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WEIGHTED HAMILTONIAN STATIONARY LAGRANGIAN SUBMANIFOLDS AND GENERALIZED LAGRANGIAN MEAN CURVATURE FLOWS IN TORIC ALMOST CALABI–YAU MANIFOLDS

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Abstract. In this paper, we generalize examples of Lagrangian mean curvature flows constructed by Lee and Wang in $\mathbb{C}^m$ to toric almost Calabi–Yau manifolds. To be more precise, we construct examples of weighted Hamiltonian stationary Lagrangian submanifolds in toric almost Calabi–Yau manifolds and solutions of generalized Lagrangian mean curvature flows starting from these examples. We allow these flows to have some singularities and topological changes.

1. Introduction. Recently, study of Lagrangian submanifolds acquire much importance in association with Mirror Symmetry. There are several classes of Lagrangian submanifolds. For example, special Lagrangian submanifolds are defined in Calabi–Yau manifolds by Harvey and Lawson [5] and they have an important role in the Strominger–Yau–Zaslow conjecture [11]. A class of Hamiltonian stationary Lagrangian submanifolds is also defined in Calabi–Yau manifolds, especially a special Lagrangian submanifold is a Hamiltonian stationary Lagrangian submanifold. In general, constructing explicit examples of special or Hamiltonian stationary Lagrangian submanifolds is difficult since these conditions are locally written by nonlinear PDE. However some examples are constructed in the case that the ambient Calabi–Yau manifold has symmetries, especially in $\mathbb{C}^m$.

First, we introduce some previously known examples of special or Hamiltonian stationary Lagrangian submanifolds and Lagrangian mean curvature flows. One of examples of special Lagrangian submanifolds in $\mathbb{C}^m$ constructed by Harvey and Lawson [5, III.3.A] is defined by

$$M_c := \{(z_1, \ldots, z_m) \in \mathbb{C}^m \mid \operatorname{Im}(z_1 \cdots z_m) = c_1,\ |z_1|^2 - |z_j|^2 = c_j \quad (j = 2, \ldots, m)\},$$

where $c = (c_1, \ldots, c_m) \in \mathbb{R}^m$. Note that the phase of M_c is i^m. We remark that if $c_1 = 0$ and $z_j = x_j e^{i\theta_j}$ for $x_j \in \mathbb{R}$ and $\theta_j \in \mathbb{R}$, then M_c is written by

$$\{\exp(\theta_2\zeta_2 + \cdots + \theta_m\zeta_m) \cdot x \in \mathbb{C}^m \mid x \in \mathbb{R}^m,\ \theta_j \in \mathbb{R},\ \langle \mu(x), \zeta_j \rangle = \frac{c_j}{2} \quad (j = 2, \ldots, m)\},$$

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where $\zeta_j := (1, 0, \ldots, 0, -1, 0, \ldots, 0) = e_1 - e_j \in \mathbb{R}^m$, $\mu(x) := \frac{1}{2}(x_1^2, \ldots, x_m^2)$ and $\exp(v) \cdot x = (x_1 e^{2\pi i v_1}, \ldots, x_m e^{2\pi i v_m})$ for $v = (v_1, \ldots, v_m) \in \mathbb{R}^m$. This is a T^{m-1}-invariant special Lagrangian submanifold in $\mathbb{C}^m$.

Next examples of special Lagrangian submanifolds in $\mathbb{C}^m$ are constructed by Joyce [6, Example 9.4]. He considered a family of T^1-invariant Lagrangian submanifolds denoted by

$$N_c^{a_1,\ldots,a_m} := \{ (x_1 e^{2\pi i a_1 \theta}, \ldots, x_m e^{2\pi i a_m \theta}) \in \mathbb{C}^m \mid \theta \in \mathbb{R},\ a_1 x_1^2 + \cdots + a_m x_m^2 = c \},$$

where $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$ and $c \in \mathbb{R}$, and he proved that $N_c^{a_1,\ldots,a_m}$ is a special Lagrangian submanifold if and only if

$$a_1 + \cdots + a_m = 0\,. \tag{1}$$

He constructed these examples by using a moment map of T^1-action on $\mathbb{C}^m$. Of course, in the same way as M_c, the Lagrangian submanifold $N_c^{a_1,\ldots,a_m}$ can be written by

$$\left\{ \exp(\theta a) \cdot x \mid x \in \mathbb{R}^m,\ \theta \in \mathbb{R},\ \langle \mu(x), a \rangle = \frac{c}{2} \right\}.$$

These two examples suggest that a torus action, a real structure and a moment map are useful to construct special Lagrangian submanifolds. From this view point, the author [13] generalized Joyce's example $N_c^{a_1,\ldots,a_m}$ in $\mathbb{C}^m$ to in an m-dimensional toric almost Calabi–Yau cone manifold. To be more precise, the author constructed examples of special Lagrangian submanifolds of the form

$$\{\exp(t\zeta) \cdot p \mid p \in M^\sigma,\ t \in \mathbb{R},\ \langle \mu(p), \zeta \rangle = c\}$$

in a toric almost Calabi–Yau cone manifold $(M, \omega, g, J, \Omega_\gamma)$, where M^σ is the real form of M, μ is a moment map of T^m-action on M, ζ is a vector in $\mathbb{R}^m$ satisfying a special condition and c is a constant. This is a T^1-invariant special Lagrangian submanifold in a toric almost Calabi–Yau cone manifold $(M, \omega, g, J, \Omega_\gamma)$.

This type of constructions is also effective to construct examples of Hamiltonian stationary Lagrangian submanifolds and its mean curvature flows. Actually, Lee and Wang [8] proved that V_t defined by

$$\left\{ (x_1 e^{2\pi i \zeta_1 s}, \ldots, x_m e^{2\pi i \zeta_m s}) \in \mathbb{C}^m \,\middle|\, 0 \le s \le 1,\ \sum_{j=1}^m \zeta_j x_j^2 = -4\pi t \sum_{j=1}^m \zeta_j,\ x = (x_1, \ldots, x_m) \in \mathbb{R}^m \right\}$$

is Hamiltonian stationary Lagrangian submanifolds for all $\zeta \in \mathbb{R}^m$ and $c \in \mathbb{R}$. Furthermore, they proved that this family $\{V_t\}_{t\in\mathbb{R}}$ is a solution of mean curvature flow and it has a singularity when $t = 0$. To be more precise, they proved that it is a solution of Brakke flow. Here Brakke flow proposed by Brakke [3] is a weak formulation of a mean curvature flow with singularities.

A mean curvature flow is one of potential approaches to find a special Lagrangian submanifold in a given Calabi–Yau manifold as the following meaning. If there exists a long time solution of a mean curvature flow starting from a given Lagrangian submanifold and the flow converges to a smooth manifold, then it is a minimal Lagrangian submanifold, that is, a special Lagrangian submanifold. Indeed, this method has more deep background related to Mirror Symmetry proposed by Thomas and Yau [12]. Roughly speaking, they introduce a stability condition on Lagrangian submanifolds and conjecture that the Lagrangian mean curvature flow starting from a stable Lagrangian submanifold exists for all time and converges to a special Lagrangian submanifold in its Hamiltonian deformation class. This conjecture is called Thomas–Yau conjecture. Recently, Joyce [7] has updated the Thomas–Yau conjectures to achieve more plausible statement. In [7], he discusses the possibility that the Lagrangian mean curvature flow develops singularities many times even if an initial Lagrangian submanifold is stable and mentions the necessity of surgeries of Lagrangian mean curvature flows. Thus it is meaningful to construct examples of Lagrangian mean curvature flows with singularities to understand the motion of Lagrangian mean curvature flows and to develop this program.

In this paper, we construct explicit examples of special or weighted Hamiltonian stationary Lagrangian submanifolds in toric almost Calabi–Yau manifolds and construct solutions of generalized Lagrangian mean curvature flows with singularities and topological changes starting from these examples. These examples can be considered as some kind of generalization of examples of Lee and Wang [8] in $\mathbb{C}^m$ to toric almost Calabi–Yau manifolds. When the ambient space is a general toric almost Calabi–Yau manifold, its topology is not simple and there are many fixed points of torus action. Hence we can get examples of special or weighted Hamiltonian stationary Lagrangian submanifolds with various topologies. Furthermore, its generalized Lagrangian mean curvature flow develops singularities many times though examples of Lee and Wang in $\mathbb{C}^m$ develops a singularity once.

Note that, in this paper, we use notions of *weighted* Hamiltonian stationary and *generalized* Lagrangian mean curvature flow. These notions are modifications of the ordinary notions of Hamiltonian stationary and Lagrangian mean curvature flow defined in Calabi–Yau manifolds to almost Calabi–Yau manifolds. See Section 4 for precise definitions.

