pencil of curves on S. The general member of this pencil is a $4:1$ cover of $\mathbb{P}^1$ branched in 2 points of order 4 and two points of order 2, thus it is isomorphic to E_i. So, the pencil of lines through P in $\mathbb{P}^2$ induces an isotrivial elliptic fibration on S whose general fiber is isomorphic to E_i. Hence there exists a curve C such that S is birational to $(C \times E_i)/(\mathbb{Z}/4\mathbb{Z})$, X to $(C \times E_i \times E_i)/(\mathbb{Z}/4\mathbb{Z})$ and the variation of the Hodge structures of X depends only on the one of the curve C.

REMARK 6.6. Let S be a K3 surface, which is a $4:1$ cover of $\mathbb{P}^2$ and let α_S be the cover automorphism. The branch locus of $S \xrightarrow{4:1} \mathbb{P}^2$ is a possibly singular quartic curve Q. Let us denote by x the number of nodes of Q, by y the number of cusps of Q and by z the number of the components of Q. Then the properties of the fixed locus of α_S are the following: D is a curve of first type and is the component of Q of highest genus, $k = z$, $N = k + x + y$, $n = 2(x + y)$, $a = y$. Hence the Calabi–Yau 3-folds of type X_4 constructed by S are the ones described in lines 2, 3, 4, 5, 7, 8, 11, ..., 19 of Table 2. In particular the Calabi–Yau 3-folds which are of the type described in Remark 6.5 are the ones of lines 12, ..., 19 of Table 2.

REMARK 6.7. There exists at least three different Calabi–Yau 3-folds of Borcea–Voisin type which are rigid and admit an automorphism acting as $\zeta_4 = i$ on ω_X (see lines 18, 19 of Table 2 and line 5 of Table 4). Since these Calabi–Yau 3-folds are rigid, $H^3(X, \mathbb{Q}) \simeq \mathbb{Q}^2$ and the Hodge structure is given by the decomposition in eigenspaces for the action of α_X (i.e. for the action of $\cdot i$ on $\mathbb{Q}^2 \simeq \mathbb{Q}(i)$). These Calabi–Yau 3-folds are associated to the same K3 surface S (the unique K3 surface admitting a non-symplectic involution, α_S^2, whose fixed locus consists of 10 rational curves). On this S there are at least 3 automorphisms $\alpha_S \in \mathrm{Aut}(S)$ of order 4 such that α_S^2 fixes 10 rational curves and for each of them the Borcea–Voisin construction gives a different Calabi–Yau 3-fold. For example, S admits both a $4:1$ cover of $\mathbb{P}^2$ branched along 4 lines and a $4:1$ cover of $\mathbb{P}^2$ branched along a quartic with three cusps. The cover automorphisms in these two cases are different and have different fixed loci (these are associated to the Calabi–Yau 3-folds in lines 18 and 19 in Table 2).

6.2. (Almost) elliptic fibrations. Let X be a Calabi–Yau 3-fold of type X_4. Let us consider the map $\mathcal{E}_4 : X \to S/\alpha_S$ induced on X by $(S \times E_i)/(\alpha_S \times \alpha_E^3) \to S/\alpha_S$. We consider the quotient map $q_S : S \to S/\alpha_S$ and we recall that S/α_S has exactly $n_1 + n_2$ singular points, the images of the $n_1 + n_2$ isolated fixed points on S. We will assume (S, α_S) to be generic in the family of K3 surfaces$S$ with the non-symplectic automorphism of order 4 α_S (i.e. $\rho(S) = 22 - 2m$).

PROPOSITION 6.8. *The map $\mathcal{E}_4 : X \to S/\alpha_S$ is an almost elliptic fibration whose general fiber is isomorphic to E_i and it is an elliptic fibration if and only if $n_1 + n_2 = 0$. The fiber F_R of $\mathcal{E}_4$ over $R \in S/\alpha_S$ is of dimension 1 if and only if R is a smooth point, and is singular if and only if R is in the branch locus of $q : S \to S/\alpha_S$. In particular:*

- *If $q^{-1}(R)$ consists of two points, the fiber F_R is of type I_0^*;*
- *If $q^{-1}(R)$ consists of one point and $R \in S/\alpha_S$ is smooth, the fiber F_R is of type III^*;*
- *If R is a singular point of S/α_S, the fiber F_R consists of 2 rational curves meeting in a point and two disjoint divisors which are $\mathbb{P}^1$-bundles over $\mathbb{P}^1$, and which intersect both the same rational curve in one point (see Figure 2).*

The Mordell–Weil group of this fibration is generically equal to $\mathbb{Z}/2\mathbb{Z}$.

PROOF. With the same notation as in the previous section, we denote by $\mathcal{E}' : X' \to S/\alpha_S^2$ the map induced on X' by $(S \times E_i)/(\alpha_S^2 \times \alpha_E^2) \to S/\alpha_S^2$ and by F'_R the fiber of $\mathcal{E}'$ over $R \in S/\alpha_S^2$. Moreover we denote by $q' : S \to S/\alpha_S^2$ the quotient map.

Let $R \in S/\alpha_S$ be such that $q^{-1}(R) = \{R_1, R_2\} \subset S$. This implies $q'(R_i) = R_i$, $i = 1, 2$

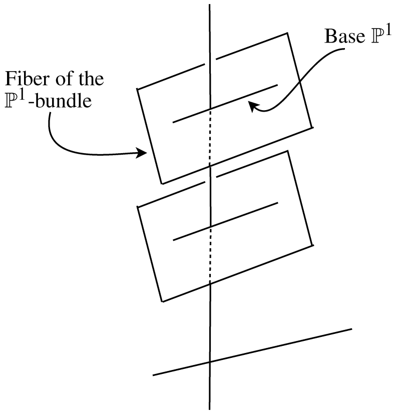


FIGURE 2. The fiber over a singular point of S/α_S.

and so the fibers F'_{R_i} are of type I_0^*, by Proposition 4.4. The automorphism induced by α_S on S/α_S^2 switches R_1 and R_2. So α' switches F_{R_1} and F_{R_2} and thus F_R is a fiber of type I_0^*.

Let $R \in S/\alpha_S$ be a smooth point such that $q^{-1}(R) = R_1 \in S$. This implies that there exists a curve $K \subset S$ such that $R_1 \in K$ and $K \subset \mathrm{Fix}_{\alpha_S}(S)$. The fiber F'_R of $\mathcal{E}'$ is of type I_0^* (by Proposition 4.4). Two of the components with multiplicity one (those which are fibers of the $\mathbb{P}^1$-bundles introduced by β_1 over $K \times Q_1$ and $K \times Q_2$ respectively) are switched by α'. The automorphism α' acts as an involution on the other 2 components of multiplicity 1 and fixes two points on each of them. So the fiber F_R of $\mathcal{E}_4$ is a fiber of type III^* and 7 of its 8 components are fibers of the seven $\mathbb{P}^1$-bundles over K introduced by the blow ups β_1 and β_2. The other component is the image of the strict transform of E_i.

Let $R \in S/\alpha_S$ be a singular point. Then $q^{-1}(R) = R_1$, R_1 is an isolated fixed point for α_S and it lies on a curve B fixed by α_S^2. The fiber F'_R of $\mathcal{E}'$ is of type I_0^* (by Proposition 4.4). Two of the components of multiplicity 1 of I_0^* are switched by α' (the ones which are fibers of the $\mathbb{P}^1$-bundles introduced by β_1 over $B \times Q_1$ and $B \times Q_2$ respectively). The other two are fibers of the $\mathbb{P}^1$-bundles introduced by β_1 over $B \times P_1$ and $B \times P_2$. These curves are fixed by α' and so β_2 is a blow up of each of these curves. Hence the fiber F_R consists of a rational curve which is the image of the strict transform of E_i, of a rational curve which is the image of the two rational curves switched by α' and of 2 divisors, which are exceptional divisors of β_2 and are $\mathbb{P}^1$-bundles over $\mathbb{P}^1$. In particular it contains 2 curves and 2 divisors (see Figure 2).

The (rational) sections are the images of $S \times P_1$ and $S \times P_2$ under the rational map $\pi_2 \circ \beta_2^{-1} \circ \pi_1 \circ \beta_1^{-1}$. One of them can be chosen as zero section. The other is a section of order 2, indeed if P_1 is the zero of the elliptic curve E_i, then P_2 has order 2 on E_i. □

REMARK 6.9. The automorphism α_X defined in Definition 3.5 is induced by the complex multiplication of order 4 on each smooth fiber of the fibration.

We now give the Weierstrass equations of $\mathcal{E}_4$ in case the base S/α_S is smooth. This is equivalent to require that α_S has no isolated fixed points, i.e. $n_1 + n_2 = 0$. We observe that in this case $\mathcal{E}_4$ is an elliptic fibration, by Proposition 6.8.

By [AS2, Proposition 1], $n_1+n_2 = 0$ implies $h = -2$ where $h = \sum_{C \in \mathrm{Fix}_{\alpha_S}(S)}(1-g(C))$ (h is the same as α in [AS2]). We already noticed that if D is of the second type $h = k$, i.e. h is the number of curves fixed by α_S. Since $k \geq 0$, if $n_1+n_2 = 0$, then D is of first type. In this case $-2 = h = k + 1 - g(D)$, which implies $g(D) \geq 3$. By [AS2, Theorem 4.1], $g(D) \leq 3$. So $n_1 + n_2 = 0$ implies that D is of the first type and $g(D) = 3$. There exist exactly two families of K3 surfaces admitting a non-symplectic automorphism of order 4 fixing a curve of genus 3, and the difference among these cases is that in one case D is hyperelliptic while in the other it is not. These families are respectively 5 and 6 dimensional. We now consider both these situations.

6.2.1. *Weierstrass equation of $\mathcal{E}_4$ if $S/\alpha_S \simeq \mathbb{P}^2$.* If D is not a hyperelliptic curve, the K3 surface is a $4:1$ cover of $\mathbb{P}^2$ branched along a smooth quartic. Let us denote by Q the smooth quartic in $\mathbb{P}^2$, which is the zero locus of $f_4(x_1 : x_2 : x_3)$. The equation of S is $t^4 = f_4(x_1 : x_2 : x_3)$. The automorphism $\alpha_S : S \to S$ is the cover automorphism $(t : x_1 : x_2 : x_3) \mapsto (it : x_1 : x_2 : x_3)$.

