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ON THE GEOMETRY OF THE CROSS-CAP IN MINKOWSKI 3-SPACE AND BINARY DIFFERENTIAL EQUATIONS

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Abstract. We initiate in this paper the study of the geometry of the cross-cap in Minkowski 3-space $\mathbb{R}^3_1$. We distinguish between three types of cross caps according to their tangential line being spacelike, timelike or lightlike. For each of these types, the principal plane which is generated by the tangential line and the limiting tangent direction to the curve of self-intersection of the cross-cap plays a key role. We obtain special parametrisations for the three types of cross-caps and consider their affine properties. The pseudo-metric on the cross-cap changes signature along a curve and the singularities of this curve depend on the type of the cross-cap. We also study the binary differential equations of the lightlike curves and of the principal curves in the parameters space and obtain their topological models as well as the configurations of their solution curves.

1. Introduction.

Whitney showed that maps $\mathbb{R}^2 \to \mathbb{R}^3$ can have a stable local singularity under smooth changes of coordinates in the source and target. A model of this local singularity under these changes of coordinates is given by $(x, y) \mapsto (x, xy, y^2)$. The image of this map is a singular surface called a cross-cap (it is also called a surface with a pinch-point or a Whitney umbrella).

Because the cross-cap is a stable singular surface, it is natural to seek to understand its geometry. The extrinsic differential geometry of the cross-cap in the Euclidean 3-space is investigated in [6, 8, 9, 10, 13, 20, 25, 27], and in [13] the authors considered its intrinsic properties. For instance, it is shown in [6, 27] that there are generically two types of cross-caps, labelled *hyperbolic cross-cap* and *elliptic cross-cap* (Figure 1 left and center), and these are characterised by the singularity type of their parabolic set in the source (see also §4 for another characterisation and [18, 20] for applications to the geometry of surfaces in $\mathbb{R}^4$). The change from an elliptic to a hyperbolic cross-cap occurs at a parabolic cross-cap, Figure 1 right.

We initiate in this paper the study of the geometry of the cross-cap in the Minkowski 3-space $\mathbb{R}^3_1$. At the cross-cap point, the tangent plane to the surface degenerates to a line which we call, following [13], the *tangential line* of the cross-cap. This line together with the limiting direction to the curve of self-intersection of the cross-cap span the *principal plane*

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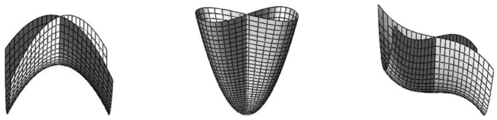

FIGURE 1. Hyperbolic, elliptic and parabolic cross-caps.

of the cross-cap. The type (spacelike, timelike, lightlike) of the tangential line and of the principal plane play a key role in this paper.

We obtain in §3 parametrisations of the cross-cap in simplified forms using smooth changes of coordinates in the source and Lorentzian transformations in the target. From these parametrisations we get pairs of quadratic forms (Q_1, Q_2) in (x, y). We show in §4 that the $\mathcal{G} = GL(2,\mathbb{R}) \times GL(2,\mathbb{R})$-class of (Q_1, Q_2) determines if the cross-cap is elliptic, hyperbolic or parabolic and obtain affine invariant properties of the cross-cap (such us its Dupin indicatrices and its focal conic).

In §5, we study the induced metric on the cross-cap and determine the generic topological configurations of the lightlike curves in the source (which we call pre-lightlike curves). We study in §6 the lines of principal curvature in the source (which we call pre-principal curves). The pre-lightlike and pre-principal curves are solutions binary differential equations (BDEs). These are implicit differential equations written in the form

$$a(x, y)dx^2 + 2b(x, y)dxdy + c(x, y)dy^2 = 0\,.$$

Some of the configurations in §5 and in §6 are obtained in §7 using the blowing-up technique on general BDEs. The configurations of the solution curves of the BDEs in this papers have all been checked using Montesinos program [17]. The solutions of a BDE form a pair of foliations in the region in $\mathbb{R}^2$ where $(b^2 - ac)(x, y) > 0$ (the set where $(b^2 - ac)(x, y) = 0$ is the *discriminant* of the BDE). In all the figures in this paper, we draw one foliation in continuous line and the other in dashed line. The discriminant curve is drawn in thick black.

The results in this paper are summarised below, where the triple (a, b, c) in the fourth and last columns are the coefficients of the BDE of the pre-lightlike curves and pre-principal curves. The discriminant of the pre-lightlike curves (resp. pre-principal curves) is denoted by PLD (resp. $PLPL$) and its column indicates the singularity type of its defining equation.

The tangential line is timelike or spacelike

Tg. line	Pr. plane	PLD	Pre-lightlike curves	$PLPL$	Pre-principal curves
Timelike	Timelike	A_1^+	$(-x^2 - y^2, 0, 1)$	A_3^-	$(y^3, -\frac{1}{2}x + by^2, y)$
Spacelike	Timelike	A_1^-	$(x^2 - y^2, 0, 1)$	A_3^-	$(y^3, -\frac{1}{2}x + by^2, y)$
	Spacelike	A_1^-	$(-x^2 + y^2, 0, 1)$	A_3^+	$(-y^3, -\frac{1}{2}x, y)$
	Lightlike	A_1^-	$(xy + y^3, 0, 1)$	A_4	$(xy + y^3 + y^4, -\frac{1}{2}x, y)$

The tangential line is lightlike

Pr. plane	d_0	PLD	Pre-Lightlike curves	$PLPL$	Pre-Principal curves
Timelike	$D_4^{\pm}$	A_2	$(x^2, \pm y, x)$	$X_{1,0}$	$(xy, -x^2, -3xy + y^2)$ $(xy, -x^2, 3xy + 3y^2)$ $(xy, x^2, -3xy + y^2)$
Lightlike	D_4^-	A_2	$(y^2, \frac{1}{2}x, y)$	$X_{1,1}^{\pm}$	$(x^2 + xy^2 + y^3, xy, \mp x^2 - 2y^2)$

Above, d_0 is the distance squared function with centre the cross-cap-point, Tg. is short for tangential and Pr. for principal. See Table 2 for the configurations of the pre-lightlike curves and Table 3 for those of the pre-principal curves.

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2. Preliminaries. The *Minkowski space* $(\mathbb{R}^3_1, \langle , \rangle)$ is the vector space $\mathbb{R}^3$ endowed with the metric given by the pseudo-scalar product

$$\langle \mathbf{u}, \mathbf{v} \rangle = -u_0 v_0 + u_1 v_1 + u_2 v_2 ,$$

for any vectors $\mathbf{u} = (u_0, u_1, u_2)$ and $\mathbf{v} = (v_0, v_1, v_2)$ in $\mathbb{R}^3$ (see for example [21], p55). We say that a non-zero vector $\mathbf{u} \in \mathbb{R}^3_1$ is *spacelike* if $\langle \mathbf{u}, \mathbf{u} \rangle > 0$, *lightlike* if $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ and *timelike* if $\langle \mathbf{u}, \mathbf{u} \rangle < 0$. The norm of a vector $\mathbf{u} \in \mathbb{R}^3_1$ is defined by $\|\mathbf{u}\| = \sqrt{|\langle \mathbf{u}, \mathbf{u} \rangle|}$.

The vector product in $\mathbb{R}^3_1$ of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} -\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_0 & u_1 & u_2 \\ v_0 & v_1 & v_2 \end{vmatrix}$$

where $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ is the canonical basis in $\mathbb{R}^3$.

A plane $\{\mathbf{v} \in \mathbb{R}^3_1 \mid \langle \mathbf{v}, \mathbf{u} \rangle = 0\}$ is said to be spacelike (resp. lightlike, timelike) if its pseudo-normal vector $\mathbf{u}$ is timelike (resp. lightlike, spacelike).

We have the following pseudo-spheres in $\mathbb{R}^3_1$ with centre $p \in \mathbb{R}^3_1$ and radius $r > 0$,

$$\begin{aligned} H^2(p, -r) &= \{\mathbf{u} \in \mathbb{R}^3_1 \mid \langle \mathbf{u} - p, \mathbf{u} - p \rangle = -r^2\} , \\ S^2_1(p, r) &= \{\mathbf{u} \in \mathbb{R}^3_1 \mid \langle \mathbf{u} - p, \mathbf{u} - p \rangle = r^2\} , \\ LC^*(p) &= \{\mathbf{u} \in \mathbb{R}^3_1 \mid \langle \mathbf{u} - p, \mathbf{u} - p \rangle = 0\} . \end{aligned}$$

We denote by $H^2(-r)$ and $S^2_1(r)$ the pseudo-spheres centred at the origin in $\mathbb{R}^3_1$.

We consider the set $\mathcal{C}$ of smooth map-germs $\mathbb{R}^2, 0 \to \mathbb{R}^3_1$ with a cross-cap singularity at the origin endowed with the Whitney C^∞-topology. We say that a property of the cross-cap is *generic* if it is satisfied by map-germs in a residual subset of $\mathcal{C}$.

Let $\phi : U \subset \mathbb{R}^2 \to \mathbb{R}^3_1$ be a representative of a map-germ with a cross-cap singularity at the origin and denote its image by M. Let

$$E = \langle \phi_x, \phi_x \rangle , \quad F = \langle \phi_x, \phi_y \rangle , \quad G = \langle \phi_y, \phi_y \rangle$$

denote the coefficients of the first fundamental form of M (the subscripts denote partial derivatives).

We label the *Pre-Locus of Degeneracy* (PLD) the set of point $(x, y) \in U$ where $(F^2 - EG)(x, y) = 0$, and by the *Locus of Degeneracy* (LD) its image by ϕ. The LD is the locus of points on M where the induced metric is degenerate.

We decompose $U = U_1 \cup U_2 \cup PLD$, where $\phi(U_1)$ is the Riemannian part of M and $\phi(U_2)$ is its Lorentzian part. One can define the de Sitter Gauss map $U_1 \to S^2_1(1)$ on the Lorentzian part of the surface and the hyperbolic Gauss map $U_2 \to H^2(-1)$ on its Riemannian part. Both maps are given by $\mathbf{N} = \phi_x \times \phi_y / ||\phi_x \times \phi_y||$. The map $A_p(\mathbf{u}) = -d\mathbf{N}_p(\mathbf{u})$ is a self-adjoint operator on $M \setminus LD$. We denote by

$$\begin{aligned} l &= -\langle \mathbf{N}_x, \phi_x \rangle = \langle \mathbf{N}, \phi_{xx} \rangle, \\ m &= -\langle \mathbf{N}_x, \phi_y \rangle = \langle \mathbf{N}, \phi_{xy} \rangle, \\ n &= -\langle \mathbf{N}_y, \phi_y \rangle = \langle \mathbf{N}, \phi_{yy} \rangle \end{aligned}$$

the coefficients of the second fundamental form on $M \setminus LD$. At points on the LD, we multiply the above coefficients by $||\phi_x \times \phi_y||$ and set

$$\bar{l} = \langle \phi_x \times \phi_y, \phi_{xx} \rangle , \quad \bar{m} = \langle \phi_x \times \phi_y, \phi_{xy} \rangle , \quad \bar{n} = \langle \phi_x \times \phi_y, \phi_{yy} \rangle .$$

The Gaussian curvature K of M at $p = \phi(q) \in M \setminus LD$ is given by

$$K(q) = \det(A_p) = \frac{ln - m^2}{EG - F^2}(q) .$$

The (closure of) the pre-parabolic set is defined as the set of points in U where $(\bar{l}\bar{n} - \bar{m}^2)(q) = 0$. Its image under ϕ is defined as the parabolic set on M (this is the closure of the set of points where the Gaussian curvature vanishes).

We are interested in the singularities of the zero set of a germ of a function $f : \mathbb{R}^2, 0 \to \mathbb{R}, 0$. For this reason, we consider the action of the contact group $\mathcal{K}$ on the set of germs of functions $f : \mathbb{R}^2, 0 \to \mathbb{R}, 0$. Two germs, at the origin, of functions f, g are $\mathcal{K}$-equivalent if $g(x, y) = k(x, y) f(h^{-1}(x, y))$, where $h : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ is a germ of a diffeomorphism and $k : \mathbb{R}^2, 0 \to \mathbb{R}$ is a germ of a function not vanishing at the origin. If two germs are $\mathcal{K}$-equivalent, then their zero sets are diffeomorphic. We shall use in this paper representatives of the following $\mathcal{K}$ orbits of this action (see [1]):

$$\begin{array}{ll} A_k & x^2 \pm y^{k+1}, \quad k \geq 0 \\ D_k & x^2 y \pm y^{k-1}, \quad k \geq 4 \\ X_{1,0} & \begin{cases} x^4 + ax^2y^2 + y^4, & a^2 - 4 \neq 0 \\ xy(x^2 + bxy + y^2), & b^2 - 4 < 0 \end{cases} \\ X_{1,1} & x^4 \pm x^2y^2 + ay^5, \quad a \neq 0. \end{array}$$

In the complex case, the singularity $X_{1,0}$ has one normal form given by $x^4 + ax^2y^2 + y^4$, $a^2 - 4 \neq 0$. However, this form does not include the case when the quartic has two real roots. This case is represented by the normal form $xy(x^2 + bxy + y^2)$, $b^2 - 4 < 0$.

3. Special parametrisations of the cross-cap. It is shown in [27] that a parametrisation of a cross-cap in Euclidean 3-space can be taken, by a suitable choice of coordinates in the source and Euclidean transformations the target, in the form

$$\phi(x, y) = (x, xy + p(y), y^2 + ax^2 + q(x, y)), \tag{1}$$

where $p \in \mathcal{M}^3(y)$ and $q \in \mathcal{M}^3(x, y)$ ($\mathcal{M}(u)$ denotes the maximal ideal in the ring of germs of functions in u). We have the following result on parametrisations of a cross-cap in $\mathbb{R}^3_1$ depending on the type of its tangential line (spacelike, timelike or lightlike). We remark that the proof is different from that given in [27] for the parametrisation (1) of the cross-cap in Euclidean 3-space.

THEOREM 3.1. *Let $\phi : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$ be a germ of a parametrisation of a cross-cap in $\mathbb{R}^3_1$. There is a Lorentzian transformation T in $\mathbb{R}^3_1$ and a germ of a diffeomorphism $\rho : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ such that $T \circ \phi \circ \rho$ is expressed as follows:*

(a) *When the tangential line is timelike:*

$$(x, y^2 + p(x), q(x, y)). \tag{2}$$

(b) *When the tangential line is spacelike:*

$$(y^2 + p(x), x, q(x, y)). \tag{3}$$

(c) *When the tangential line is lightlike:*

$$(x, ax + y^2 + p(x), bx + q(x, y)) \tag{4}$$

where $p \in \mathcal{M}^2(x)$, $q \in \mathcal{M}^2(x, y)$ and a, b are constants with $a^2 + b^2 = 1$.