Here we give a short description of the main results of this paper. Let $(M, \omega, g, J, \Omega_\gamma)$ be a real $2m$-dimensional toric almost Calabi–Yau manifold with torus T^m action. To be more precise, that is a toric Kähler manifold with a nonvanishing holomorphic $(m, 0)$-form Ω_γ. We see in Section 4 that Ω_γ is constructed by a vector γ in $\mathbb{Z}^m$ which is canonically determined by the toric structure of (M, J). Note that we do not assume that (M, ω, g, J) is Ricci-flat. Since (M, ω, g, J) is a toric Kähler manifold, there exist a moment map $\mu : M \to \Delta$ with a moment polytope Δ and an anti-holomorphic and anti-symplectic involution $\sigma : M \to M$, see Section 2 for more precise settings. We denote the fixed point set of σ by M^σ and call it the real form of M. This is a real m-dimensional submanifold in M. Fix an integer n with $0 \leq n \leq m$. Take a set of n vectors $\zeta = \{\zeta_1, \dots, \zeta_n\} \subset \mathbb{Z}^m$ and a set of n constants $c = \{c_1, \dots, c_n\} \subset \mathbb{R}$ and consider the set

$$M^\sigma_{\zeta,c} := \{ p \in M^\sigma \mid \langle \mu(p), \zeta_i \rangle = c_i,\ i = 1, \dots, n\}$$

We assume that $M^{\sigma}_{\zeta,c}$ is a real $(m-n)$-dimensional submanifold in M^{σ} and $T_{\zeta} := V_{\zeta}/(V_{\zeta} \cap \mathbb{Z}^m)$ is isomorphic to a subtorus T^n in T^m, where $V_{\zeta} := \mathrm{Span}_{\mathbb{R}}\{\zeta_1, \ldots, \zeta_n\}$. Then we put a real m-dimensional manifold as

$$L_{\zeta,c} := M^{\sigma}_{\zeta,c} \times T_{\zeta} \tag{2}$$

and define a map $F_{\zeta,c} : L_{\zeta,c} \to M$ by

$$F_{\zeta,c}(p, [v]) := \exp v \cdot p\,.$$

Then the main theorems in this paper are the following.

THEOREM 1.1. *$F_{\zeta,c} : L_{\zeta,c} \to M$ is a T^n-invariant weighted Hamiltonian stationary Lagrangian submanifold for all ζ and c, and its Lagrangian angle $\theta_{\zeta,c} : L_{\zeta,c} \to \mathbb{R}/\pi\mathbb{Z}$ is given by $\theta_{\zeta,c}(p,[v]) = 2\pi\langle\gamma, v\rangle + \frac{\pi}{2}n\,(\mathrm{mod}.\,\pi)$. Thus $F_{\zeta,c} : L_{\zeta,c} \to M$ is a special Lagrangian submanifold if and only if $\langle\gamma, \zeta_i\rangle = 0$ for all $i = 1, \ldots, n$.*

THEOREM 1.2. *The family of the images of $\{F_{\zeta,c(t)} : L_{\zeta,c(t)} \to M\}_{0\le t\le T}$ is a solution of generalized Lagrangian mean curvature flow with singularities and topological changes with initial condition $F_{\zeta,c}$, where $c(t) := \{c_1(t), \ldots, c_n(t)\}$ and each $c_j(t)$ is given by $c_j(t) := c_j - 2\pi t\langle\gamma, \zeta_j\rangle$. Here T is the first time that $M^{\sigma}_{\zeta,c(t)}$ becomes empty set.*

Theorem 1.1 is a summary of Theorem 4.2, Corollary 4.3 and Theorem 4.5. Theorem 1.2 is a part of Theorem 5.2.

The definitions of Lagrangian angle and weighted Hamiltonian stationary are given in Section 4. The meaning of weighted Hamiltonian stationary is explained in Appendix A. The notion of generalized Lagrangian mean curvature flow with singularities and topological changes is defined in Section 5. Roughly speaking, this flow is parametrized by a smooth flow except for some m-dimensional Hausdorff measure zero sets. In Example 6.1 of Section 6, we see that our construction is a kind of generalization of the example of Lee and Wang [8]. In Example 6.2, we give a concrete example of generalized Lagrangian mean curvature flow with singularities and topological changes in $K_{\mathbb{P}^2}$, the total space of the canonical bundle over $\mathbb{P}^2$.

We note that the example M_c of Harvey and Lawson is in the case when $n = m - 1$, and $N_c^{a_1,\ldots,a_m}$ of Joyce, V_t of Lee and Wang and the previous work of the author in [13] are in the case when $n = 1$. After finishing my work, the author learned from H. Konno that Mironov and Panov [10] constructed examples of T^n-invariant Hamiltonian stationary Lagrangian submanifolds in m-dimensional toric varieties for $0 \le n \le m$. First, Mironov [9] constructed T^n-invariant Hamiltonian stationary or minimal Lagrangian submanifolds in $\mathbb{C}^m$ and $\mathbb{CP}^m$. These examples can be written as the form (2) in $\mathbb{C}^m$. In [10], they used a Kähler quotient of $\mathbb{C}^m$ to construct new examples in toric varieties. We remark that our method is different from theirs in the point that we use the real form and a moment map rather than Kähler quotient, and furthermore we study the motion of generalized Lagrangian mean curvature flows starting from these examples.

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2. Toric Kähler manifold. In this section, we fix our notations of toric Kähler geometry and introduce an anti-holomorphic involution and its properties. Let $T^m \cong (S^1)^m$ be an m-dimensional real torus and (M, ω, g, J) be a toric Kähler manifold with complex dimension m. Then T^m acts on M effectively and the Kähler form ω is invariant under the action. Let $\mu : M \to \mathfrak{g}^*$ be a moment map and $\Delta := \mu(M)$ be a moment polytope, where $\mathfrak{g}$ is a Lie algebra of T^m and $\mathfrak{g}^*$ is its dual. Since (M, J) is a toric variety, there is a complex torus $T^m_{\mathbb{C}} \cong (\mathbb{C}^\times)^m$ which is a complexification of T^m and $T^m_{\mathbb{C}}$ acts on (M, J) as biholomorphic automorphisms. Then M has an open dense $T^m_{\mathbb{C}}$-orbit and we denote the fan of (M, J) by Σ. Let $\Sigma(1) := \{\, \rho \in \Sigma \mid \dim \rho = 1 \,\}$ be the set of 1-dimensional cones in Σ. We assume that $\Sigma(1)$ is a finite set and write $\Sigma(1) = \{\rho_1, \dots, \rho_d\}$. Let λ_i be the primitive element in $\mathbb{Z}^m$ that generates ρ_i for $i = 1, \dots, d$, that is, $\rho_i = \mathbb{R}^+\lambda_i$. Note that, in general, Δ is not a closed subset in $\mathfrak{g}^*$. For example, if we consider a toric Kähler manifold constructed by removing all fixed points of torus action from some toric Kähler manifold, then its moment polytope has a shape that all vertices are removed from the original polytope and this is not a closed subset.

We assume that there exist κ_i in $\mathbb{R}$ for $i = 1, \dots, d$ so that the closure of Δ is given by

$$\overline{\Delta} = \bigcap_{i=1}^{d} H^+_{\lambda_i,\kappa_i} \,.$$

Here for a nonzero vector λ in $\mathfrak{g}$ and κ in $\mathbb{R}$, we define the affine hyperplane $H_{\lambda,\kappa}$ and closed half-space $H^+_{\lambda,\kappa}$ by

$$H_{\lambda,\kappa} := \{y \in \mathfrak{g}^* \mid \langle y, \lambda\rangle = \kappa\} \quad \text{and} \quad H^+_{\lambda,\kappa} := \{y \in \mathfrak{g}^* \mid \langle y, \lambda\rangle \geq \kappa\} \,.$$

A subset $F \subset \overline{\Delta}$ is called a face of $\overline{\Delta}$ if and only if there exist a vector v in $\mathfrak{g}$ and a constant c such that

$$\overline{\Delta} \subset H^+_{v,c} \quad \text{and} \quad F = \overline{\Delta} \cap H_{v,c} \,.$$

We denote the set of all faces of $\overline{\Delta}$ by $\mathcal{F}$. Then we assume that there exists a subset $\mathcal{G}$ of $\mathcal{F}$ such that Δ is of the form

$$\overline{\Delta} - \bigcup_{F \in \mathcal{G}} F \,.$$

For a point y in Δ, we define $\mathfrak{z}_y$ a subspace of $\mathfrak{g}$ by

$$\mathfrak{z}_y := \mathrm{Span}_{\mathbb{R}}\{\lambda_i \mid y \in H_{\lambda_i,\kappa_i}\}.$$

For example, if y is in the interior of Δ then $\mathfrak{z}_y$ is $\{0\}$. For a point p in M, if we denote the stabilizer at p by $Z_p = \{\, t \in T^m \mid t \cdot p = p \,\}$, then the Lie algebra of Z_p coincides with $\mathfrak{z}_{\mu(p)}$. Thus, if $\mu(p)$ is in the interior of Δ then torus action is free at p, and if μ maps p to a vertex of Δ then p is a fixed point.