The variables $Y := vt^9$, $X := ut^6$, defined on $S \times E_i$, are invariant for $\alpha_S \times \alpha_E^3$ and satisfy

$$Y^2 = X^3 + X f_4^3(x_1 : x_2 : x_3)\,. \tag{6.3}$$

Moreover, the map $((v,u),(t, x_0 : x_1 : x_2)) \to ((X,Y),(x_1 : x_2 : x_3))$ is $4:1$ on $S \times E_i$ and thus (6.3) is the Weierstrass equation of the elliptic fibration $\mathcal{E}_4 : X \to S/\alpha_S$ described in Proposition 6.8. Since the discriminant locus has equation $\Delta = f_4^9(x_1 : x_2 : x_3)$ all the singular fibers are of type III^*. This in fact agrees with Proposition 6.8.

6.2.2. *Weierstrass equation of $\mathcal{E}_4$ if $S/\alpha_S \simeq \mathbb{F}_4$.* We now assume that α_S fixes a hyperelliptic curve of genus 3 on S. We first briefly recall the construction of S as $4:1$ cover of $\mathbb{F}_4$ given in [A, Section 2]. Then we deduce an equation for the elliptic fibration $\mathcal{E}_4$ in this case.

Let C be a hyperelliptic curve of genus 3. Then C is defined by an equation of the form $y^2 = f_8(s : t)$ with $f_8 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(8))$ in the line bundle $\mathcal{O}_{\mathbb{P}^1}(4)$, where $(s : t)$ are coordinates on $\mathbb{P}^1$ while y is a coordinate on the fiber. We embed $\mathcal{O}_{\mathbb{P}^1}(4)$ in $\mathbb{P}^5$ as the cone on the rational normal curve in $\mathbb{P}^4 = \{x_5 = 0\}$ with vertex $(0:0:0:0:0:1)$. Blowing up this cone in the singular point gives us the Hirzebruch surface $\mathbb{F}_4$. The toric resolution of $\mathcal{O}_{\mathbb{P}^1}(4)$ (with coordinates $(s : t : y)$) is $\mathbb{F}_4$ with coordinates $(s : t : y : z)$ as in Section 2.3. With this coordinates, an equation for C is

$$C : y^2 = f_8(s : t) z^2\,.$$

Let $\widetilde{\mathbb{F}_4}$ be the double cover of $\mathbb{F}_4$ branched along C, and S the double cover of $\widetilde{\mathbb{F}_4}$ branched along the strict transforms of C and D_z. Then S is a K3 surface, which is a $4:1$ covering of $\mathbb{F}_4$. Moreover the covering automorphism fixes the strict transform of C and swithces the strict transforms of D_z.

Introducing a new coordinate η, an equation for $\widetilde{\mathbb{F}_4}$ is $\eta^2 = y^2 - f_8(s:t)z^2$ and so an equation for S is

$$\begin{cases} \xi^2 = \eta z \\ \eta^2 = y^2 - f_8(s:t)z^2 \,. \end{cases}$$

The covering automorphism is $(\eta, \xi, (s:t:y:z)) \longmapsto (-\eta, i\xi, (s:t:y:z))$.

The surface S admits a model S' in $\mathcal{O}_{\mathbb{F}_4}(2D_t + D_z)$, with equation $w^4 = (y^2 - f_8(s:t)z^2)z^2$, and the birational morphism

$$\begin{array}{ccc} S & \to & S' \\ (\eta, \xi, (s:t:y:z)) & \longmapsto & (w, (s:t:y:z)) = (\xi, (s:t:y:z)) \end{array}$$

is compatible with α_S and the covering transformation $\alpha_{S'} : (w, (s:t:y:z)) \longmapsto (iw, (s:t:y:z))$.

On the product $S' \times E$ the functions $Y := \frac{vw^9}{z^3}$ and $X := \frac{uw^6}{z^2}$ are invariant under the action of the morphism α. Moreover they satisfy the relation

$$Y^2 = X^3 + (y^2 - f_8(s:t)z^2)^3 z^2 X \,,$$

which is an equation for the Weierstrass model of $\mathcal{E}_4 : X \to S/\alpha_S$ in $\mathbb{P}(\mathcal{O}_{\mathbb{F}_4}(-2K_{\mathbb{F}_4}) \oplus \mathcal{O}_{\mathbb{F}_4}(-3K_{\mathbb{F}_4}) \oplus \mathcal{O}_{\mathbb{F}_4})$.

7. Quotient of order 6. Let us consider the elliptic curve E_{ζ_3} introduced in Section 2.1. We already observed that its Weierstrass equation is $v^2 w = u^3 + w^3$ and that it has an automorphism $\gamma_E : (u:v:w) \to (\zeta_3^2 u : -v : w)$, whose square is $\gamma_E^2 = \alpha_E$.

We consider a K3 surface S which admits a purely non-symplectic automorphism γ_S of order 6. As shown in Proposition 3.2 there exists a desingularization of $(S \times E_{\zeta_3})/(\gamma_S \times \gamma_E^5)$ which is a Calabi–Yau. Observe that, since S admits both a non-symplectic automorphism of order 3 (γ_S^2) and one of order 2 (γ_S^3), one can construct from S special members of the families of Calabi–Yau 3-folds of type X_2 and of type X_3 described in Sections 4.1 and 5.1 respectively. In [R3], the author analyzes the consequence of this fact on Calabi–Yau 3-folds constructed from some special K3 surfaces S.

First we recall some properties of the loci with non-trivial stabilizer for γ_E and γ_S. The automorphism γ_E fixes only the point 0_E. The automorphism γ_E^3 fixes four points 0_E, P_1, P_2 and P_3 and the points P_i, $i = 1, 2, 3$ form an orbit for the action of γ_E. The automorphism γ_E^2 fixes 3 points: 0_E, Q_1 and Q_2 and $\gamma_E(Q_1) = Q_2$.

7.1. Preliminaries on purely non-symplectic automorphisms of order 6 on a K3 surface. The purely non-symplectic automorphism of order 6 are not completely classified. Some results on the fixed locus of a non-symplectic automorphism of order 6 are given in [D1]. Here we need to give a more precise description not only of the fixed locus of the purely non-symplectic automorphism, but also of the loci with non-trivial stabilizer for the action of the automorphism.

Let $\gamma_S \in \mathrm{Aut}(S)$ be a purely non-symplectic automorphism of order 6 on S. Following [D1], we observe that the fixed locus of γ_S consists of disjoint curves and of isolated fixed points. We will say that an isolated fixed point for γ_S is of type (r, s) if the action of γ_S

near the point linearizes to $\mathrm{diag}(\zeta_6^r, \zeta_6^s)$. Up to the power of γ_S and the choice of the local coordinates for the linearization, there are only two different types of isolated fixed point for γ_S: the type $(2, 5)$ and the type $(3, 4)$. There is at most a curve with positive genus in the fixed locus of γ_S and we denote the curve with highest genus by D.

The loci with non-trivial stabilizer for γ_S can be described in the following way. We call:

(1) curves of first type: the curves which are fixed by γ_S (and thus of course also by γ_S^2 and by γ_S^3); we denote by L_i these curves and by l their number;
(2) curves of second type: the curves which are fixed by γ_S^3 and are invariant for γ_S; we denote by U_i these curves and by u their number. The automorphism γ_S could have some isolated fixed points on these curves and this point could be both of type $(2, 5)$ and of type $(3, 4)$;
(3) curves of third type: the curves fixed by γ_S^3 and sent to another curve by γ_S; we denote by (A_i', A_i'', A_i''') the triples of these curves, assuming that $\gamma_S(A_i') = A_i''$ and $\gamma_S(A_i'') = A_i'''$, and we denote by a the number of these triples;
(4) curves of the fourth type: the curves fixed by γ_S^2 and invariant for γ_S; we denote by W_i these curves and by w their number. The automorphism γ_S could have some isolated fixed points on these curves and all of them are of type $(3, 4)$;
(5) curves of fifth type: the curves fixed by γ_S^2 and sent to another curve by γ_S; we denote by (B_i', B_i'') the pairs of these curves, assuming that $\gamma_S(B_i') = B_i''$, and we denote by b the numbers of these pairs;
(6) points of fifth type: the isolated points fixed by γ_S^2 and switched by γ_S. We denote by $2n'$ their number.

The properties of the locus with non-trivial stabilizer for the action of γ_S on S are described by the numbers $(N, g(F_1), g(F_2), k, n, g(G), l, p_{2,5}, p_{3,4}, g(D), u, w)$ defined here:

DEFINITION 7.1. The automorphism γ_S^3 fixes N curves. We denote by F_1 and F_2 the curves with highest genus fixed by γ_S^3.

The automorphism γ_S^2 fixes k curves and n isolated points. We denote by G the curve with highest genus fixed by γ_S^2.

The automorphism γ_S fixes l curves, $p_{2,5}$ points of type $(2, 5)$ and $p_{3,4}$ points of type $(3, 4)$. We denote by D the curve of highest genus fixed by γ_S.

The isolated fixed points of γ_S lie on the u curves (of second type) which are fixed by γ_S^3 and are invariant for γ_S.

The isolated fixed points of γ_S of type $(3, 4)$ lie also on the w curves (of fourth type) which are fixed by γ_S^2 and are invariant for γ_S.

The isolated fixed points of γ_S of type $(2, 5)$ are isolated fixed points also for γ_S^2.

REMARK 7.2. The isolated fixed points of type $(3, 4)$ lie on the intersection among curves of the second type and curves of the fourth type.

We have $N = l + u + 3a$ and $k = l + 2b + w$, $n = p_{2,5} + 2n'$.

By [D1, Theorem 4.1], $2p_{2,5} + p_{3,4} - 6l = 6$.

Since γ_S^3 is an involution, if $g(F_1) \neq 0$ and $g(F_2) \neq 0$, then $g(F_1) = g(F_2) = 1$. Moreover if $g(D) \neq 0$, then $D \equiv G \equiv F_1$.