PROOF. (a) When the tangential line is timelike, we can make a Lorentzian transformation in the target and take it to be along $(1, 0, 0)$. We can then write the new parametrisation in the form $\phi(x, y) = (x, f(x, y), g(x, y))$, where f and g are germs of functions with zero 1-jets. As the singularity is a cross-cap, we have $\frac{\partial^2 f}{\partial y^2}(0, 0) \neq 0$ or $\frac{\partial^2 g}{\partial y^2}(0, 0) \neq 0$. We can suppose that $\frac{\partial^2 f}{\partial y^2}(0, 0) \neq 0$ (if it vanishes, we make the Lorenztian transformation $(u, v, w) \mapsto (u, w, v)$ in the target to get back to the case where it does not vanish).

We consider $f(x, y)$ as a 1-parameter unfolding of the function $f(0, y)$. Since $\frac{\partial^2 f}{\partial y^2}(0, 0) \neq 0$, $f(x, y)$ is $\mathcal{R}^+$-equivalent to the germ y^2, that is, there exist a germ of a diffeomorphism $H : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ of the form $H(x, y) = (h(x), k(x, y))$ and a germ of a function c such that

$$y^2 = f(h(x), k(x, y)) + c(x)$$

(see [1]). We have $\phi \circ H(x, y) = (h(x), y^2 - c(x), g(h(x), k(x, y)))$.

Let K be a change of coordinates in the source with $K(s, t) = (x, y) = (h^{-1}(s), t)$. Then

$$\phi \circ H \circ K(s, t) = (u, v^2 - c(h^{-1}(s)), g(s, k(h^{-1}(s), t)))\,.$$

We revert back to the original notation and write x for s and y for t, so that the cross-cap is parametrised in the form

$$(x, y^2 + p(x), q(x, y))\,,$$

where p and q are germs of functions with zero 1-jets.

(b) We can take, by applying a Lorentzian transformation in the target if necessary, the spacelike tangential direction along $(0, 1, 0)$ and write $\phi(x, y) = (f(x, y), x, g(x, y))$, where f and g are germs of functions with zero 1-jets. As for the case (a), we have $\frac{\partial^2 f}{\partial y^2}(0, 0) \neq 0$ or $\frac{\partial^2 g}{\partial y^2}(0, 0) \neq 0$. We can suppose that $\frac{\partial^2 f}{\partial y^2}(0, 0) \neq 0$ (if it vanishes, we make the Lorentzian transformation $(u, v, w) \mapsto (u \cosh(t) + w \sinh(t), v, u \sinh(t) + w \cosh(t))$ in the target, for any $t \neq 0$, and get back to the case where $\frac{\partial^2 f}{\partial y^2}(0, 0) \neq 0$). We then proceed as for the case (a).

(c) When the tangential direction is lightlike we can take a parametrisation of the surface in the form $\phi(x, y) = (x, ax + f(x, y), bx + g(x, y))$, where f and g have zero 1-jets and a, b are constants with $a^2 + b^2 = 1$. As for the case (a), we can assume that $\frac{\partial^2 f}{\partial y^2}(0, 0) \neq 0$ (if it is, we make the Lorenztian transformation $(u, v, w) \mapsto (u, w, v)$ and it becomes non-zero). We then proceed as in (a). $\square$

In the rest of the paper, we write the homogeneous parts of degree n of p and q in Theorem 3.1 in the form

$$p_{n0}x^n\,,$$
$$q_{n0}x^n + q_{n1}x^{n-1}y + \cdots + q_{nn}y^n\,,$$

with $q_{21} \neq 0$ as the surface has a cross-cap singularity.

Let η denotes the null direction at the cross-cap (for the parametrisations in Theorem 3.1, $\eta = (0, 1)$). The *principal plane* at the cross-cap parametrised by $\phi : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$ is defined in [13] as the plane spanned by the tangential direction at the cross-cap together with the direction $\eta^2\phi(0, 0)$. The principal plane is in fact the plane spanned by the tangential direction and the limiting tangent direction to the curve of self-intersection of the cross-cap surface at the cross-cap point ([13], Proposition 9).

PROPOSITION 3.2. (i) *When the tangential direction is timelike the limiting tangent direction to the curve of self-intersection is spacelike, so the principal plane is timelike.*

(ii) *When the tangential direction is spacelike the principal plane is lightlike* (*resp. spacelike*, *timelike*) *if and only if the limiting tangent direction to the curve of self-intersection is lightlike* (*resp. spacelike*, *timelike*). *For a parametrisation as in* (3), *the limiting tangent direction to the curve of self-intersection is lightlike if and only if* $q_{22}^2 - 1 = 0$. *It is spacelike* (*resp. timelike*) *if and only if* $q_{22}^2 - 1 > 0$ (*resp.* $q_{22}^2 - 1 < 0$).

(iii) *When the tangential direction is lightlike the principal plane* (*resp. limiting tangent direction to the curve of self-intersection*) *is either timelike or lightlike* (*resp. spacelike*). *The principal plane is lightlike if and only if the limiting tangent direction to the curve of self-intersection is pseudo-orthogonal to the tangential direction. For a parametrisation as in* (4), *the principal plane is lightlike if and only if* $bq_{22} + a = 0$.

PROOF. As observed above, the principal plane is generated by the tangential direction and the limiting tangent direction to the curve of self-intersection of the cross-cap surface at the cross-cap point ([13], Proposition 9). The property of the principal plane and of the limiting tangent direction to the curve of self-intersection being spacelike, timelike or lightlike is invariant under Lorentzian transformations and reparametrisations of the surface. Therefore, we can take a parametrisation of the cross-cap as in (2), (3) or (4).

A point p is on the curve of self-intersection if there exists two distinct points (x_1, y_1) and (x_2, y_2) in the source such that $p = \phi(x_1, y_1) = \phi(x_2, y_2)$. For ϕ as in (2), (3) or (4), one can show that $x_1 = x_2 = 0$ and $y_2 = -y_1$, so the double point curve in the source is parametrised by $(0, y)$ and its image, the curve of self-intersection on the cross-cap, is given by $\phi(0, y)$. For ϕ as in (3), the limiting tangent direction to the curve of self-intersection is along $(1, 0, q_{22})$ and for ϕ as in (2) and (4) it is along $(0, 1, q_{22})$.

The pseudo-normal direction to the principal plane is along $(0, -q_{22}, 1)$ for ϕ as in (2), along $(q_{22}, 0, 1)$ for ϕ as in (3) and along $(-aq_{22} + b, -q_{22}, 1)$ for ϕ as in (4). The result follows by comparing the type (spacelike, timelike or lightlike) of this vector to that of the limiting tangent direction to the curve of self-intersection. For example, for (4) the pseudo normal to the principal plane is parallel to

$$\mathbf{v} = (1, a, b) \times (0, 1, q_{22}) = (-aq_{22} + b, -q_{22}, 1)$$

and we have

$$\begin{aligned}\langle \mathbf{v}, \mathbf{v} \rangle &= -(-aq_{22} + b)^2 + q_{22}^2 + 1 \\ &= (1 - a^2)q_{22}^2 + 2abq_{22}^2 + 1 - b^2 \\ &= b^2 q_{22}^2 + 2abq_{22}^2 + a^2 \\ &= (bq_{22} + a)^2 \geq 0 \,.\end{aligned}$$

Thus, the principal plane is timelike or lightlike. It is lightlike if and only if $bq_{22} + a = \langle (1, a, b), (0, 1, q_{22}) \rangle = 0$. □

4. Affine properties of the cross-cap. We take the parametrisations of the cross-cap as in (2), (3) and (4). Let π denotes the projection of the tangent space of $\mathbb{R}^3_1$ to the quotient space $\mathbb{R}^3_1/\mathbb{R}.\phi_x(0, 0)$. We associate to the parametrisations (2), (3) and (4) the pair of quadratic

TABLE 1. The $\mathcal{G}$-orbits of pairs of quadratic forms.

$\mathcal{G}$-orbit	Name
$(y^2 + x^2, xy)$	hyperbolic
$(y^2 - x^2, xy)$	elliptic
(x^2, xy)	parabolic
$(x^2 \pm y^2, 0)$	inflection
$(x^2, 0)$	degenerate inflection
$(0, 0)$	degenerate inflection

forms

$$j^2(\pi \circ \phi) = (Q_1(x, y), Q_2(x, y)) = (y^2 + p_{20}x^2, q_{20}x^2 + q_{21}xy + q_{22}y^2)\,.$$

We consider the action of the group $\mathcal{G} = GL(2,\mathbb{R}) \times GL(2,\mathbb{R})$ on the pairs of binary forms $j^2(\pi \circ \phi) = (Q_1, Q_2)$, where $GL(2,\mathbb{R})$ denotes the general linear group (see for example [11]). If $H = (h, k) \in \mathcal{G}$, then $H.\,(Q_1, Q_2) = k \circ (Q_1 \circ h^{-1}, Q_2 \circ h^{-1})$. The $\mathcal{G}$-orbits are listed in Table 1.

LEMMA 4.1. *For ϕ as in* (2), (3) *or* (4), *$j^2(\pi \circ \phi) = (Q_1, Q_2)$ is*

$$\begin{aligned} \textit{hyperbolic} &\Leftrightarrow q_{21}^2 p_{20} + (q_{22}p_{20} - q_{20})^2 > 0\,,\\ \textit{elliptic} &\Leftrightarrow q_{21}^2 p_{20} + (q_{22}p_{20} - q_{20})^2 < 0\,,\\ \textit{parabolic} &\Leftrightarrow q_{21}^2 p_{20} + (q_{22}p_{20} - q_{20})^2 = 0\,. \end{aligned}$$

PROOF. The action in the target by $(u, v) \mapsto (u, v - q_{22}u)$ gives

$$(Q_1, Q_2) \sim_{\mathcal{G}} (y^2 + p_{20}x^2, (q_{20} - p_{20}q_{22})x^2 + q_{21}xy)\,.$$

The action by $(x, y) \mapsto (x, y - \frac{q_{20}-p_{20}q_{22}}{q_{21}}x)$ in the source followed by an action in the target of the form $(u, v) \mapsto (u - \alpha v, \frac{1}{q_{21}}v)$ gives

$$(Q_1, Q_2) \sim_{\mathcal{G}} \left(y^2 + \frac{q_{21}^2 p_{20} + (q_{22}p_{20} - q_{20})^2}{q_{21}^2}x^2, xy\right)$$

and the result follows. □

A cross-cap is hyperbolic/elliptic/parabolic if the $\mathcal{K}$-singularity type of its pre-parabolic set is $A_1^+/A_1^-/A_2$ ([6, 27]). We call here a parabolic cross-cap a cross-cap whose pre-parabolic set has an $A_{\geq 2}$-singularity. The singularity of the pre-parabolic set depends on the contact of the surface with planes, so is affine invariant (in particular, they do not depend on the metric in the ambient space). We have the following for the cross-cap in $\mathbb{R}^3_1$; see [20] for an analogous result for a cross-cap in Euclidean 3-space.

PROPOSITION 4.2. *The cross-cap is hyperbolic/elliptic/parabolic if and only if its associated pair of quadratic forms (Q_1, Q_2) is elliptic/hyperbolic/parabolic.*

PROOF. The 2-jet of $\bar{m}^2 - \bar{n}\bar{l}$ for the three cross-caps in Theorem 3.1 is given by

$$-4q_{21}(p_{20}q_{21}x^2 + 2(q_{22}p_{20} - q_{20})xy - q_{21}y^2)\,.$$

We have $q_{21} \neq 0$, so the discriminant of the above quadratic form is given, up to a non-zero factor, by $q_{21}^2 p_{20} + (q_{22}p_{20} - q_{20})^2$. The result follows by Lemma 4.1. (When $q_{21}^2 p_{20} + (q_{22}p_{20} - q_{20})^2 = 0$, the parabolic set has an A_k-singularity, with $k \geq 2$, as the $j^2(\bar{m}^2 - \bar{n}\bar{l})$ cannot be identically zero.) □

In view of Proposition 4.2, we label the property hyperbolic/elliptic/parabolic of a cross-cap as its affine property. This property can also be detected by considering the following curves in the source. We consider the intersection of the cross-cap parametrised by $\phi : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$ with the planes $f(x, y, z) = ax + by + cz - d = 0$, $d \neq 0$, parallel to a plane containing the unique tangent direction to the cross-cap. We analyse the limit of the curves $f \circ \phi(x, y) = 0$ as d tends to zero and call them the *Dupin indicatrices in the source* associated to the tangent plane $ax + by + cz = 0$. These are approximated by the zero set of the 2-jet of $f \circ \phi$.

Consider the case of the timelike tangential direction with ϕ as in (2) (the other two cases follow similarly). Then $a = 0$ and the Dupin indicatrices in the source are given by

$$bQ_1(x, y) + cQ_2(x, y) - d = 0\,.$$

We identify a quadratic form $Q = Ax^2 + Bxy + Cy^2$ by its coefficients $(A : B : C)$ in the projective plane $\mathbb{R}P^2$. We denote by Γ the conic $\{Q : B^2 - 4AC = 0\}$ of degenerate quadratic forms. Then the $\mathcal{G}$-orbit of a pair of quadratic forms (Q_1, Q_2) is completely determined by the pencil $bQ_1(x, y) + cQ_2(x, y)$ in $\mathbb{R}P^2$. The pair (Q_1, Q_2) is hyperbolic (resp. elliptic) if and only if its associated pencil intersects the conic Γ in 2 (resp. 0) points. It is parabolic if the pencil is tangent to Γ.

PROPOSITION 4.3. (i) *At a hyperbolic cross-cap, the Dupin indicatrices in the source associated to any tangent plane are hyperbolae.*

(ii) *At an elliptic cross-cap, there are two tangent planes whose associated Dupin indicatrices is a pair of parallel lines. The remaining Dupin indicatrices are either hyperbolae or ellipses.*

(iii) *At an parabolic cross-cap, there is a unique tangent plane whose associated Dupin indicatrices is a pair of parallel lines. The remaining Dupin indicatrices are all hyperbolae.*

REMARK 4.4. The height function on the cross-cap along a normal direction (a, b, c) is given by $\langle \phi(x, y), (a, b, c) \rangle$. On a hyperbolic cross-cap the singularities of the height function in any normal direction is A_1^-. On an elliptic cross-cap, there are two normal directions along which the singularity of the height function is $A_k, k \geq 2$, and for the remaining directions it is A_1^+ or A_1^-. On a parabolic cross-cap, there is a unique normal direction along which the singularity of the height function is $A_k, k \geq 2$, and for the remaining directions it is A_1^- ([6, 27]). As the 2-jet of $\langle \phi(x, y), (a, b, c) \rangle$ is the pencil associated to the pair of quadratic forms (Q_1, Q_2), the statement of Proposition 4.3 is a reformulation of the result on the singularities height functions in terms of the Dupin indicatrices in the source.