Since (M, J) is a toric variety, there exists the intrinsic anti-holomorphic involution $\sigma : M \to M$ determined by the fan Σ, that is, $\sigma^2 = id$ and $\sigma_* J = -J\sigma_*$, where J is the complex structure on M. This involution satisfies $\sigma(u \cdot p) = \overline{u} \cdot \sigma(p)$, where $u \in T^m_{\mathbb{C}}$ acts on p. Let $M^\sigma := \{ p \in M \mid \sigma(p) = p \}$ be the set of fixed points of σ, that is a submanifold of M with real dimension m, we call it the real form of M.

PROPOSITION 2.1. *The involution $\sigma : M \to M$ is anti-symplectic, and consequently σ is isometry.*

PROOF. Let U be an open dense $T^m_{\mathbb{C}}$-orbit. For $(w^1, \dots, w^m) \in U \cong (\mathbb{C}^\times)^m$, we take the logarithmic holomorphic coordinates $(z^1, \dots, z^m)$ with $e^{z^i} = w^i$. Since ω is T^m-invariant and the action of T^m is Hamiltonian, there exists a function $F \in C^\infty(\mathbb{R}^m)$ with the property

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{m} \frac{\partial^2 F}{\partial x^i \partial x^j} dz^i \wedge d\overline{z}^j \quad \text{on } U\,, \tag{3}$$

where $z^i = x^i + \sqrt{-1}y^i$. (See Theorem 3.3 in Appendix 2 of [4].) On U, the involution σ coincides with the standard complex conjugate $\sigma(z) = \overline{z}$, where $\overline{z} = (\overline{z}^1, \dots, \overline{z}^m)$. Since ω is T^m-invariant, note that F is independent of the coordinates $(y^1, \dots, y^m)$. Thus we have $\sigma^*\omega = -\omega$ on U. Since U is open and dense in M, thus we have $\sigma^*\omega = -\omega$ on M. □

3. Lagrangian submanifold. In this section, we construct our examples of Lagrangian submanifold. First of all, let n be an integer with $0 \leq n \leq m$. Next, take a set of n vectors $\zeta = \{\zeta_i\}_{i=1}^n \subset \mathfrak{g}$ and a set of n constants $c = \{c_i\}_{i=1}^n \subset \mathbb{R}$. If $n = 0$, we take no vectors and no constants. We assume that $\{\zeta_i\}_{i=1}^n$ is linearly independent. Then the intersection of n affine hyperplanes H_{ζ_i, c_i} defines an $(m-n)$-dimensional affine plane. We assume that this affine plane intersects in the interior of Δ, and we define $\Delta_{\zeta,c}$ a subset of Δ by

$$\begin{aligned} \Delta_{\zeta,c} &:= \Delta \cap \left(\bigcap_{i=1}^{n} H_{\zeta_i, c_i} \right) \\ &= \{ y \in \Delta \mid \langle y, \zeta_i \rangle = c_i,\ (i = 1, \dots, n) \}\,. \end{aligned}$$

DEFINITION 3.1. Let $V_\zeta := \mathrm{Span}_{\mathbb{R}}\{\zeta_1, \dots, \zeta_n\} \subset \mathfrak{g}$. We call a point y in Δ a ζ-singular point if and only if $V_\zeta \cap \mathfrak{z}_y \neq \{0\}$, and if $V_\zeta \cap \mathfrak{z}_y = \{0\}$ we call y a ζ-regular point. We denote the set of all ζ-singular points and all ζ-regular points in Δ by $\Delta_{\zeta sing}$ and $\Delta_{\zeta reg}$ respectively. Note that $\Delta_{\zeta reg}$ is open dense in Δ.

For a point p in M, a vector v in $\mathfrak{g}$ generates a tangent vector at p denoted by

$$v_p = \left. \frac{d}{dt} \right|_{t=0} \exp(tv) \cdot p\,.$$

This map $\mathfrak{g} \to T_pM$ is a homomorphism. Then it is clear that y is a ζ-regular point if and only if the restricted homomorphism $V_\zeta \to T_pM$ is injective for p in $\mu^{-1}(y)$. For example, vertices of Δ are always ζ-singular points and interior points are always ζ-regular points.

DEFINITION 3.2. We call a point p in M^σ a ζ-singular point if and only if $\mu(p)$ is a ζ-singular point, and if not, we call p a ζ-regular point. We denote the set of all ζ-singular points and all ζ-regular points in M^σ by $M^\sigma_{\zeta sing}$ and $M^\sigma_{\zeta reg}$ respectively. Note that $M^\sigma_{\zeta reg}$ is open dense in M^σ.

DEFINITION 3.3. We denote the restriction of the moment map on the real form by $\mu^\sigma : M^\sigma \to \mathbb{R}^m$. We define a subset of M^σ as the pull-back of $\Delta_{\zeta,c}$ by μ^σ by

$$\begin{aligned} M^\sigma_{\zeta,c} &:= (\mu^\sigma)^{-1}(\Delta_{\zeta,c}) \\ &= \{p \in M^\sigma \mid \langle \mu(p), \zeta_i \rangle = c_i,\ i = 1, \dots, n\}. \end{aligned}$$

PROPOSITION 3.4. *If $\Delta_{\zeta,c}$ is contained in $\Delta_{\zeta reg}$, then $M^\sigma_{\zeta,c}$ is a smooth submanifold of M^σ with* $\dim_{\mathbb{R}} M^\sigma_{\zeta,c} = m - n$.

PROOF. We define n functions f_i $(i = 1, \dots, n)$ on M^σ by

$$f_i(p) := \langle \mu(p), \zeta_i \rangle - c_i .$$

Then $M^\sigma_{\zeta,c} = \{\, p \in M^\sigma \mid f_i(p) = 0,\ i = 1, \dots, n \,\}$. By a property of the moment map, for all p in $M^\sigma_{\zeta,c}$, we have

$$df_i(p) = d\langle \mu, \zeta_i \rangle (p) = -\omega(\zeta_{i,p}, \cdot) .$$

Since every point in $\Delta_{\zeta,c}$ is ζ-regular, the restricted homomorphism $V_\zeta \to T_pM$ is injective for all p in $M^\sigma_{\zeta,c}$. Thus $\{df_i\}_{i=1}^n$ are linearly independent 1-forms on $M^\sigma_{\zeta,c}$. This means that $M^\sigma_{\zeta,c}$ is a smooth submanifold of M^σ by the implicit function theorem. □

In this section, we assume that $\Delta_{\zeta,c}$ is contained in $\Delta_{\zeta reg}$. Then $M^\sigma_{\zeta,c}$ is a smooth submanifold of M^σ. Let $\exp : \mathfrak{g} \to T^m$ be the exponential map. Let $\mathbb{Z}_{\mathfrak{g}} (\cong \mathbb{Z}^m)$ be a integral lattice of $\mathfrak{g}$, that is a kernel of $\exp : \mathfrak{g} \to T^m$ and $\mathfrak{g}/\mathbb{Z}_{\mathfrak{g}} \cong T^m$. Let $\frac{1}{2}\mathbb{Z}_{\mathfrak{g}}$ be the set of all elements y in $\mathfrak{g}$ such that $2y$ is in $\mathbb{Z}_{\mathfrak{g}}$. Then $\frac{1}{2}\mathbb{Z}_{\mathfrak{g}}/\mathbb{Z}_{\mathfrak{g}} \cong \{1, -1\}^m$ is a subgroup of T^m considered as all elements t in T^m such that $t^2 = e$ identity element. Let $V_\zeta = \mathrm{Span}_{\mathbb{R}}\{\zeta_1, \dots, \zeta_n\} \subset \mathfrak{g}$. Now we construct a manifold $L_{\zeta,c}$ with real dimension m.

(I) GENERIC CASE. For a generic case, let U be an open small ball in V_ζ centered at 0 such that U and $\frac{1}{2}\mathbb{Z}_{\mathfrak{g}}$ intersect only at 0. Then we define an m-dimensional manifold $L_{\zeta,c}$ and a map $F_{\zeta,c} : L_{\zeta,c} \to M$ by

$$L_{\zeta,c} = M^\sigma_{\zeta,c} \times U \quad \text{and} \quad F_{\zeta,c}(p, v) := \exp(v) \cdot p ,$$

for p in $M^\sigma_{\zeta,c}$ and v in U. Then $F_{\zeta,c}$ is injective and its image is

$$L'_{\zeta,c} := \{\exp(v) \cdot p \mid v \in U,\ p \in M^\sigma, \langle \mu(p), \zeta_j \rangle = c_j,\ j = 1, \dots, n\} . \tag{4}$$

(II) UNIMODULAR CASE. If the set of vectors $\zeta = \{\zeta_i\}_{i=1}^n$ satisfies the following unimodular condition then we can take $L_{\zeta,c}$ as explained below.