7.2. The construction of Calabi–Yau 3-folds of type X_6. We now construct an explicit crepant resolution of $(S \times E_{\zeta_3})/(\gamma_S \times \gamma_E^5)$ considering two following quotients, using a procedure similar to the one of Section 6.1. Let us denote by $\gamma := \gamma_S \times \gamma_E^5$ and by $b_1 : \widetilde{S \times E_{\zeta_3}} \to S \times E_{\zeta_3}$ the blow up of $S \times E_{\zeta_3}$ in the fixed locus of γ^3, which consists of disjoint rational curves. We claim that γ induces an automorphism $\widetilde{\gamma}$ on $\widetilde{S \times E_{\zeta_3}}$. As in Section 6.1, it suffices to consider the definition of $\widetilde{\gamma}$ locally near the locus blown up by b_1.

Since the local action of γ^3 near the fixed locus is represented by the diagonal matrix $\mathrm{diag}(-1, 1, -1)$, locally we are blowing up $\mathbb{C}^3_{(x,y,z)}$ in $x = z = 0$. Hence, locally the blow up is identified with $\widetilde{\mathbb{C}^3_{(x,y,z)}} := V(bx = az) \subset \mathbb{C}^3_{(x,y,z)} \times \mathbb{P}^1_{(a:b)}$. If a curve C is fixed by γ, then b_1 introduces an exceptional divisor on it and since γ respects the tangent directions it extends to an automorphism $\widetilde{\gamma}$ on the exceptional divisor. Equivalently, one can observe that the local action of γ is $(x, y, z) \mapsto (\zeta_6 x, y, \zeta_6^5 z)$ which induces the map $((x, y, z); (a : b)) \mapsto ((\zeta_6 x, y, \zeta_6^5 z); (a : \zeta_6^4 b))$ on $\widetilde{C^3_{(x,y,z)}}$. If C is a curve fixed by γ^3 and leaved invariant by γ, then either γ has no fixed points on C or it has some fixed points. In the first case since γ respects the tangent directions and move all the points of C, it lifts to an automorphism $\widetilde{\gamma}$ of the exceptional divisor over C. In the latter case, the local action of γ near the fixed points is diagonalized either by $\mathrm{diag}(\zeta_6^5, \zeta_6^2, \zeta_6^5)$ or by $\mathrm{diag}(\zeta_6^3, \zeta_6^4, \zeta_6^5)$. In both the cases one observes that γ induces an automorphism $\widetilde{\gamma}$ on the exceptional divisors lifting the action of γ respectively to $((x, y, z); (a : b)) \mapsto ((\zeta_6^5 x, \zeta_6^2 y, \zeta_6^5 z); (a : b))$ or to $((x, y, z); (a : b)) \mapsto ((\zeta_6^3 x, \zeta_6^4 y, \zeta_6^5 z); (a : \zeta_6^2 b))$ on $\widetilde{\mathbb{C}^3_{(x,y,z)}}$. If C is a curve fixed by γ^3 and $\gamma(C) = C' \neq C$, then C' is a curve fixed by γ^3. So b_1 introduces an exceptional divisor both on C and C' and $\widetilde{\gamma}$ is well defined, since it send the exceptional divisor over C to the one over C', respecting the tangent directions. So γ defines an automorphism $\widetilde{\gamma}$ on $\widetilde{S \times E_{\zeta_3}}$. Let us now denote by X' the quotient $\widetilde{S \times E_{\zeta_3}}/\widetilde{\gamma}^3$. Thus, X' is a crepant resolution of $(S \times E_{\zeta_3})/\gamma^3$ and it carries an automorphism of order 3 induced by $\widetilde{\gamma}$ and denoted by γ'. The automorphism γ' is an order 3 automorphism of X' which preserves the period of X', which is 3-fold with trivial canonical bundle. In Section 5.1 we already constructed a crepant resolution of the quotient of smooth threefold with trivial canonical bundle by automorphism of order 3 which preserves the period. Applying this construction to (X', γ'), we obtain a crepant resolution of X'/γ', which will be denoted by X and will be called a 3-fold of type X_6. By construction this is a crepant resolution also of the singular 3-fold $(S \times E_{\zeta_3})/\gamma$. The construction of X is summarized by the following diagram:

$$\begin{array}{ccccccc}
S\times E_{\zeta_3} & \xleftarrow{\ b_1\ } & \widetilde{S\times E_{\zeta_3}} & & & & \\
\downarrow & & \downarrow{\scriptstyle \pi_1} & & & & \\
S\times E_{\zeta_3}/\gamma^3 & \longleftarrow & \widetilde{S\times E_{\zeta_3}}/\tilde{\gamma}^3 = X' & \xleftarrow{\ b_2\ } & \widetilde{X'} & & \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \pi_2} & & \\
S\times E_{\zeta_3}/\gamma & \xleftarrow{\ a_1\ } & X'/\gamma' & \xleftarrow{\ a_2\ } & \widetilde{X'}/\tilde{\gamma}' & \xrightarrow{\ \phi\ } & X
\end{array}$$

where b_1 is the blow up of $S \times E_{\zeta_3}$ in $\mathrm{Fix}_{\gamma^3}(S \times E_{\zeta_3})$, b_2 is the blow up of X' in $\mathrm{Fix}_{\gamma'}(X')$, the maps π_1 and π_2 are $2:1$ and $3:1$ respectively and ϕ is a contraction. Indeed, in order to construct a crepant resolution of the singularities of type A_2, we introduce three exceptional divisors, we make the quotient and then we contract the image of one of these divisors (cf. Section 5.1). The maps a_1 and a_2 are defined by the commutativity of the diagram. The map $\phi \circ a_2^{-1} \circ a_1^{-1} : (S \times E_{\zeta_3})/\gamma \to X$ is the crepant resolution of the quotient $(S \times E_{\zeta_3})/\gamma$.

PROPOSITION 7.3. *Let us denote by* $m := \dim(H^2(S,\mathbb{Z})_{-\zeta_3}) = \dim(H^2(S,\mathbb{Z})_{-\zeta_3^2})$ *and* $r := \dim(H^2(S,\mathbb{Z})^{\gamma_S})$. *With the notation above, the Hodge numbers of X are the following:*

$$\begin{array}{rl}
h^{j,0} = & 1 \text{ if } j = 0,3,\ h^{j,0} = 0 \text{ if } j = 1,2\,,\\
h^{1,1} = & r + 1 + 2l + 2N - 2a + 4k - 2b + 3n' + 3p_{2,5} + p_{3,4}\\
h^{2,1} = & \left\{\begin{array}{ll} m - 1 + 8g(D) + g(F_2/\gamma_S) + g(F_2), & \text{if } g(D) \geq 1\\ m - 1 + 2g(G/\gamma_S) + 2g(G) + g(F_1/\gamma_S) & \\ +g(F_1) + g(F_2/\gamma_S) + g(F_2), & \text{if } g(D) = 0\,. \end{array}\right.
\end{array}$$

PROOF. The proof is similar to the one of Proposition 6.3: we denote by D_{tot} the divisor which is the sum of all the exceptional divisors of the crepant resolution X of $(S\times E_{\zeta_3})/\gamma$. By (6.2) and (3.1), we have $h^{1,1}(X) = r + 1 + h^{0,0}(D_{tot})$ and $h^{2,1}(X) = m - 1 + h^{1,0}(D_{tot})$. In order to compute $h^{i,j}(D_{tot})$ we describe explicitly the divisors introduced by the resolution.

Step 1: Description of the resolution of the singularities.

Let L be a **curve of the first type** on $S \times E_{\zeta_3}$, i.e. a fixed curve for γ. Then it is a fixed curve for γ^3 and so it is in the locus blown up by b_1. This introduces a divisor D_1 on $\widetilde{S \times E_{\zeta_3}}$, which is the projectivization of the normal bundle $\mathcal{N}_{L/(S\times E_i)}$ and is a $\mathbb{P}^1$-bundle over L. Since locally near L the action of γ is $\mathrm{diag}(\zeta_6, 1, \zeta_6^5)$, the action restricted to the normal bundle has two different eigenvalues, ζ_6 and ζ_6^5. This splits the normal bundle in two eigenspaces which can not be interchanged by γ. Each of these two eigenspaces determines a section of the exceptional divisor D_1, which is a fixed curve for the action of $\widetilde{\gamma}$ on $D_1 \subset \widetilde{S \times E_{\zeta_3}}$. The automorphism $\widetilde{\gamma}^3$ acts as the identity on D_1, so $\pi_1 : D_1 \to \pi_1(D_1)$ is a $1:1$ map and $\pi_1(D_1)$ is isomorphic to D_1. The action of γ' on $\pi_1(D_1)$ coincides with the action of $\widetilde{\gamma}$ on D_1 and hence γ' fixes two curves on $\pi_1(D_1)$ which are section of a $\mathbb{P}^1$-bundle over L, so they are two copies of L and we call them $L^{(1)}$ and $L^{(2)}$. These two curves are fixed by γ', which is an automorphism of order 3. Hence b_2 introduces 3 exceptional divisors on $L^{(i)}$, $i = 1, 2$, denoted by $D_1'^{(i)}$, $D_1''^{(i)}$, $D_1'''^{(i)}$ which are $\mathbb{P}^1$-bundles over $L^{(i)} \simeq L$. The action of $\widetilde{\gamma}'$ on $D_1'^{(i)}$, $D_1''^{(i)}$ and $D_1'''^{(i)}$ is the identity, so $\pi_2(D_1'^{(i)}) \simeq D_1'^{(i)}$, $\pi_2(D_1''^{(i)}) \simeq D_1''^{(i)}$ and $\pi_2(D_1'''^{(i)}) \simeq D_1'''^{(i)}$. As we discussed in Section 5.1, one of the three divisors $D_1'^{(i)}$ $D_1''^{(i)}$, $D_1'''^{(i)}$ is contracted by ϕ (in fact the divisors which has non-trivial intersection with both the others, say $D_1'''^{(i)}$). Hence for each curve L of the first type on $S \times E_{\zeta_3}$ we have the following 5 divisors on X, $\phi(\pi_2((b_2^{-1}(\pi_1(D_1))))$, $\phi(\pi_2(D_1'^{(1)}))$, $\phi(\pi_2(D_1''^{(1)}))$, $\phi(\pi_2(D_1'^{(2)}))$, $\phi(\pi_2(D_1''^{(2)}))$ which are $\mathbb{P}^1$-bundles over a curve isomorphic to L and are mapped by $b_1 \circ \pi_1^{-1} \circ b_2 \circ \pi_2^{-1} \circ \phi^{-1}$ to the curve L.