We turn now to an aspect of the contact of a cross-cap parametrised by $\phi : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$ with pseudo-spheres in $\mathbb{R}^3_1$. This contact is measured by the $\mathcal{K}$-singularity type of the pseudo-distance squared function $d_u : \mathbb{R}^2, 0 \to \mathbb{R}$, given by

$$d_u(x, y) = \langle \phi(x, y) - u, \phi(x, y) - u \rangle ,$$

with $u \in \mathbb{R}^3_1$. Varying u gives the family d of pseudo-distance squared functions. We consider the $\mathcal{K}$-singularity type of d_u at the cross-cap point, that is, at the origin in the xy-plane.

The plane orthogonal to the tangential line at the cross-cap is called the *normal plane to the cross-cap*. It is not difficult to show that the function d_u is singular at the origin if and only if u is on the normal plane to the cross-cap.

THEOREM 4.5. (i) *When the tangential line is spacelike or timelike the singularities of d_u are always of type A_k, $k \geq 1$. There is a conic in the normal plane of the cross-cap, called* the focal conic, *where the singularities of d_u are of type A_k, $k \geq 2$. The focal conic contains the cross-cap point and is as follows:*

$$\begin{aligned} &\textit{an ellipse} &&\Leftrightarrow\ j^2(\pi \circ \phi) \textit{ is elliptic},\\ &\textit{a hyperbola} &&\Leftrightarrow\ j^2(\pi \circ \phi) \textit{ is hyperbolic},\\ &\textit{a parabola} &&\Leftrightarrow\ j^2(\pi \circ \phi) \textit{ is parabolic}. \end{aligned}$$

(ii) *When the tangential line is lightlike the singularity of d_0 at the cross-cap point is of type D_4 if and only if*

$$\Gamma = (q_{21}^2 - 4q_{20}q_{22})b^2 - 4(p_{20}q_{22} + q_{20})ab - 4a^2 p_{20} \neq 0, \tag{5}$$

with the singularity being of type D_4^- if $\Gamma > 0$ and D_4^+ if $\Gamma < 0$. The singularities of d_u, $u \neq 0$, are of type A_k, $k \geq 1$. The focal conic is

$$\begin{aligned} &\textit{an isolated point} &&\Leftrightarrow\ j^2(\pi \circ \phi) \textit{ is elliptic},\\ &\textit{a pair of transverse lines} &&\Leftrightarrow\ j^2(\pi \circ \phi) \textit{ is hyperbolic},\\ &\textit{a double line} &&\Leftrightarrow\ j^2(\pi \circ \phi) \textit{ is parabolic}. \end{aligned}$$

PROOF. (i) We consider only the timelike tangential direction and take ϕ as in (2). The case of the spacelike tangential direction follows similarly and we get the same conditions which identify the focal conic. We write $u = (u_0, u_1, u_2)$, so

$$\begin{aligned} d_u(x, y) = &-(u_0 - x)^2 + (u_1 - y^2 - p_{20}x^2 - p(x))^2 \\ &+\big(u_2 - q_{20}x^2 - q_{21}xy - q_{22}y^2 - q\,(x, y)\big)^2 . \end{aligned}$$

We have $j^1 d_u(x, y) = -u_0^2 + u_1^2 + u_2^2 + 2u_0 x$, so d_u is singular at the origin if and only if $u_0 = 0$, that is, if and only if u is on the normal plane of the cross-cap.

We take now $u_0 = 0$. Then the 2-jet of d_u, without the constant terms, is given by

$$-(1 + 2u_1 p_{20} + 2u_2 q_{20})x^2 - 2u_2 q_{21} xy - 2(u_1 - u_2 q_{22})y^2 . \tag{6}$$

The quadratic form (6) can never vanish identically as $q_{21} \neq 0$, so the singularities of d_u are always of type A_k, $k \geq 1$.

The singularity of d_u is of type A_k, $k \geq 2$, if and only if the quadratic form (6) is degenerate, that is,

$$(7) \qquad 2p_{20}u_1^2 + 2(q_{20} + p_{20}q_{22})u_1u_2 + \left(2q_{20}q_{22} - \frac{1}{2}q_{21}^2\right)u_2^2 + u_1 + u_2q_{22} = 0\,.$$

The above equation is that of a non-degenerate conic (the focal conic), in the normal plane $(0, u_1, u_2)$. The discriminant of its quadratic part is

$$\delta = p_{20}q_{21}^2 + (q_{20} - p_{20}q_{22})^2\,.$$

The focal conic is a parabola if and only if $\delta = 0$, equivalently, (Q_1, Q_2) is parabolic (Lemma 4.1).

When $\delta \neq 0$, the linear terms in (7) can be removed by a translation to obtain a new equation in the form

$$2p_{20}U_1^2 + 2(q_{20} + p_{20}q_{22})U_1U_2 + \left(2q_{20}q_{22} - \frac{1}{2}q_{21}^2\right)U_2^2 = \frac{1}{8}\frac{q_{21}^2}{p_{20}q_{21}^2 + (q_{20} - p_{20}q_{22})^2}\,.$$

This is a hyperbola if and only if $p_{20}q_{21}^2 + (q_{20} - p_{20}q_{22})^2 > 0$ and an ellipse if and only if $p_{20}q_{21}^2 + (q_{20} - p_{20}q_{22})^2 < 0$. The interpretation of these inequalities in terms of the pair of quadratic forms (Q_1, Q_2) is given in Lemma 4.1. (It is worth observing that for a spacelike tangential direction, the focal conic is tangent to a lightlike line if and only if the limiting tangent direction to the curve of self-intersection is lightlike.)

(ii) When the tangential direction is lightlike, we take ϕ as in (4), so that

$$\begin{aligned} d_u = &-(u_0 - x)^2 + (u_1 - ax - y^2 - p_{20}x^2 - p(x))^2 \\ &+(u_2 - bx - q_{20}x^2 - q_{21}xy - q_{22}y^2 - q(x, y))^2\,. \end{aligned}$$

We have $j^1d_u(x, y) = -u_0^2 + u_1^2 + u_2^2 + 2(u_0 - u_1a - u_2b)x$, so d_u is singular at the origin if and only if $u_0 - u_1a - u_2b = -\langle(1, a, b), (u_0, u_1, u_2)\rangle = 0$, that is, if and only if u is on the lightlike normal plane of the cross-cap.

We suppose now that $u_0 = u_1a + u_2b$. Then the 2-jet of d_u, without the constant term, is given by

$$(8) \qquad -2(u_1p_{20} + u_2q_{20})x^2 - 2u_2q_{21}xy - 2(u_2q_{22} + u_1)y^2\,.$$

The quadratic form (8) vanishes identically if and only if $u_1 = u_2 = 0$, that is, $u = 0$. Then the 3-jet of d_0, without the constant term, is given by $2x((ap_{20} + bq_{20})x^2 + bq_{21}xy + (a + bq_{22})y^2)$. Thus, d_0 has a D_4-singularity if and only if

$$\Gamma = (q_{21}^2 - 4q_{20}q_{22})b^2 - 4(p_{20}q_{22} + q_{20})ab - 4a^2p_{20} \neq 0\,.$$

Suppose that $u \neq 0$. Then the singularity of d_u is of type A_k, $k \geq 1$. It is of type A_k, $k \geq 2$ if and only if the quadratic form (8) is degenerate, that is, if and only if

$$(9) \qquad -4p_{20}u_1^2 - 4(q_{20} + p_{20}q_{22})u_1u_2 + (q_{21}^2 - 4q_{20}q_{22})u_2^2 = 0\,.$$

The focal conic given by (9) is an isolated point/a pair of transverse lines/a double line if and only if $q_{21}^2 p_{20} + (q_{22}p_{20} - q_{20})^2 < 0/ > 0/ = 0$, and by Lemma 4.1 this is equivalent to $j^2(\pi \circ \phi)$ being elliptic/hyperbolic/parabolic. (Observe that (9) cannot vanish identically as $q_{21} \neq 0$.) □

We have the following result about more degenerate singularities of the distance squared functions (these are not affine invariant).

PROPOSITION 4.6. *When the tangential direction is timelike or a spacelike, there are generically* 1, 3 *or* 5 *points on the focal conic where d_u generically has an A_3-singularity at the cross-cap point. One of these points is always at the cross-cap point.*

When the tangential direction is lightlike, the cross-cap point is a D_4-singularity of d_0 when condition (5) *is satisfied. Generically, there is one point u_i $(i = 1, 2)$ on each line of the focal conic where d_{u_i} has generically an A_3-singularity at the cross-cap point.*

The A_2 and A_3-singularities of d_u, $u \neq 0$, at the cross-cap point are versally unfolded by the family d. The singularity of d_0 at the cross-cap point is not versally unfolded by the family d.

PROOF. We take u on the focal conic. Then the 2-jet of d_u is a perfect square L^2. The singularity of d_u is of type A_k, $k \geq 3$ if and only if L divides C, where C is the homogeneous cubic part of d_u. When the tangential direction is timelike, we take $L = u_2q_{21}x + 2(u_1 - u_2q_{22})y$ (up to a constant factor) if $u \neq 0$, see the proof of Theorem 4.5. (When $u = 0$, we take $L = -x^2$ and show that d_0 has always an A_3-singularity.) The cubic C divides L if and only if $C(2(u_1 - u_2q_{22}), -u_2q_{21}) = 0$, that is, if and only if

$$\begin{aligned} 8p_{30}u_1^4 & -8(3p_{30}q_{22} + q_{30})u_1^3u_2 + 4(q_{21}q_{31} - 6p_{30}q_{22}^2 - 6q_{30}q_{22})u_1^2u^2 \\ & +2(4q_{31}q_{21}q_{22} - q_{21}^2q_{32} - 12q_{30}q_{22}^2 - 4p_{30}q_{22}^3)u_1u_2^3 \\ & +(q_{21}^3q_{33} + 4q_{31}q_{21}q_{22}^2 - 2q_{32}q_{21}^2q_{22} - 8q_{30}q_{22}^3)u_2^4 = 0\,. \end{aligned}$$

This is a homogeneous quartic in u_1, u_2, so is generically a union of 0, 2, 4 real lines meeting at the origin. Thus, the singularity of d_u is of type A_k, $k \geq 3$ if and only if $u = (0, u_1, u_2)$ is a point of intersection of these lines with the conic (7), so we get generically 1, 3 or 5 such points, and those away from the origin generically give singularities of type A_3. We proceed similarly for the case when the tangential direction is spacelike.

For the lightlike tangential direction, proceeding as above, we show that d_u has an A_k, $k \geq 3$-singularity if and only if u is a point of intersection of the conic (9) with the following non-homogeneous quartic

$$\begin{aligned} Q(u_0, u_2) = & (-8q_{30}q_{22}^3 + 4q_{21}q_{31}q_{22}^2 + q_{33}q_{21}^3 - 2q_{21}^2q_{32}q_{22})u_2^4 \\ & +(-2q_{21}^2q_{32} + 8q_{21}q_{31}q_{22} - 8p_{30}q_{22}^3 - 24q_{30}q_{22}^2)u_1u_2^3 \\ & +(-24p_{30}q_{22}^2 - 24q_{30}q_{22} + 4q_{21}q_{31})u_1^2u_2^2 \\ & +(-24p_{30}q_{22} - 8q_{30})u_2u_1^3 - 8p_{30}u_1^4 \\ & +(-2q_{21}^2bq_{22}^2 + 8bq_{20}q_{22}^3 + 8ap_{20}q_{22}^3 + 2q_{21}^2aq_{22})u_2^3 \end{aligned}$$

$$+(24ap_{20}q_{22}^2 + 2q_{21}^2 a - 6q_{21}^2 bq_{22} + 24bq_{20}q_{22}^2)u_2^2 u_1$$
$$+(24ap_{20}q_{22} - 4q_{21}^2 b + 24bq_{20}q_{22})u_2 u_1^2 + (8bq_{20} + 8ap_{20})u_1^3 .$$

Denote by $u_0 = \lambda_i u_2$, $i = 1, 2$, the lines of the conic (9). Then $Q(\lambda_i u_2, u_2) = u_2^3(A_i u_2 + B_i)$, $i = 1, 2$, where A_i and B_i depend on λ_i, p_{j0} and q_{jk}, $j, k = 2, 3$. Generically, $A_i \neq 0$ and $B_i \neq 0$, so we have a single point $u \neq 0$ on each line of the focal conic where the singularity of d_u at the origin is generically of type A_3. (When $u = 0$, the singularity of type D_4.)

The statement about the versality of the family d follows by standard calculations and are omitted (see for example [8] for detailed calculations for the cross-cap in Euclidean 3-space). □

REMARK 4.7. Porteous defined *ridge curves* on a smooth surface in Euclidean 3-space as the (closure of the) locus of points where the distance squared function d_u has an A_3-singularity for some $u \in \mathbb{R}^3$. We can define in a similar way ridge curves on surfaces (smooth or singular) in $\mathbb{R}^3_1$. Then Proposition 4.6 gives the number of ridge curves on the cross-cap passing through the cross-cap point.

5. The first fundamental form. Let E, F, G denote, as in §2, the coefficients of the first fundamental form of a cross-cap parametrised by ϕ. The induced pseudo-metric on the cross-cap is given by $ds^2 = Edx^2 + 2Fdxdy + Gdy^2$. It is Riemannian at points where $F^2 - EG < 0$, Lorentzian at points where $F^2 - EG > 0$ and degenerate at points where $F^2 - EG = 0$. The Pre-Locus of Degeneracy (PLD) is defined as the set of points (x, y) in the source where $(F^2 - EG)(x, y) = 0$.