DEFINITION 3.5. We say that ζ satisfies the unimodular condition if there exists a set of n vectors $v = \{v_j\}_{j=1}^n$ in $V_\zeta \cap \mathbb{Z}_{\mathfrak{g}}$ such that v is a base of V_ζ and v is a generator of $V_\zeta \cap \mathbb{Z}_{\mathfrak{g}}$ over $\mathbb{Z}$.

If ζ satisfies the unimodular condition, we replace U in the case (I) by $T_\zeta := V_\zeta/(V_\zeta \cap \mathbb{Z}_\mathfrak{g})$ and we define an m-dimensional manifold $L_{\zeta,c}$ and a map $F_{\zeta,c} : L_{\zeta,c} \to M$ by

$$L_{\zeta,c} = M^\sigma_{\zeta,c} \times T_\zeta \quad \text{and} \quad F_{\zeta,c}(p,[v]) := \exp(v) \cdot p\,,$$

for p in $M^\sigma_{\zeta,c}$ and $[v]$ in $T_\zeta = V_\zeta/(V_\zeta \cap \mathbb{Z}_\mathfrak{g})$, this map is well defined. Since $T_\zeta \cong T^n$ which is a subtorus of T^m, the product manifold $L_{\zeta,c}$ is diffeomorphic to $M^\sigma_{\zeta,c} \times T^n$. We denote the subgroup $(V_\zeta \cap \frac{1}{2}\mathbb{Z}_\mathfrak{g})/(V_\zeta \cap \mathbb{Z}_\mathfrak{g})$ of T_ζ by K_ζ. Then, of course, K_ζ acts on T_ζ freely and K_ζ also acts on $M^\sigma_{\zeta,c}$ as

$$[k] \cdot p := \exp(k) \cdot p$$

for $[k]$ in K_ζ and p in $M^\sigma_{\zeta,c}$. Thus K_ζ acts on $L_{\zeta,c} = M^\sigma_{\zeta,c} \times T_\zeta$ as a diagonal action and this action is free. Hence we have an m-dimensional manifold $\tilde{L}_{\zeta,c}$ by

$$\tilde{L}_{\zeta,c} := (M^\sigma_{\zeta,c} \times T_\zeta)/K_\zeta\,.$$

In this case (II), $F_{\zeta,c} : L_{\zeta,c} \to M$ is not injective and one can show that $F_{\zeta,c}(p_1,[v_1]) = F_{\zeta,c}(p_2,[v_2])$ if and only if there exists a $[k]$ in K_ζ such that $[k] \cdot (p_1,[v_1]) = (p_2,[v_2])$. Thus the image of $F_{\zeta,c}$ written by

$$L'_{\zeta,c} := \{\exp(v) \cdot p \mid v \in V_\zeta,\ p \in M^\sigma, \langle \mu(p), \zeta_j \rangle = c_j,\ j = 1,\dots,n\} \tag{5}$$

is diffeomorphic to $\tilde{L}_{\zeta,c}$. Note that $\tilde{L}_{\zeta,c}$ is a T^n-bundle over a smooth $(m-n)$-dimensional manifold $M^\sigma_{\zeta,c}/K_\zeta$.

REMARK 3.6. Here we explain the meaning of $L_{\zeta,c}$ and the number n, that is the number of vectors in ζ. In an m-dimensional toric Kähler manifold M, there are two typical Lagrangian submanifolds, one is the real form M^σ and the other is a torus fiber T^m, and these two Lagrangians M^σ and T^m intersect transverse and orthogonal just like $\mathbb{R}^m$ and $i\mathbb{R}^m$ in $\mathbb{C}^m$. First, if we take $n = 0$ then we take no vectors ζ and no constants c. Then $L_{\zeta,c}$ becomes the real form M^σ, hence $L_{\zeta,c}$ has no torus factors. On the other hand, if n is full, that is, $n = m$, then $M^\sigma_{\zeta,c} = \{pt\}$, thus $L_{\zeta,c}$ is diffeomorphic to a torus fiber T^m. Hence, roughly speaking, $L_{\zeta,c}$ is a hybrid (or interpolation) of the real form M^σ and a torus fiber T^m, and n is the dimension of torus factors in $L_{\zeta,c}$.

From now, we consider both cases (I) and (II) above.

THEOREM 3.7. *$F_{\zeta,c} : L_{\zeta,c} \to M$ is a Lagrangian immersion.*

PROOF. In this proof, we write $F_{\zeta,c}$ by F for short. Since the case (II) is locally diffeomorphic to the cace (I), it is clear that we only have to prove in the case (I). First we prove that F is an immersion map. Fix a point $x = (p,v)$ in $L_{\zeta,c} = M^\sigma_{\zeta,c} \times U$. Then we have a decomposition

$$T_x L_{\zeta,c} = T_p M^\sigma_{\zeta,c} \oplus T_v U\,,$$

and note that $T_v U \cong V_\zeta$ since U is an open ball in a vector space V_ζ. Take tangent vectors X, X_1, X_2 in $T_p M^\sigma_{\zeta,c}$. We have

$$F_* X = t_{v*} X\,,$$

where we put $t_v := \exp(v)$ for short, and we identify an element t_v in T^m with a left transition map $t_v : M \to M$. Take vectors Y, Y_1, Y_2 in $T_v U \cong V_\zeta$. We have

$$F_* Y = t_{v*} Y_p \,.$$

Here Y_p is the tangent vector at p generated by $Y \in V_\zeta \subset \mathfrak{g}$. Since g is torus-invariant, that is, $t_v^* g = g$, we have

$$g(F_* X, F_* Y) = g(t_{v*} X, t_{v*} Y_p) = (t_v^* g)(X, Y_p) = g(X, Y_p)\,. \tag{6}$$

Note that $\sigma_* X = X$ since X is tangent to the real form, and $\sigma_* Y_p = -Y_p$ since the direction of the curve of the exponential map generated by Y is reversed by σ because of the reration $\sigma(u \cdot p) = u^{-1} \cdot p$ for all u in T^m. Since σ is isometry, that is $\sigma^* g = g$, by Proposition 2.1, we have

$$g(X, Y_p) = (\sigma^* g)(X, Y_p) = g(\sigma_* X, \sigma_* Y_p) = -g(X, Y_p)\,,$$

and this means that $g(X, Y_p) = 0$ and also $g(F_* X, F_* Y) = 0$ by (6). Thus $F_*(T_p M^\sigma_{\zeta,c})$ and $F_*(T_v U)$ are orthogonal to each other. It is clear that F_* restricted on $T_p M^\sigma_{\zeta,c}$ is injective and F_* restricted on $T_v U$ is also injective. Thus F_* is injective on $T_x L_{\zeta,c}$ and F is an immersion map.

Next we prove that F is a Lagrangian, that is, $F^*\omega = 0$. It is easy to see $(F^*\omega)(X_1, X_2) = 0$ and $(F^*\omega)(Y_1, Y_2) = 0$ since the real form and a torus fiber are typical Lagrangians. We can also prove that $(F^*\omega)(X, Y) = 0$ easily. Since ω is torus-invariant and $\omega(\cdot, Y_p) = d\langle \mu, Y\rangle$, we have

$$(F^*\omega)(X, Y) = \omega(F_* X, F_* Y) = \omega(X, Y_p) = X(\langle \mu, Y\rangle)\,.$$

Since Y is in $T_v U \cong V_\zeta = \mathrm{Span}_{\mathbb{R}}\{\zeta_1, \ldots, \zeta_n\}$, we can write Y as $Y = a^1 \zeta_1 + \cdots + a^n \zeta_n$ for some coefficients $a^k \in \mathbb{R}$, and we have

$$\langle \mu, Y\rangle = a^1 \langle \mu, \zeta_1\rangle + \cdots + a^n \langle \mu, \zeta_n\rangle\,.$$

By the definition of $M^\sigma_{\zeta,c}$, this function $\langle \mu, Y\rangle$ is a constant

$$a^1 c_1 + \cdots + a^n c_n$$

on $M^\sigma_{\zeta,c}$, and now X is a tangent vector on $M^\sigma_{\zeta,c}$, thus it is clear that

$$X(\langle \mu, Y\rangle) = 0\,.$$

Hence we have $F^*\omega = 0$. □

4. Lagrangian angle. In above sections, the ambient space (M, ω, g, J) is a toric Kähler manifold. From this section, we assume that the canonical line bundle K_M of (M, J) is trivial. This condition is equivalent to that there exists a vector γ in $\mathbb{Z}^*_{\mathfrak{g}}$ such that $\langle \gamma, \lambda_i\rangle = 1$ for all $i = 1, \ldots, d$, where λ_i is a primitive generator of a 1-dimensional cone of fan Σ of M, see Section 2. In fact, if such a vector $\gamma = (\gamma_1, \ldots, \gamma_m)$ exists, a holomorphic $(m, 0)$-form

$$\Omega_\gamma := e^{\gamma_1 z^1 + \cdots + \gamma_m z^m} dz^1 \wedge \cdots \wedge dz^m \tag{7}$$

written by logarithmic holomorphic coordinates on an open dense $(\mathbb{C}^*)^m$-orbit can be extend over M as a nowhere vanishing holomorphic $(m, 0)$-form. We call this $(M, \omega, g, J, \Omega_\gamma)$ a toric almost Calabi–Yau manifold.