Let U be a **curve of the second type** on $S \times E_{\zeta_3}$, i.e., it is fixed by γ^3 and γ restricts to an order 3 automorphism on U. We say that an isolated fixed point for γ on $S \times E_{\zeta_3}$ is of type (a, b) if the local action of γ near the point is $\mathrm{diag}(\zeta_6^a, \zeta_6^b, \zeta_6^5)$. The automorphism γ restricted to U has $(p_{2,5})_U$ fixed points which are of type $(2, 5)$ for γ and $(p_{3,4})_U$ fixed points which are of type $(3, 4)$ for γ. The blow up b_1 introduces a divisor D_2 over U which is a $\mathbb{P}^1$-bundle over U. The local action of $\widetilde{\gamma}$ on D_2 was computed before: $\widetilde{\gamma}$ acts on the basis of the $\mathbb{P}^1$-bundle D_2 as γ acts on U. Let $V \subset U$ be an isolated fixed point for γ. If V is of type $(2, 5)$, then it is an isolated fixed point for γ and $\widetilde{\gamma}$ fixes all the fiber over V. Hence $\widetilde{\gamma}$ fixes $(p_{2,5})_U$ fibers of D_2. Since $\widetilde{\gamma}^3$ is the identity on D_2, $D_2 \simeq \pi_1(D_2)$ and γ' fixes $(p_{2,5})_U$ rational curves on $\pi_1(D_2)$. For each of these curves b_2 introduces 3 exceptional divisors which are preserved by π_2 and ϕ contracts one of them, so $\phi \circ \pi_2 \circ b_2^{-1}$ introduces 2 divisors for each fibers of D_2 fixed by γ'. If V is of type $(3, 4)$, then V lies on a curve of fourth type. One can explicitly consider the local action of $\widetilde{\gamma}$ on D_2 and near the fiber over V: we already observed that it is $((x, y, z); (a : b)) \mapsto ((\zeta_6^3 x, \zeta_6^4 y, \zeta_6^5 z); (a : \zeta_6^2 b))$ on $\widetilde{\mathbb{C}^3_{(x,y,z)}} := V(bx = az) \subset \mathbb{C}^3_{(x,y,z)} \times \mathbb{P}^1_{(a:b)}$. So $\widetilde{\gamma}$ has exactly two fixed points (i.e. $(0 : 1)$, $(1 : 0) \subset \mathbb{P}^1_{(a:b)}$) on the fiber of D_2 over $V \in U$. One of this point, $(1 : 0)$, lies on a curve in $\widetilde{S \times E_{\zeta_3}}$ which is the strict transform of a curve of fourth type on $S \times E_{\zeta_3}$, locally given by $y = z = 0$. Since we will blow up this curve, we do not have to blow up $(1 : 0)$. The point $(0 : 1)$ is an isolated fixed point for $\widetilde{\gamma}^2$ and so it will be an isolated fixed point for γ' on X'. Hence, this is an isolated fixed point for an automorphism of order 3 on X'. We already observed in Section 5.1 that it suffices to blow up once this type of points in order to have a smooth quotient. So b_2 introduces an exceptional divisor for every point of type $(3, 4)$ on U and this divisor is isomorphic to $\mathbb{P}^2$.

To recap, for every curve U of the second type, we introduced the following divisors in order to resolve the singularities of $(S \times E_{\zeta_3})/\gamma$: $\pi_2(\beta_2^{-1}(\pi_1(D_2)))$ which is a $\mathbb{P}^1$-bundle over U/γ, $2(p_{2,5})_U$ divisors which are $\mathbb{P}^1$-bundles over $\mathbb{P}^1$ and $(p_{3,4})_U$ divisors which are isomorphic to $\mathbb{P}^2$.

Let (A', A'', A''') be a **triple of curves of the third type** on $S \times E_{\zeta_3}$, i.e. A', A'' and A''' are fixed by γ^3 and $\{A', A'', A'''\}$ is an orbit for γ. The blow up b_1 introduces three divisors D_3', D_3'' and D_3''' over A', A'' and A''' respectively and all of them are $\mathbb{P}^1$-bundles over $A' \simeq A'' \simeq A'''$. These divisors are interchanged by $\widetilde{\gamma}$ and fixed by $\widetilde{\gamma}^3$. So the divisors $b_2^{-1}(\pi_1(D_3')) \subset X'$, $b_2^{-1}(\pi_1(D_3'')) \subset X'$ and $b_2^{-1}(\pi_1(D_3''')) \subset X'$ are isomorphic to D_3', D_3'', D_3''' and they are permuted by γ'. Hence $\phi(\pi_2(b_2^{-1}(\pi_1(D_3')))) = \phi(\pi_2(b_2^{-1}(\pi_1(D_3'')))) = \phi(\pi_2(b_2^{-1}(\pi_1(D_3'''))))$ is a divisor on X, isomorphic to D_3' and so it is $\mathbb{P}^1$-bundle over $A' \simeq A'' \simeq A'''$. Thus in order to resolve the singularities of $(S \times E_{\zeta_3})/\gamma$ over the image of the triple (A', A'', A''') we introduced one divisor $\phi(\pi_2(b_2^{-1}(\pi_1(D_3'))))$, which is a $\mathbb{P}^1$-bundle over A'.

Let W be a **curve of the fourth type** on $S \times E_{\zeta_3}$, i.e. a curve fixed by γ^2 and invariant for the action of γ. Let $\widetilde{W}$ be the strict transform of W in $\widetilde{S \times E_{\zeta_3}}$. The automorphism $\widetilde{\gamma} \subset \mathrm{Aut}(\widetilde{S \times E_{\zeta_3}})$ restricts to an involution of $\widetilde{W}$. Let $W' \subset X'$ be the image of $\widetilde{W}$ under π_1. It is a curve isomorphic to W/γ. The curve W' is a fixed curve for γ' (which is an order 3 automorphism on X'), so b_2 introduces three divisors on it, D_4', D_4'' and D_4'''. Each of them is a $\mathbb{P}^1$-bundle over $W/\gamma \simeq W'$ and γ' acts as the identity on each of them. So $\pi_2(D_4') \simeq D_4'$, $\pi_2(D_4'') \simeq D_4''$ and $\pi_2(D_4''') \simeq D_4'''$, and ϕ is the contraction of one of these divisors (say D_4'''). Thus, there are the two divisors $\phi(\pi_2(D_4')) \simeq D_4'$ and $\phi(\pi_2(D_4'')) \simeq D_4''$ which arise from the desingularization of the image of W in $(S \times E_{\zeta_3})/\gamma$. Both these divisors are $\mathbb{P}^1$-bundles over W/γ.

Let (B', B'') be a **pair of curves of the fifth type** on $S \times E_{\zeta_3}$, i.e. both B' and B'' are fixed by γ^2 and $\gamma(B') = B''$. Let us denote by $\widetilde{B'}$ and $\widetilde{B''}$ the strict transform of B' and B'' on $\widetilde{S \times E_{\zeta_3}}$. Since γ (and γ^3) interchanges B' and B'', $\widetilde{B'} \simeq \widetilde{B''}$ and $\pi_1(\widetilde{B'}) = \pi_1(\widetilde{B''})$ is a curve in X', denoted by B'. We observe that $B' \simeq B''$. Now the automorphism γ' is an order 3 automorphism on X' which fixes the curve B'. So b_2 introduces three divisors D_5', D_5'' and D_5''' on B' and then $\phi \circ \pi_2$ contracts one divisors among $\pi_2(D_5')$, $\pi_2(D_5'')$ and $\pi_2(D_5''')$ (say $\pi_2(D_5''')$). Hence in order to resolve the singularities of $(S \times E_{\zeta_3})/\gamma$ which are image of B', we introduced two divisors $\phi(\pi_2(D_5'))$ and $\phi(\pi_2(D_5''))$. Both of them are $\mathbb{P}^1$-bundles over a curve isomorphic to B'.

Let (V', V'') be a **pair of points of fifth type** on $S \times E_{\zeta_3}$, i.e. both V' and V'' are isolated fixed points for γ^2 and $\gamma(V') = V''$. The point $\pi_1(b_1^{-1}(V'))(= \pi_1(b_1^{-1}(V''))) \in X'$ is an isolated fixed points for γ'. So b_2 introduces a divisor, isomorphic to $\mathbb{P}^2$ on $\pi_1(b_1^{-1}(V'))$. The automorphism γ' is the identity on this divisor, so it is preserved by $\phi \circ \pi_2$. Hence, for each pair of points of the fifth type we introduced a divisor isomorphic to $\mathbb{P}^2$ in order to resolve the singularities of $(S \times E_{\zeta_3})/\gamma$.

Step 2: Computation of the exceptional divisors and the Hodge numbers of X.