PROPOSITION 5.1. (i) *If the tangential line is timelike* (*resp. spacelike*), *then the PLD has an A_1^+* (*resp. A_1^-*)*-singularity.*

(ii) *If the tangential line is lightlike*, *then the PLD has a singularity more degenerate than A_1. It has precisely an A_2-singularity if and only if the distance squared function d_0 has a D_4-singularity.*

PROOF. (i) We compute the coefficients of the first fundamental form. Suppose that the tangential line is timelike and take a parametrisation of the surface as in (2). Then,

$$\begin{aligned} j^2E &= -1 + 4(p_{20}^2 + q_{20}^2)x^2 + 4q_{21}q_{20}xy + q_{21}^2 y^2 ,\\ j^2F &= 2q_{20}x^2 q_{21} + (4p_{20} + q_{21}^2 + 4q_{20}q_{22})xy + 2q_{21}q_{22}y^2 ,\\ j^2G &= q_{21}^2 x^2 + 4q_{22}q_{21}xy + 4(1 + q_{22}^2)y^2 \end{aligned} \tag{10}$$

so that

$$j^2(F^2 - EG) = q_{21}^2 x^2 + 4q_{21}q_{22}xy + 4(1 + q_{22}^2)y^2 .$$

The discriminant of the above quadratic form is $-16q_{21}^2$ and is strictly negative, so the PLD has a Morse singularity of type A_1^+, i.e., it is an isolated point.

When the tangential line is spacelike we take a parametrisation of the surface as in (3). Then,

$$
\begin{aligned}
j^2E &= 1 - 4(p_{20}^2 - q_{20}^2)x^2 + 4q_{21}q_{20}xy + q_{21}^2y^2\,,\\
j^2F &= 2q_{20}q_{21}x^2 + (4q_{20}q_{22} - 4p_{20} + q_{21}^2)xy + 2q_{21}q_{22}y^2\,,\\
j^2G &= q_{21}^2x^2 + 4q_{22}q_{21}xy - 4(1 - q_{22}^2)y^2
\end{aligned}
\tag{11}
$$

and

$$
j^2(F^2 - EG) = -q_{21}^2x^2 - 4q_{21}q_{22}xy + 4(1 - q_{22}^2)y^2\,.
$$

The discriminant of the above quadratic form is $16q_{21}^2$ so the PLD has a Morse singularity of type A_1^-, i.e., it is a pair of transverse crossing curves.

(ii) For a cross-cap as in (4) with a lightlike tangential direction, we have

$$
\begin{aligned}
j^2E &= 4(ap_{20} + bq_{20})x + 2bq_{21}y + 2(3ap_{30} + 3bq_{30} + 2p_{20}^2 + 2q_{20}^2)x^2\\
&\quad +4(bq_{31} + q_{20}q_{21})xy + (q_{21}^2 + 2bq_{32})y^2\,,\\
j^2F &= bq_{21}x + 2(a + bq_{22})y + (bq_{31} + 2q_{20}q_{21})x^2\\
&\quad +(2bq_{32} + 4p_{20} + q_{21}^2 + 4q_{20}q_{22})xy + (2q_{21}q_{22} + 3bq_{33})y^2\\
j^2G &= q_{21}^2x^2 + 4q_{22}q_{21}xy + 4(1 + q_{22}^2)y^2\,.
\end{aligned}
\tag{12}
$$

We have $j^2(F^2 - EG) = (bq_{21}x + 2(a + bq_{22})y)^2$ which is not identically zero as $q_{21} \neq 0$ and $a^2 + b^2 = 1$. Therefore, the singularity of the PLD is of type $A_{\geq 2}$. If $b \neq 0$, we change x by $x - 2(a + bq_{22})/(bq_{21})y$, and the coefficients of y^3 in the 3-jet of $F^2 - EG$ becomes

$$
-\frac{1}{q_{21}b^3}((q_{21}^2 - 4q_{20}q_{22})b^2 - 4(p_{20}q_{22} + q_{20})ab - 4a^2p_{20})\,.
$$

It is not zero if and if only if condition (5) is satisfied. Thus, the PLD has an A_2-singularity if and only if d_0 has a D_4-singularity at the cross-cap point (see Theorem 4.5). If $b = 0$, then $a^2 = 1$ and the relevant part of $j^3(F^2 - EG)$ is $4y^2 - 4ap_{20}q_{21}^2x^3$. This is an A_2-singularity if and only if condition (5) is satisfied (that is, $p_{20} \neq 0$ in this case). □

REMARK 5.2. The singularities of the PLD can be explained geometrically as follows. There is a pencil of planes containing the tangential line of the cross-cap which are tangent to the cross-cap. When the tangential direction is timelike all the planes in the pencil are timelike so all nearby tangent planes to the surface are timelike, i.e., the PLD must be an isolated point. When the tangential direction is spacelike (resp. lightlike), there are two (resp. one) tangent planes in the pencil which are lightlike and this indicates that there are two (resp. one) branches of the PLD.

We consider the integral curves of the lightlike directions on a cross-cap, which we label the lightlike curves. (These are the isotropic geodesics, i.e., those with identically zero length [22].) The lightlike curves are the images by the parametrisation ϕ of the solution curves of the binary quadratic differential equation (BDE)

$$
\omega : Gdy^2 + 2Fdxdy + Edx^2 = 0\,.
\tag{13}
$$

We identify a BDE by its coefficients and write $\omega = (G, F, E)$. We call the solutions of (13) the pre-lightlike curves in the source. There are two pre-lightlike curves at each point in the region of the plane mapped by ϕ to the Lorentzian region of the cross-cap and none at points mapped to its Riemannian region. The PLD, which is the *discriminant curve* of the BDE (13) (the discriminant curve of a BDE is the set of points where the equation determines a unique solution direction) separates the two regions. We have the following result about the generic configurations of the pre-lightlike curves at the cross-cap point. (See [15] for the generic configurations of the lightlike curves on a smooth surface.)

THEOREM 5.3. *The BDE* (13) *of the pre-lightlike curves of a cross-cap in* $\mathbb{R}^3_1$ *is topologically equivalent to one of the following normal forms.*

(i) *When the tangential direction is timelike:* $(-x^2 - y^2, 0, 1)$.

(ii) *When the tangential direction is spacelike and*

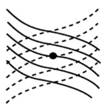
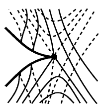
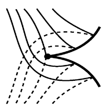
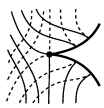
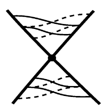
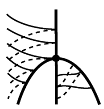
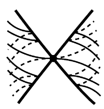
$$\begin{array}{ll} \textit{the principal plane is spacelike:} & (-x^2 + y^2, 0, 1)\,, \\ \textit{the principal plane is timelike:} & (x^2 - y^2, 0, 1)\,, \\ \textit{the principal plane is lightlike:} & (xy + y^3, 0, 1)^{(*)}\,. \end{array}$$

(iii) *When the tangential direction is lightlike and*

$$\begin{array}{ll} \textit{the principal plane is timelike:} & (x^2, y, x) \textit{ if } d_0 \textit{ has a } D^+\textit{-singularity}\,, \\ & (x^2, -y, x) \textit{ if } d_0 \textit{ has a } D^-\textit{-singularity}\,, \\ \textit{the principal plane is lightlike:} & (y^2, \tfrac{1}{2}x, y)\,. \end{array}$$

See Table 2 *for figures.* ($(*)$*: a generic condition is need, see proof for details.*)

TABLE 2. Pre-lightlike curves on cross-caps in $\mathbb{R}^3_1$.

Tg. line \ Pr. plane	Timelike	Lightlike	Spacelike
Timelike			
Lightlike	 (D_4^+)  (D_4^-)		
Spacelike			

PROOF. (i) We are interested in the local topological configurations of the solution curves of the BDE (13). Thus, multiplying equation (13) by a germ of a non-zero function or making smooth changes of coordinates in the plane will not alter the topological type of the configuration.

The 2-jets of the coefficients of the first fundamental form of a cross-cap as in (2) are given in (10). The change of y by $y + \lambda x$, with $\lambda = -q_{21}q_{22}/(2(1 + q_{22}^2))$, changes dy to $dy + \lambda dx$ and eliminates the term xy in the coefficient of dy^2 in equation (13). Dividing the new BDE by the coefficient of dx^2 gives a new BDE with coefficients having the following 2-jets:

$$
\begin{aligned}
A &= \mathrm{Coeff}(dy^2) = -\frac{q_{21}^2}{(1+q_{22}^2)}x^2 - 4(1+q_{22}^2)y^2\,,\\
B &= \mathrm{Coeff}(dxdy) = \frac{2q_{21}}{(1+q_{22}^2)^2}(2p_{20}q_{22}^3 - 2q_{20}q_{22}^2 + (2p_{20} + q_{21}^2)q_{22} - 2q_{20})x^2\\
&\quad - \frac{2}{1+q_{22}^2}(4q_{20}q_{22}^3 + (4p_{20} - q_{21}^2)q_{22}^2 + 4q_{20}q_{22} + q_{21}^2 + 4p_{20})xy\,,\\
C &= \mathrm{Coeff}(dx^2) = 1\,.
\end{aligned}
$$

We change now x by $x + \alpha_1 x^2 y + \alpha_2 x y^2$ which changes dx to $(1 + 2\alpha_1 xy + \alpha_2 y^2)dx + (\alpha_1 x^2 + 2\alpha_2 xy)dy$ and get a new BDE with a 2-jet

$$
A dy^2 + (B + 2\alpha_1 x^2 + 4\alpha_2 xy)dxdy + (1 + 4\alpha_1 xy + 2\alpha_2 y^2)dx^2\,,
$$

with A, B as above. Clearly, we can choose α_1, α_2 so that the coefficient of $dxdy$ becomes zero. Then, dividing by the coefficient of dx^2 yields a BDE with a 2-jet

$$
\left(-\frac{q_{21}^2}{(1+q_{22}^2)}x^2 - 4(1+q_{22}^2)y^2\right)dy^2 + dx^2\,.
$$

It is shown in Proposition 2.7 in [4] that BDEs with one of the coefficients not vanishing at the origin can be written locally in the form $dx^2 + f(x, y)dy^2 = 0$. When the discriminant of such a BDE, which is given by $f(x, y) = 0$, has a Morse singularity (of type A_1^+ or A_1^-) then the BDE is topologically determined by the 2-jet of its coefficients and is topologically equivalent to $dx^2 + (\varepsilon_1 y^2 + \varepsilon_2 x^2)dy^2 = 0$, where $\varepsilon_1 = \mathrm{sign}(f_{yy}(0, 0))$ and $\varepsilon_2 = \pm$ (see Theorem 2.7 in [2]). Such BDEs are called *Morse Type 1* $A_1^\pm$ (Type 1 for the case when at least one of the coefficients of the BDE is not zero at the origin, and $A_1^\pm$ for the type of the Morse singularity of the discriminant of the BDE). For each type A_1^- and A_1^+ there are two topological models depending on the sign of $f_{yy}(0, 0)$. If $f_{yy}(0, 0) < 0$ (resp. $f_{yy}(0, 0) > 0$), then the BDE is called of *Morse Type 1* $A_1^\pm$ *saddle type* (resp. *focus type*) as two folded saddles (resp. focus) singularities appear in the bifurcation in a generic 1-parameter families of such BDEs, [2].

It follows that the BDE of the lightlike curves on a cross-cap with a timelike tangential direction is topologically equivalent to $dx^2 + (-y^2 - x^2)dy^2 = 0$ (Morse Type 1 A_1^- saddle type). Thus, the configuration of the pre-lightlike curves are as in Table 2.

We observe that the topological type $dx^2 + (y^2 + x^2)dy^2 = 0$ (Morse Type 1 A_1^- focus type) with empty solution curves does not occur in this case because the surface is Lorentzian away from the singular point, so there are two lightlike curves passing through each point.

(ii) Proceeding as in (i) and using the 2-jets of the coefficients of the first fundamental form in (10), we can transform the 2-jet of the BDE (13) to

$$(q_{21}^2 x^2 + 4q_{22}q_{21}xy - 4(1 - q_{22}^2)y^2)dy^2 + dx^2\,.$$

As the discriminant of the BDE (13) has an A_1^- singularity (Proposition 5.1), it follows by Theorem 2.7 in [2] that the BDE (13) is topologically equivalent to $dx^2 \pm (x^2 - y^2)dy^2 = 0$ if and only if $q_{22}^2 - 1 \neq 0$, that is, the principal plane is not lightlike (Proposition 3.2). When this is the case, we can reduce further the 2-jet of the BDE (13) to

$$\left(4(q_{22}^2 - 1)y^2 - \frac{q_{21}^2}{(q_{22}^2 - 1)}x^2\right)dy^2 + dx^2\,.$$

We have the Morse Type 1 A_1^- saddle type model $dx^2 + (-x^2 + y^2)dy^2 = 0$ (resp. the focus type model $dx^2 + (x^2 - y^2)dy^2 = 0$) if and only if $q_{22}^2 - 1 > 0$ (resp. $q_{22}^2 - 1 < 0$), that is, if the principal plane is spacelike (resp. timelike). The configuration of the pre-lightlike curves are as in Table 2.

When $q_{22}^2 - 1 = 0$ (the principal plane is lightlike), we can reduce the 3-jet of the BDE (following the procedure in (i)) to one of the form $g(x, y)dy^2 + dx^2$, with

$$\begin{aligned} g(x, y) = {} & q_{21}^2 x^2 + 4q_{22}q_{21}xy \\ & + 2q_{21}q_{31}x^3 + 4(q_{31}q_{22} + q_{21}q_{32})x^2 y \\ & + 2(4q_{32}q_{22} + 3q_{21}q_{33})xy^2 + 12q_{22}q_{33}y^3\,. \end{aligned}$$

Then, by Proposition 3.3 in [26], the BDE (13) is topologically equivalent to $(xy + y^3)dy^2 + dx^2 = 0$ if and only if $g_{xy}(0, 0)g_{yyy}(0, 0) \neq 0$, that is, if and only if $q_{33} \neq 0$ (as $q_{22}^2 = 1$ and $q_{21} \neq 0$).

A BDE which is topologically equivalent to $(xy + y^3)dy^2 + dx^2 = 0$ is said to have a *Non-transverse Morse singularity* ([26]) because its discriminant has a Morse A_1^--singularity and the unique direction determined by the BDE at the singular point is tangent to one of the branches of the discriminant.