In general, an m-dimensional Kähler manifold (M, ω, g, J) with nowhere vanishing holomorphic $(m, 0)$-form Ω is called an almost Calabi–Yau manifold, and for a Lagrangian immersion $F : L \to M$ we can define the Lagrangian angle $\theta_F : L \to \mathbb{R}/\pi\mathbb{Z}$ as follows. For a point x in L, take a local chart $(U, (x^1, \dots, x^m))$ around x, then $F^*\Omega$ is a $\mathbb{C}^*$-valued m-form on U, so there exists a $\mathbb{C}^*$-valued function h_U on U such that

$$F^*\Omega = h_U(x^1, \dots, x^m)dx^1 \wedge \cdots \wedge dx^m$$

on U, and we define the Lagrangian angle $\theta_F : L \to \mathbb{R}/\pi\mathbb{Z}$ by

$$\theta_F(x) := \arg(h_U(x)) \mod \pi\,.$$

This definition is independent of the choice of local charts. It is clear that if L is oriented we can lift θ_F to an $\mathbb{R}/2\pi\mathbb{Z}$-valued function $\theta_F : L \to \mathbb{R}/2\pi\mathbb{Z}$. If we can lift θ_F to an $\mathbb{R}$-valued function $\theta_F : L \to \mathbb{R}$ then $F : L \to M$ is called Maslov zero, and furthermore if θ_F is constant θ_0 then $F : L \to M$ is called a special Lagrangian submanifold with phase $e^{i\theta_0}$. Note that the definition of special Lagrangian condition depends on the choice of holomorphic volume form Ω.

In [1], Behrndt introduced the notion of the generalized mean curvature vector field K for a Lagrangian immersion $F : L \to M$ in an almost Calabi–Yau manifold. The generalized mean curvature vector field K is defined by

$$K := H - m\nabla\psi^{\perp}\,, \tag{8}$$

where H is the mean curvature vector field of the immersion $F : L \to (M, g)$, ψ is a function on M defined by the following equation;

$$e^{2m\psi}\frac{\omega^m}{m!} = (-1)^{\frac{m(m-1)}{2}}\left(\frac{i}{2}\right)^m \Omega \wedge \overline{\Omega}\,, \tag{9}$$

and $\nabla\psi^{\perp}$ is the normal part of the gradient of ψ. By the definition of K, if M is a Calabi–Yau manifold, that is, $\psi \equiv 0$, then the generalized mean curvature vector field K coincides with the mean curvature vector field H. In Proposition 4.8 in [2], Behrndt proved the relation between K and θ_F which is written by

$$K = J\nabla\theta_F. \tag{10}$$

Thus $K \equiv 0$ is equivalent to that L is a special Lagrangian submanifold.

Furthermore, in this paper, we introduce the notion of *weighted Hamiltonian stationary* for a Lagrangian immersion $F : L \to M$ into an almost Calabi–Yau manifold $(M, \omega, g, J, \Omega)$ with ψ defined by (9).

DEFINITION 4.1. Let θ_F be the Lagrangian angle of $F : L \to M$. If $\Delta_f\theta_F = 0$ then we call $F : L \to M$ a weighted Hamiltonian stationary Lagrangian submanifold.

Here f is a function on L defined by $f := -mF^*\psi$ and Δ_f is the weighted Laplacian on Riemannian manifold (L, F^*g). In general, for a Riemannian manifold (N, h) with a function f, the weighted Laplacian with respect to f is defined by $\Delta_f u := \Delta u + \langle \nabla u, \nabla f \rangle$. Thus, if M is a Calabi–Yau manifold, that is, $\psi = 0$, then the notion of weighted Hamiltonian stationary is equivalent to the Hamiltonian stationary condition, namely $\Delta\theta_F = 0$. For the meaning of the weighted Hamiltonian stationary condition, see Appendix A. Note that Δ_f is the standard Laplace operator on L with respect to a Riemannian metric $F^*(e^{2\psi}g)$.

In this section, we compute the Lagrangian angle of our example $F_{\zeta,c} : L_{\zeta,c} \to M$ constructed in Section 3, and show some properties of $F_{\zeta,c} : L_{\zeta,c} \to M$.

Let $(M, \omega, g, J, \Omega_\gamma)$ be an m-dimensional toric almost Calabi–Yau manifold and $F_{\zeta,c} : L_{\zeta,c} \to M$ be a Lagrangian immersion constructed by $\zeta = \{\zeta_1, \dots, \zeta_n\} \subset \mathfrak{g}$ and $c = \{c_1, \dots, c_n\} \subset \mathbb{R}$, explained in Section 3.

THEOREM 4.2. *The Lagrangian angle θ of $F_{\zeta,c} : L_{\zeta,c} \to M$ is given by*

$$\theta(x) = 2\pi\langle \gamma, v\rangle + \frac{\pi}{2}n \mod \pi$$

for $x = (p, v)$ in $L_{\zeta,c} = M^\sigma_{\zeta,c} \times U$ in the case (I) *and for $x = (p, [v])$ in $L_{\zeta,c} = M^\sigma_{\zeta,c} \times T_\zeta$ in the case* (II).

PROOF. In this proof, we write $F_{\zeta,c}$ by F for short. It is clear that we only have to prove in the case (I). Let M^σ be the real form of M and $\mathfrak{g}$ be a Lie algebra of T^m. We define a map $\tilde{F} : M^\sigma \times \mathfrak{g} \to M$ by

$$\tilde{F}(p, v) := \exp(v) \cdot p\,.$$

Remember that $L_{\zeta,c} = M^\sigma_{\zeta,c} \times U$, and $M^\sigma_{\zeta,c}$ is an $(m-n)$-dimensional submanifold in M^σ and U is an n-dimensional submanifold in $\mathfrak{g}$. Thus we have the inclusion map $L_{\zeta,c}$ into $M^\sigma \times \mathfrak{g}$ by

$$\iota = (\iota_1, \iota_2) : L_{\zeta,c} = M^\sigma_{\zeta,c} \times U \hookrightarrow M^\sigma \times \mathfrak{g}\,.$$

Then the map $F : L_{\zeta,c} \to M$ coincides with $\tilde{F} \circ \iota$ by the definition of F, so we compute $\iota^*(\tilde{F}^*\Omega_\gamma)$ to compute $F^*\Omega_\gamma$. It is enough to prove this theorem on an open dense $(\mathbb{C}^*)^m$-orbit, so we take a logarithmic holomorphic coordinates $(z^1, \dots, z^m)$, then $(x^1, \dots, x^m)$ define local coordinates on the real form M^σ, where $z^j = x^j + iy^j$. Let $(t^1, \dots, t^m)$ be coordinates of $\mathfrak{g} \cong \mathbb{R}^m$, then we have a local expression of a map $\tilde{F} : M^\sigma \times \mathfrak{g} \to M$ by

$$\tilde{F}(x^1, \dots, x^m, t^1, \dots, t^m) = (x^1 + 2\pi i t^1, \dots, x^m + 2\pi i t^m)\,.$$

Since $\Omega_\gamma = e^{\gamma_1 z^1 + \cdots + \gamma_m z^m} dz^1 \wedge \cdots \wedge dz^m$, we have

$$\tilde{F}^*\Omega_\gamma = e^{(\gamma_1 x^1 + \cdots + \gamma_m x^m) + 2\pi i(\gamma_1 t^1 + \cdots + \gamma_m t^m)}(dx^1 + 2\pi i dt^1) \wedge \cdots \wedge (dx^m + 2\pi i dt^m)\,.$$

Since $L_{\zeta,c} = M^\sigma_{\zeta,c} \times U$, and $M^\sigma_{\zeta,c}$ is an $(m-n)$-dimensional submanifold in M^σ and U is an n-dimensional submanifold in $\mathfrak{g}$, in the expansion of $(dx^1 + 2\pi i dt^1) \wedge \cdots \wedge (dx^m + 2\pi i dt^m)$, differential forms such as

$$(2\pi i)^n dx^I \wedge dt^J$$

with $\sharp I = m - n$ and $\sharp J = n$ do not vanish after pull-back by ι, and other forms vanish, where I and J are multi-indices. Thus the argument of $F^*\Omega_\gamma = \iota^*(\tilde{F}^*\Omega_\gamma)$ at (p, v) is the argument of

$$(2\pi i)^n e^{\langle\gamma, p\rangle + 2\pi i\langle\gamma, v\rangle},$$

that is, $2\pi\langle\gamma, v\rangle + \frac{\pi}{2}n \mod \pi$. □

Then the following corollary is clear.