In order to compute the Hodge numbers of X, we need to count the number of curves and points of each type that appear on $S \times E_i$:

(1) for each curve $L \subset S$ which is of the first type for γ_S, the curve $L \times 0_E \subset (S \times E_{\zeta_3})$ is of the first type for γ, the triple $(L \times P_1, L \times P_2, L \times P_3) \subset (S \times E_{\zeta_3})$ is a triple of curves of third type for γ, the pair $(L \times Q_1, L \times Q_2) \subset (S \times E_{\zeta_3})$ is a pair of curves of fifth type for γ;

(2) for each curve $U \subset S$ which is of the second type for γ_S, the curve $U \times 0_E$ is of the second type for γ, the triple $(U \times P_1, U \times P_2, U \times P_3) \subset (S \times E_{\zeta_3})$ is a triple of

curves of the third type for γ, there is no power γ^j of γ such that the curves $U \times Q_1$ and $U \times Q_2$ are fixed by γ^j;

(3) for each triple $(A', A'', A''') \subset S$ of curves of the third type for γ_S, the 4 triples $(A' \times 0_E, A'' \times 0_E, A''' \times 0_E)$, $(A' \times P_1, A'' \times P_2, A''' \times P_3)$, $(A' \times P_2, A'' \times P_3, A''' \times P_1)$, $(A' \times P_3, A'' \times P_1, A''' \times P_2)$ are triples of curves of the third type for γ;

(4) for each curve $W \subset S$ of the fourth type for γ_S, the curve $W \times 0_E$ is of the fourth type for γ, and the pair $(W \times Q_1, W \times Q_2)$ is a pair of curves of the fifth type for γ;

(5) for each pair $(B', B'') \subset S$ of curves of fifth type for γ_S, the 3 pairs $(B' \times 0_E, B'' \times 0_E)$, $(B' \times Q_1, B'' \times Q_2)$ and $(B' \times Q_2, B'' \times Q_1)$ are pairs of curves of the fifth type for γ;

(6) for each point $V \in S$ which is an isolated fixed point for γ_S, the point $V \times 0_E$ is an isolated fixed points for γ, $\{V \times P_1, V \times P_2, V \times P_3\}$ is an orbit for γ, and if V is of type $(2, 5)$ then the pair $(V \times Q_1, V \times Q_2)$ is a pair of points of the fifth type for γ;

(7) for each pair of points $(Z', Z'') \subset S$ of fifth type, i.e., switched by γ_S and which are isolated fixed points for γ_S^2, the pair of points $(Z' \times 0_E, Z'' \times 0_E)$, $(Z' \times Q_1, Z'' \times Q_2)$ and $(Z' \times Q_2, Z'' \times Q_1)$ are pairs of points of fifth type.

So the locus with non-trivial stabilizer for the action of γ on $S \times E_{\zeta_3}$ is the following: there are l curves of the first type; u curves of the second type, $l + u + 4a$ triples of curves of the third type; w curves of the fourth type; $l + w + 3b$ pairs of curves of the fifth type.

Moreover there are $p_{2,5} + p_{3,4}$ isolated fixed points for γ and $p_{2,5} + 3n'$ pairs of points of the fifth type.

So the number of divisors of X which arise by the desingularization of $(S \times E_i)/\alpha$ is $(5l)+(u+2p_{2,5}+p_{3,4})+(l+u+4a)+(2w)+2(l+w+3b)+p_{2,5}+3n' = 8l+2u+3p_{2,5}+p_{3,4}+4a+4w+6b+3n' = 8l+2(N-l-3a)+3p_{2,5}+p_{3,4}+4a+4(k-l-2b)+6b+3n' = 2l + 2N - 2a + 4k - 2b + 3p_{2,5} + p_{3,4} + 3n'$. So,

$$h^{1,1}(X) = r + 1 + 2l + 2N - 2a + 4k - 2b + 3p_{2,5} + p_{3,4} + 3n'.$$

In order to compute $h^{2,1}(X)$, we recall that it is the sum of the dimension of the invariant part of $H^{2,1}(S \times E_{\zeta_3})$ for γ and the dimension of $H^{1,0}(D_{tot})$. Since the divisors which are $\mathbb{P}^1$-bundles over $\mathbb{P}^1$ and which are $\mathbb{P}^2$ have a trivial $h^{1,0}$, it suffices to identify the divisors in D_{tot} which are $\mathbb{P}^1$-bundles over curves of positive genus (and in this case, $h^{1,0}$ of a $\mathbb{P}^1$-bundle is the genus of its basis). They arise form the resolution of singularities caused by curves in S which have a non-trivial stabilizer for γ_S and have a positive genus.

Let us assume that γ_S fixes one curve of positive genus, denoted by D. If $g(D) \geq 2$, then γ is the unique curve of positive genus with a non-trivial stabilizer for γ_S and $D/\gamma_S \simeq D$. If $g(D) = 1$, then γ_S^3 fixes either D as unique curve of positive genus, or D and F_2, both of genus 1. The curve $D \times 0_E \subset S \times E_{\zeta_3}$ is a curve of the first type and in order to resolve the singularity of $(S \times E_{\zeta_3})/\gamma$ image of $D \times 0_E$ we introduce 5 divisors which are $\mathbb{P}^1$-bundles over D. The triple of curves $(D \times P_1, D \times P_2, D \times P_3)$ is of the third type, and in order to resolve the singularity of $(S \times E_{\zeta_3})/\gamma$ which is the common image of $D \times P_i$, $i = 1, 2, 3$ we introduce 1 divisor which is a $\mathbb{P}^1$-bundle over D. The pair of curves $(D \times Q_1, D \times Q_2)$ is of the fifth type, and in order to resolve the singularity of $(S \times E_{\zeta_3})/\gamma$ which is the common image of $D \times Q_j$, $j = 1, 2$ we introduce 2 divisors which are $\mathbb{P}^1$-bundles over D. Moreover, $F_2 \times 0_E$ is a curve of second type for γ and thus the resolution introduces one divisor which is a $\mathbb{P}^1$-bundle over F_2/γ_S (and could introduce other divisors which have for sure $h^{1,0} = 0$); the triple $(F_2 \times P_1, F_2 \times P_2, F_2 \times P_3) \subset (S \times E_{\zeta_3})$ is a triple of curves of the third type for γ and thus the resolution introduces 1 divisor which is a $\mathbb{P}^1$-bundle over F_2. So if γ_S fixes a curve D of positive genus (and possibly γ_S^3 fixes F_2 of genus either 1 or 0) then

$$h^{2,1}(X) = m - 1 + 8g(D) + g(F_2/\gamma_S) + g(F_2)\,.$$

If γ_S does not fix curves of positive genus, then we denote by G the curve with highest genus fixed by γ_S^2 and by F_1 and F_2 the two curves with highest genus fixed by γ_S^3. The curve G is a curve of the fourth type on S, so: $G \times 0_E$ is of fourth type and thus the resolution of $(S \times E_{\zeta_3})/\gamma$ introduces 2 $\mathbb{P}^1$-bundles over G/γ to resolve the singularity which is the image of $G \times 0_E$; the pair $(G \times Q_1, G \times Q_2)$ is a pair of curves of fifth type for γ and thus the resolution of $(S \times E_{\zeta_3})/\gamma$ introduces 2 $\mathbb{P}^1$-bundles over G to resolve the singularity which is the image of $G \times Q_1$. The curves F_i, $i = 1, 2$ are of second type for γ_S, so $F_i \times 0_E$ are of second type and thus the resolution of $(S \times E_{\zeta_3})/\gamma$ introduces 1 $\mathbb{P}^1$-bundle over F_i/γ (and possibly other divisors with trivial $h^{1,0}$) to resolve the singularity which is the image of $F_i \times 0_E$; the triple $(F_i \times P_1, F_i \times P_2, F_i \times P_3)$ is a triple of curves of third type for γ and thus the resolution of $(S \times E_{\zeta_3})/\gamma$ introduces 1 $\mathbb{P}^1$-bundle over F_i to resolve the singularity which is the image of $F_i \times P_1$. So, in this case

$$h^{2,1}(X) = m - 1 + 2g(G/\gamma_S) + 2g(G) + g(F_1/\gamma_S) + g(F_1) + g(F_2/\gamma_S) + g(F_2)\,.$$

□

REMARK 7.4. We observe that $h^{2,1}(X) = m - 1$ if and only if both γ_S^2 and γ_S^3 do not fix curves of positive genus. This happens for at least one family of K3 surfaces, the one constructed in Example 7.9. If $h^{2,1}(X) = m - 1$, then the family of X is Borcea–Voisin maximal and does not admit maximal unipotent monodromy (cf. Proposition 3.7).

7.3. (Almost) elliptic fibrations. We consider the map $\mathcal{E}_6 : X \to S/\gamma_S$ and we denote by F_Z the fiber of such a fibration over a point $Z \in S/\gamma_S$. We consider the map $\mathcal{E}' : X' \to S/\gamma_S^3$ and we denote by F'_Z the fiber of such a fibration over a point $Z \in S/\gamma_S^3$. Moreover we consider the quotient maps $q_S : S \to S/\gamma_S$ and $q' : S \to S/\gamma_S^3$. We observe that S/γ_S has $p_{2,5} + n'$ singularities of type A_2.

We will assume (S, γ_S) to be generic in the family of K3 surfaces S with the non-symplectic automorphism of order 6 γ_S (i.e. $\rho(S) = 22 - 2m$).

PROPOSITION 7.5. *The map $\mathcal{E}_6 : X \to S/\gamma_S$ is an almost elliptic fibration whose general fiber is isomorphic to E_{ζ_3}. The fiber F_Z of $\mathcal{E}_6$ over $Z \in S/\gamma_S$ has dimension 1 if and only if Z is a smooth point which is not the image of a point of type* (3, 4) *and is singular if and only if Z is in the branch locus of $q : S \to S/\gamma_S$. In particular:*

- *If $q^{-1}(Z)$ consists of 3 points, F_Z is of type I_0^*;*

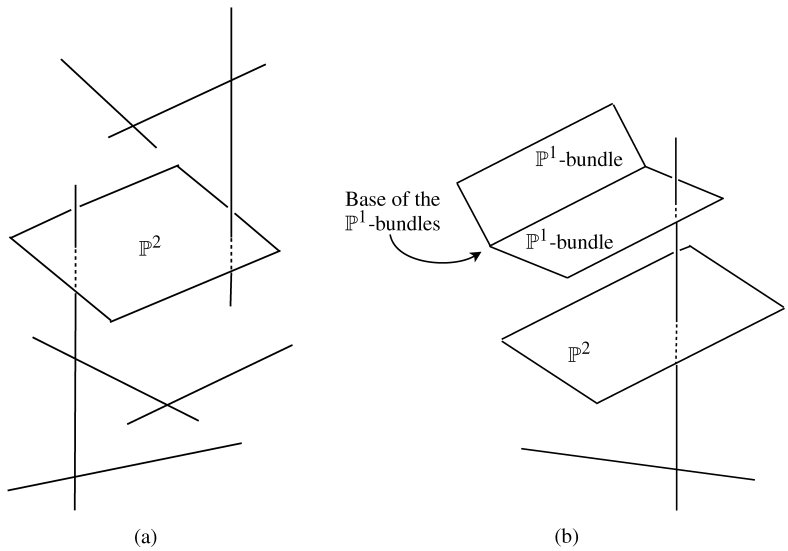


FIGURE 3. Non-Kodaira fibers of $\mathcal{E}_6$ over the image of a point of type (3, 4) and (2, 5) respectively.