(iii) Here the 2-jets of the coefficients of the BDE (13) are as in (12). We observe that all the coefficients of the BDE vanish at the origin and the 1-jet of the BDE is given by

$$2((bq_{21}x + (a + bq_{22})y))dxdy + (4(ap_{20} + bq_{20})x + 2bq_{21}y)dx^2\,. \tag{14}$$

Suppose that $a + bq_{22} \neq 0$, that is, the principal plane is not lightlike. For simplicity we take $b \neq 0$ (the case $b = 0$ follows similarly and is omitted). Then the linear change of coordinates $(x, y) \mapsto (\lambda x, \mu x + \eta y)$ with $\lambda = -2(a+bq_{22}))/(bq_{21})\mu$ gives a new BDE with 1-jet

$$\frac{8\mu}{bq_{21}}(a+bq_{22})^2\big(-\eta^2 y dx dy + \mu^2 q_{21}^2\big((q_{21}^2 - 4q_{20}q_{22})b^2 - 4(p_{20}q_{22}+q_{20})ab - 4a^2 p_{20}\big)x dx^2\big)\,.$$

We can choose μ and η to reduce the 1-jet of the BDE (13) to $xdx^2 \pm ydxdy$ if and only if condition (5) is satisfied, that is, if and only if d_0 has a D_4-singularity at the cross-cap point (see Theorem 4.5). We suppose that this is the case. Then the 1-jet of the BDE (13) is equivalent to $xdx^2 + ydxdy$ (resp. $xdx^2 - ydxdy$) if $\Gamma < 0$ (resp. $\Gamma > 0$), with Γ as in (5) (see Theorem 4.5).

Using Theorem 7.1 in Section 7, we deduce that the BDE (13) is generically topologically equivalent to $xdx^2+2ydxdy+x^2dy^2 = 0$ if $\Gamma < 0$ and to $xdx^2-2ydxdy+x^2dy^2 = 0$ if $\Gamma > 0$.

If the principal plane is lightlike, the 1-jet (14) becomes

$$2bq_{21}xdxdy + (4b(q_{20} - p_{20}q_{22})x + 2bq_{21}y)dx^2\,.$$

If $q_{20} - p_{20}q_{22} \neq 0$, the change of coordinate $(x, y) \mapsto \big(\frac{q_{21}}{q_{22}p_{20}-q_{20}}x, x + y\big)$ and multiplication by a non-zero constant reduces the 1-jet to $xdxdy + ydx^2$. The coefficient of y^2 in Coeff(dy^2) of the new equation is equal to $2(1 + q_{22}^2)(q_{22}p_{20} - q_{20})^2/(2bq_{21}^3)$ so is never zero. Therefore, by Theorem 3.4 in [19], the BDE is topologically equivalent to $y^2dy^2 + xdxdy + ydx^2$. If $q_{20} - p_{20}q_{22} = 0$, the coefficient of y^2 in Coeff(dy^2) is equal to $2(1 + q_{22}^2)/(bq_{21})$ so the BDE is also topologically equivalent to $y^2dy^2 + xdxdy + ydx^2$. (Observe that when the principal plane is lightlike, $\Gamma = b^2q_{21}^2 > 0$, so the singularity of d_0 is of type D_4^-.) □

REMARK 5.4. Asymptotic directions can be characterised in terms of the contact of the surface with lines. This contact is affine invariant [3] so does not depend on the metric in $\mathbb{R}^3$. This means that the asymptotic curves on a cross-cap in Minkowski 3-space are the same as those on the cross-cap in Euclidean 3-space (see [25] for their study).

6. The lines of principal curvature. When the shape operator A_p has real eigenvalues at a point $p \in M \setminus LD$, we call them the *principal curvatures* and their associated eigenvectors the *principal directions* of M at p. (There are always two principal curvatures at each point on the Riemannian part of M but this is not always true on its Lorentzian part.) The lines of principal curvature, which are the integral curves of the principal directions, are the images by the parametrisation ϕ of the solutions of the BDE

$$(Gm - Fn)dy^2 + (Gl - En)dydx + (Fl - Em)dx^2 = 0\,. \tag{15}$$

One can extend the lines of principal curvature across the LD as follows ([15]). As equation (15) is homogeneous in l, m, n, we substitute these by $\bar{l}, \bar{m}, \bar{n}$. This substitution does not alter the pair of foliations on $M \setminus LD$. The new equation is defined on the LD and defines the same pair of foliations associated to the de Sitter (resp. hyperbolic) Gauss map on the

Riemannian (resp. Lorentzian) part of M. The extended lines of principal curvature are the images by ϕ of the solution curves of the BDE

$$(G\bar{m} - F\bar{n})dy^2 + (G\bar{l} - E\bar{n})dydx + (F\bar{l} - E\bar{m})dx^2 = 0\,. \tag{16}$$

We call the solutions of the BDE (16) the *pre-lines of principal curvature* and label its discriminant the *Pre-Lightlike Principal Locus* ($PLPL$). The image of the $PLPL$ by ϕ is labelled the *Lightlike Principal Locus* (LPL) (see [14, 15] for smooth Lorentzian surfaces and smooth surfaces with varying signature metric).

The $PLPL$ is the zero set of the function

$$(G\bar{l} - E\bar{n})^2 - 4(G\bar{m} - F\bar{n})(F\bar{l} - E\bar{m})\,, \tag{17}$$

and we have the following about its singularities.

PROPOSITION 6.1. (i) *If the tangential line is timelike, then the $PLPL$ has an A_3^--singularity.*

(ii) *If the tangential line is spacelike, then the $PLPL$ has: an A_3^--singularity if and only if the principal plane is timelike; an A_3^+-singularity if and only if the principal plane is spacelike; generically an A_4-singularity if the principal plane is lightlike.*

For both spacelike and timelike tangential lines, when the $PLPL$ has an A_3^--singularity, its two branches are tangent to the double point curve in the source.

(iii) *Suppose that the cross-cap is not parabolic. If the tangential line is lightlike and the principal plane is timelike, then the $PLPL$ has generically and $X_{1,0}$-singularity with two or four real branches. One of the branches has generically an ordinary tangency with the double point curve. When the principal plane is lightlike, the $PLPL$ has an $X_{1,1}^{\pm}$-singularity ($+$ for a hyperbolic cross-cap and $-$ for an elliptic one). The double point curve is tangent to the cusp component of the $PLPL$.*

PROOF. (i) We take a parametrisation of the cross-cap as in (2) and compute the relevant jets of $\bar{l}, \bar{m}, \bar{n}$ and find that the 3-jet of the $PLPL$ is given by $4q_{21}^2x^2 + 8q_{21}q_{31}x^3 - 24q_{21}q_{33}xy^2$. Therefore, its singularity is of type $A_{\geq 3}$. We eliminate the term xy^2 by a change of coordinates of the form $x \mapsto x + ay^2$ and find that the 4-jet of the $PLPL$ is $\mathcal{K}$-equivalent to $4q_{21}^2(x^2 - 16(1 + q_{22}^2)y^4)$, so the $PLPL$ has an A_3^--singularity.

(ii) We parametrise the cross-cap as in (3), and show, by a similar calculation to the case (i), that the 4-jet of the $PLPL$ is $\mathcal{K}$-equivalent to $4q_{21}^2(x^2 - 16(1 - q_{22}^2)y^4)$. This is an A_3^- (resp. A_3^+) if and only if $1 - q_{22}^2 > 0$ (resp. $1 - q_{22}^2 < 0$) (see Proposition 3.2 for interpretation in terms of the principal plane).

When $1 - q_{22}^2 = 0$, the principal plane is lightlike (Proposition 3.2) and the 5-jet of the $PLPL$ is $\mathcal{K}$-equivalent to $4q_{21}^2(x^2 + 96q_{33}y^5)$. Thus, the singularity of the $PLPL$ is of type A_4 if and only if $q_{33} \neq 0$.

(iii) We parametrise the cross-cap as in (4). The 4-jet of the $PLPL$ is given by

$$16q_{21}x[q_{21}(4(ap_{20}+bq_{20})^2+q_{21}^3b^2p_{20})x^3+6bq_{21}^2(b(q_{20}+q_{22}p_{20})+2ap_{20})yx^2 \\ +3q_{21}(4p_{20}(a+bq_{22})^2+b^2q_{21}^2)xy^2 \\ +4(a+bq_{22})(2(q_{22}p_{20}-q_{20})(a+bq_{22})+bq_{21}^2)y^3]\,.$$

Here, the $PLPL$ has an $X_{1,0}$-singularity and consists of two or four curves intersecting transversally at the origin if and only if

$$(q_{22}p_{20}-q_{20})^2+q_{21}^2p_{20}\neq 0 \text{ and } (a+bq_{22})(2(q_{22}p_{20}-q_{20})(a+bq_{22})+bq_{21}^2)\neq 0\,.$$

Observe that the first condition above means that the cross-cap is not parabolic. The quartic of the 4-jet of the $PLPL$ has always a real root with tangent direction $x=0$, so it is tangent to the double point curve. A calculation shows that the tangency is ordinary if and only if the coefficient of y^5 in the Taylor expansion of the $PLPL$ is not zero, that is, if and only if

$$\Lambda=(-2q_{21}(q_{22}^2+1)+3q_{33}(bq_{22}+a))(2(q_{22}p_{20}-q_{20})(bq_{22}+a)+bq_{21}^2)\neq 0\,. \tag{18}$$

When $a+bq_{22}=0$, the principal plane is lightlike and the 4-jet of the $PLPL$ becomes

$$16b^2q_{21}^2x^2((q_{21}^2p_{20}+4(q_{22}p_{20}-q_{20})^2)x^2-6q_{21}(q_{22}p_{20}-q_{20})xy+3q_{21}^2y^2)\,,$$

with $b\neq 0$. We can make a linear change of coordinate of the form $y\mapsto y+\alpha x$ to reduce this 4-jet to

$$16b^2q_{21}^2x^2((q_{21}^2p_{20}+(q_{22}p_{20}-q_{20})^2)x^2+3q_{21}^2y^2)\,.$$

Then the coefficient of y^5 is given by $128(1+q_{22}^2)q_{21}^3b$. Therefore the singularity of the $PLPL$ is of type $X_{1,1}^{\pm}$ (i.e., it is $\mathcal{R}$-equivalent to $x^4\pm x^2y^2+a_0y^5$, $a_0\neq 0$) provided that $q_{21}^2p_{20}+(q_{22}p_{20}-q_{20})^2\neq 0$, that is, provided that the cross-cap is not parabolic. Observe that the singularity of the $PLPL$ is $\mathcal{R}$-equivalent to $x^4+x^2y^2+a_0y^5$ (resp. $x^4-x^2y^2+a_0y^5$) if and only if the cross-cap is hyperbolic (resp. elliptic). Then the $PLPL$ consists of a cusp (resp. a cusp and two transverse lines). The limiting tangent direction to the cusp is tangent to the double point curve in the source. □

We seek to determine the generic topological configurations of the pre-lines of principal curvature and their images on the cross-cap. We start with the cases where the tangential line is timelike or spacelike. Then the $PLPL$, which is the discriminant of the BDE (16), has an $A_3^{\pm}$-singularity when the principal plane is not lightlike and an A_4-singularity when it is (Proposition 6.1).

Suppose that the $PLPL$ has an $A_3^{\pm}$-singularity. For parametrisations of the surface as in Theorem 3.1, the 1-jet of the coefficients of the BDE (16) is $(0, b_0x, y)$, with $b_0=-1/2$. It follows by Proposition 3.2 in [25] that the 3-jet of the BDE (16) is equivalent to $(a_3y^3, b_0x+b_2y^2+b_3y^3, y)$, and Theorem 3.3 in [25] states that if the discriminant has an $A_3^{\pm}$-singularity, then this BDE is topologically equivalent to

$$(\mp y^3, b_0x+b_2y^2, y)\,,$$

with (b_0, b_2) a fixed value in an open region delimited by some exceptional curves in the b_0b_2-plane. The exceptional curves are the parabola $1+b_0-b_2^2=0$ and the lines $b_0=$

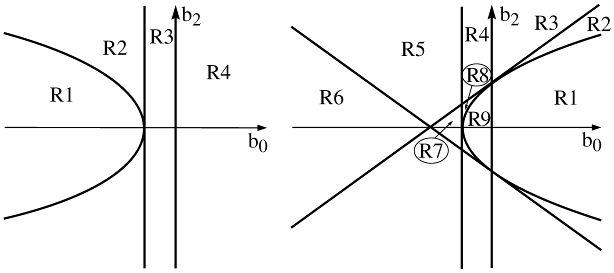


FIGURE 2. Partition of the (b_0, b_2)-plane, A_3^+ left and A_3^- right. The topological type for $(b_0, -b_2)$ is the same as that for (b_0, b_2).

$-1, b_0 = 0, 2 + b_0 - 2b_2 = 0, 2 + b_0 + 2b_2 = 0$ (Figure 2). There are 4 generic topological models when the singularity is A_3^+ and 9 when it is A_3^-.

THEOREM 6.2. (i) *Suppose that the tangential line of the cross-cap is timelike or spacelike and the $PLPL$ has an A_3^--singularity. Then the BDE* (16) *of the pre-lines of principal curvature is topologically equivalent to*

$$\left(y^3, -\frac{1}{2}x + b_2 y^2, y\right)$$

if $b_2 \neq 3/4, \pm\sqrt{2}/2$, where $b_2 = 3q_{33}/(4q_{21}\sqrt{1+q_{22}^2})$ when the tangential line is timelike and $b_2 = 3q_{33}/(4q_{21}\sqrt{1-q_{22}^2})$ when the tangential line is spacelike. The topological configuration of the pre-lines of principal curvature is as in

Table 3 first figure in the appropriate block if $|b_2| < \frac{\sqrt{2}}{2}$ (R9 in Figure 2 right),

Table 3 second figure if $-\frac{3}{4} < b_2 < -\frac{\sqrt{2}}{2}$ or $\frac{\sqrt{2}}{2} < b_2 < \frac{3}{4}$ (R8 in Figure 2 right),

Table 3 third figure (R4 in Figure 2 right) *if* $|b_2| > \frac{3}{4}$.

(ii) *Suppose that the tangential line of the cross-cap is spacelike and the $PLPL$ has an A_3^+-singularity. Then the BDE* (16) *is topologically equivalent to* $(-y^3, -\frac{1}{2}x, y)$ (R3 in Figure 2 left)*; see* Table 3.