COROLLARY 4.3. *$F_{\zeta,c} : L_{\zeta,c} \to M$ is a special Lagrangian submanifold if and only if $\langle\gamma, \zeta_i\rangle = 0$ for all $i = 1, \ldots, n$.*

REMARK 4.4. It is clear that the real form M^σ, that is the case of $n = 0$, is always a special Lagrangian submanifold, and every torus fiber, that is the case of $n = m$, is not a special Lagrangian submanifold with respect to this holomorphic volume form Ω_γ. If $M = \mathbb{C}^m$, we take $\gamma = (1, \ldots, 1)$, see also Example 6.1. Then the special Lagrangian condition (1) by Joyce introduced in Section 1 coincides with the condition $\langle\gamma, a\rangle = 0$ in Corollary 4.3.

THEOREM 4.5. *$F_{\zeta,c} : L_{\zeta,c} \to M$ is weighted Hamiltonian stationary.*

PROOF. In this proof, we write $F_{\zeta,c}$ by F for short. We only have to prove that $\Delta_f\theta = 0$ in the case (I) that $L_{\zeta,c} = M^\sigma_{\zeta,c} \times U$. As noted above, Δ_f is the standard Laplace operator on L with respect to a Riemannian metric $F^*(e^{2\psi}g)$. Since g is invariant under the torus action and it is easily seen that ψ is also torus invariant by the equation (7) and (9), so the metric $e^{2\psi}g$ is also a torus invariant metric on M. Since $F : L_{\zeta,c} \to M$ is given by $F(p, v) := \exp(v) \cdot p$ and $e^{2\psi}g$ is a torus invariant metric on M, the metric $F^*(e^{2\psi}g)$ on L is independent of the U-factor of $L_{\zeta,c}$. Furthermore, in the proof of Theorem 3.7 we prove that $F_*(TM^\sigma_{\zeta,c})$ and $F_*(TU)$ are orthogonal, thus $F^*(e^{2\psi}g)$ is a product metric over $M^\sigma_{\zeta,c}$ and U locally. By Theorem 4.2, the Lagrangian angle is given by $\theta(p, v) = 2\pi\langle\gamma, v\rangle + \frac{\pi}{2}n$, it is independent of $M^\sigma_{\zeta,c}$-factor of $L_{\zeta,c}$ and affine on U-factor. Then one can easily prove that $\Delta_f\theta = 0$. □

5. Mean curvature flow. In this section, we consider generalized Lagrangian mean curvature flows. In general, a generalized Lagrangian mean curvature flow is defined in an almost Calabi–Yau manifold $(M, \omega, g, J, \Omega)$. Let $F_0 : L \to M$ be a Lagrangian immersion, then a one parameter family of Lagrangian submanifolds $F : L \times I \to M$ is called a solution of a generalized Lagrangian mean curvature flow with initial condition F_0, if it moves along its generalized Lagrangian mean curvature vector field K defined in (8), that is,

$$\left(\frac{\partial F}{\partial t}\right)^\perp = K_t \quad \text{and} \quad F(\cdot, 0) = F_0, \tag{11}$$

where K_t is the generalized Lagrangian mean curvature vector field of immersion $F_t : L \to M$ defined by $F_t(p) := F(p, t)$. Of course, if M is a Calabi–Yau manifold then a generalized Lagrangian mean curvature flow is an ordinary Lagrangian mean curvature flow. It is clear that $K = 0$ on a special Lagrangian submanifold by the equation (10), thus a special Lagrangian submanifold is a stationary solution of a generalized Lagrangian mean curvature flow. In general, a generalized Lagrangian mean curvature flow develops some singularities in a finite time, so here we define a notion of a generalized Lagrangian mean curvature flow with some singularities and topological changes.

DEFINITION 5.1. Let $(M, \omega, g, J, \Omega)$ be a real $2m$-dimensional almost Calabi–Yau manifold and $\{L_t\}_{t\in I}$ be a one parameter family of subsets in M. Then we call $\{L_t\}_{t\in I}$ a solution of a generalized Lagrangian mean curvature flow with singularities and topological changes if there exists a real m-dimensional manifold L and a solution of a generalized Lagrangian mean curvature flow $F : L \times I \to M$ such that $F_t : L \to M$ is an embedding into L_t and m-dimensional Hausdorff measure of $L_t \setminus F_t(L)$ is zero, i.e.

$$F_t(L) \subset L_t \quad \text{and} \quad \mathcal{H}^m(L_t \setminus F_t(L)) = 0\,. \tag{12}$$

It means that $\{L_t\}_{t\in I}$ is almost parametrized by a smooth solution of a generalized Lagrangian mean curvature flow.

The purpose of this section is to observe how our concrete examples $F_{\zeta,c} : L_{\zeta,c} \to M$ move along the generalized Lagrangian mean curvature flow. Let $(M, \omega, g, J, \Omega_\gamma)$ be a toric almost Calabi–Yau manifold and $F_{\zeta,c} : L_{\zeta,c} \to M$ be a Lagrangian submanifold constructed in Section 3 by data $\zeta = \{\zeta_1, \dots, \zeta_n\} \subset \mathfrak{g}$ and $c = \{c_1, \dots, c_n\} \subset \mathbb{R}$. Let

$$c_i(t) := c_i - 2\pi\langle\gamma, \zeta_i\rangle t$$

for $t \in \mathbb{R}$ and we denote $c(t) := \{c_1(t), \dots, c_n(t)\}$. We define an open interval I by

$$I := \left\{ t \in \mathbb{R} \,\middle|\, \mathrm{Int}\Delta \cap \left(\bigcap_{i=1}^{n} H_{\zeta_i, c_i(t)}\right) \neq \emptyset \right\},$$

by the assumption of ζ and c we have $0 \in I$.

THEOREM 5.2. *A one parameter family of subsets* $\{L'_{\zeta,c(t)}\}_{t\in I}$ *defined by* (4) *in the case* (I) *or by* (5) *in the case* (II) *is a solution of a generalized Lagrangian mean curvature flow with singularities and topological changes.*

PROOF. It is sufficient to prove this theorem in the case (I). First we define

$$\Delta''_{\zeta,c(t)} := \mathrm{Int}\Delta \cap \left(\bigcap_{i=1}^{n} H_{\zeta_i, c_i(t)}\right).$$

Remember that $\Delta_{\zeta,c(t)}$ is defined by

$$\Delta_{\zeta,c(t)} := \Delta \cap \left(\bigcap_{i=1}^{n} H_{\zeta_i, c_i(t)}\right).$$

Since $\Delta_{\zeta,c(t)} \setminus \Delta''_{\zeta,c(t)}$ is contained in $\partial\Delta_{\zeta,c(t)}$, it is clear that $(m-n)$-dimensional Hausdorff measure of $\Delta_{\zeta,c(t)} \setminus \Delta''_{\zeta,c(t)}$ is zero. Since each $\Delta''_{\zeta,c(t)}$ is an $(m-n)$-dimensional connected convex affine open subset in $\mathbb{R}^m$, all $\Delta''_{\zeta,c(t)}$ are diffeomorphic to each other.