- *If $q^{-1}(Z)$ consists of 2 points and $Z \in S/\gamma_S$ is smooth, F_Z is of type IV^*;*
- *If $q^{-1}(Z)$ consists of 2 points and $Z \in S/\gamma_S$ is singular, F_Z contains three disjoint copies of $\mathbb{P}^2$ and a rational curve meeting each $\mathbb{P}^2$ in a point (see Figure 1);*
- *If $q^{-1}(Z)$ consists of 1 point and $Z \in S/\gamma_S$ is a smooth point which is not the image of a point of type* (3, 4)*, F_Z is of type II^*;*
- *If $q^{-1}(Z)$ consists of 1 point and $Z \in S/\gamma_S$ is the image of a point of type* (3, 4)*, F_Z contains seven rationals curves and a divisor $D \simeq \mathbb{P}^2$ (see Figure 3(a));*
- *If $q^{-1}(Z)$ consists of 1 point and $Z \in S/\gamma_S$ is singular, F_Z contains two rational curves and three divisors, one of them is isomorphic to $\mathbb{P}^2$, the other are $\mathbb{P}^1$-bundles over $\mathbb{P}^1$ (see Figure 3(b)).*

The Mordell–Weil group of this fibration is generically trivial.

PROOF. If $q^{-1}(Z) = \{Z_1, Z_2, Z_3\}$, then the stabilizer of $Z_i \in S$ is the involution γ_S^3. So the fibers of $\mathcal{E}' : X' \to S/\gamma^3$ over $q'(Z_i)$, $i = 1, 2, 3$, are 3 fibers of type I_0^* constructed as in Proposition 4.4. They are identified in the quotient by γ'.

If $q^{-1}(Z) = \{Z_1, Z_2\}$, then $\gamma_S^3(Z_1) = Z_2$ and $q'(Z_1) = q'(Z_2)$. So the fiber $F'_{q'(Z_1)}$ coincides with the fiber $F'_{q'(Z_2)}$ and they are isomorphic to E_{ζ_3}. Now the order 3 automorphism γ' fixes the point $q'(Z_1) = Z$ and the fiber over this point can be constructed as in Proposition 5.5. In particular, if $Z \in S/\gamma_S$ is smooth, F_Z is a fiber of type IV^*, while if $Z \in S/\gamma_S$ is singular, it contains three copies of $\mathbb{P}^2$.

If $q^{-1}(Z) = \{Z_1\}$, then $\gamma_S^3(Z_1) = Z_1$. So the fiber $F'_{q'(Z_1)}$ is a fiber of type I_0^*. Let us denote by E the component of multiplicity 2 of this fiber (it is the quotient by γ^3 of the strict transform by b_1 of $Z_1 \times E_{\zeta_3} \subseteq S \times E_{\zeta_3}$). Three of the four components with multiplicity 1 are permuted by γ' (the ones corresponding to the points $Z_1 \times P_i \in Z_1 \times E_{\zeta_3}$, $i = 1, 2, 3$), the other is invariant (it is the one corresponding to the point $Z_1 \times O \in Z_1 \times E_{\zeta_3}$). We denote by C_1, C_2, C_3 and C_4 the four simple components of the fiber of type I_0^* and we assume that $\gamma'(C_1) = C_1$. We now describe F_Z according with the following three possibilities: i) $Z \in S/\gamma_S$ is smooth and not the image of a point of type $(3, 4)$; ii) $Z \in S/\gamma_S$ is the image of a point o type $(3, 4)$; iii) $Z \in S/\gamma_S$ is singular.

Let us now assume that $Z \in S/\gamma_S$ is smooth and that it is not the image of a point of type $(3, 4)$. On the invariant component γ' acts as an automorphism of order 3 with 2 fixed points, one of them is the intersection point with the component of multiplicity 2. So b_2 blows up these two points. Since Z is smooth, these points lie on curves fixed by γ', so we introduce 2 rational curves on each point. Moreover there is a point of E which is fixed by γ' (it is the image of the point $Z_1 \times R_1 \in Z_1 \times E_{\zeta_3}$) and again it lies on a curve fixed in X'. So, in order to resolve it, we introduce two other rational curves in this fiber. To recap, the fiber F_Z contains the image of the strict transform of E. It intersects: a rational curve which is the image of C_2 (and of C_3 and C_4); a tree of 5 rational curves which corresponds to C_1; a tree of 2 rational curves which corresponds to the point $Z_1 \times R_1 \in Z_1 \times E_{\zeta_3}$. The fiber F_Z is then of type II^*.

Let $Z \in S/\gamma_S$ be the image of a point of type $(3, 4)$. The automorphism γ' fixes 2 points on C_1, one of them is the intersection of C_1 and E and is an isolated fixed point. The other lies on a curve of fixed points. Hence b_2 introduces a copy of $\mathbb{P}^2$ over the isolated fixed point and a tree of two rational curves (fibers of $\mathbb{P}^1$-bundles) over the other fixed point on C_1. Moreover, there is a point of E, which is fixed by γ' and which lies on a curve of fixed points intersecting transversally E. So, b_2 introduces a tree of two rational curves (fibers of $\mathbb{P}^1$-bundles) on this point. To recap: the fiber F_Z contains the image of the strict transform of E. It intersects one rational curve, image of C_2 (and of C_3 and C_4); a copy of $\mathbb{P}^2$, which intersects also a tree of 3 rational curves and which corresponds to C_1; a tree of 2 rational curves which correspond to the point $Z_1 \times R_1 \subset Z_1 \times E_{\zeta_3}$. Thus, we get a configuration of curves as shown in Figure 3(a).

Finally, let us now assume that $Z \in S/\gamma_S$ is singular. The automorphism γ' acts as the identity on C_1. So b_2 blows up this rational curve introducing divisors which are all contained in the fiber F_Z. In particular, on X we have 2 $\mathbb{P}^1$-bundles over $\mathbb{P}^1$ meeting along the base. Moreover there is a point of E, which is fixed by γ' (it is the image of the point $Z_1 \times R_1 \in Z_1 \times E_{\zeta_3}$) and it is an isolated fixed point for γ' on X'. So in order to resolve it we introduce a copy of $\mathbb{P}^2$. To recap, the fiber F_Z contains the image of the strict transform of E, which intersects: one rational curve which is the image of C_2 (and of C_3 and C_4); two $\mathbb{P}^1$-bundles which correspond to C_1; one $\mathbb{P}^2$ which corresponds to the point $Z_1 \times R_1 \in Z_1 \times E_{\zeta_3}$ (see Figure 3(b)).

The (rational) section is the image of $S \times O$ under the rational map $\phi \circ \pi_2 \circ b_2^{-1} \circ \pi_1 \circ b_1^{-1}$. □

REMARK 7.6. We observe that $\mathcal{E}_6$ would be an elliptic fibration if and only if $n' =$

$p_{3,4} = p_{2,5} = 0$, but this never happens. Indeed by [D1, Theorem 4.1], $p_{3,4} = p_{2,5} = 0$ implies $6l = -6$, which is absurd.

REMARK 7.7. This is the first case where we get 2-dimensional fibers over smooth points of the base surface. The reason why this happens is the following: around a point of type (3, 4), a local equation for the Weierstrass model of the fibration is

$$y^2 = x^3 + s^3 t^4\,,$$

and so the origin is a singular point, which is not of cDV type ([Re, Cor. 2.10]). But then to resolve it we must introduce divisors in the corresponding fibre.

REMARK 7.8. The automorphism γ_X induced on X by $\gamma_S \times \mathrm{id}$ (cf. Definition 3.5) is induced by the complex multiplication of order 6 on each smooth fiber of the fibration.

7.3.1. *Weierstrass equation of $\mathcal{E}_6$ if $S/\gamma_S \simeq \mathbb{P}^2$.* The almost elliptic fibration $\mathcal{E}_6$ has a smooth base if and only if $n = 0$ (where n is the number of isolated fixed points of the non-symplectic automorphism of order 3, i.e. $\alpha_S = \gamma_S^2$). Hence we are interested in particular in this situation. We already observed (Section 5.2.1) that there are two 9-dimensional families of K3 surfaces admitting a non-symplectic automorphism of order 3 without isolated fixed points. We now focus our attention on one of these families (the other will be analyzed in the next section).

Let us assume $\alpha_S = \gamma_S^2$ fixes only 1 curve. Then it has genus 4, and S admits a projective model as complete intersection of a quadric and a cubic in $\mathbb{P}^4$ as in (5.1). Since γ_S^2 has no isolated fixed points, $p_{2,5} = 0$ and thus, by [D1, Theorem 4.1], we obtain $l = 0$, $p_{3,4} = 6$. So γ_S fixes exactly 6 points. We now apply the Lefschetz fixed points formula in order to compute the dimension of the family of K3 surfaces admitting a purely non-symplectic automorphism of order 6 whose square fixes only one curve. We observe that γ_S^* acts on $H^2(S, \mathbb{C})$ and we denote by $m := \dim(H^2(S, \mathbb{C}))_{\zeta_6^i}$, $i = 1, 5$, $r := \dim(H^2(S, \mathbb{C}))_1$, $a := \dim(H^2(S, \mathbb{C}))_{\zeta_6^j}$, $j = 2, 4$, $b := \dim(H^2(S, \mathbb{C}))_{-1}$. The Lefschetz fixed points formula states that

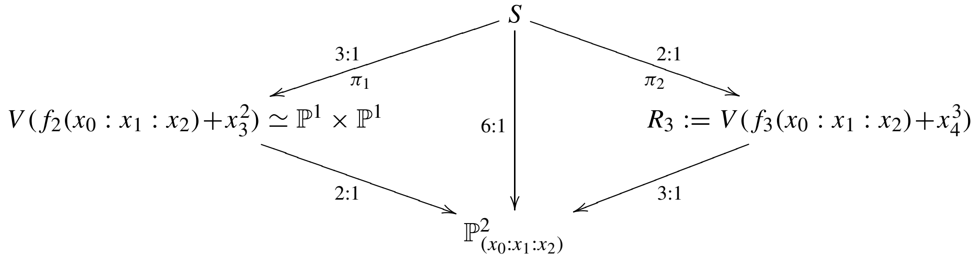
$$\sum_{i=0}^{4} \mathrm{tr}\left(\gamma^*_{S\,|H^i(S,\mathbb{C})}\right) = \chi(\mathrm{Fix}_{\gamma_S}(S))\,. \tag{7.1}$$

We obtain $2 + m - a - b + r = 6$. Moreover we apply the same formula to γ_S^2 and obtain $2 - m - a + b + r = -6$. Since $\dim(H^2(S, \mathbb{C})) = 22$, we have also $2m + 2a + b + r = 22$. We recall that m, a, b, r are non negative integers (they are the dimension of vector spaces), $m \geq 1$ (because $H^{2,0}(S) \subset H^2(S, \mathbb{C})_{\zeta_6}$) and $r \geq 1$ because there exists at least one invariant class. So the unique possibilities for (m, a, b, r) are $(7, 3, 1, 1)$ and $(6, 4, 0, 2)$. Since the dimension of the moduli space of the K3 surfaces with the required properties is $m - 1$, we proved that it is at most 6 (in particular not all the K3 surfaces admitting a non-symplectic automorphism α_S of order 3 fixing one curve admit a purely non-symplectic automorphism γ_S of order 6 such that $\gamma_S^2 = \alpha_S$).