PROOF. We deal with the case when the $PLPL$ has an A_3^--singularity and the tangential line is timelike. For a parametrisation ϕ of the cross-cap as in (2), we can reduce the 3-jet of the BDE (16) following similar steps to those in the proof of Theorem 5.3 to

$$\left(4(1+q_{22}^2)y^3, -\frac{1}{2}x + \frac{3q_{33}}{2q_{21}}y^2 + \beta y^3, y\right),$$

where β is a constant depending on $j^4\phi$. We divide the new BDE by $4(1+q_{22}^2)$ and make smooth changes of coordinates in the source of the form $x = 2\sqrt{1+q_{22}^2}X, \quad y = Y$. This results in a BDE with a 3-jet

$$(Y^3, b_0 X + b_2 Y^2 + \tilde{\beta} Y^3, Y), \tag{19}$$

where $b_0 = -1/2$, $b_2 = 3q_{33}/4q_{21}\sqrt{1+q_{22}^2}$ and $\tilde{\beta}$ is a new constant. The result follows by apply Theorem 3.3 in [25]. Similar calculations give the result for the case when the tangential line is spacelike and the $PLPL$ has an $A_3^{\pm}$-singularity. □

THEOREM 6.3. *Suppose that the tangential line is spacelike and the principal plane is lightlike* (*generically the $PLPL$ has an A_4-singularity*). *Then the BDE of the pre-lines of principal curvature is topologically equivalent to* $(xy + y^3 + y^4, -\frac{1}{2}x, y)$. *See* Table 3.

PROOF. The 1-jet of the BDE is given by $(0, -q_{21}x, 2q_{21}y)$. We make successive changes of coordinates (and divide by $2q_{21}$) to reduce the 4-jet of the BDE to the form $a(x, y)dy^2 - xdxdy + ydx^2$, where a has a zero 1-jet. We have $a_{21} = a_{xy}(0, 0) = 2q_{33}/q_{21}$ and the discriminant has an A_4-singularity if and only if $q_{33} \neq 0$. The result follows by applying Theorem 7.3 in Section 7. □

We turn now to the lightlike cross-cap.

THEOREM 6.4. *When the tangential line is lightlike and the principal plane is timelike, the BDE* (16) *of the pre-lines of principal curvature of a cross-cap parametrised as in* Theorem 3.1(c) *is topologically equivalent to one of the following normal forms:*

$(xy, -x^2, 3xy + 3y^2)$	*if* $\Gamma > 0$,	Table 3 first figure in appropriate box if $\lvert c\rvert > 2$
$(xy, -x^2, 3xy + y^2)$	*if* $\Gamma > 0$,	Table 3 second figure if $\lvert c\rvert < 2$
$(xy, x^2, -3xy + y^2)$	*if* $\Gamma < 0$,	Table 3 third figure

with $\Lambda\Gamma \neq 0$ and $c = \frac{2}{|\Gamma|} b(bq_{21}^2 + 2(bq_{22} + a)(-q_{20} + q_{22}p_{20}))\sqrt{|\Gamma|/(b^2q_{21}^2)} \neq 0$, *where Γ is as in* (5) *and Λ is as in* (18).

PROOF. The 2-jets of the coefficients of the BDE (16) are

$$\begin{aligned}
A &= \mathrm{Coeff}(dy^2) = 2bq_{21}^2x^2 + 4q_{21}(bq_{22} + a)xy\,,\\
B &= \mathrm{Coeff}(dxdy) = 8(bq_{20} + ap_{20})q_{21}x^2 + 4bq_{21}^2xy\,,\\
C &= \mathrm{Coeff}(dx^2) = -2bq_{21}^2p_{20}x^2 - 4q_{21}(2p_{20}bq_{22} + 3ap_{20} + bq_{20})xy\\
&\qquad -4(2(bq_{22} + a)(q_{22}p_{20} - q_{20}) + bq_{21}^2)y^2\,.
\end{aligned}$$

Suppose that $b \neq 0$ (the case $b = 0$ follows similarly). As the principal plane is timelike, $(bq_{22} + a) \neq 0$. We follow similar steps of the proof of Theorem 7.2 and make the linear change of coordinates $(x, y) \mapsto (\lambda x, x + \eta y)$ with $\lambda = -2(bq_{22} + a)/(bq_{21})$. We multiply the new equation by $-8\eta^3(bq_{22} + a)^2/b$ and take $\eta = \sqrt{|\Gamma|/(q_{21}^2b^2)}$. This gives a new BDE with a 2-jet

$$\begin{aligned}
&xydy^2 - 2x^2dxdy + (3xy + cy^2)dx^2 \quad \text{if } \Gamma > 0\\
&xydy^2 + 2x^2dxdy + (-3xy + cy^2)dx^2 \quad \text{if } \Gamma < 0
\end{aligned}$$

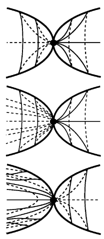
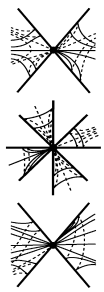
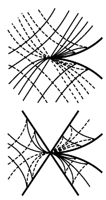
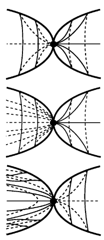
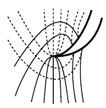
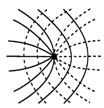
with c is as in the statement of the theorem. The result follows by applying Theorem 7.2 in Section 7. □

THEOREM 6.5. *Suppose the tangential line is lightlike and the principal plane is lightlike, but the cross-cap is not parabolic (i.e., it is elliptic or hyperbolic). Then the BDE* (16) *of the pre-lines of principal curvature of a cross-cap parametrised as in* Theorem 3.1(c) *is generically topologically equivalent to*

$(x^2 + xy^2 + y^3, xy, x^2 - 2y^2)$ *for an elliptic cross-cap*, Table 3 first figure in appropriate box

$(x^2 + xy^2 + y^3, xy, -x^2 - 2y^2)$ *for a hyperbolic cross-cap*, Table 3 second figure.

TABLE 3. Pre-Principal curves on cross-caps in $\mathbb{R}^3_1$.

Tg. line \ Pr. plane	Timelike	Lightlike	Spacelike
Timelike			
Lightlike			
Spacelike			

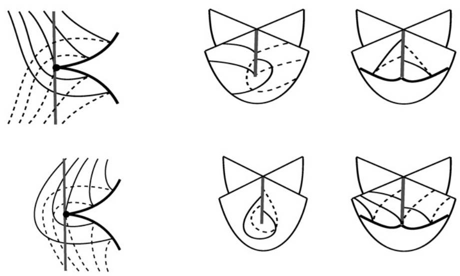


FIGURE 3. Pre-lightlike and lightlike curves (with the cross-cap viewed from two opposite drections) when the tangential line is lightlike and the principal plane is timelike (this is the D_4^- case, in the appropriate box, in Table 2). There are two possible configurations on the cross-cap depending of the relative position of the double point curve with the separatrice. In the top left figure, the double point curve intersects a given pre-lightlike curve only once whereas in the bottom left one it intersects it twice.

PROOF. We can make linear changes of coordinates and reduce the 3-jet the coefficients of the BDE (16) to

$$j^3A = x^2 + a_{30}x^3 + a_{31}x^2y + \frac{6(2q_{22}^3p_{20} - 2q_{22}^2q_{20} + q_{22}(q_{21}^2 + 2p_{20}) - 2q_{20})}{q_{21}^2 b}xy^2 + \frac{4(1+q_{22}^2)}{q_{21}b}y^3\,,$$

$$j^3B = 2xy + b_{30}x^3 + b_{31}x^2y + b_{32}xy^2 + b_{33}y^3\,,$$

$$j^3C = -\frac{(q_{22}p_{20} - q_{20})^2 + q_{21}^2p_{20}}{q_{21}^2}x^2 - 2y^2 + c_{30}x^3 + c_{31}x^2y + c_{32}xy^2 + c_{33}y^3$$

with the coefficients a_{ij}, b_{ij}, c_{ij} depending on the 3-jets of p and q in Theorem 3.1(c). The result follows by applying Theorem 7.4 in Section 7.

The genericity condition is $2q_{22}^3p_{20} - 2q_{22}^2q_{20} + q_{22}(q_{21}^2 + 2p_{20}) - 2q_{20} \neq 0$ (i.e., the coefficient of xy^2 in A is not zero, see the proof of Theorem 7.4). □

REMARK 6.6 (Configurations on the cross-cap). One can make a cross-cap (ignoring the metric) from a rectangular piece of paper as follows. Label one side of the paper A and the other B. Draw a line parallel to one side of the rectangle that divides the piece of paper into two equal rectangles. This line is the double point curve. Cut the piece of paper along half of the double point curve. Fold one free edge of the cut and seller tape it to the other fixed half of the double point curve on the side A of the paper. Take the remaining free edge and fold it along the fixed half of the double point curve on the side B of the paper.

When every pre-lightlike (resp. pre-principal) curve intersects the double point curve in at most one point, their images on the cross-cap do not self-intersect. Then one can draw the pre-lightlike (resp. pre-principal) curves on a piece of paper and determine by the above procedure the configurations of the lightlike (resp. principal) curves on the cross-cap itself (see the example in Figure 3 top figures). If there are pre-lightlike (resp. pre-principal) curves which intersect the double point curve twice (see the example in Figure 3 bottom figures) one needs to show that these two points are not mapped to the same image on the cross-cap (see for example [25] for proofs for some pairs of foliations on a cross-cap). We conjecture that this case for all the cases in this paper.

7. Normal forms of certain BDEs. We obtain here topological normal forms of BDEs needed in the previous sections. A germ of a BDE is an equation in the form

$$\omega:\ a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2 = 0\,,$$

where a, b, c are germs of smooth functions (say, at the origin) $\mathbb{R}^2, 0 \to \mathbb{R}$. We denote a BDE by $\omega = (a, b, c)$. BDEs are extensively studied with many applications including control theory and differential geometry; see for example [7, 16] and [23] for a survey article. A BDE determines a pair of transverse foliations away from the discriminant curve which is the set of points where the function $\delta = b^2 - ac$ vanishes. The pair of foliations together with the discriminant curve are called the *configuration* of the solutions of the BDE.

Following the notation in [12], let $f_i(w)$, $i = 1, 2$, denote the foliation associated to ω which is tangent to the vector field

$$\xi_i(\omega) = a\frac{\partial}{\partial u} + (-b + (-1)^i\sqrt{b^2 - ac}\,)\frac{\partial}{\partial v}\,.$$

If ψ is a diffeomorphism and $\lambda(x, y)$ is a non-vanishing real valued function, then ([12]) for $k = 1, 2$,

1. $\psi(f_k(w)) = f_k(\psi^*(\omega))$ if ψ is orientation preserving;
2. $\psi(f_k(w)) = f_{3-k}(\psi^*(\omega))$ if ψ is orientation reserving;
3. $f_k(\lambda w) = f_k(\omega)$ if $\lambda(x, y)$ is positive;
4. $f_k(\lambda w) = f_{3-k}(\omega)$ if $\lambda(x, y)$ is negative.

7.1. BDEs with 1-jet $(0, \pm y, x)$. We consider BDEs ω with 1-jet equivalent to $(0, \pm y, x)$ and whose discriminants have an A_2-singularity (see Section 5). We shall take $j^1\omega = (0, \pm y, x)$. Similar calculation to those carried out in [4, 5, 23] show that any k-jet, $k \geq 3$, of ω can be reduced by smooth changes of coordinates in $\mathbb{R}^2, 0$ and multiplication by a non-zero polynomial to one in the form

$$(M_1(x), \pm y, x + M_2(y))\,, \tag{20}$$

where $M_1(x) = a_2x^2 + a_3x^3 + \cdots + a_kx^k$ and $M_2(y) = b_3y^3 + \cdots + b_ky^k$. As the discriminant is supposed to have an A_2-singularity, $a_2 \neq 0$, so we can re-scale and set $a_2 = 1$.

THEOREM 7.1. *Suppose that $j^1\omega = (0, \varepsilon y, x)$, $\varepsilon = \pm 1$ and that the discriminant of ω has an A_2-singularity. Then ω is topologically determined by the 2-jet of its coefficients and is topologically equivalent to one of the following normal forms*

(i) (x^2, y, x) *Figure* 4, *bottom left*,
(ii) $(x^2, -y, x)$ *Figure* 4, *bottom right.*

PROOF. We consider the blowing-up $x = u$, $y = uv$ and $x = uv$, $y = v$.

The blowing-up $x = u$, $y = uv$:

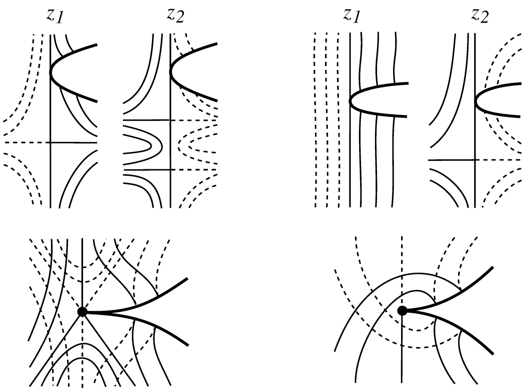


FIGURE 4. Configurations of the integral curves of a BDE ω with $j^1\omega = (0, \varepsilon y, x)$ and whose discriminant has an A_2-singularity, together with their associated blowing up models: $\varepsilon = -1$ left and $\varepsilon = 1$ right.

We have $dx = du$ and $dy = udv + vdu$. We denote $\omega_0 = (u, v)^*\omega$ the BDE obtained by applying the blowing-up transformation to ω. If $\omega = (a, b, c)$, then $\omega_0 = (\bar{a}, \bar{b}, \bar{c})$ is given by

$$\omega_0 : a(u, uv)(udv + vdu)^2 + 2b(u, uv)du(udv + vdu) + c(u, uv)du^2 = 0\,,$$

so that

$$\begin{aligned}\bar{a}(u, v) &= u^2 a(u, uv)\,,\\ \bar{b}(u, v) &= uva(u, uv) + ub(u, uv)\,,\\ \bar{c}(u, v) &= v^2 a(u, uv) + c(u, uv)\,.\end{aligned}$$

For ω as in (20) we have

$$\begin{aligned}\bar{a} &= u^2 M_1(u)\,,\\ \bar{b} &= uv(\varepsilon v + M_1(u))\,,\\ \bar{c} &= u + 2\varepsilon uv^2 + M_1(u)v^2 + M_2(uv)\,.\end{aligned}$$

We can write $\omega_0 = u(u^2 A_1, uB_1, C_1)$ with

$$\begin{aligned}A_1 &= uN_1(u)\,,\\ B_1 &= \varepsilon v + uvN_1(u)\,,\\ C_1 &= 1 + 2\varepsilon v^2 + u(N_1(u)v^2 + N_2(uv))\end{aligned}$$

where $M_1(u) = u^2 N_1(u)$ and $M_2(uv) = u^2 N_2(uv)$.

The quadratic form $\omega_1 = (u^2 A_1, uB_1, C_1)$ is a product of two 1-forms, and to these 1-forms are associated the vectors fields

$$Z_i = \Big(-uB_1 + (-1)^i u\sqrt{B_1^2 - A_1C_1}\Big)\frac{\partial}{\partial u} + C_1\frac{\partial}{\partial v}\,, \quad i = 1, 2\,.$$

The blowing-up transformation is orientation preserving if u is positive and orientation reserving if u is negative. As we factored out u once, it follows that Z_1 is tangent to the foliation associated to $f_1(\omega)$ and Z_2 is tangent to the foliation associated to $f_2(\omega)$.