Next we define

$$M''^{\sigma}_{\zeta,c(t)} := (\mu^\sigma)^{-1}(\Delta''_{\zeta,c(t)}) \quad \text{and} \quad L''_{\zeta,c(t)} := M''^{\sigma}_{\zeta,c(t)} \times U\,.$$

Then $M''^{\sigma}_{\zeta,c(t)}$ is an $(m-n)$-dimensional open dense submanifold in M, and $L''_{\zeta,c(t)}$ is an m-dimensional open dense submanifold in $L_{\zeta,c(t)}$. As same as $\Delta''_{\zeta,c(t)}$, all $M''^{\sigma}_{\zeta,c(t)}$ are diffeomorphic to each other, and $(m-n)$-dimensional Hausdorff measure of $M^{\sigma}_{\zeta,c(t)} \setminus M''^{\sigma}_{\zeta,c(t)}$ is zero, and m-dimensional Hausdorff measure of $L_{\zeta,c(t)} \setminus L''_{\zeta,c(t)}$ is also zero. Thus we can take a one parameter family of diffeomorphisms

$$G_t : M''^{\sigma}_{\zeta,c} \to M''^{\sigma}_{\zeta,c(t)}\,,$$

for all $t \in I$, and G_t induces a one parameter family of diffeomorphisms

$$\tilde{G}_t : L''_{\zeta,c} \to L''_{\zeta,c(t)}$$

by $\tilde{G}_t(p,v) := (G_t(p), v)$. Then we have a one parameter family of maps $F : L''_{\zeta,c} \times I \to M$ by

$$F_t(p,v) := F_{\zeta,c(t)} \circ \tilde{G}_t(p,v) = \exp(v)\cdot G_t(p)\,.$$

It is clear that

$$F_t(L''_{\zeta,c}) = F_{\zeta,c(t)}(\tilde{G}_t(L''_{\zeta,c})) = F_{\zeta,c(t)}(L''_{\zeta,c(t)}) \subset L'_{\zeta,c(t)}\,,$$

where remember that

$$L'_{\zeta,c(t)} = \{\exp(v)\cdot p \mid v \in U,\ p \in M^{\sigma}, \langle \mu(p), \zeta_j\rangle = c_j(t),\ j = 1,\dots,n\}\,.$$

Since torus action is free on $M''^{\sigma}_{\zeta,c(t)}$, one can easily prove that F_t is embedding for all t, and m-dimensional Hausdorff measure of $L'_{\zeta,c(t)} \setminus F_t(L''_{\zeta,c})$ is zero.

Hence the remainder we have to prove is to prove that $F : L''_{\zeta,c} \times I \to M$ is a solution of a generalized Lagrangian mean curvature flow. Since both K_t and the normal part of $\partial F/\partial t$ are sections of normal bundle and $F_t : L''_{\zeta,c} \to M$ is a Lagrangian submanifold, it is enough to prove

$$\omega\left(\frac{\partial F}{\partial t}, F_{t*}Z\right) = \omega(K_t, F_{t*}Z) \tag{13}$$

for all tangent vectors Z on $L''_{\zeta,c}$ to prove the equation (11). Fix a point $x = (p,v)$ in $L''_{\zeta,c} = M''^{\sigma}_{\zeta,c} \times U$. Since we have a decomposition

$$T_x L''_{\zeta,c} = T_p M''^{\sigma}_{\zeta,c} \oplus T_v U$$

and note that $T_v U \cong V_\zeta$, a tangent vector Z is written by $Z = X + Y$ for some tangent vectors X in $T_p M''^{\sigma}_{\zeta,c}$ and Y in V_ζ. For X and Y, we have

$$F_{t*}X = \exp(v)_*(G_{t*}X) \qquad \text{and} \qquad F_{t*}Y = \exp(v)_*(Y_{G_t(p)})\,.$$

For X, we have

$$\omega\left(\frac{\partial F}{\partial t}, F_{t*}X\right) = \omega\left(\exp(v)_*\left(\frac{\partial G}{\partial t}\right), \exp(v)_*(G_{t*}X)\right) = \omega\left(\frac{\partial G}{\partial t}, G_{t*}X\right) = 0\,.$$

The second equality follows from the torus invariance of ω, and the third equality follows from that both $\partial G/\partial t$ and $G_{t*}X$ are tangent to real form and it is a Lagrangian. If we use the equation (10), we have

$$\omega(K_t, F_{t*}X) = \omega(J\nabla\theta_{F_t}, F_{t*}X) = -g(\nabla\theta_{F_t}, F_{t*}X) = -X\theta_{F_t} = 0\,,$$

since $\theta_{F_t}(p, v) = 2\pi\langle\gamma, v\rangle + \frac{\pi}{2}n$ by Theorem 4.2 and it is independent of $M''^{\sigma}_{\zeta,c}$ part. Thus the equation (13) holds for X. Next, for Y, we have

$$\begin{aligned}\omega\left(\frac{\partial F}{\partial t}, F_{t*}Y\right) &= \omega\left(\frac{\partial G}{\partial t}, Y_{G_t(p)}\right) = \frac{\partial G}{\partial t}\langle\mu, Y\rangle = \frac{\partial}{\partial t}\langle\mu\circ G_t, Y\rangle\\ &= \frac{\partial}{\partial t}\langle\mu\circ G_t, a^1\zeta_1 + \cdots + a^n\zeta_n\rangle\\ &= \frac{\partial}{\partial t}(a^1c_1(t) + \cdots + a^nc_n(t))\\ &= -2\pi\langle\gamma, Y\rangle\,.\end{aligned}$$

The second equality follows from the assumption of the moment map μ. In the fourth equality we put $Y = a^1\zeta_1 + \cdots + a^n\zeta_n$ for some coefficients a^i and the fifth equality follows from the definition of $M''^{\sigma}_{\zeta,c(t)}$. In the last equality, remember that $c_i(t)$ is defined by $c_i(t) := c_i - 2\pi\langle\gamma, \zeta_i\rangle t$. If we use the equation (10), we have

$$\omega(K_t, F_{t*}Y) = \omega(J\nabla\theta_{F_t}, F_{t*}Y) = -g(\nabla\theta_{F_t}, F_{t*}Y) = -Y\theta_{F_t} = -2\pi\langle\gamma, Y\rangle\,.$$

Thus the equation (13) holds for Y and it is proved that $F : L''_{\zeta,c} \times I \to M$ is a solution of a generalized Lagrangian mean curvature flow. □

6. Examples. In this section, we give some examples of our main theorems. First we explain that if the ambient space M is $\mathbb{C}^m$ then our examples coincide with those constructed by Lee and Wang in [8].

EXAMPLE 6.1. Let $(\mathbb{C}^m, \omega, g, J, \Omega)$ be a standard complex plane with a holomorphic volume form $\Omega = dw_1 \wedge \cdots \wedge dw_m$ by the standard coordinates w. If we write $w_i = e^{z_i}$ where $w_i \neq 0$, then Ω is written by $\Omega = e^{z_1+\cdots+z_m}dz_1 \wedge \cdots \wedge dz_m$. Hence we can take γ as $\gamma = (1, \dots, 1)$. A moment map is given by $\mu(w) = \frac{1}{2}(|w_1|^2, \dots, |w_m|^2)$ and a moment polytope is given by

$$\Delta = \{y \in \mathbb{R}^m \mid \langle y, \lambda_i\rangle \geq 0,\ i = 1, \dots, m\}\,,$$

where $\lambda_i := e_i$, the i-th standard base, and then we have $\langle\gamma, \lambda_i\rangle = 1$ for all i. The real form of $\mathbb{C}^m$ is $\mathbb{R}^m$ and note that $\mathbb{R}^m$ can be constructed by gluing from 2^m-copies of Δ. Take one vector $\zeta = (\zeta_1, \dots, \zeta_m) \in \mathbb{R}^m$ satisfying $\langle\gamma, \zeta\rangle > 0$ and $c = 0$. Since

$$c(t) = c - 2\pi\langle\gamma, \zeta\rangle t = -2\pi t\langle\gamma, \zeta\rangle = -2\pi t\sum_{j=1}^{m}\zeta_j$$

and $\Delta_{\zeta,c(t)} = \{\, y \in \Delta \mid \langle y, \zeta\rangle = c(t)\,\}$, we have

$$M^{\sigma}_{\zeta,c(t)} = (\mu|_{\mathbb{R}^m})^{-1}(\Delta_{\zeta,c(t)})$$

$$= \left\{ x \in \mathbb{R}^m \;\middle|\; \sum_{j=1}^m \zeta_j x_j^2 = -4\pi t \sum_{j=1}^m \zeta_j \right\},$$

and $L'_{\zeta,c(t)}$, the image of $F_{\zeta,c(t)} : L_{\zeta,c} \to \mathbb{C}^m$, is given by

$$L'_{\zeta,c(t)} = \left\{ (x_1 e^{2\pi i \zeta_1 s}, \dots, x_m e^{2\pi i \zeta_m s}) \in \mathbb{C}^m \;\middle|\; 0 \le s \le 1, \right.$$
$$\left. \sum_{j=1}^m \zeta_j x_j^2 = -4\pi t \sum_{j=1}^m \zeta_j,\ x = (x_1, \dots, x_m) \in \mathbb{R}^m \right\}.$$

This $L'_{\zeta,c(t)}$ coincides with V_t in Theorem 1.1 in [8], and Lee and Wang proved that V_t is Hamiltonian stationary and $\{V_t\}_{t\in\mathbb{R}}$ forms an eternal solution for Brakke flow. Hence our theorems can be considered as a kind of generalization of example of Lee and Wang to toric almost Calabi–Yau manifolds.