Here we show that there exists at least a 6-dimensional family of K3 surfaces with a purely non-symplectic automorphism of order 6 whose square fixes one curve of genus 4, specializing the family given in (5.1). Indeed the complete intersections in $\mathbb{P}^4$ given by the system

$$(7.2) \qquad \begin{cases} f_2(x_0:x_1:x_2) & +x_3^2 & & =0 \\ f_3(x_0:x_1:x_2) & & +x_4^3 & =0 \end{cases}$$

are generically smooth and hence K3 surfaces. The projective dimension of such a family is 6 and every member of this family clearly admits an automorphism of order 6, induced by $\gamma_{\mathbb{P}^4} : (x_0 : x_1 : x_2 : x_3 : x_4) \to (x_0 : x_1 : x_2 : -x_3 : \zeta_3^2 x_4)$. We will denote by S the generic K3 surface with equation (7.2) and by γ_S the automorphism induced by $\gamma_{\mathbb{P}^4}$ on S. Observe that $V(f_2(x_0 : x_1 : x_2) + x_3^2)$ is a quadric in $\mathbb{P}^3$ and thus it is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Denote by $R_3 := V(f_3(x_0 : x_1 : x_2) + x_4^3)$. We have the following diagram:



Hence $S/\gamma_S \simeq \mathbb{P}^2$. The fixed locus $\mathrm{Fix}_{\gamma_S}(S)$ consists of 6 points, as we expect. The fixed locus of γ_S^2 consists of a curve of genus 4, as we required, and the fixed locus of γ_S^3 is the $3:1$ cover of the rational curve $V(f_2(x_0 : x_1 : x_2)) \subset \mathbb{P}^2_{(x_0:x_1:x_2)}$ branched along the six points $V(f_2(x_0 : x_1 : x_2)) \cap V(f_3(x_0 : x_1 : x_2))$.

In order to exhibit S as $6 : 1$ cover of $\mathbb{P}^2$ we introduce the variable $w := ix_3x_4$ and thus an equation of S in $\mathcal{O}_{\mathbb{P}^2}(2)$ is

$$w^6 = f_2(x_0 : x_1 : x_2)^3 f_3(x_0 : x_1 : x_2)^2$$

and $\gamma_S : (w, (x_0 : x_1 : x_2)) \mapsto (-\zeta_3^2 w, (x_0 : x_1 : x_2))$.

From the equation of S we deduce a Weierstrass equation for the almost elliptic fibration described in Proposition 7.5:

$$Y^2 = X^3 + f_2(x_0 : x_1 : x_2)^3 f_3(x_0 : x_1 : x_2)^4 ,$$

where the functions $Y := \frac{vw^{15}}{f_2^6 f_3^3}$, $X := \frac{uw^{10}}{f_2^4 f_3^2}$ are invariant for $\gamma_S \times \gamma_E^5$.

7.3.2. *Weierstrass equation of $\mathcal{E}_6$ if $S/\alpha_S \simeq \mathbb{F}_{12}$.* The second family of K3 surfaces admitting a non-symplectic automorphism of order 6 such that $n = 0$, consists of the family of K3 surfaces, S, admitting an isotrivial elliptic fibration whose equation is $y^2 = x^3 + f_{12}(t)$ where $f_{12}(t)$ is a generic polynomial of degree 12 without multiple roots. The non-symplectic automorphism we consider is $\gamma_S : (x, y, t) \to (\zeta_3^2 x, -y, t)$. We analyze the generic case (i.e. $n = 0$, i.e. $f_{12}(t)$ has no multiple roots) showing that in this case S is a $6 : 1$ cover of $\mathbb{F}_{12}$ and giving an equation for the almost elliptic fibration $\mathcal{E}_6$ in this case.

TABLE 1. Fixed loci of γ_S on singular fibers.

$\mu_{\bar{t}}$	fiber $F_{\bar{t}}$	$\mathrm{Fix}_{\gamma_S}(S)$	$\mathrm{Fix}_{\gamma_S^2}(S)$	$\mathrm{Fix}_{\gamma_S^3}(S)$
1	II	1 pt.	-	-
2	IV	1 pt.	1 pt.	-
3	I_0^*	2 pts.	1 pt.	1 curve
4	IV^*	3 pts.	1 curve, 3 pts.	1 curve
5	II^*	1 curve, 7 pts.	2 curves, 4 pts.	4 curves

Let S have equation $y^2z = x^3 + f_{12}(s:t)z^3$ in $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1})$, and consider the morphism to $\mathbb{F}_{12}$ given by $(s:t:x:y:z) \longmapsto (s:t:x^3:z^3)$. Observe that this is the composition of the $2:1$ covering $S \longrightarrow \mathbb{F}_4$ described in Section 4.3 with the $3:1$ covering $\mathbb{F}_4 \longrightarrow \mathbb{F}_{12}$ described in Section 2.3, hence it is a $6:1$ covering. Observe also that S admits a birational model S' in $\mathcal{O}_{\mathbb{F}_{12}}(10D_T + D_Z)$, with equation $w^6 = X^2(X + f_{12}(S:T)Z)^3Z$: the birational morphism

$$\begin{array}{ccc} S & \longrightarrow & S' \\ (s:t:x:y:z) & \longmapsto & (w,(S:T:X:Z)) = (xyz,(s:t:x^3:z^3)) \end{array}$$

is compatible with γ_S, since it induces on S' the covering automorphism $\gamma_{S'}: w \to -\zeta_3^2 w$.

The functions $\eta := \frac{vw^{15}}{X^3(X+f_{12}(S:T)Z)^6}$, $\xi := \frac{uw^{10}}{X^2(X+f_{12}(S:T)Z)^4}$ defined on $S' \times E_{\zeta_3}$ are invariant for $\gamma_{S'} \times \gamma_E^5$ and satisfy

$$\eta^2 = \xi^3 + X^4(X + f_{12}(S:T)Z)^3Z^5$$

which is a Weierstrass equation for X in $\mathbb{P}(\mathcal{O}_{\mathbb{F}_{12}}(-2K_{\mathbb{F}_{12}}) \oplus \mathcal{O}_{\mathbb{F}_{12}}(-3K_{\mathbb{F}_{12}}) \oplus \mathcal{O}_{\mathbb{F}_{12}})$.

7.4. Invariants of the fixed loci of γ_S^j for some K3 surfaces. In [D1] some non-symplectic automorphisms of order 6 on K3 surfaces are classified. In particular, Dillies considers the case where S is an elliptic K3 surface with the following Weierstrass equation $y^2 = x^3 + f_{12}(t)$, where $f_{12}(t)$ is a polynomial of degree 12 which does not admit roots of multiplicity greater than 5, and the automorphism of order 6 is $(x, y, t) \mapsto (\zeta_3 x, -y, t)$. In order to compute the Hodge numbers of the Calabi–Yau X of type X_6, we want to describe the fixed loci of γ_S, γ_S^2 and γ_S^3 according to the variation of $f_{12}(t)$. Let $\bar{t}$ be a zero of $f_{12}(t)$, $\mu_{\bar{t}}$ the multiplicity of such zero, $F_{\bar{t}}$ the fiber of $y^2 = x^3 + f_{12}(t)$ over $\bar{t}$. The automorphism γ_S fixes the zero section; the automorphism γ_S^2 fixes the zero section and the bisection $y^2 = f_{12}(t)$ (in some cases this can split in 2 distinct sections); the automorphism γ_S^3 fixes the zero section and the trisection $x^3 = -f_{12}(t)$ (in some cases this can split either in a section and a bisection or in three sections). All the other fixed curves are components of the reducible fibers. In Table 1 we give the number of isolated points and of components which are fixed by γ_S^j on the singular fibers. We do not compute the points which are fixed on the fiber but lie on curves (sections or multi-sections) fixed by the automorphism.

Moreover we need to compute explicitly the numbers $r := \dim(H^2(S, \mathbb{C})^{\gamma_S})$ and $m := \dim(H^2(S, \mathbb{C}))_{-\zeta_3}$. This computation is done by applying the Lefschetz fixed points formula to γ_S, γ_S^2 and γ_S^3 (cf. Section 7.3.1). With all these information, one can compute the invariants related to the automorphism described in [D1]. The results of these computations are shown in Table 5.

7.5. A family of Calabi–Yau 3-folds of type X_6 without maximal unipotent monodromy. We now construct another example of a K3 surface admitting a non-symplectic automorphism of order 6 by specializing $y^2 = x^3 + f_{12}(t)$. The peculiarity of this example is that the family of Calabi–Yau 3-folds constructed does not admit maximal unipotent monodromy, since it satisfies the condition of Remark 7.4.