The fields $Z_i, i = 1, 2$, are defined in the region where $B_1^2 - A_1C_1 > 0$. The set $B_1^2 - A_1C_1 = 0$ is a smooth curve tangent to the exceptional fibre at $u = 0$ and we have $(B_1^2 - A_1C_1)(0, v) = v^2$, so the whole exceptional fibre is an integral curve for both Z_1 and Z_2.

We study the vector fields Z_i in a neighbourhood of the exceptional fibre $u = 0$. The singularities of Z_i on $u = 0$ occur when $1 + 2\varepsilon v^2 = 0$. Thus, the vector fields Z_1 and Z_2 have singularities at $v = \pm\sqrt{2}/2$ when $\varepsilon = -1$ and have no singularities when $\varepsilon = 1$.

Consider $\varepsilon = -1$. At $v = \sqrt{2}/2$, we have $B_1(0, \sqrt{2}/2) = -\sqrt{2}/2$, so that

$$\begin{aligned}-uB_1 - u\sqrt{B_1^2 - A_1C_1} &= -uB_1 + uB_1\sqrt{1 - A_1C_1/B_1^2} \\ &= \frac{A_1C_1}{2B_1} + C_1^2 g(u, v)\end{aligned}$$

for some germ of a smooth function g with a zero 1-jet at the origin. Therefore Z_1 is singular along the curve $C_1(u, v) = 0$. We replace Z_1 with the vector field $\tilde{Z}_1 = Z_1/C_1$, which is regular along the exceptional fibre.

At $v = -\sqrt{2}/2$, the eigenvalues of the linear part of Z_1 are $2\sqrt{2}$ and $-\sqrt{2}$, so Z_1 has a saddle singularity at this point.

Similar calculations to those for Z_1 show that Z_2 has a saddle singularity at $v = \sqrt{2}/2$ and is regular at $v = -\sqrt{2}/2$.

The blowing-up $x = uv$, $y = v$:

This yields a new BDE $\omega_0 = (u, v)^*\omega = (\bar{a}, \bar{b}, \bar{c})$ with

$$\begin{aligned}\bar{a} &= u^3v + 2\varepsilon uv + u^2M_2(v) + M_1(uv)\,, \\ \bar{b} &= u^2v^2 + \varepsilon v^2 + uvM_2(v)\,, \\ \bar{c} &= uv^3 + v^2M_2(v)\,.\end{aligned}$$

We can write $\omega_0 = v(A_1, vB_1, v^2C_1)$ with

$$\begin{aligned}A_1 &= u^3 + 2\varepsilon u + v(u^2N_2(v) + N_1(uv))\,, \\ B_1 &= u^2 + \varepsilon + vuN_2(v)\,, \\ C_1 &= u + vN_2(v)\end{aligned}$$

where $M_1(uv) = v^2N_1(uv)$ and $M_2(v) = v^2N_2(v)$.

The quadratic form $\omega_1 = (A_1, vB_1, v^2C_1)$ is a product of two 1-forms, and to these 1-forms are associated the vectors fields

$$Z_i = \big(-B_1 + (-1)^i\sqrt{B_1^2 - A_1C_1}\big)\frac{\partial}{\partial u} + vC_1\frac{\partial}{\partial v}\,, \quad i = 1, 2$$

with

$$\begin{aligned}A_1 &= u^3 + 2\varepsilon u + v(N_1(uv) + u^2N_2(v))\,, \\ B_1 &= u^2 + \varepsilon + v(uN_2(v))\,, \\ C_1 &= u + vN_2(v)\,.\end{aligned}$$

We only need to study the vector fields Z_i at origin. Similar calculations to the first blowing-up show that Z_1 has a saddle singularity (resp. has no singularity) and Z_2 has no singularity (resp. has a saddle singularity) when $\varepsilon = -1$ (resp. $\varepsilon = 1$). Therefore, the integral curves of Z_1 and Z_2 are in as Figure 4, top figures. Blowing down yields the configuration in Figure 4, bottom figures. □

7.2. BDEs with 2-jet $(xy, \varepsilon x^2, -3\varepsilon xy + cy^2)$. We take $j^2\omega = (xy, \varepsilon x^2, -3\varepsilon xy + cy^2)$, $\varepsilon = \pm 1$. Then the 4-jet of the discriminant of ω is given by

$$x(x^3 + 3\varepsilon xy^2 - cy^3)\,.$$

The discriminant has an $X_{1,0}$-singularity if $c \neq 0, \pm 2$ when $\varepsilon = -1$, and if $c \neq 0$ when $\varepsilon = 1$. We write

$$\omega = (a, b, c) = (xy + M_1(x, y), \varepsilon x^2 + M_2(x, y), -3\varepsilon xy + cy^2 + M_3(x, y))\,, \tag{21}$$

where $M_i(x, y)$, $i = 1, 2, 3$, are germs of smooth functions with zero 2-jets at the origin. We set

$$\begin{aligned} j^3 M_1 &= a_{30}x^3 + a_{31}x^2y + a_{32}xy^2 + a_{33}y^3\,,\\ j^3 M_2 &= b_{30}x^3 + b_{31}x^2y + b_{32}xy^2 + b_{33}y^3\,,\\ j^3 M_3 &= c_{30}x^3 + c_{31}x^2y + c_{32}xy^2 + c_{33}y^3\,. \end{aligned}$$

THEOREM 7.2. *Suppose that $j^2\omega = (xy, \varepsilon x^2, -3\varepsilon xy + cy^2)$, $\varepsilon = \pm 1$, $c \neq 0, \pm 2$ if $\varepsilon = -1$, $c \neq 0$ if $\varepsilon = 1$. Suppose further that $(b_{33} - 2ca_{33}) \neq 0$. Then ω is topologically determined by the 2-jet of its coefficients and is topologically equivalent to one of the following normal forms*

(i) $(xy, -x^2, 3xy + 3y^2)$ Table 3 first figure in appropriate box,

(ii) $(xy, -x^2, 3xy + y^2)$ Table 3 second figure,

(iii) $(xy, x^2, -3xy + y^2)$ Table 3 third figure.

PROOF. We start with the case $\varepsilon = -1$. We consider the blowing-up $x = uv$, $y = v$ and $x = u$, $y = uv$.

The blowing-up $x = uv$, $y = v$:

This gives a new BDE $\omega_0 = (u, v)^*\omega = (\bar a, \bar b, \bar c)$ with

$$\begin{aligned} \bar a &= (u^3 + cu^2 + u)v^2 + M_3(uv, v)u^2 + 2M_2(uv, v)u + M_1(uv, v)\,,\\ \bar b &= (2u^2 + cu)v^3 + M_2(uv, v)v + M_3(uv, v)uv\,,\\ \bar c &= (3u + c)v^4 + M_3(uv, v)v^2\,. \end{aligned}$$

We can write $\omega_0 = v^2(A_1, vB_1, v^2C_1)$ with

$$\begin{aligned} A_1 &= u^3 + cu^2 + u + v(N_1(u, v) + 2N_2(u, v)u + N_3(u, v)u^2)\,,\\ B_1 &= u(2u + c) + v(N_2(u, v) + N_3(u, v)u)\,,\\ C_1 &= 3u + c + vN_3(u, v) \end{aligned}$$

where $M_i(uv, u) = v^2 N_i(u, v)$, $i = 1, 2, 3$. The quadratic form $\omega_1 = (A_1, vB_1, v^2C_1)$ is a product of two 1-forms, and to these 1-forms are associated the vectors fields

$$Y_i = A_1 \frac{\partial}{\partial u} + \big(- vB_1 + (-1)^i |v| \sqrt{(B_1^2 - A_1C_1)} \big) \frac{\partial}{\partial v}\,, \quad i = 1, 2\,. \tag{22}$$

Here, as we factored out v twice, it follows that Y_1 is tangent to the foliation associated to $f_1(\omega)$ if v is positive and to that associated to $f_2(\omega)$ if v is negative; while Y_2 is tangent to the foliation associated to $f_2(\omega)$ if v is positive and to $f_1(\omega)$ if v is negative.

We study the vector fields Y_i in a neighbourhood of the exceptional fibre $v = 0$. The fields Y_i are only defined in the regions where the discriminant $\delta = B_1^2 - A_1C_1 \geq 0$. On $v = 0$, this means that

$$u(u^3 - 3u - c) \geq 0\,.$$

The above segment of the exceptional fibre is an integral curve of both fields Y_i, $i = 1, 2$. The discriminant δ has two roots if $|c| > 2$ and four roots if $|c| < 2$.

We start with the case $|c| > 2$ and take $c > 2$ (the case $c < -2$ is topologically equivalent to the case $c > 2$). The singularities of Y_1 on $v = 0$ occur when $A_1(u, 0) = 0$, that is, when

$$u(u^2 + cu + 1) = 0\,.$$

Thus, Y_1 has singularities at $u = 0$ and $u_\pm = (-c \pm \sqrt{c^2 - 4})/2$.

At $u_+ = (-c + \sqrt{c^2 - 4})/2$, we have $B_1(u_+, 0) = \sqrt{c^2 - 4}\,(-c + \sqrt{c^2 - 4})/2 < 0$, so that

$$\begin{aligned} -vB_1 - |v|\sqrt{B_1^2 - A_1C_1} &= -vB_1 - |vB_1|\sqrt{1 - A_1C_1/B_1^2} \\ &= -vB_1 + |v|B_1 - \frac{A_1C_1|v|}{2B_1} + A_1^2 g(u, v) \end{aligned}$$

for some germ of a smooth function g with a zero 1-jet at the origin. When $v > 0$, Y_1 is singular along the curve $A_1(u, v) = 0$. We consider the vector field $\tilde{Y}_1 = Y_1/A_1$. Then $\tilde{Y}_1$ has no singularity. When $v < 0$, Y_1 has a saddle singularity at $(u_+, 0)$.

Similarly, Y_1 has a saddle singularity at $(u_-, 0)$ if $v > 0$ and no singularities if $v < 0$.

The singularity of Y_1 at $u = 0$ occur at the point of intersection of the exceptional fibre with the branches of the blown-up discriminant. We change variables and set $t = v$, $s^2 = B_1^2 - A_1C_1$, with $s \geq 0$. (The map $(u, v) \mapsto (s, t)$ is a fold map, so is a local diffeomorphism from the uper half-plane $s > 0$ to the set in the (u, v)-plane with $B_1^2 - A_1C_1 > 0$. It is a 1-1 map on the closure of these sets, i.e., including boundaries.)

The 2-jet of the vector field $(s, t)^* Y_1$ is equivalent to

$$\begin{aligned} &(-s + \Lambda_1 t)\frac{\partial}{\partial s} + (2ts)\frac{\partial}{\partial t} \quad \text{if} \quad v > 0 \\ &(s + \Lambda_1 t)\frac{\partial}{\partial s} + (2ts)\frac{\partial}{\partial t} \quad \text{if} \quad v < 0\,, \end{aligned} \tag{23}$$

where $\Lambda_1 = b_{33} - 2ca_{33}$. The singularity of $(s, t)^* Y_1$ is a saddle-node provided $\Lambda_1 \neq 0$, and its integral curves are as in Figure 5.

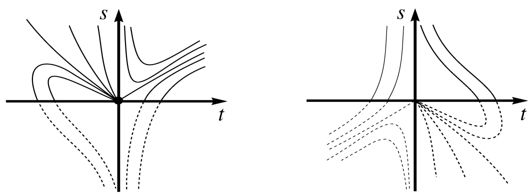


FIGURE 5. Integral curves of $(s,t)^*Y_1(s \geq 0)$, $\Lambda_1 > 0$ left, and $\Lambda_1 < 0$ right. (The continuous curves are those of interest as $s \geq 0$.)

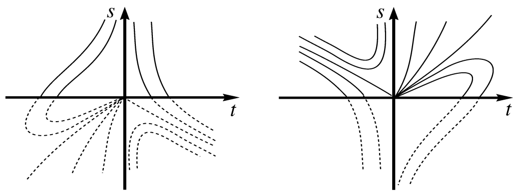


FIGURE 6. Integral curves of $(s,t)^*Y_2(s \geq 0)$, $\Lambda_1 > 0$ left, and $\Lambda_1 < 0$ right. (The continuous curves are those of interest as $s \geq 0$.)

The singularities of Y_2 on $v = 0$ occur when $A_1(u, 0) = 0$, that is, when

$$u(u^2 + cu + 1) = 0.$$

Therefore, Y_2 has singularities at $u = 0$ and $u_\pm = (-c \pm \sqrt{c^2 - 4})/2$.

At $u_+ = (-c + \sqrt{c^2 - 4})/2$, we have $B_1(u_+, 0) = \sqrt{c^2 - 4}\,(-c + \sqrt{c^2 - 4})/2 < 0$. Following the same arguments for Y_1, we can write

$$-vB_1 - |v|\sqrt{B_1^2 - A_1C_1} = -vB_1 + |v|B_1 - \frac{A_1C_1|v|}{2B_1} + A_1^2 g(u, v)$$

for some germ of a smooth function g with a zero 1-jet at the origin. When $v < 0$, Y_2 is singular along the curve $A_1(u, v) = 0$. We consider the vector field $\tilde{Y}_2 = Y_2/A_1$. Then $\tilde{Y}_2$ has no singularities. When $v > 0$, Y_2 has a saddle singularity at $(u_+, 0)$.

Similarly, Y_2 has a saddle singularity at $(u_-, 0)$ if $v < 0$ and no singularities if $v > 0$.

At $u = 0$, we change variables and set $t = v$, $s^2 = B_1^2 - A_1C_1$, with $s \geq 0$. The 2-jet of the vector field $(s,t)^*Y_2$ is equivalent to

$$(s + \Lambda_1 t)\frac{\partial}{\partial s} + (2ts)\frac{\partial}{\partial t} \quad \text{if} \quad v > 0$$

$$(-s + \Lambda_1 t)\frac{\partial}{\partial s} + (2ts)\frac{\partial}{\partial t} \quad \text{if} \quad v < 0 \tag{24}$$

where $\Lambda_1 = b_{33} - 2ca_{33}$, as for Y_1. The singularity of $(s,t)^*Y_2$ is a saddle-node provided $\Lambda_1 \neq 0$, and its integral curves are as in Figure 6.