EXAMPLE 6.2. Let $M = K_{\mathbb{P}^2}$ be the total space of the canonical line bundle of $\mathbb{P}^2$. Then a moment polytope is given by $\Delta = \{ y \in \mathbb{R}^3 \mid \langle y, \lambda_i \rangle \ge \kappa_i,\ i = 1, \dots, 4 \}$, where

$$\lambda_1 = (0, 0, 1), \quad \lambda_2 = (1, 0, 1), \quad \lambda_3 = (0, 1, 1), \quad \lambda_4 = (-1, -1, 1)$$

and $\kappa_1 = \kappa_2 = \kappa_3 = 0$, $\kappa_4 = -1$. Of course, M is a toric almost Calabi–Yau manifold since we can take $\gamma = (0, 0, 1)$ so that $\langle \gamma, \lambda_i \rangle = 1$ for all i. For example, take $n = 1$, and take one vector and one constant as

$$\zeta = (3, 1, 5) \quad \text{and} \quad c = 5.$$

Then $\Delta_{\zeta,c(t)}$ is written by

$$\Delta_{\zeta,c(t)} = \{ y \in \Delta \mid \langle y, \zeta \rangle = 5 - 10\pi t \},$$

since $c(t) = c - 2\pi \langle \gamma, \zeta \rangle t$ and $t \ge 0$. We write each facet of Δ by $F_i := \{ y \in \Delta \mid \langle y, \lambda_i \rangle = \kappa_i \}$ for $i = 1, 2, 3, 4$.

By simple calculation, one can easily see the following.

- On $0 \le t < \frac{1}{5\pi}$, $\Delta_{\zeta,c(t)}$ intersects with F_2, F_3 and F_4, so $\Delta_{\zeta,c(t)}$ is a triangle.
- At $t = \frac{1}{5\pi}$, $\Delta_{\zeta,c(t)}$ across $(1, 0, 0)$, a vertex of Δ, and a topological change happens.
- On $\frac{1}{5\pi} < t < \frac{2}{5\pi}$, $\Delta_{\zeta,c(t)}$ intersects with F_1, F_2, F_3 and F_4, so $\Delta_{\zeta,c(t)}$ is a square.
- At $t = \frac{2}{5\pi}$, $\Delta_{\zeta,c(t)}$ across $(0, 1, 0)$, a vertex of Δ, and a topological change happens.
- On $\frac{2}{5\pi} < t < \frac{1}{2\pi}$, $\Delta_{\zeta,c(t)}$ intersects with F_1, F_2 and F_3, so $\Delta_{\zeta,c(t)}$ is a triangle.
- At $t = \frac{1}{2\pi}$, $\Delta_{\zeta,c(t)}$ is one point $\{(0, 0, 0)\}$, this means that $\Delta_{\zeta,c(t)}$ vanishes.

Hence a solution $\{L'_{\zeta,c(t)}\}_{t\in I}$ of a generalized Lagrangian mean curvature flow with singularities and topological changes exists for $t \in I = [0, \frac{1}{2\pi})$. It forms singularities and topological changes when $t = \frac{1}{5\pi}$ and $t = \frac{2}{5\pi}$, and vanishes when $t = \frac{1}{2\pi}$.

One can see the topology of $L_{\zeta,c(t)} = M^{\sigma}_{\zeta,c(t)} \times S^1$ (since now $T_\zeta \cong S^1$) by the same argument as explained in the proof of Proposition A.3 in [13]. In fact the topology of $M^{\sigma}_{\zeta,c(t)}$ is S^2 when $0 \le t < \frac{1}{5\pi}$, is T^2 when $\frac{1}{5\pi} < t < \frac{2}{5\pi}$, is S^2 when $\frac{2}{5\pi} < t < \frac{1}{2\pi}$.

Appendix A. In Section 4, we introduce the notion of the weighted Hamiltonian stationary. In this appendix, we explain the meaning of it. Let $(M, \omega, g, J, \Omega)$ be a $2m$-dimensional almost Calabi–Yau manifold with the function ψ defined by (9) and $F : L \to M$ be a Lagrangian immersion with the Lagrangian angle θ_F. Then we say that $F : L \to M$ is a weighted Hamiltonian stationary if $\Delta_f \theta_F = 0$. Here f is a function on L defined by $f := -mF^*\psi$ and Δ_f is the weighted Laplacian on Riemannian manifold (L, F^*g) defined by $\Delta_f u := \Delta u + \langle \nabla u, \nabla f \rangle$, where Δ is the standard Laplacian on L with respect to a metric F^*g.

Let $\tilde{g} := e^{2\psi} g$ be a conformal rescaling of g on M, then we get a new Riemannian manifold $(M, \tilde{g})$. For an immersion $F : L \to M$, we define a weighted volume functional Vol_ψ by

$$\mathrm{Vol}_\psi(F) := \int_L dV_{F^*\tilde{g}}\,,$$

where $dV_{F^*\tilde{g}}$ is the volume form on L with respect to a metric $F^*\tilde{g}$. Note that the relation between $dV_{F^*\tilde{g}}$ and dV_{F^*g} is given by

$$dV_{F^*\tilde{g}} = e^{mF^*\psi} dV_{F^*g} = e^{-f} dV_{F^*g}\,.$$

Then we consider a symplectic manifold (M, ω) with the weighted volume functional Vol_ψ. The following proposition is the meaning of the weighted Hamiltonian stationary.

PROPOSITION A.1. *A Lagrangian immersion $F : L \to M$ is weighted Hamiltonian stationary if and only if F is a critical point of the weighted volume functional* Vol_ψ *along Hamiltonian deformations with respect to ω.*

PROOF. Let $\{F_t : L \to M\}_t$ be a Hamiltonian deformation of F with Hamiltonian functions $\{h_t : L \to \mathbb{R}\}_t$, that is, $F_0 = F$ and

$$\omega\left(\frac{\partial F}{\partial t}, \cdot\right) = -dh_t\,. \tag{14}$$

If L is non-compact, we assume that each h_t has a compact support. Then the first variation of Vol_ψ at F along $\{F_t : L \to M\}_t$ is derived by the first variation formula as

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} \mathrm{Vol}_\psi(F_t) &= \frac{d}{dt}\bigg|_{t=0} \int_L e^{mF_t^*\psi} dV_{F_t^*g} \\
&= -\int_L g\left(e^{mF^*\psi} H - m e^{mF^*\psi} \nabla\psi^\perp, \frac{\partial F}{\partial t}\bigg|_{t=0}\right) dV_{F^*g} \\
&= -\int_L g\left(H - m\nabla\psi^\perp, \frac{\partial F}{\partial t}\bigg|_{t=0}\right) e^{-f} dV_{F^*g}\,.
\end{aligned}$$

Next we remember the definition of the generalized mean curvature vector filed K, see (8), and use the equation (10), then we have

$$-\int_L g\left(H - m\nabla\psi^{\perp}, \frac{\partial F}{\partial t}\bigg|_{t=0}\right)e^{-f}dV_{F^*g} = -\int_L g\left(K, \frac{\partial F}{\partial t}\bigg|_{t=0}\right)e^{-f}dV_{F^*g}$$
$$= -\int_L g\left(J\nabla\theta_F, \frac{\partial F}{\partial t}\bigg|_{t=0}\right)e^{-f}dV_{F^*g} .$$

Since the equation (14) is equivalent to $\frac{\partial F}{\partial t} = J\nabla h_t$, we have

$$-\int_L g\left(J\nabla\theta_F, \frac{\partial F}{\partial t}\bigg|_{t=0}\right)e^{-f}dV_{F^*g} = -\int_L g(J\nabla\theta_F, J\nabla h_0)e^{-f}dV_{F^*g}$$
$$= -\int_L \langle d\theta_F, dh_0\rangle_{F^*g}e^{-f}dV_{F^*g}$$
$$= -\int_L (\Delta_f\theta_F)h_0 e^{-f}dV_{F^*g}$$
$$= -\int_L (\Delta_f\theta_F)h_0 dV_{F^*\tilde{g}} .$$

In the third equality, we use the another definition of $\Delta_f u = \delta_f(du)$, where δ_f is the formal adjoint of d with respect to a weighted measure $e^{-f}dV_{F^*g}$. One can easily show that $\delta_f(du) = \Delta u + \langle\nabla u, \nabla f\rangle_{F^*g}$. Now we can take any h_0, thus it is clear that the first variation of Vol_ψ at F along all Hamiltonian deformations is zero if and only if $\Delta_f\theta_F = 0$. □

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