EXAMPLE 7.9. Let us assume $f_{12}(t) = t^4(t-1)^3(t+1)^3(t-\lambda)^2$ with $\lambda \neq 0, \pm 1$. The elliptic K3 surface whose Weierstrass equation is $y^2 = x^3 + t^4(t-1)^3(t+1)^3(t-\lambda)^2$ has 4 reducible fibers: F_0 is of type IV^*, F_1 and F_{-1} are of type I_0^*, F_λ is of type IV. Applying the results described in Section 7.4, one obtains that γ_S fixes one rational curve (the zero section) and 8 isolated points (3 on F_0, 2 on F_i for $i = 1, -1$ and 1 on F_λ). The automorphism γ_S^2 fixes 3 rational curves (the zero section, the bisection $y^2 = t^4(t-1)^3(t+1)^3(t-\lambda)^2$ and one component of F_0) and 6 isolated points (3 on F_0, 1 on F_i for $i = 1, -1, \lambda$). The automorphism γ_S^3 fixes 5 rational curves (the zero section, the trisection $x^3 = -t^4(t-1)^3(t+1)^3(t-\lambda)^2$, 1 component of each fiber F_i, $i = 0, 1, -1$). So the Euler characteristic of the fixed loci of γ_S, γ_S^2 and γ_S^3 are 8, 12, 10 respectively. Applying the Lefschetz fixed points formula one computes $r = 11$ and $m = 2$. So the family of K3 surfaces admitting a non-symplectic automorphism of order 6, such that the fixed loci of all its powers are as described, is 1-dimensional. Since also the family of K3 surfaces admitting an elliptic fibration with equation $y^2 = x^3 + t^4(t-1)^3(t+1)^3(t-\lambda)^2$ is 1-dimensional, these two families coincide.

We observe that the family of K3 surfaces described is obtained also as the minimal model of the quotient $(E_{\zeta_3} \times C)/(\gamma_E \times \gamma_C)$, where C is a $6:1$ cover of $\mathbb{P}^1_t$ with equation $w^6 = t^4(t-1)^3(t+1)^3(t-\lambda)^2$ and γ_C is the cover automorphism $\gamma_C : (w, t) \mapsto (-\zeta_3 w, t)$. As in Remark 5.3 this implies that the variation of the Hodge structures both of the family of S and of the family of X depend only on the variation of the Hodge structures of C.

8. Appendix: the Tables. In this appendix we summarize the properties of the Calabi–Yau constructed above, in some tables. In all the tables we give a reference to where the K3 surface S and the associated automorphism α_S (or γ_S) are constructed, we list the properties of the fixed locus of the automorphism and of its powers (we follow the notation introduced before), we compute the Hodge numbers and we say whenever the family of Calabi–Yau constructed does not admit maximal unipotent monodromy (this is denoted by MUM) and whenever the family is "new", in the sense that it is not contained in the list [J] of known Calabi–Yau 3-folds. We omit the cases $n = 2, 3$ since they were already analyzed in previous papers. We underline that if the column MUM is empty, this means that our argument is not sufficient to conclude whether there is or not a point with maximal unipotent monodromy (so we are not stating any result on the presence of points with maximal unipotent monodromy). Clearly, if the column MUM contains "no" this means that there is no point with maximal unipotent monodromy in the family.

8.1. Order 4. In Table 2 we assume that the curve of highest genus, $g(D)$, fixed by α_S^2 is also fixed by α_S. The first line corresponds to the assumption that α_S^2 fixes two elliptic curves.

TABLE 2. Hodge numbers of X_4 in case 1) Proposition 6.3.

	Ref K3	m	r	n_1	k	a	$g(D)$	$h^{1,1}(X)$	$h^{2,1}(X)$	MUM	new
1	[AS2, Table 1, l. 1]	6	6	4	1	0	1	25	13		
2	[AS2, Table 1, l. 2]	5	7	4	1	0	1	29	11		
3	[AS2, Table 1, l. 3]	4	10	6	2	0	1	46	10		
4	[AS2, Table 1, l. 4]	4	8	4	1	1	1	34	10		
5	[AS2, Table 1, l. 5]	3	9	4	1	2	1	39	9		
6	[AS2, Table 1, l. 6]	2	10	4	1	3	1	44	8		
7	[AS2, Table 2, l. 1]	7	1	0	1	0	3	9	27		
8	[AS2, Table 2, l. 2]	6	4	2	1	0	2	19	19		
9	[AS2, Table 2, l. 3]	6	2	0	1	1	3	14	26		
10	[AS2, Table 2, l. 4]	5	5	2	1	1	2	24	18		
11	[AS2, Table 2, l. 5]	4	6	2	1	2	2	29	17		
12	[AS2, Table 3, l. 1]	4	10	6	1	0	0	39	3	no	
13	[AS2, Table 3, l. 2]	3	13	8	2	0	0	56	2	no	
14	[AS2, Table 3, l. 3]	3	11	6	1	1	0	44	2	no	X
15	[AS2, Table 3, l. 4]	2	16	10	3	0	0	73	1	no	
16	[AS2, Table 3, l. 5]	2	14	8	2	1	0	61	1	no	
17	[AS2, Table 3, l. 6]	2	12	6	1	2	0	49	1	no	X
18	[AS2, Table 3, l. 7]	1	19	12	4	0	0	90	0	no	
19	[AS2, Table 3, l. 8]	1	13	6	1	3	0	54	0	no	X

In Table 3 we assume that α_S is an involution on the curve of highest genus, $g(D)$, fixed by α_S^2.

In [AS2, Table 6] a list of admissible lattices associated to certain fixed loci is given. For some of them, the corresponding family of K3 surfaces is also constructed. In particular, all the cases listed in [AS2, Table 6] and with $g = 0$ are associated to a family of K3 surfaces, constructed in [AS2, Example 7.2]. If also α_S^2 does not fix curves with positive genus, then we are in the assumption of Remark 6.4. In Table 3 we list the Calabi–Yau of type X_4 corresponding to these assumptions.

REMARK 8.1. We observe that the Hodge numbers of the Calabi–Yau in Table 2, line 11 are the same of the Calabi–Yau in Table 3, line 4 and are mirrors of the ones of the Calabi–Yau in Table 3, line 1. The Hodge numbers of the Calabi–Yau in Table 2, line 9 are the same of the Calabi–Yau in Table 3, line 2. The Hodge numbers of the Calabi–Yau in Table 2 line 8 are self mirror.

TABLE 3. Hodge numbers of X_4 in case 2) Proposition 6.3.

	Ref. K3	m	r	n_1	n_2	k	a	$g(D)$	$h^{1,1}(X)$	$h^{2,1}(X)$	MUM	new
1	[AS2, Table 5, l. 1]	10	2	2	2	0	0	10	17	29		
2	[AS2, Table 5, l. 2]	10	2	0	4	0	0	9	14	26		
3	[AS2, Table 5, l. 3]	8	6	2	4	1	0	7	32	20		
4	[AS2, Table 5, l. 4]	8	6	0	6	1	0	6	29	17		
5	[AS2, Table 5, l. 5]	6	10	6	2	2	0	6	53	17		
6	[AS2, Table 5, l. 6]	6	10	4	4	2	0	5	50	14		
7	[AS2, Table 5, l. 7]	6	10	2	6	2	0	4	47	11		
8	[AS2, Table 5, l. 8]	6	10	0	8	2	0	3	44	8		
9	[AS2, Table 5, l. 9]	4	14	6	4	3	0	3	68	8		
10	[AS2, Table 5, l. 10]	4	14	4	6	3	0	2	65	5		
11	[AS2, Table 5, l. 11]	2	18	10	2	4	0	2	89	5		
12	[AS2, Table 5, l. 12]	2	18	8	4	4	0	1	86	2		

TABLE 4. α fixes only isolated points, α^2 fixes only rational curves.

	Ref. K3	m	r	n_1	n_2	k	a	$g(D)$	$h^{1,1}(X)$	$h^{2,1}(X)$	MUM	new
1	[AS2, Table 6, l. 13]	5	7	2	2	0	0	0	22	4	no	X
2	[AS2, Table 6, l. 18]	4	8	2	2	0	1	0	27	3	no	X
3	[AS2, Table 6, l. 23]	3	9	2	2	0	2	0	32	2	no	X
4	[AS2, Table 6, l. 27]	2	10	2	2	0	3	0	37	1	no	X
5	[AS2, Table 6, l. 30]	1	11	2	2	0	4	0	42	0	no	X

TABLE 5. Hodge numbers of X_6.

	n	n'	k	a	$g(G)$	$p_{3,4}$	$p_{2,5}$	l	N	b	$g(F)$	r	m	$h^{1,1}$	$h^{2,1}$	MUM	new
1	0	0	2	0	5	12	0	1	2	0	10	2	10	29	29		
2	1	0	2	0	4	10	1	1	2	0	9	3	9	31	25		
3	2	0	2	0	3	8	2	1	2	0	8	4	8	33	21		
4	3	0	2	0	2	6	3	1	2	0	7	5	7	35	17		
5	3	1	3	0	3	10	1	1	3	0	7	6	7	43	19		
6	4	0	2	0	1	4	4	1	2	0	6	6	6	37	13		
7	4	1	3	0	2	8	2	1	3	0	6	7	6	45	15		
8	4	0	4	0	3	10	4	2	6	0	6	10	6	65	17		
9	5	0	2	0	0	2	5	1	2	0	5	7	5	39	9		
10	5	1	3	0	1	6	3	1	3	0	5	8	5	47	11		
11	5	0	4	0	2	8	5	2	6	0	5	11	5	67	13		
12	6	1	3	0	0	4	4	1	3	0	4	9	4	49	7		
13	6	0	4	0	1	6	6	2	6	0	4	12	4	69	9		
14	7	0	4	0	0	4	7	2	6	0	3	13	3	71	5		
15	7	1	5	0	1	8	5	2	7	0	3	14	3	72	7		
16	8	1	5	0	0	6	6	2	7	0	2	15	2	81	3		
17	8	0	6	0	1	8	8	3	10	0	2	18	2	101	5		
18	9	0	6	0	0	6	9	3	10	0	1	19	1	103	1		
19	6	1	3	0	0	4	4	1	5	0	0	11	2	55	1	no	X

8.2. Order 6. In Table 5 we compute the Hodge numbers of Calabi–Yau 3-folds of type X_6 as in Proposition 7.3 if the K3 surface S is one the the K3 surfaces listed in [D1, Table 1 (column 1–11)]. The last line corresponds to the K3 surface constructed in Example 7.9. We omit $g(D)$ since it is zero for any K3 considered in the table.

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