The blowing-up $x = u$, $y = uv$:

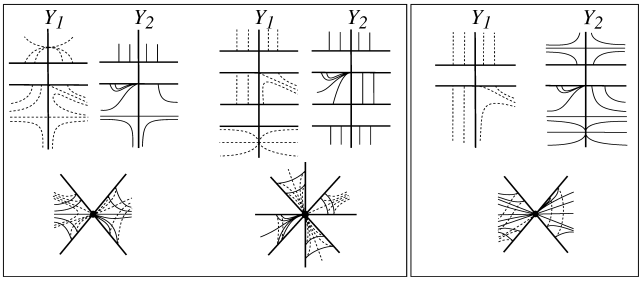


FIGURE 7. Configurations of the integral curves of the BDEs when $\varepsilon = -1$: $|c| > 2$ left, and $|c| < 2$ right and their associated blowing-up.

We take the coefficients of the BDE as in (21). The blowing-up yields new BDE given by $\omega_0 = u^2(u^2 A_1, uB_1, C_1)$ with

$$
\begin{aligned}
A_1 &= v + uN_1(u, uv)\,,\\
B_1 &= v^2 - 1 + u(N_1(u, uv)v + N_2(u, uv))\,,\\
C_1 &= v^3 + cv^2 + v + u(N_1(u, uv)v^2 + 2N_2(u, uv)v + N_3(u, uv))
\end{aligned}
$$

where $M_i(uv, u) = v^2 N_i(u, v)$, $i = 1, 2, 3$. The quadratic form $\omega_1 = (u^2 A_1, uB_1, C_1)$ is a product of two 1-forms, and to these 1-forms are associated the vectors fields

$$X_i = (u^2 A_1)\frac{\partial}{\partial u} + \big(-uB_1 + (-1)^i |u|\sqrt{(B_1^2 - A_1 C_1)}\big)\frac{\partial}{\partial v}\,, \quad i = 1, 2\,.$$

These vector fields are tangent to the foliations defined by ω_1. It is clear that we can factor out the term u in X_i, with an appropriate sign change when $u < 0$. The vector fields

$$Y_i = (uA_1)\frac{\partial}{\partial u} + \big(-B_1 + (-1)^i \sqrt{(B_1^2 - A_1 C_1)}\big)\frac{\partial}{\partial v}\,, \quad i = 1, 2\,,$$

are then considered. It is easy to see that Y_1 (resp. Y_2) has a node singularity (resp. has no singularities) at the origin.

We can now draw the integral curves of the fields Y_1 and Y_2, as illustrated in Figure 7, top figures, and blow down to obtain the configurations of the integral curves of the associated BDE (Figure 7, bottom figures, left box).

We consider now the case $|c| < 2$. The singularities of Y_i, $i = 1, 2$, on $v = 0$ occur only at $u = 0$. At $u = 0$, the vector fields Y_i have a saddle-node singularity as in (23) and (24). The configurations of the integral curves of Y_i are as in Figure 7 right, top figures (left box). Blowing-down yields the configuration of the integral curves of the original BDE.

The case $\varepsilon = 1$ follows similarly and the configuration of the integral curves is given in Figure 7, right box. $\square$

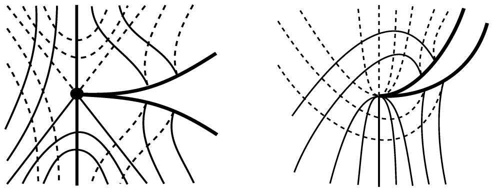


FIGURE 8. Configuration of the integral curves of a BDE topologically equivalent to $(y, -\frac{1}{2}x, xy + y^3 + y^4)$ right figure, and its associated blowing-up left figure.

7.3. BDEs with 1-jet $\left(0, -\frac{1}{2}x, y\right)$ and with a discriminant with an A_4-singularity.

THEOREM 7.3. *Suppose that $j^1\omega = (0, -\frac{1}{2}x, y)$ and that the discriminant has an A_4-singularity. Then ω is topologically determined by the 4-jet of its coefficients and is topologically equivalent to $(xy + y^3 + y^4, -\frac{1}{2}x, y)$, with configuration as in Figure 8.*

PROOF. One can reduce the k-jet of such BDEs to the form $j^k\omega = (a(x, y), -1/2x, y)$, with $j^1a = 0$. If we denote by a_{ij} the coefficient of $x^{i-j}y^j$ in the Taylor expansion of a, then the discriminant of ω has an A_4-singularity if and only if

$$a_{22} = 0\,, \quad a_{33} - a_{21}^2 = 0\,, \quad 4a_{20}a_{21}^2 + 2a_{32}a_{21} + a_{44} \neq 0\,.$$

We consider the blowing-up $x = uv$ and $y = v$. We get, after dividing by v^2 a BDE $(u, v)^*\omega/v^2$ whose discriminant has a cusp singularity (an A_2) at the origin. Both vector fields are regular along $v = 0$ and away from $u = 0$. The 2-jet of $(u, v)^*\omega/v^2$ at the origin is given by

$$(a_{21}u - a_{21}^2v + a_{20}u^2 + a_{32}uv + a_{44}v^2)dv^2 + ududv + vdu^2\,.$$

When $a_{21} \neq 0$, we can make changes of coordinates to show that the 2-jet of $(u, v)^*\omega/v^2$ is equivalent to

$$\left(\frac{2}{9a_{21}^2}(4a_{20}a_{21}^2 + 2a_{32}a_{21} + a_{44})u^2 + \alpha_1uv + \alpha_2v^2\right)dv^2 - 2vdudv + udu^2\,,$$

where α_1 and α_2 depend on the coefficients of c. It follows now from Theorem 3.3 in [24], that $(u, v)^*\omega/v^2$ is topologically equivalent to $u^2dv^2 - 2vdudv + udu^2 = 0$ and the configuration of its integral curves are as in Figure 8, left figure. (We observe that in Theorem 3.3 in [24], the 1-jet is taken in the form $(0, -v + b_2u, u)$ with $b_2 \neq 0$. In fact we can take $b_2 = 0$, the only difference in the proof there is that discriminant appears when blowing up in the v-direction, so the condition $b_2 \neq 0$ is redundant.)

Blowing up in the v-direction does not give any extra information. Therefore, the configuration of the BDE is as shown in Figure 8, right figure. The conditions on the coefficients of c that we need are $a_{21} \neq 0$ together with those above for the discriminant to have an A_4-singularity. We take $a_{21} = 1$ (so $a_{33} = 1$), $a_{44} = 1$ and set the remaining coefficients to be zero. □

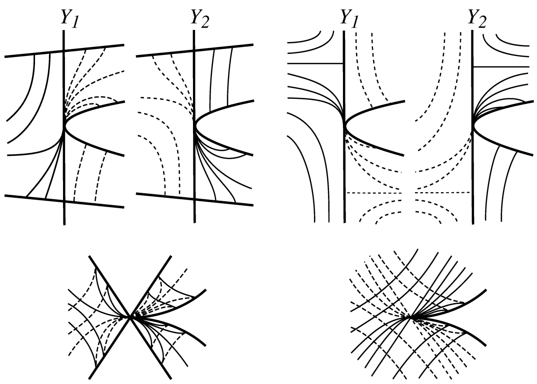


FIGURE 9. Configuration of the integral curves of a BDE topologically equivalent to $(x^2 + xy^2 + y^3, xy, x^2 - 2y^2)$ (resp. $(x^2 + xy^2 + y^3, xy, -x^2 - 2y^2)$) bottom left figure (resp. bottom right), and its associated blowing-up top figure.

7.4. BDEs with 1-jet $(x^2, xy, c_{20}x^2 - 2y^2)$ and whose discriminant has an $X_{1,1}$-singularity.

THEOREM 7.4. *Suppose that the $j^2\omega = (x^2, xy, c_{20}x^2 - 2y^2)$ and that its discriminant has an $X^{\pm}_{1,1}$-singularity at the origin. Then the BDE is locally topologically determined by its 3-jet and is topologically equivalent to*

$$X^-_{1,1}: \quad (x^2 + xy^2 + y^3, xy, x^2 - 2y^2), \quad \textit{Figure } 9, \textit{ bottom left},$$
$$X^+_{1,1}: \quad (x^2 + xy^2 + y^3, xy, -x^2 - 2y^2), \quad \textit{Figure } 9, \textit{ bottom right}.$$

PROOF. We write the coefficients of the BDE $\omega = (a, b, c)$ in the form

$$\begin{aligned} a &= x^2 + a_{30}x^3 + a_{31}x^2y + a_{32}xy^2 + a_{33}y^3 + O(3), \\ b &= \tfrac{1}{2}(2xy + b_{30}x^3 + b_{31}x^2y + b_{32}xy^2 + b_{33}y^3 + O(3)), \\ c &= c_{20}x^2 - 2y^2 + c_{30}x^3 + c_{31}x^2y + c_{32}xy^2 + c_{33}y^3 + O(3). \end{aligned}$$

The discriminant $b^2 - ac$ has an $X_{1,1}$-singularity if and only if $c_{20} \neq 0$ and $a_{33} \neq 0$. We have an $X^+_{1,1}$(resp. $X^-_{1,1}$) if and only if $c_{20} < 0$ (resp. $c_{20} > 0$).

We proceed as in the previous subsection. (The blowing-up $x = u$ and $y = uv$ does not give any extra information). With blowing up $x = uv$ and $y = v$, we get a new BDE $(u, v)^*\omega/v^2 = (A, vB, v^2C)$ with,

$$\begin{aligned} A &= a_{33}v + u^2 + (a_{32} + b_{33})uv + a_{44}v^2 + O(3), \\ B &= \tfrac{1}{2}(-2u + b_{33}v + (2c_{33} + b_{32})uv + b_{44}v^2 + O(3)), \\ C &= -2 + c_{33}v + c_{20}u^2 + c_{32}uv + a_{44}v^2 + O(3). \end{aligned}$$

The discriminant is given by $v^2(B^4 - AC)$ with

$$B^2 - AC = 8a_{33}v - 4u^2(c_{20}u^2 - 3) + vg(u, v)\,,$$

where g a smooth function with a zero 1-jet. Thus, on the exceptional fibre $v = 0$ there is one smooth component of the blown up discriminant at $u = 0$ which has an ordinary tangency with the exceptional fibre provided $a_{33} \neq 0$ and two extra smooth branches transverse to the exceptional fibre at $u^{\pm} = \pm\sqrt{3/c_{20}}$ if $c_{20} > 0$ (i.e. when the discriminant of ω has an $X^{-}_{1,1}$) and no extra branches if $c_{20} < 0$ (see Figure 9, top figures).

We consider the vector fields Y_i, $i = 1, 2$, as in (22). They are singular on the exceptional fibre (away from $u = 0$, and $u^{\pm}$) occur when $A(u, 0) = u^2(c_{20}u^2 + 1) = 0$. When the discriminant of ω has an $X^{-}_{1,1}$-singularity, the vector fields Y_i, $i = 1, 2$, have no singularities on the exceptional fibre away from $u = 0$. At $u^{\pm}$, they are transverse to the blown-up branches of the discriminant (see Figure 9, first two top left figures).

When the singularity is an $X^{+}_{1,1}$ the vector fields Y_1 has a saddle singularity on the exceptional fibre at $u = -\sqrt{-c_{20}}$ (resp. $u = 1/\sqrt{-c_{20}}$) when $v \geq 0$ (resp. $v \leq 0$) and is regular at $u = -\sqrt{-c_{20}}$ (resp. $u = 1/\sqrt{-c_{20}}$) when $v \leq 0$ (resp. $v \geq 0$). The vector field Y_2 has mirror image behaviour to Y_1 with respect to the exceptional at $\pm 1/\sqrt{-c_{20}}$. As the blowing up is orientation preserving if $v > 0$ and orientation reversing if $v < 0$, the configuration of the foliations associated to Y_i are as Figure 9.

We can now blow down $(u, v)^*\omega/v^2$ to obtain the configuration of the integral curves of ω (Figure 9). □

At $u = 0$, we proceed as in the proof of Theorem 7.2. By the implicit function theorem, $(B^2 - 4AC)(u, v) = s^2$ gives $v = g(u, s)$. We make the change of variables $u = t$ and $v = g(t, s)$, with $s \geq 0$, in Y_i, $i = 1, 2$, at $u = v = 0$. The difference here with the case of Theorem 7.2 is that the new vector fields $(t, s)^*Y_i$ have more degenerate singularities at $t = s = 0$. In fact,

$$j^3(t, s)^*Y_1 = \frac{1}{16a_{33}}s(s - 2t)(s + 2t)\frac{\partial}{\partial t} + \frac{1}{8a_{33}}(s - 2t)(-s^2 + 6st + 24t^2)\frac{\partial}{\partial s}\,,$$
$$j^3(t, s)^*Y_2 = \frac{1}{16a_{33}}s(s - 2t)(s + 2t)\frac{\partial}{\partial t} + \frac{1}{8a_{33}}(s + 2t)(s^2 + 6st - 24t^2)\frac{\partial}{\partial s}\,.$$

We blow up the singularity at the origin ($t = T, s = ST$) and obtain the configuration in Figure 10. On the exceptional fibre $T = 0$, $(T, S)^*((s, t)^*Y_1)$ has a saddle singularity at $S = \sqrt{12}$ and $S = -4$, a node singularity at $-\sqrt{12}$ and a saddle-node singularity at $S = 2$ provided that $a_{32} \neq 0$. The vector field $(T, S)^*((s, t)^*Y_2)$ has a saddle singularity at $S = -\sqrt{12}$ and $S = 4$, a node singularity at $\sqrt{12}$ and a saddle-node singularity at $S = -2$ provided that $a_{32} \neq 0$. Blowing down gives the configuration of $(s, t)^*Y_i$. Blowing down again gives the configuration $(u, v)^*\omega/v^2$ as in Figure 10.

We can now blow down $(u, v)^*\omega/v^2$ to obtain the configuration of the integral curves of ω (Figure 9). □

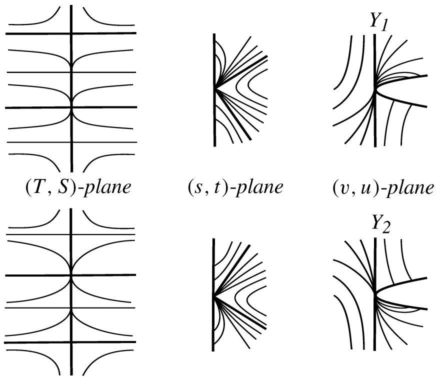


FIGURE 10. Change of variable followed by a blowing up of the singularities of Y_i, $i = 1, 2$ at $u = v = 0$.

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